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ECO6108A/SYS5140A Economic System Design
Professor: Nguyen Van Quyen
Assignment 1
Due Date: 30 September, 2019

1. Open Seeds Ltd has developed a new variety of plant and is now planning to sell its seeds for the first time. Each year a small proportion of the seeds produced will be retained for production of more seeds. In the earlier stages of development, there is a trade-off between retaining a larger proportion of the seeds harvested and a smaller sales volume or retaining a smaller proportion of the seeds harvested, but a larger sales volume.

The plant is an annual, sown in the spring and harvested in the fall. Each plant is obtained from one seed, and produces γ seeds upon harvest. A small amount of seeds has been developed and tested to start production. The development cost was sunk. All the seeds produced within a growing season will be sold or planted to produce seeds for the following season. Suppose that the marketing plan stretches over m growing seasons, say 0, 1, ..., $m-1$. Here m can be interpreted as the number of years that the seeds are popular, and that at the end of year $m-1$ the plant developed by Open Seeds Ltd will be supplanted by a new competitor.

Let x_t be the amount of seeds sown in season t, $t = 0, \ldots, m-1$. The harvest at the end of season t is γx_t. If α_t is the proportion of the harvest retained for sowing in the next season, then the volume of seed sales in period t is $(1-\alpha_t)\,\gamma\, x_t$, and the profit obtained at the end of that growing season is $\pi_t = p\,(1-\alpha_t)\,\gamma\, x_t - c\, x_t$, where p is the price of seeds and c is the unit cost of sowing seeds.

The problem of Open Seeds Ltd is to find a sequence $(\alpha_t)_{t=0,\ldots,m-1}$ to maximize the following discounted profit: $\sum_{t=0}^{m-1} \frac{\pi_t}{(1+r)^t}$. Here r is the market rate of interest.

The values of the parameters are

$\gamma = 8,\ c = 4,\ p = 1,\ r = 0.05.$

Also, the initial stock of seeds available for sowing is $x_0 = 1$, and the time horizon is $m = 4$.

(a) *Use dynamic programming* to solve this profit maximization problem.
(b) Find the shadow price of seeds at the beginning of each period along the optimal trajectory.

```
In[ ]:= data = {γ → 8, c → 4, p → 1, r → 0.05}
Out[ ]= {γ → 8, c → 4, p → 1, r → 0.05}

In[ ]:= v[3] = -c x[3] + p (1 - α[3]) γ x[3]
Out[ ]= -c x[3] + p γ x[3] (1 - α[3])

In[ ]:= α[3] = 0
Out[ ]= 0
```

```
In[]:= v[3] = v[3] /. data
Out[]= 4 x[3]

In[]:= λ[3] = D[v[3], x[3]]
                偏导
Out[]= 4

In[]:= x[3] = α[2] γ x[2]
Out[]= γ x[2] α[2]

In[]:= v[3]
Out[]= 4 γ x[2] α[2]
```

In[]:= $f[2] = -c\,x[2] + p\,(1 - \alpha[2])\,\gamma\,x[2] + \frac{1}{1+r}\,v[3]$

Out[]= $-c\,x[2] + p\,\gamma\,x[2]\,(1 - \alpha[2]) + \frac{4\,\gamma\,x[2]\,\alpha[2]}{1+r}$

```
In[]:= f[2] = f[2] /. data
Out[]= -4 x[2] + 8 x[2] (1 - α[2]) + 30.4762 x[2] α[2]

In[]:= f[2] = f[2] // FullSimplify
                      完全简化
Out[]= x[2] (4. + 22.4762 α[2])

In[]:= α[2] = 1
Out[]= 1

In[]:= v[2] = f[2]
Out[]= 26.4762 x[2]
```

