A Tutorial on EM for HMM

A try on YuChen's LaTeX Beamer template

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Abstract

A Tutorial on the Expectation Maximization(EM) Alogrithm for Hidden Markov Model with single Guassian. Nearly all material of this tutorial is from KaiYu's thesis 'Adaptive Training for Large Vocabulary Continuous Speech Recognition'. If you have a problem during the presentation, please ask directly. I hope that I didn't make a mistake and at the end everyone will understand as much as I do.

A quick introduction to MLE

We use a Guassian distrubution

$$\mathcal{N}(x|\mu,\Sigma) = \frac{1}{(2\pi)^{D/2}}\frac{1}{|\Sigma|^{1/2}}exp\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\} \quad (1)$$

We want to estimate the parameters μ and Σ, however, we only have some observance $x_1, x_2, ..., x_N$ that are drawn independently from the distribution.

A quick introduction to MLE

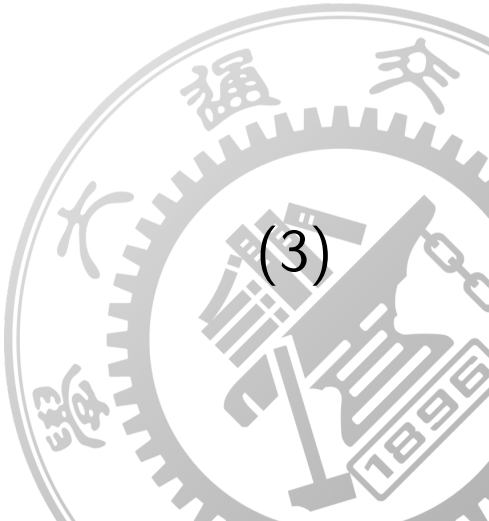
MLE means maximum likelyhood estimation. We maximize $\prod_i \mathcal{N}(x_i|\mu, \Sigma)$. This is equivalent to maximize the log likelyhood function.(Note that now I only consider μ, so others are considered constant and thrown away)

$$M(\mu, \Sigma) = log \prod_i \mathcal{N}(x_i|\mu, \Sigma) = \sum_i (-\frac{1}{2}(x_i - \mu)^T \Sigma^{-1}(x_i - \mu)) \quad (2)$$

To maximize with respect to μ,we set $\frac{\partial M}{\partial \mu} = 0$, that is

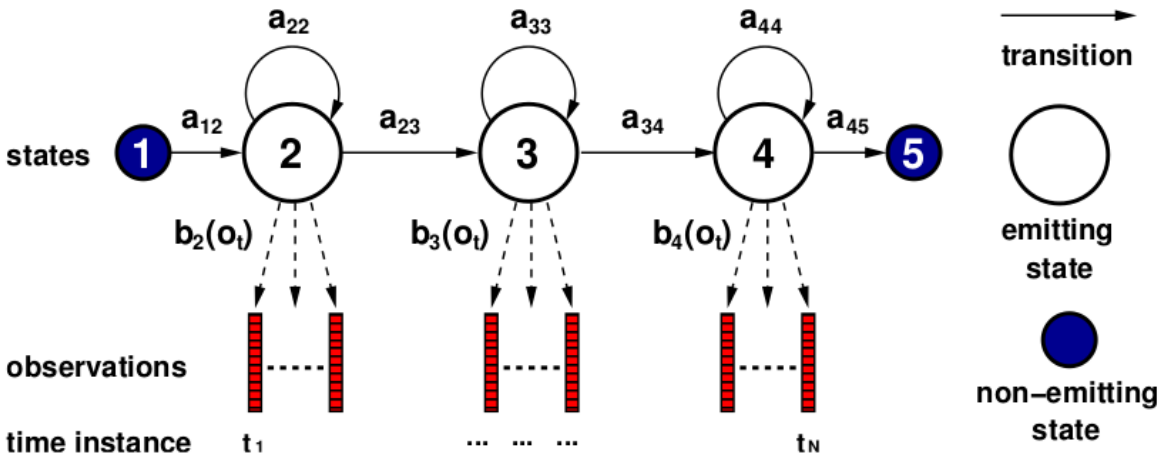
$$\sum_{i=1}^{N} \Sigma^{-1}(x_i - \mu) = 0 \quad (3)$$

