Blog Notes 5-Option Pricing On bonds

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Introduction

- Option Pricing on Bonds

Problem

Problem: Consider a T-Bond option, it has a bond maturity T. Assume the option maturity is S, with $S < T$. The bond option has a strike price K. Assume the short rate process is $r(t)$, the T-bond price at time t is denoted by $B(t, T)$. The forward bond price at t is denoted by $B(t; S, T) = \frac{B(t,T)}{B(t,S)}$. Define a discount process given by $\frac{1}{B(t)} = \exp\{-\int_0^t r(u)du\}$, which gives you the present value for one unit of currency which will be received in a future time t.

At the option expiry time S, the European call option on the bond has a payoff $V(S) = (B(S, T) - K)^+$, which is the bond option price at time S. Calculate the bond option price at time t.

Risk-Neutral Option Pricing formula

According the generic pricing formula, a derivative security process can be replicated with a portfolio process, where the discounted value of the portfolio process is a martingale under the risk-neutral probability measure $\mathbb{Q}$. Then, the discounted value of the derivative security process is also a martingale.

We have an option pricing formula for the European call option on T-bond, with the time S payoff $(B(S,T)-K)^+$. That is $\frac{V(t)}{B(t)} = E_t^{\mathbb{Q}}[\frac{V(S)}{B(S)}] = E^{\mathbb{Q}}[\frac{(B(S,T)-K)^+}{B(S)}|\mathcal{F}_t]$.

The arbitrage pricing theorem states that, under the risk-neutral probability measure $\mathbb{Q}$, The return on any traded asset with positive price process $S(t)$ is an Itô process with a risk-free short rate drift $r(t)$ and volatility $\sigma(t)$; In differential form, it is $\frac{dS(t)}{S(t)} = r(t)dt + \sigma(t)d\widetilde{W}(t)$, where $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral measure.

Bond price process is a positive process, according to the arbitrage pricing theorem, in the risk-neutral probability measure $\mathbb{Q}$, the bond is a trading asset with the risk-free drift $r(t)$ and the T-bond volatility $\Sigma(t, T)$, satisfying
$dB(t, T) = r(t)B(t, T)dt + \Sigma(t, T)B(t, T)d\widetilde{W}(t)$.

Under the risk-neurtal measure, assuming the short rate process $r(t)$ follows $dr(t) = \mu(t, r(t))dt + \sigma_r(t, r(t))d\widetilde{W}(t)$, we can find the T-bond volatility $\Sigma(t, T)$ satisfies
$\Sigma(t, T) = -\sigma_r(t, r(t)) \cdot G(t, T)$, with
$G(t, T) = \int_t^T e^{-\kappa(u-t)}du = e^{\kappa t}\int_t^T e^{-\kappa u}du = e^{\kappa t} \cdot \frac{e^{-\kappa t} - e^{-\kappa T}}{\kappa} = e^{\kappa t} \cdot e^{\kappa t}\frac{1-e^{-\kappa(T-t)}}{\kappa} = \frac{1-e^{-\kappa(T-t)}}{\kappa}$. The equation works for both Vasicek model and Hull-White model. It can be tested that the derivation of $\Sigma(t, T)$ are the same for both models. The proof is done as follows.

Proof:

Assuming the T-bond price at time t has the form $B(t,T) = e^{f(t,T)+r(t)g(t,T)} = F(t, r(t))$.

Calculating $dB(t,T) = dF(t,r(t)) = dF(t,X)$ using Itô formula, we have $F_t(t,X) = (\frac{\partial f}{\partial t} + X \cdot \frac{\partial g}{\partial t})F(t,X)$, $F_X(t,X) = g(t,T)F(t,X)$, $F_{X,X}(t,X) = g^2(t,T)F(t,X)$, we have $dB(t,T) = dF(t,r(t)) = F_t(t,X)dt + F_X(t,X)dX(t) + \frac{1}{2}F_{X,X}(t,X)(dX(t))^2 = \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial X}dX(t) + \frac{1}{2}\frac{\partial^2 B}{\partial X^2}(dX(t))^2$

