STAT 153 - Introduction to Time Series
Lecture Sixteen

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Aditya Guntuboyina

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1 Different Regimes of the $AR(1)$ Model

The $AR(1)$ model is

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + Z_t \tag{1}$$

Depending on the value of ϕ_1, the $AR(1)$ model is classified into three regimes:

1. **Causal Stationary** $AR(1)$ **Model**: This corresponds to $|\phi_1| < 1$.
2. **Non-causal Stationary** $AR(1)$ **Model**: This corresponds to $|\phi_1| > 1$.
3. **Non-Stationary** $AR(1)$ **Model**: This corresponds to $|\phi_1| \neq 1$.

We shall see today the intuition behind this classification.

2 Causal Stationary $AR(1)$

Here $|\phi_1| < 1$. Assuming that the recursion (1) starts at $t = 1$, we can write every Y_t (for $t \geq 1$) in terms of $Y_0, Z_t, \dots, Z_1$ as follows:

$$\begin{aligned}
Y_t &= \phi_0 + \phi_1 Y_{t-1} + Z_t \\
&= \phi_0 + \phi_1 \left(\phi_0 + \phi_1 Y_{t-2} + Z_{t-1}\right) + Z_t \\
&= \phi_0 \left(1 + \phi_1\right) + Z_t + \phi_1 Z_{t-1} + \phi_1^2 Y_{t-2} \\
&= \phi_0 \left(1 + \phi_1\right) + Z_t + \phi_1 Z_{t-1} + \phi_1^2 \left(\phi_0 + \phi_1 Y_{t-3} + Z_{t-2}\right) \\
&= \phi_0 \left(1 + \phi_1 + \phi_1^2\right) + Z_t + \phi_1 Z_{t-1} + \phi_1^2 Z_{t-2} + \phi_1^3 Y_{t-3} \\
&= \dots \\
&= \phi_0 \left(1 + \phi_1 + \dots + \phi_1^{t-1}\right) + \sum_{j=0}^{t-1} \phi_1^j Z_{t-j} + \phi_1^t Y_0 \\
&= \phi_0 \frac{1 - \phi_1^t}{1 - \phi_1} + \sum_{j=0}^{t-1} \phi_1^j Z_{t-j} + \phi_1^t Y_0.
\end{aligned}$$

Because $|\phi_1| < 1$, when t gets large, the first term above converges quickly to $\frac{\phi_0}{1-\phi_1}$, and the third term converges quickly to zero (if we assume that Y_0 is a fixed constant). The second term becomes

$$\sum_{j=0}^{t-1} \phi_1^j Z_{t-j} = Z_t + \phi_1 Z_{t-1} + \phi_1^2 Z_{t-2} + \dots + \phi_1^{t-1} Z_1 \to \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}$$

when t becomes large. Thus when t is large,

$$Y_t \approx \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}.$$

Note that $\sum_{j=0}^{\infty} \phi_1^j Z_{t-j}$ involves negative indices of Z so we assume that we have a doubly infinite sequence $\dots, Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2, \dots$ of i.i.d $N(0, \sigma^2)$ random variables. Let us denote

$$Y_t^* := \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j Z_{t-j} \qquad \text{for } t = \dots, -2, -1, 0, 1, 2, \dots \tag{2}$$

This random variable is well-defined for every t (the infinite series $\sum_{j=0}^{\infty} \phi_1^j Z_{t-j}$ is well-defined because ϕ_1^j decays rapidly to zero so effectively this infinite sum acts like a finite sum). This sequence of random variables $Y_t^*, t = \dots, -2, -1, 0, 1, 2, \dots$ satisfies the following properties:

1. $\{Y_t^*\}$ satisfies the AR(1) recursion (1) for all $t = \dots, -2, -1, 0, 1, 2, \dots$. This can be easily verified.

