Theorem. *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0, EX_i^2 = \sigma^2$, and $E|X_i|^3 = \rho < \infty$. If $F_n(x)$ is the distribution of $(X_1 + \cdots + X_n)/\sigma\sqrt{n}$ and $\mathcal{N}(x)$ is the standard normal distribution, then*

$$|F_n(x) - \mathcal{N}(x)| \leq 3\rho/\sigma^3\sqrt{n}$$

Proof Let φ_F and φ_G be the ch.f.'s of F and G. Applying Lemma 3.4.19 to $F_L = F * H_L$ and $G_L = G * H_L$, gives

$$|F_L(x) - G_L(x)| \leq \frac{1}{2\pi}\int |\varphi_F(t)\omega_L(t) - \varphi_G(t)\omega_L(t)| \frac{dt}{|t|} \leq \frac{1}{2\pi}\int_{-L}^{L} |\varphi_F(t) - \varphi_G(t)| \frac{dt}{|t|}$$

since $|\omega_L(t)| \leq 1$. Using Lemma 3.4 .18 now, we have

$$|F(x) - G(x)| \leq \frac{1}{\pi}\int_{-L}^{L} |\varphi_F(\theta) - \varphi_G(\theta)| \frac{d\theta}{|\theta|} + \frac{24\lambda}{\pi L}$$

where $\lambda = \sup_x G'(x)$. Plugging in $F = F_n$ and $G = \mathcal{N}$ gives

$$|F_n(x) - \mathcal{N}(x)| \leq \frac{1}{\pi}\int_{-L}^{L} \left|\varphi^n(\theta/\sqrt{n}) - \psi(\theta)\right| \frac{d\theta}{|\theta|} + \frac{24\lambda}{\pi L}$$

and it remains to estimate the right-hand side. This phase of the argument is fairly routine, but there is a fair amount of algebra. To save the reader from trying to improve the inequalities along the way in hopes of getting a better bound, we would like to observe that we have used the fact that $C = 3$ to get rid of the cases $n \leq 9$, and we use $n \geq 10$ in (e). To estimate the second term in (3.4.1)*, we observe that*

(a)

$$\sup_x G'(x) = G'(0) = (2\pi)^{-1/2} = 0.39894 < 2/5$$

For the first, we observe that if $|\alpha|, |\beta| \leq \gamma$

(b)

$$|\alpha^n - \beta^n| \leq \sum_{m=0}^{n-1} \left|\alpha^{n-m}\beta^m - \alpha^{n-m-1}\beta^{m+1}\right| \leq n|\alpha - \beta|\gamma^{n-1}$$

Using (3.3.3) now gives (recall we are supposing $\sigma^2 = 1$)

(c) $\left|\varphi(t) - 1 + t^2/2\right| \leq \rho|t|^3/6$ *so if* $t^2 \leq 2$

(d)

$$|\varphi(t)| \leq 1 - t^2/2 + \rho|t|^3/6$$

Let $L = 4\sqrt{n}/3\rho$. If $|\theta| \leq L$, then by (d) and the fact $\rho|\theta|/\sqrt{n} \leq 4/3$

$$|\varphi(\theta/\sqrt{n})| \leq 1 - \theta^2/2n + \rho|\theta|^3/6n^{3/2} \leq 1 - 5\theta^2/18n \leq \exp\left(-5\theta^2/18n\right)$$

since $1 - x \leq e^{-x}$. *We will now apply (b) with*

$$\alpha = \varphi(\theta/\sqrt{n}) \quad \beta = \exp\left(-\theta^2/2n\right) \quad \gamma = \exp\left(-5\theta^2/18n\right)$$

Since we are supposing $n \geq 10$

(e)

$$\gamma^{n-1} \leq \exp\left(-\theta^2/4\right)$$

For the other part of (b), we write

$$n|\alpha - \beta| \leq n\left|\varphi(\theta/\sqrt{n}) - 1 + \theta^2/2n\right| + n\left|1 - \theta^2/2n - \exp\left(-\theta^2/2n\right)\right|$$

To bound the first term on the right-hand side, observe (c) implies

$$n\left|\varphi(\theta/\sqrt{n}) - 1 + \theta^2/2n\right| \leq \rho|\theta|^3/6n^{1/2}$$

For the second term, note that if $0 < x < 1$*, then we have an alternating series with decreasing terms so*

$$\left|e^{-x} - (1-x)\right| = \left|-\frac{x^2}{2!} + \frac{x^3}{3!} - \ldots\right| \leq \frac{x^2}{2}$$

Taking $x = \theta^2/2n$*, it follows that for* $|\theta| \leq L \leq \sqrt{2n}$

$$n\left|1 - \theta^2/2n - \exp\left(-\theta^2/2n\right)\right| \leq \theta^4/8n$$

Combining this with our estimate on the first term gives

(f)

$$n|\alpha - \beta| \leq \rho|\theta|^3/6n^{1/2} + \theta^4/8n$$

Using (f) and (e) in (b) gives

$$\frac{1}{|\theta|}\left|\varphi^n(\theta/\sqrt{n}) - \exp\left(-\theta^2/2\right)\right| \leq \exp\left(-\theta^2/4\right)\left\{\frac{\rho\theta^2}{6n^{1/2}} + \frac{|\theta|^3}{8n}\right\} \leq \frac{1}{L}\exp\left(-\theta^2/4\right)\left\{\frac{2\theta^2}{9} + \frac{|\theta|^3}{18}\right\}$$

since $\rho/\sqrt{n} = 4/3L$*, and* $1/n = 1/\sqrt{n} \cdot 1/\sqrt{n} \leq 4/3L \cdot 1/3$*, since* $\rho \geq 1$ *and* $n \geq 10$*. Using the last result and (a) in Lemma 3.4 .19 gives*

$$\pi L\left|F_n(x) - \mathcal{N}(x)\right| \leq \int \exp\left(-\theta^2/4\right)\left\{\frac{2\theta^2}{9} + \frac{|\theta|^3}{18}\right\} d\theta + 9.6$$

Recalling $L = 4\sqrt{n}/3\rho$*, we see that the last result is of the form* $|F_n(x) - \mathcal{N}(x)| \leq C\rho/\sqrt{n}$*. To evaluate the constant, we observe*

$$\int (2\pi a)^{-1/2} x^2 \exp\left(-x^2/2a\right) dx = a$$

and writing $x^3 = 2x^2 \cdot x/2$ *and integrating by parts*

$$2\int_0^\infty x^3 \exp\left(-x^2/4\right) dx = 2\int_0^\infty 4x \exp\left(-x^2/4\right) dx = -\left.16e^{-x^2/4}\right|_0^\infty = 16$$

This gives us

$$|F_n(x) - \mathcal{N}(x)| \leq \frac{1}{\pi} \cdot \frac{3}{4} \left(\frac{2}{9} \cdot 2 \cdot \sqrt{4\pi} + \frac{16}{18} + 9.6 \right) \frac{\rho}{\sqrt{n}} < 3 \frac{\rho}{\sqrt{n}}$$

For the last step, you have to get out your calculator or trust Feller.