$$\hat{\mu}_{ML} = \frac{1}{N} \sum_{i=1}^{N} x_i$$



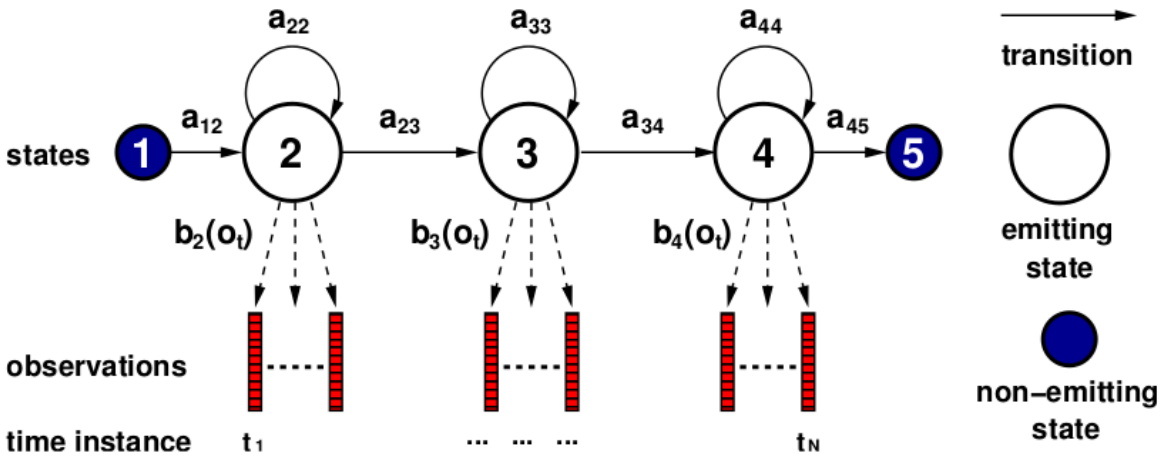
For maximizing with respect to Σ,I encourage readers to PRML $2.3.4.

A quick introduction to Hidden Markov Model



We use a HMM to model a *context dependent phone*. a_{ij} is the state transition probability. $b(o)$ is a single Guassian distribution that models the observance that this state.
We call $O = (o_1, o_2, ..., o_T)$ the observance of a HMM, and $w = (w_1, w_2, ..., w_T)$ the hidden state sequence.

HMM for a context dependent phone



For example, If we want to model the *wa* in $ae + wa + n + t$ for 'I want'. We create a HMM called 'ae + wa + n', where the 'wa' is called the center phone. State 2 for ae, State 3 for wa, and State 4 for n.