2. $\mathbb{E} Y_t^*$ is the same for all values of t. This is because:

$$\begin{aligned}\mathbb{E} Y_t^* &= \mathbb{E}\left(\frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}\right) \\ &= \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \mathbb{E}(Z_{t-j}) = \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \times 0 = \frac{\phi_0}{1-\phi_1}.\end{aligned}$$

3. The covariance between Y_t^* and Y_{t+h}^* depends only on h (i.e., it is the same for all t). This is because

$$\begin{aligned}\mathrm{Cov}\left(Y_t^*, Y_{t+h}^*\right) &= \mathrm{Cov}\left(\frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}, \frac{\phi_0}{1-\phi_1} + \sum_{k=0}^{\infty} \phi_1^k Z_{t+h-k}\right) \\ &= \mathrm{Cov}\left(\sum_{j=0}^{\infty} \phi_1^j Z_{t-j}, \sum_{k=0}^{\infty} \phi_1^k Z_{t+h-k}\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi_1^{j+k} \mathrm{Cov}\left(Z_{t-j}, Z_{t+h-k}\right).\end{aligned}$$

Because Z_t are independent, $\mathrm{Cov}(Z_t, Z_s)$ equals 0 for $t \neq s$ (and equals σ^2 when $s = t$). We represent this using indicator notation as $\mathrm{Cov}(Z_t, Z_s) = \sigma^2 I\{t = s\}$. Thus

$$\begin{aligned}\mathrm{Cov}\left(Y_t^*, Y_{t+h}^*\right) &= \sigma^2 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi_1^{j+k} I\{t - j = t + h - k\} \\ &= \sigma^2 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi_1^{j+k} I\{k = j + h\}\end{aligned}$$

which clearly does not depend on t. We can get a more explicit expression for the covariance in the following way. Suppose $h \geq 0$, then $I\{k = j + h\}$ is only non-zero for $k = j + h$ so

$$\mathrm{Cov}\left(Y_t^*, Y_{t+h}^*\right) = \sigma^2 \sum_{j=0}^{\infty} \phi_1^{j+(j+h)} = \sigma^2 \phi_1^h \sum_{j=0}^{\infty} \phi_1^{2j} = \frac{\sigma^2}{1-\phi_1^2}\phi_1^h.$$

When $h < 0$, we can replace j by $k - h$ to obtain the same expression as above with h replaced by $-h$. We have thus proved that

$$\mathrm{Cov}\left(Y_t^*, Y_{t+h}^*\right) = \frac{\sigma^2}{1-\phi_1^2}\phi_1^{|h|}.$$

This implies that

$$\mathrm{var}(Y_t^*) = \mathrm{Cov}\left(Y_t^*, Y_t^*\right) = \frac{\sigma^2}{1-\phi_1^2} \qquad \text{for every } t$$

so that the correlation between Y_t^* and Y_{t+h}^* is given by

$$\mathrm{corr}(Y_t^*, Y_{t+h}^*) = \frac{\mathrm{Cov}(Y_t^*, Y_{t+h}^*)}{\sqrt{\mathrm{var}(Y_t^*)}\sqrt{\mathrm{var}(Y_{t+h}^*)}} = \phi_1^{|h|} \qquad \text{for every } t \text{ and } h.$$

The above properties of $\{Y_t^*\}$ are the reason why this is referred to as causal and stationary. Specifically, the definition of stationarity is the following:

Definition 2.1 (Stationarity). *A doubly infinite sequence of random variables $\{X_t\}$ is said to be stationary if*

1. *The mean of X_t is the same for all times t*
2. *The covariance between X_t and X_{t+h} only depends on h.*

Thus Y_t^* is clearly stationary according to the above definition. The causality just refers to the fact that Y_t^* only depends on $Z_t, Z_{t-1}, Z_{t-2}, \ldots$. In other words, only the present and past values of $\{Z_t\}$ determine Y_t^* so one can say that $\{Z_t\}$ causes Y_t^*.

