IOE 511/MATH 562: Homework № 1

Due: 1/20 (by 11:59pm EST)

1. Show that the following functions are norms:
   (a) $\|x\|_1 = \sum_{i=1}^n |x_i|$
   (b) $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$

2. Show (by deriving appropriate constants for each case) that the following norms on $\mathbb{R}^n$ are equivalent:
   (a) $\|\cdot\|_1$ and $\|\cdot\|_2$
   (b) $\|\cdot\|_2$ and $\|\cdot\|_\infty$
   (c) $\|\cdot\|_\infty$ and $\|\cdot\|_1$
   (d) Derive the result for general norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\alpha, \beta \geq 0$.

3. Show that $\{x : Ax = 0\}$ is closed (here, $A \in \mathbb{R}^{m \times n}$).

4. Show that $f(x) = \|x\|_2^2 : \mathbb{R}^n \to \mathbb{R}$ is continuous everywhere using the "$\epsilon - \delta$" definition.

5. Consider the following sequences $\{x_k\}_{k \geq 0}$. Using the definitions from class, determine whether the following sequences: (1) converge to a finite or infinite limit, (2) have a limit point, and (3) are bounded.
   (a) $x_k = \frac{1}{k}$;
   (b) $x_k = \rho^k$, for $0 < \rho < 1$;
   (c) $x_k = \sum_{j=1}^k \rho^j$, for $0 < \rho < 1$;
   (d) $x_k = k^{3/2}$;
   (e) $x_k = k^{(-1)^k}$.

6. Consider the following sequences $\{x_k\}_{k \geq 0}$. Using the definitions of the different convergence rates, determine whether the following sequences converge Q-linearly, Q-superlinearly, and/or Q-quadratically.
   (a) $x_k = 3 + \left(\frac{1}{3}\right)^{2k}$;
   (b) $x_k = 3 + \left(\frac{1}{3}\right)^{2^k}$;
   (c) $x_k = 1 - k^{-2}$.

7. Suppose that A is a symmetric matrix. Show that the following two statements are equivalent:
   - $x^T A x > 0$ for all $x \neq 0$.
   - All eigenvalues of A are positive.

Note: These are two equivalent definitions of "A is positive definite".

8. An outer product of two vectors $a, b \in \mathbb{R}^n$ is defined as ab^T. What is the rank of the matrix ab^T? (Please give a detailed answer)

9. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) = x_1 e^{-\frac{1}{2}(x_1^2+x_2^2)}.$$

   (a) Write down the gradient and the Hessian matrix of $f(x)$.

   (b) Plot the function. In MATLAB you can use the following commands:

```
[X1,X2] = meshgrid(-3:.2:3, -3:.2:3);
F = X1 .* exp(-0.5*(X1.^2 + X2.^2));
surf(X1,X2,F);
```

   (c) Derive the second-order Taylor expansion of $f(x)$ at the point $\bar{x} = [-1, \frac{1}{2}]^T$.

   (d) Plot the Taylor second-order Taylor expansion in the same plot as $f(x)$. (The `hold on` command in MATLAB will be handy).

   The goal is to demonstrate that the Taylor expansion approximates f(x) very well close to $\bar{x}$; for this, you will need to "zoom in" a bit (e.g., use `[X1,X2] = meshgrid(-1.5:.1:-0.5, -1:.1:1);`? or something like that).

10. Let $f(x) = \frac{1}{2}x^T Qx + q^T x$, where $Q \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$.

   (a) Derive expressions for the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$.

   (b) Assume that Q is symmetric. Derive expressions for the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$.

   (c) Assume that Q is symmetric, positive semi-definite. Using the definition of convexity, prove that $f(x)$ is convex.

11. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be convex functions. Show that $f(g(x))$ is a convex function. (*Hint:* You may need an additional assumption to prove this result.)

12. Describe a problem (ideally, in your field of study), other than those described in class or the textbook, that can be modeled as a mathematical optimization problem. Your description should be high-level but specific: describe the source/meaning of the problem, give the decision variables, describe the objective function and constraints, and indicate what kind of functions are used to represent these (linear, nonlinear, etc.). Discuss the data required for this problem (availability, size, etc). Also, mention whether modeling the "real" problem mathematically this way requires making assumptions or simplifications. Your description should not exceed a couple of paragraphs.