A quick introduction to Hidden Markov Model

composite HMM

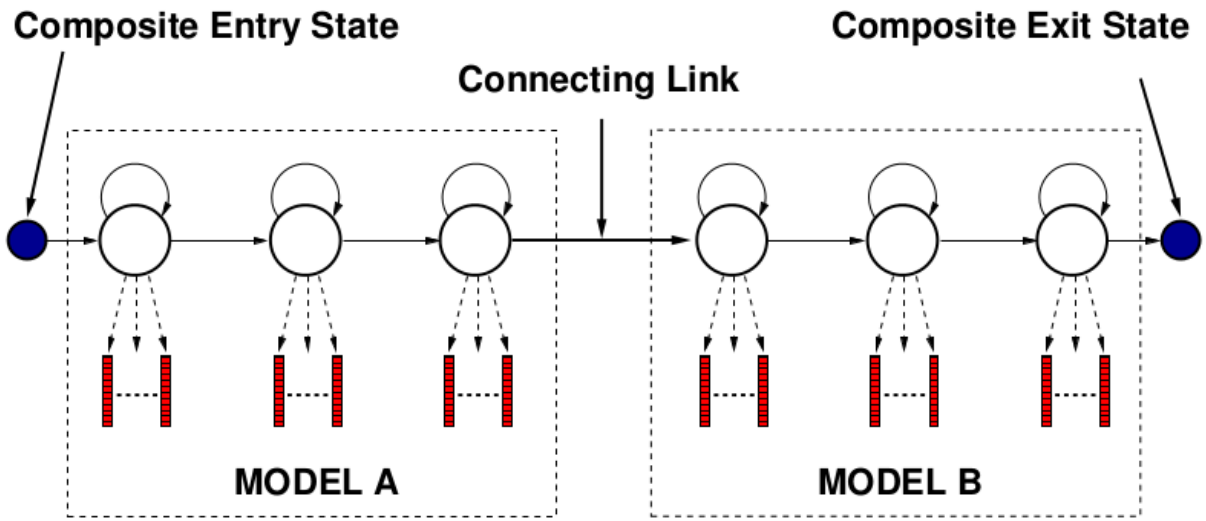
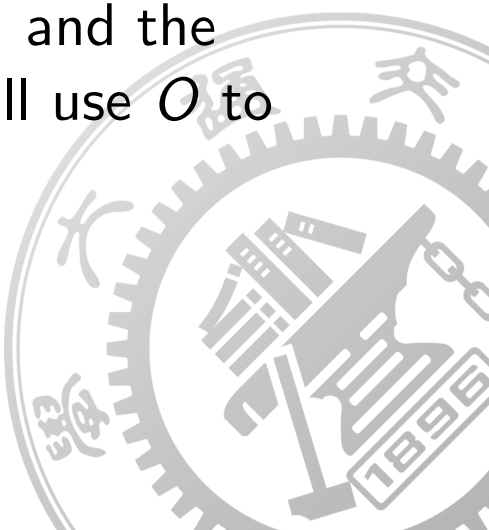


Figure 2.2 *A composite HMM constructed from two individual HMMs*

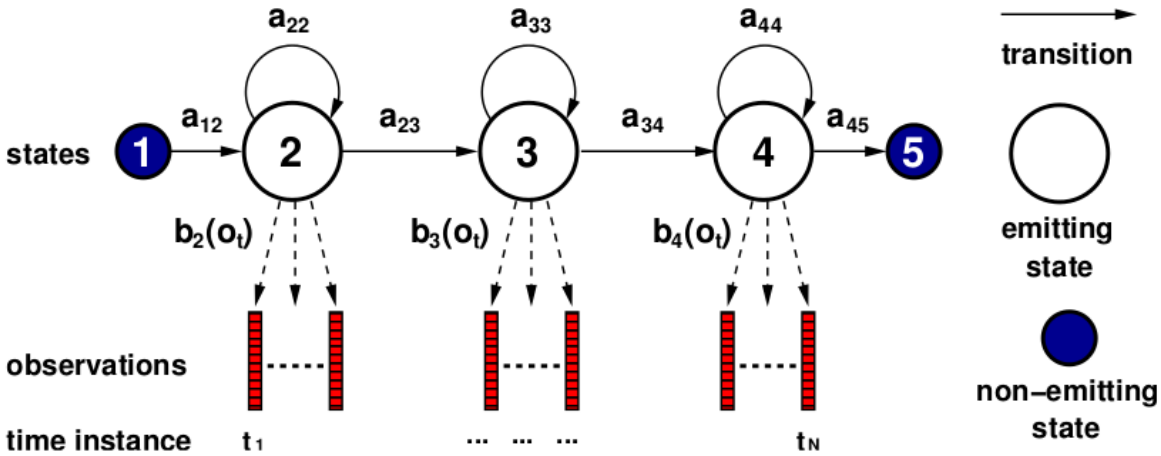
We can easily link HMM models to form a composite HMM. And we can use composite HMM to model a sentence.