To summarize: when $|\phi_1| < 1$, the AR(1) recursion (1) admits the causal and stationary solution Y_t^* given by (2). If Y_t is instead initialized at Y_0 and defined by (1) for $t \geq 1$, then Y_t would be quite close to Y_t^* when t is large.

Remeber also that if the fitted AR(1) model to a particular dataset corresponds to $|\phi_1| < 1$, then future predictions will converge to the constant value $\phi_0/(1 - \phi_1)$.

3 Non-Causal Stationary $AR(1)$

Here $|\phi_1| > 1$. It is then easy to see that

$$Y_t' := \frac{\phi_0}{1-\phi_1} - \sum_{j=1}^{\infty} \frac{Z_{t+j}}{\phi_1^j}$$

satisfies the $AR(1)$ difference equation for all t. Further this process is also stationary (as can be checked by calculating the mean of Y_t' and the covariance between Y_t' and Y_{t+h}'). But it depends on the future values $Z_{t+1}, Z_{t+2}, \ldots$ of the $\{Z_t\}$ process which is why this is called Non-Causal. Thus when $|\phi_1| > 1$, there exists a stationary solution to the $AR(1)$ difference equation that is non-causal (i.e., it depends on future values of $\{Z_t\}$).

4 Non-Stationary $AR(1)$

Here $|\phi_1| = 1$ (i.e., either $\phi_1 = 1$ or $\phi_1 = -1$). Here it turns out that no stationary solution exists for the $AR(1)$ difference equation. To see this, consider the case $\phi_1 = 1$ (the case $\phi_1 = -1$ is similar) where

$$Y_t = \phi_0 + Y_{t-1} + Z_t$$

This implies that

$$Y_t - Y_0 = t\phi_0 + Z_1 + \dots Z_t$$

When $\phi_0 \neq 0$, clearly Y_t and Y_0 have different means so the process cannot be stationary. But even if $\phi_0 = 0$, we have

$$\text{var}(Y_t - Y_0) = \text{var}(Z_1 + \cdots + Z_t) = t\sigma^2$$

which approaches ∞ as $t \uparrow \infty$. But if $\{Y_t\}$ were stationary, we would have

$$\text{var}(Y_t - Y_0) \leq 2\text{var}(Y_t) + 2\text{var}(Y_0) \leq \text{constant}.$$

5 Regimes for $AR(p)$

Similar to $AR(1)$, every $AR(p)$ model has regimes corresponding to stationarity and causality. These are characterized by the roots of the characteristic equation. Specifically:

1. **Causal stationary** $AR(p)$: this corresponds to the case when all the roots of the characteristic equation have modulus strictly larger than one

2. **Non-Causal stationary** $AR(p)$: this corresponds to the case when all the roots of the characteristic equation have modulus different from one but at least one root has modulus less than one

3. **Non-stationary** $AR(p)$: this corresponds to the case when at least one root of the characteristic equation has modulus equal to one.

We shall see some justification for these in the next class.

6 The $MA(q)$ model

We saw that the causal stationary $AR(1)$ model is given by (2) which is written as a linear combination of $Z_t, Z_{t-1}, \dots$. Motivated by this, we can consider more general linear combinations of $Z_t, Z_{t-1}, \dots$ as a time series model. This leads to the $MA(q)$ model:

$$Y_t = \theta_0 + Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}.$$

The parameters in this model are $\theta_0, \theta_1, \dots, \theta_q$. We shall study the fitting of these models to observed time series data later.

7 Recommended Reading for Today

1. For definitions of stationarity, see Section 1.4 of the book by Shumway and Stoffer titled *Time Series Analysis and its applications* (Fourth Edition).

2. For the different regimes of the $AR(1)$ model, see Example 3.1, 3.2, 3.3 and 3.4 of the book by Shumway and Stoffer titled *Time Series Analysis and its applications* (Fourth Edition).