In[]:= $\lambda[2] = \partial_{x[2]}\,v[2]$

Out[]= 26.4762

```
In[]:= x[2] = α[1] γ x[1]
Out[]= γ x[1] α[1]
```

In[]:= $f[1] = -c\,x[1] + \frac{1}{1+r}\,v[2]$

Out[]= $-c\,x[1] + \frac{26.4762\,\gamma\,x[1]\,\alpha[1]}{1+r}$

```
In[]:= f[1] = f[1] /. data
Out[]= -4 x[1] + 201.723 x[1] α[1]

In[]:= α[1] = 1
Out[]= 1
```

In[∘]:= `v[1] = f[1]`

Out[∘]= $197.723\, x[1]$

In[∘]:= $\lambda[1] = \partial_{x[1]} v[1]$

Out[∘]= 197.723

In[∘]:= $x[1] = \alpha[0]\, \gamma\, x[0]$

Out[∘]= $\gamma\, x[0]\, \alpha[0]$

In[∘]:= $f[0] = -c\, x[0] + \frac{1}{1+r} v[1]$

Out[∘]= $-c\, x[0] + \frac{197.723\, \gamma\, x[0]\, \alpha[0]}{1+r}$

In[∘]:= `f[0] = f[0] /. data`

Out[∘]= $-4\, x[0] + 1506.46\, x[0]\, \alpha[0]$

In[∘]:= $\alpha[0] = 1$

Out[∘]= 1

In[∘]:= `v[0] = f[0]`

Out[∘]= $1502.46\, x[0]$

In[∘]:= $\lambda[0] = \partial_{x[0]} v[0]$

Out[∘]= 1502.46

In[∘]:= `Table[α[i], {i, 0, 3}]`
表格

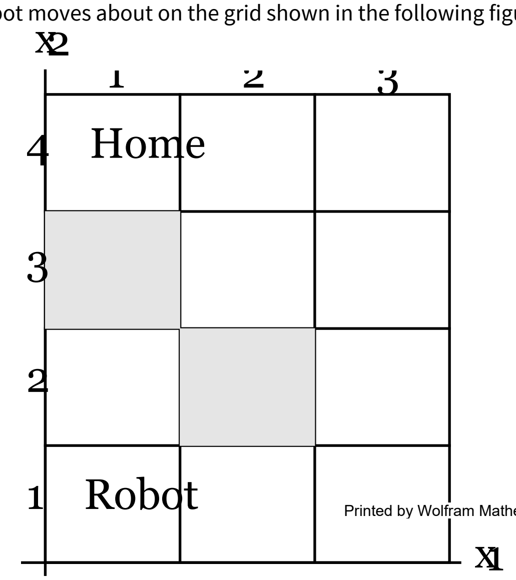
Out[∘]= {1, 1, 1, 0}

In[∘]:= `Table[λ[i], {i, 0, 3}]`
表格

Out[∘]= {1502.46, 197.723, 26.4762, 4}

In[∘]:= `ClearAll[data, f, v, λ, α, x]`
清除全部

2. A robot moves about on the grid shown in the following figure.



The robot is attempting to find its way in this environment to the "home square" located in the upper left-hand corner. There are two shaded squares that may or may not contain barriers that the robot is not allowed to pass through. Initially, the robot does not know whether or not the barriers are actually present. So part of its control strategy is to learn about the presence or absence of these barriers. The other part, of course, is to move toward Home.

At the beginning, the robot knows that there is a probability of 0.4 that a barrier exists in the square located at $x_1 = 1,\ x_2 = 3$ and that there is a probability of 0.5 that a barrier exists at $x_1 = 2,\ x_2 = 2$.

The robot can always see one move ahead; that is, if the robot is within one move of a barrier location, it can always determine with certainty whether or not a barrier is there. For a price of 0.3 moves, the robot can make an observation of all the squares that are 2 moves away, where a move is defined to be either one horizontal or one vertical square away from the robot's current location. In other words, the robot can move or observe diagonally only in 2 moves. The robot's objective is to get to the Home square while minimizing the expected value of the sum of actual moves and penalties for observation.