Out data

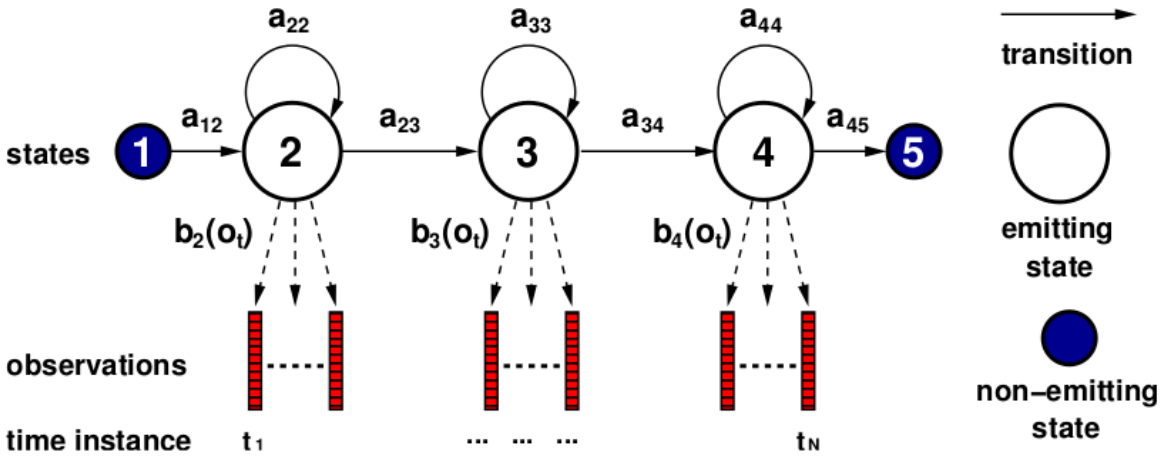
sentence: I want an apple.
phone:sil ae w an t ae n ae pl e sil
wave:...!..!!!.....!...!!!.....!....
We can link 9 HMMs(one for each context dependent phone) together to model this sentence. And use the wave to train our models.
From now on, I will forget about the composite HMM and the data for complicity. I just have observance O and I will use O to trian a HMM by EM.



Notations



M means all the parameters on the HMM, including the transition probability a_{ij} and μ_j and Σ_j for the Guassian distribution b_j.
EM is a iterative method, M_k means the result we get in the kth iteration.
H is the hypothesis that our data is indeed from a HMM.



We can easily see that

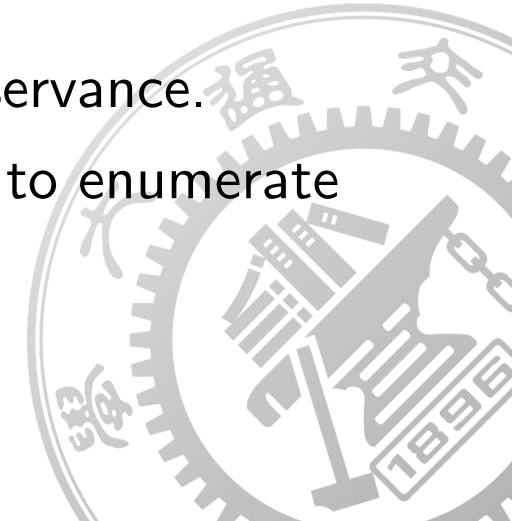
$$P(w|H, M) = \prod_t a_{w_{t-1}w_t} \tag{4}$$

The Object function

- Now we apply MLE to HMM, so we maximize the log likelyhood function,

$$\begin{aligned} logp(O|H,M) &= log \sum_{w} p(O|w,H,M)p(w|H,M) \\ &= log \sum_{w} a_{w_0 w_1} \prod_{t} a_{w_{t-1} w_t} b_{w_t}(o_t) \end{aligned} \tag{5}$$

  where w is the state sequence, and O is the observance.
- However, equation 5 is untractable, for we have to enumerate all w.