Substitute $dX(t) = dr(t) = \mu(t,r(t))dt + \sigma_r(t,r(t))dW(t)$, $(dX(t))^2 = \sigma_r^2(t,r(t))dt$, we have $dB(t,T) = [(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial X}\mu(t,r(t)) + \frac{1}{2}\frac{\partial^2 B}{\partial X^2}\sigma_r^2(t,r(t))]dt + \frac{\partial B}{\partial X}\sigma_r(t,r(t))d\widetilde{W}(t)$

Equating the $dB(t,T)$ to the other form of $dB(t,T) = r(t)B(t,T)dt + \Sigma(t,T)B(t,T)d\widetilde{W}(t)$.

Denote $r(t)$ by $X = r(t)$. Equating the Brownian motion term $d\widetilde{W}(t)$, the T-bond volatility at time t is
$\Sigma(t,T) = \frac{\partial B}{\partial X}\sigma_r(t,X)\frac{1}{B} = \sigma_r(t,X)\frac{\partial lnB(t,T)}{\partial X} = \sigma_r(t,X)\cdot g(t,T)$

$g(t,T)$ is calculated by equating the dt term of the two form of $dB(t,T)$, we have
$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial X}\mu(t,r(t)) + \frac{1}{2}\frac{\partial^2 B}{\partial X^2}\sigma_r^2(t,r(t) = r(t)B(t,T)$

Substitute $\frac{\partial B}{\partial t}$, $\frac{\partial B}{\partial X}$, $\frac{\partial^2 B}{\partial X^2}$, simplify the equation, we have
$(\frac{\partial f}{\partial t}(t,T) + X\cdot\frac{\partial g}{\partial t}(t,T)) + g(t,T)\mu(t,X) + \frac{1}{2}g^2(t,T)\sigma_r^2(t,X) = X$

In the Vasicek model, under the risk-neutral probability measure $\mathbb{Q}$, the dynamics of the short rate process is
$dr(t) = (\theta - \kappa r(t))dt + \sigma d\widetilde{W}(t)$, with $\mu(t,r(t)) = (\theta - \kappa r(t))$, $\sigma_r(t,r(t)) = \sigma$. We have $\mu(t,X) = (\theta - \kappa X)$, $\sigma_r(t,X) = \sigma$.
Substitute $\mu(t,X)$, $\sigma_r(t,X)$,

and equate the coefficient of X on both sides of the equation, we have an equation $\frac{\partial g}{\partial t}(t, T) - \kappa g(t, T) = 1$, together with a bound condition $g(T, T) = 0$, since $1 = B(T, T) = e^{f(T,T)+r(t)g(T,T)}$.

In Hull-White model, in the risk-neutral probability measure $\mathbb{Q}$, the dynamics of the short rate process is
$dr(t) = (\theta(t) - \kappa r(t))dt + \sigma(t)d\widetilde{W}(t)$, with
$\mu(t, r(t)) = (\theta(t) - \kappa r(t))$, $\sigma_r(t, r(t)) = \sigma(t)$. We have
$\mu(t, X) = (\theta(t) - \kappa X)$, $\sigma_r(t, X) = \sigma(t)$. Substitute $\mu(t, X)$, $\sigma_r(t, X)$, and equate the coefficient of X on both sides of the equation, we have the equation $\frac{\partial g}{\partial t}(t, T) - \kappa g(t, T) = 1$, together with a bound condition $g(T, T) = 0$, since
$1 = B(T, T) = e^{f(T,T)+r(t)g(T,T)}$.

In both Vasicek model and Hull-White model, the value of $g(t, T)$ is calculated by

$$\begin{cases} \frac{\partial g}{\partial t}(t,T) - \kappa g(t,T) &= 1 \\ g(T,T) &= 0 \end{cases}$$

it is the same as solving $g(t,T)$ satisfying

$$\begin{cases} d(e^{-\kappa t} \cdot g(t,T)) &= e^{-\kappa t} \\ g(T,T) &= 0 \end{cases}$$

$$\leftrightarrow \begin{cases} e^{-\kappa t} \cdot g(t,T) &= \int_0^t e^{-\kappa u} du + C \\ g(T,T) &= 0 \end{cases} \Rightarrow C = -\int_0^T e^{-\kappa u} du$$