Work out the optimal control policy for the robot, assuming that at the beginning the robot finds itself at the co-ordinates $x_1 = 1,\ x_2 = 1$.

```
In[]:= p[0, 0] = (1 - 0.4) (1 - 0.5)
Out[]= 0.3

In[]:= p[1, 0] = 0.4 (1 - 0.5)
Out[]= 0.2

In[]:= p[0, 1] = (1 - 0.4) 0.5
Out[]= 0.3

In[]:= p[1, 1] = 0.4 × 0.5
Out[]= 0.2

In[]:= v[right] = 0.5 × 7 + 0.5 × 5
Out[]= 6.

In[]:= v[up, 1] = 3
Out[]= 3

In[]:= v[up, 2] = 5
Out[]= 5

In[]:= v[up, 3] = 3
Out[]= 3

In[]:= v[up, 4] = 9
Out[]= 9

In[]:= v[up] = p[0, 0] v[up, 1] + p[1, 0] v[up, 2] + p[0, 1] v[up, 3] + p[1, 1] v[up, 4]
Out[]= 4.6
```

In[ ]:=
```
v[observe, 1] = 3.3
```
Out[ ]=
```
3.3
```
In[ ]:=
```
v[observe, 2] = 5.3
```
Out[ ]=
```
5.3
```
In[ ]:=
```
v[observe, 3] = 3.3
```
Out[ ]=
```
3.3
```
In[ ]:=
```
v[observe, 4] = 7.3
```
Out[ ]=
```
7.3
```
In[ ]:=
```
v[observe] =
 p[0, 0] v[observe, 1] + p[1, 0] v[observe, 2] + p[0, 1] v[observe, 3] + p[1, 1] v[observe, 4]
```
Out[ ]=
```
4.5
```
In[ ]:=
```
v[up]
```
Out[ ]=
```
4.6
```
In[ ]:=
```
v[right]
```
Out[ ]=
```
6.
```
In[ ]:=
```
ClearAll[p, v]
```
清除全部

The expected cost is 6 moves if the robot chooses to move horizontally to the right. The expected cost is 4.6 moves if the robot chooses to move vertically upward. If the robot observes before moving, then it can expect to pay 4,
如果
5 moves to get home. Thus the optimal decision for the robot is to observe first, and then depending on the result of observation, chooses the path home that has the least cost.

3. A woman has 5 years to find a husband. In year 0, she goes out and finds a man who asks for her hand. Let x_0 be the quality of the suitor. If she accepts the marriage proposal, then the problem ends. If she rejects the marriage proposal, then she has to wait until next year to find another man. Let x_1 be the quality of the man who proposes marriage to her in year 1. If she accepts the marriage proposal, then the problem ends. Otherwise, she has to wait one more year to receive a marriage proposal in year 2...

Let x_t, $t = 0, \ldots, 4$, be the quality of the man who makes the marriage proposal in year t. Suppose that x_t, $t = 0, \ldots, 4$, are independent random variables drawn from the uniform distribution on the unit interval [0, 1], and that the uncertainty concerning x_t is only resolved in year t.

Let ξ_t, $t = 0, \ldots, 4$, denote the reservation quality for the suitor in year t. That is, ξ_t is the critical quality in year t such that the woman only accepts the marriage proposal in that year if the quality of the suitor is at least ξ_t. Otherwise, if $x_t < \xi_t$, then the suitor in year t whose quality is x_t will be rejected, and the woman will wait for the following year in the hope of finding a better suitor. Find the sequence of reservation qualities ξ_t, $t = 0, \ldots, 4$.