Some simplification

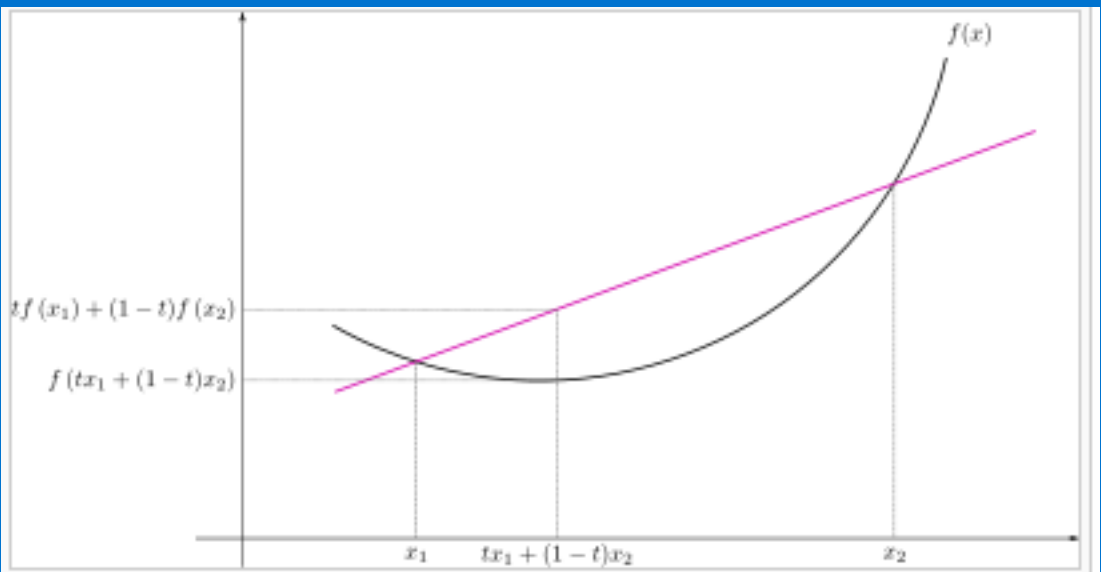
- For the remaining part of this tutorial, I will omit the omnipresent hypothesis H.
- We will now assume that $b(o)$ is Guassian (instead of Guassian Mixture) for simplicity.

Jensen's Inequality

push the log in

Jensen's Inequality

If X is a random variable and φ is a convex function. Then $\varphi(E[X]) \leq E[\varphi(X)]$.



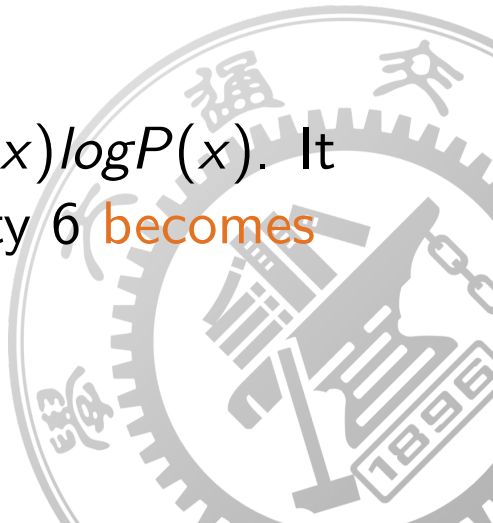
Jensen's inequality generalizes the statement that a secant line of a convex function lies above the graph.

Apply the Jensen inequailty

Note:$logp(O|M) = log \sum_w p(O, w|M)$
Remembering that $-log$ is convex and $\varphi(E[X]) \leq E[\varphi(X)]$, we can select any q. So

$$\begin{aligned} logp(O|M) &= log \sum_w \frac{q(w)p(O, w|M)}{q(w)} \\ &\geq < logp(O, w|M) >_{q(w)} + H(q(w)) \end{aligned} \tag{6}$$

Where
$< f(x) >_{P(x)} = \sum_x f(x)P(x)$ and $H(P(x)) = -\sum_x P(x)logP(x)$. It can be shown [A.P.Dempster, 1977] that the inequality 6 becomes an equality when $q(w) = P(w|O, M)$.



How do we choose M_{k+1}

The anxiliary function

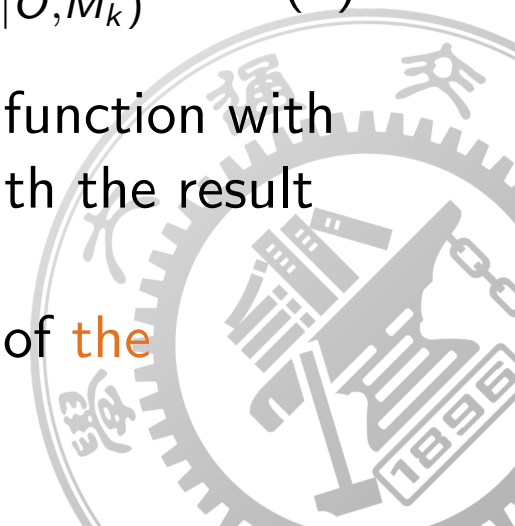
Equation 6 gives a lower bound for $P(O|M_{k+1})$ by setting $q(w) = p(w|O, \hat{M}_k)$

$$logp(O|M_{k+1}) \geq < logp(O, w|M_{k+1}) >_{p(w|O,\hat{M}_k)} + H(P(w|O, \hat{M}_k)) \tag{7}$$

Now, let us define the auxiliary function:

$$Q_{ML}(M_{k+1}; \hat{M}_k) = < logp(O, w|M_{k+1}) >_{p(w|O,\hat{M}_k)} \tag{8}$$

So, out task is now clearly to maximize the anxiliary function with respect to M_{k+1}, and suppose we find the $\hat{M}_{k+1}$. With the result of [A.P.Dempster, 1977] and the fact that $Q_{ML}(\hat{M}_{k+1}; \hat{M}_k) \geq Q_{ML}(\hat{M}_k; \hat{M}_k)$, we are able to proof the lowerbond of $P(O|\hat{M}_{k+1})$ is greater than $P(O|\hat{M}_k)$.



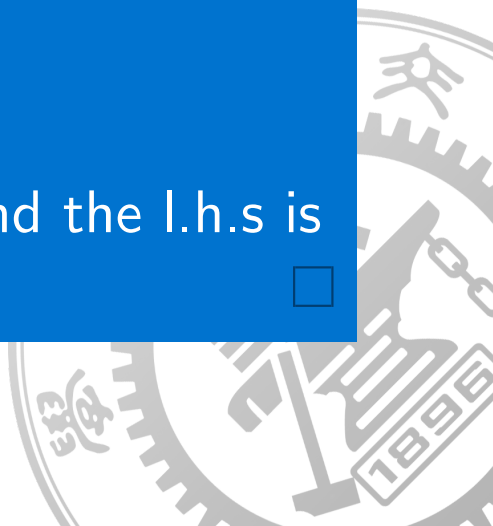
Proof for $P(O|\hat{M}_{k+1}) \geq P(O|\hat{M}_k)$

Theorem

$P(O|\hat{M}_{k+1})$ is greater than $P(O|\hat{M}_k)$, if $\hat{M}_{k+1}$ maximizes the anxiliary function.

Proof.

From the assumption we have $Q_{ML}(\hat{M}_{k+1}; \hat{M}_k) \geq Q_{ML}(\hat{M}_k; \hat{M}_k)$
We add a constant to both sides
$Q_{ML}(\hat{M}_{k+1}; \hat{M}_k) + H(P(w|O, \hat{M}_k)) \geq$
$Q_{ML}(\hat{M}_k; \hat{M}_k) + H(P(w|O, \hat{M}_k))$
Now we know that the r.h.s is exactly $logP(O|\hat{M}_k)$ and the l.h.s is a lowerbond of $P(O|\hat{M}_{k+1})$, which gives the result. □



What do E and M mean?

From

$$logp(O|M_{k+1}) \geq < logp(O, w|M_{k+1}) >_{p(w|O,\hat{M}_k)} + H(P(w|O, \hat{M}_k)) \tag{9}$$

we now know that in order to maximize the lowerbound, we only need to maximize the auxiliary function
$Q_{ML}(M_{k+1}; \hat{M}_k) = < logp(O, w|M_{k+1}) >_{p(w|O,\hat{M}_k)}$.
Now we can see the outline for the E-M algorithm. We use the state distribution from the last iteration to estimate the performance of this new iteration, and then maximize it.

A joke

We want to maximize $(3 + 2)x, x \leq 5$.
In the E step, we compute 3+2.
In the M step, we maximize $5x$ with $x \leq 5$.

E:Unfold the anxiliary function

$$
\begin{aligned}
Q_{ML}(M_{k+1}; \hat{M}_k) &= \sum_w log p(O, w|M_{k+1}) p(w|O, \hat{M}_k) \\
&= \sum_w log(a_{w_0 w_1} \prod_t a_{w_{t-1} w_t} b_{w_t}(o_t)) P(w|O, \hat{M}_k) \qquad (10) \\
Q_{ML}(M_{k+1}; \hat{M}_k) &= \sum_{t,j} \gamma_j(t) log b_j(o_t) + \sum_{t,i,j} \varepsilon_{ij}(t) log a_{ij}
\end{aligned}
$$

Where $\gamma_j(t) = P(w_t = j|O, \hat{M}_k)$ and $\varepsilon_{ij}(t) = P(w_{t-1} = i, w_t = j|O, \hat{M}_k)$. And the transistion is simply via take each a and b out and add all out their cooresponding w.