In[ ]:=
```
ξ[4] = 0
```
Out[ ]=
```
0
```

In[∘]:= `v[4] = Max[x[4], ξ[4]]`
最大值

Out[∘]= `Max[0, x[4]]`

In[∘]:= $\xi[3] = \int_0^\infty e^{-x[4]}\, x[4]\, dx[4]$

Out[∘]= 1

In[∘]:= `v[3] = Max[x[3], ξ[3]]`
最大值

Out[∘]= `Max[1, x[3]]`

In[∘]:= $\xi[2] = \int_0^\infty e^{-x[3]}\, v[3]\, dx[3]$

Out[∘]= $\frac{1 + e}{e}$

In[∘]:= $f[a\_] := a + \frac{1}{e^a}$

In[∘]:= `NestList[f, ξ[4], 4]`
嵌套列表

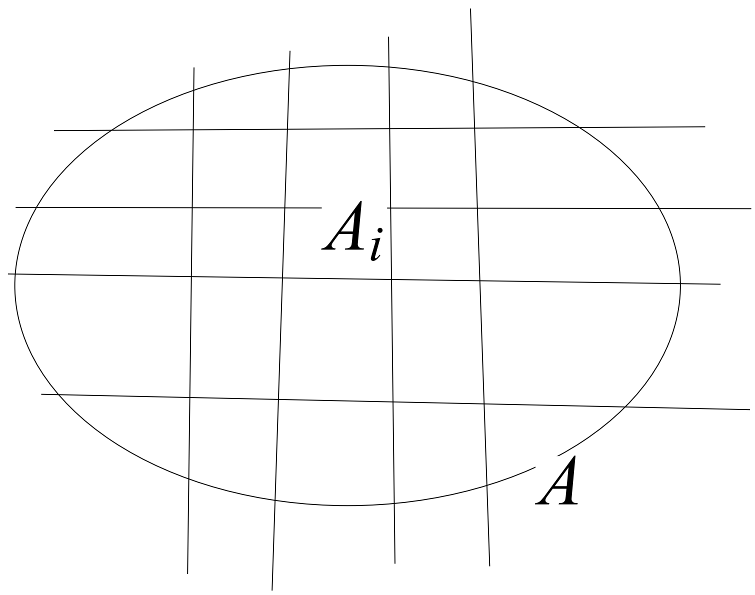
Out[∘]= $\left\{0, 1, 1 + \frac{1}{e}, 1 + \frac{1}{e} + e^{-1-\frac{1}{e}}, 1 + \frac{1}{e} + e^{-1-\frac{1}{e}} + e^{-1-\frac{1}{e}-e^{-1-\frac{1}{e}}}\right\}$

In[∘]:= `% // N`
数值运算

Out[∘]= `{0., 1., 1.36788, 1.62253, 1.81993}`

4. A jet liner went missing when it was flying over the Mediterranean sea, and is suspected to have crashed into the sea. A search to find the wreckage is being conducted.

Based on the information about the flight, the search is concentrated in an area *A* of the Mediterranean sea. The following figure depicts the search area.



The search area is partitioned into N cells: $A_1, A_2, \ldots, A_N$. Let a denote the total effort devoted to the search. Here the search is carried out by sending sonar signals down the ocean floor, and a can be interpreted as the total amount of time spent searching for the missing jet liner. A search program is a list $(x_1, \ldots, x_i, \ldots x_N)$ that satisfies the constraint $x_1 + \ldots + x_N = a$, where x_i is the time spent in cell A_i.

Let p_i denote the probability that the jet liner is under water in cell A_i. Presumably, the probabilities $p_1, \ldots, p_i, \ldots, p_N$ are the prior probabilities obtained by using the information from the radars of the air traffic control centers that were communicating with the pilots of the jet liner before it disappeared from the radar screen. If the missing jet liner is under water in cell A_i, and if x_i is the effort spent searching in this cell, then the probability of discovering the wreckage, given that it lies hidden in cell A_i, is assumed to be given by $1 - e^{-\lambda_i x_i}$. Here $\lambda_i > 0$ is a parameter that represents the search technology and the characteristics of cell A_i. The probability of discovering the wreckage is then given by $\sum_{i=1}^{N} p_i \left(1 - e^{-\lambda_i x_i}\right)$.