E:The Forward-Backward Algorithm

To calculate γ_j and ε_{ij}, we need to calculate
$\alpha_j(t) = p(o_{1-t}, w_t = j|\hat{M}_k)$ and $\beta_j(t) = p(o_{t+1-T}|w_t = j, \hat{M}_k)$
They can be calculated efficiently:

$$\alpha_j(t) = (\sum_{i=2}^{N-1} \alpha_i(t-1)a_{ij})b_j(o_t) \tag{11}$$

$$\beta_j(t) = \sum_{i=2}^{N-1} a_{ji}\, b_i(o_{t+1})\beta_i(t+1)$$

E:Use α and β to get γ and ε

Theorem

$$\gamma_j(t) = \frac{\alpha_j(t)\beta_j(t)}{p(O|\hat{M}_k)} \text{ and } \varepsilon_{ij}(t) = \frac{\alpha_i(t-1)a_{ij}b_j(o_t)\beta_j(t)}{p(O|\hat{M}_k)}$$

Proof.

I only proof the first equation. Plug the definition in.

$$\frac{\alpha_j(t)\beta_j(t)}{p(O|\hat{M}_k)} = \frac{p(o_{1-t}, w_t{=}j|\hat{M}_k)p(o_{t+1-T}|w_t{=}j,\hat{M}_k)}{p(O|\hat{M}_k)}$$

$$= \frac{p(o_{1-t}|w_t{=}j,\hat{M}_k)p(w_t{=}j|\hat{M}_k)p(o_{t+1-T}|w_t{=}j,\hat{M}_k)}{p(O|\hat{M}_k)}$$

note the condition $w_t = j$ makes the two probability independent.

$$= \frac{p(o_{1-T}|w_t{=}j,\hat{M}_k)p(w_t{=}j|\hat{M}_k)}{p(O|\hat{M}_k)}$$

$$= p(w_t = j|O,\hat{M}_k)$$

□

M:Use the Lagrange Multiplier

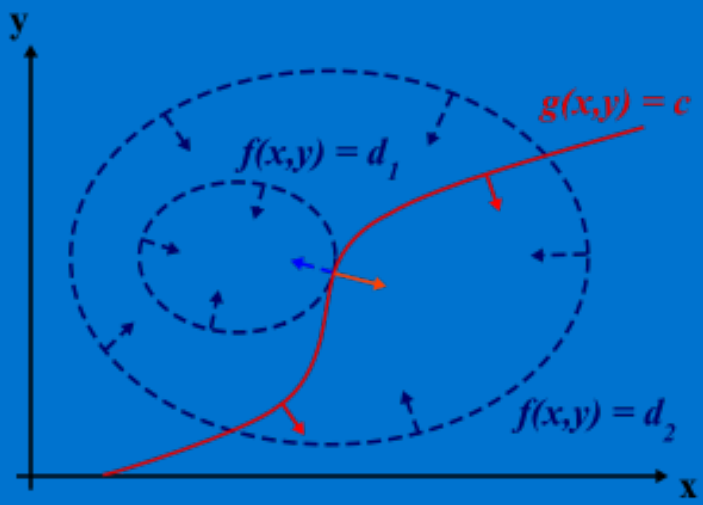
Having got the γ and ε, we finally could try to maximize

$$Q_{ML}(M_{k+1}; \hat{M}_k) = \sum_{t,j} \gamma_j(t) log b_j(o_t) + \sum_{t,i,j} \varepsilon_{ij}(t) log a_{ij} \qquad (12)$$

First let's consider a_{i*}, now we should maxmize $\sum_{t,j} \varepsilon_{ij}(t) log a_{ij}$ subject to $\sum_j a_{ij} = 1$

The Lagrange Multiplier

We want to maximize $f(x, y)$ subject $g(x, y) = c$.
Let $\Lambda(x, y\lambda) = f(x, y) + \lambda(g(x, y) - c)$.Then if $(x_0, y_0)$is a maximum of the original f, there exists (x_0, y_0, λ_0) is a stationary point for the Λ function.



The contour lines of f and g touch when the tangent vectors of the contour lines are parallel. Since the gradient of a function is perpendicular to the contour lines, this is the same as saying that the gradients of f and g are parallel.

The Lagrange Multiplier

So $\nabla_{x,y} f = -\lambda \nabla_{x,y} g$.
Combining with the constraint, we get $\nabla_{x,y,\lambda} \Lambda = 0$

Now we apply the Lagrange Multiplier method to maxmize $\sum_{t,j} \varepsilon_{ij}(t) log a_{ij}$ subject to $\sum_j a_{ij} = 1$. We get(I'll do the computing on the blackboard)

$$\hat{a}_{ij} = \frac{\sum_{t=2}^{T} \varepsilon_{ij}(t)}{\sum_{t=1}^{T} \gamma_i(t)} \tag{13}$$

Now we maximize the other component $\sum_{t,j} \gamma_j(t) log b_j(o_t)$.We set $G(\mu_j, \Sigma_j) = \sum_t \gamma_j(t) log b_j(o_t)$. And we maximize it by solving $\frac{\partial G}{\partial \mu_j} = 0$.
With the experience of MLE for Guassian distribution in the introduction part, this should be rather easy, I omit the computing here.

$$\hat{\mu_j} = \frac{\sum_t (\gamma_j(t) o_t)}{\sum_t \gamma_j(t)} \tag{14}$$

Sorry that I didn't compute the $\hat{\Sigma}_j$.

Extending to GMM

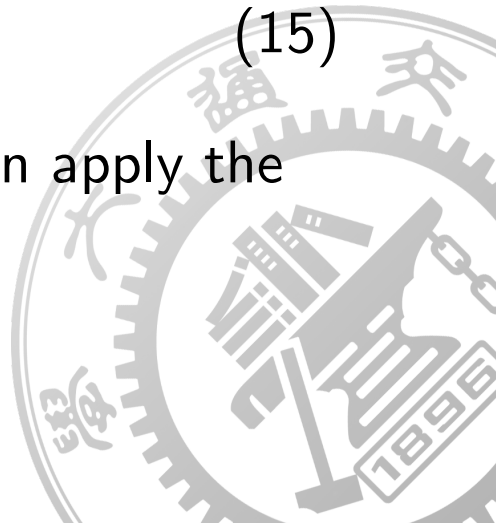
Extending single Guassian to Guassian Mixture is not part of this tutorial, and these two slices are poorly written. I encourage reader to refer to PRML $9.2 for knowledge of GMM, then you can use the technique introduced here to extend to GMM yourself!

Extending to GMM

To get the value of the Guassians, we first need to introduce sub-states by extending w.

$$\begin{aligned} Q_{ML}(M_{k+1}; \hat{M}_k) &= \sum_w log p(O, w|M_{k+1}) p(w|O, \hat{M}_k) \\ &= \sum_w log(a_{w_0 w_1} \prod_t a_{w_{t-1} w_t} c_{w_t d_t} b_{w_t d_t}(o_t)) P(w|O, \hat{M}_k) \end{aligned} \tag{15}$$

Where c is weight of the component. And now we can apply the Lagrange Multiplier method.



Extending to GMM

If we set

$$\gamma_{jm}(t) = \frac{\sum_{i=2}^{N-1} \alpha_i(t-1) a_{ij} c_{jm} b_{jm}(o_t) \beta_j(t)}{p(O|M_k)} \tag{16}$$

Then

$$\begin{aligned} \hat{c}_{jm} &= \frac{\sum_t \gamma_{jm}(t)}{\sum_{m,t} \gamma_{jm}(t)} \\ \hat{\mu}_{jm} &= \frac{\sum_t \gamma_{jm}(t) o_t}{\sum_t \gamma_{jm}(t)} \end{aligned} \tag{17}$$

Bibliography

- Adaptive Training for Large Vocabulary Continuous Speech Recognition
- Maximum likelihood from incomplete data via the EM algorithm