(a) Find the optimal search program, i.e., the search program that maximizes the probability of discovering the wreckage.
(b) What is the shadow price of effort?

$$\max \sum_{i=1}^{N} p_i \left[1 - e^{-\lambda_i x_i}\right]$$

$$\underbrace{\text{F.O.C}}_{\because \underbrace{}_{\text{常量}}} \Longrightarrow \sum_{i=1}^{N} 1 - e^{-\lambda_i x_i}$$

$$1 = \sum_{i=1}^{N} e^{-\lambda_i x_i}$$

since $x_1 + ... + x_N = a$

$$1 = \sum_{i=1}^{N} e^{-\lambda_i a}$$

$$\frac{1}{a} = \sum_{i=1}^{N} e^{-\lambda_i}$$

$$\sum_{i=1}^{N} -\lambda_i = \underbrace{\text{Log}}_{\text{对数}}\left[\frac{1}{a}\right] = \underbrace{\text{Log}}_{\text{对数}}\left[\frac{1}{x_1 + ... + x_N}\right]$$

5. Consider a polluted lake with volume V expressed in m^3. The rate of water outflow is r, assumed to be constant. Assume that the rate of water inflow is equal to the rate of water outflow so that the volume of the lake remains constant.

Let x_t, $t \geq 0$, denote the pollutant concentration in the lake at time t, with x_t being the number of grams of pollutants in one m^3 of water. The mass of pollutants in the lake at time t is then given by $x_t V$. The pollutant concentration at time t, namely x_0, is known.

Suppose that at time $t = 0$ the activities that pollute the lake is stopped. As time goes by, the concentration of the pollutants in the lake, which is reduced by the water outflow, will decline. The water inflow - runoff, snow melt - still continue to maintain the volume of the lake at V. The differential equation that governs the evolution of the stock of pollutants in the lake is

$$\frac{d(x_t V)}{dt} = -r x_t,$$

x_0 is known.

(a) Solve the preceding differential equation for x_t.

(b) You are given the following data on the volume and the rate of water outflows of the three Great Lakes:

$$\begin{pmatrix} \text{Lake} & V(\text{in } m^3) & r(\text{in } m^3/\text{day}) \\ \text{Erie} & 458 \times 10^9 & 479\,582\,208 \\ \text{Michigan} & 4871 \times 10^9 & 433\,092\,096 \\ \text{Superior} & 12\,221 \times 10^9 & 178\,619\,904 \end{pmatrix}.$$

How long does it take for the pollutant concentration in each each lake to fall to 5 % of its initial value?

```
In[•]:= s[0] = DSolve[{V x'[t] == -r x[t], x[0] == x0}, x, t]
               求解微分方程
        s[1] = DSolve[{V x'[t] == -r x[t], x[0] == x0}, x, t] // Flatten
               求解微分方程                                       压平
        x[t] /. s[1]
        x[0] /. s[1]
        eq[0] = 0.05 x0 == (x[t] /. s[1])
        s[2] = Solve[eq[0], t, Reals] // Flatten
               解方程          实数域    压平
        s[2] /. {V → 458 × 10^9 , r → 479 582 208 }
        s[2] /. {V → 4871 × 10^9 , r → 433 092 096 }
        s[2] /. {V → 12 221 × 10^9 , r → 178 619 904 }
        ClearAll[s, eq]
        清除全部
```

Out[•]= $\left\{\left\{x \to \text{Function}\left[\{t\}, e^{-\frac{r t}{V}} x0\right]\right\}\right\}$

Out[•]= $\left\{x \to \text{Function}\left[\{t\}, e^{-\frac{r t}{V}} x0\right]\right\}$

Out[•]= $e^{-\frac{r t}{V}} x0$

Out[•]= $x0$

Out[•]= $0.05\, x0 == e^{-\frac{r t}{V}} x0$

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[•]= $\left\{t \to \frac{2.99573\, V}{r}\right\}$

lake Erie : {t → 2860.9180207280083` }

lake Michigan : {t → 33693.09216042929` }

lake Superior : {t → 204965.08673021864` }