$$\Rightarrow e^{-\kappa t} \cdot g(t,T) = \int_0^t e^{-\kappa u} du - \int_0^T e^{-\kappa u} du$$

$$\Rightarrow g(t,T) = e^{-\kappa t} \cdot \left(-\int_t^T e^{-\kappa u} du\right) = -\int_t^T e^{-\kappa(u-t)} du = -G(t,T)$$

with $G(t,T) = \int_t^T e^{-\kappa(u-t)} du = e^{\kappa t} \int_t^T e^{-\kappa u} du =$
$e^{\kappa t} \cdot \frac{e^{-\kappa t} - e^{-\kappa T}}{\kappa} = e^{\kappa t} \cdot e^{\kappa t} \frac{1 - e^{-\kappa(T-t)}}{\kappa} = \frac{1 - e^{-\kappa(T-t)}}{\kappa}$.

To summary, in the Hull-White model, the bond price volatility is $\Sigma(t,T) = -\sigma_r(t, r(t)) \cdot G(t,T)$, with $\sigma_r(t, r(t)) = \sigma(t)$.

In the Vasicek model, the bond price volatility is $\Sigma(t,T) = -\sigma_r(t, r(t)) \cdot G(t,T)$, with $\sigma_r(t, r(t)) = \sigma$.

Back to the option pricing formula for the European call option on T-bond, with the time S payoff $(B(S,T) - K)^+$. That is $\frac{V(t)}{B(t)} = E^{\mathbb{Q}}[\frac{(B(S,T)-K)^+}{B(S)}|\mathcal{F}_t]$.
Substitute $B(t) = e^{\int_0^t r(u)du}$, $B(S) = e^{\int_0^S r(u)du}$, we have $V(t) = E^{\mathbb{Q}}[e^{\int_t^S r(u)du} \cdot (B(S,T) - K)^+|\mathcal{F}_t]$.

Define S-forward measure $\mathcal{P}_S \sim \mathbb{Q}$ with the Radon-Nikodym density process $Z_t = E_t^{\mathbb{Q}}[\frac{d\mathbb{P}_S}{dQ}] = e^{-\int_0^t r(u)du} \frac{B(t,S)}{B(0,S)}$, such that the Radon Nikodym random variable $Z = Z_S = E_S^{\mathbb{Q}}[\frac{d\mathbb{P}_S}{dQ}] = \frac{d\mathbb{P}_S}{dQ}$ is a positive random variable and its expectation under the measure $\mathbb{Q}$ is 1.

Consider a $\mathcal{F}_S$-measurable random variable X, the Radon Nikodym derivative random variable Z_S is $\mathcal{F}_S$-measurable Using the Bayes' rule, $E_t^{\mathcal{P}_S}[X] = \frac{E_t^{\mathbb{Q}}[Z_S X]}{Z_t} = \frac{E_t^{\mathbb{Q}}[e^{-\int_0^S r(u)du} \frac{B(S,S)}{B(0,S)} X]}{e^{-\int_0^t r(u)du} \frac{B(t,S)}{B(0,S)}} = \frac{E_t^{\mathbb{Q}}[e^{-\int_t^S r(u)du} X]}{B(t,S)}$, then, we have $B(t,S) \cdot E_t^{\mathcal{P}_S}[X] = E_t^{\mathbb{Q}}[e^{-\int_t^S r(u)du} X]$, with X is $\mathcal{F}_S$-measurable.

Therefore, with a probability measure change from risk-neutral measure to an equivalent S-forward measure $\mathcal{P}_S \sim \mathbb{Q}$ with the defined Radon-Nikodym density process $Z_t = E_t^{\mathbb{Q}}[\frac{d\mathbb{P}_S}{dQ}] = e^{-\int_0^t r(u)du} \frac{B(t,S)}{B(0,S)}$, the European call option on T-bond, with the payoff $(B(S,T) - K)^+$ at time S, the option value at time t is $V(t) = E^{\mathbb{Q}}[e^{\int_t^S r(u)du} \cdot (B(S,T) - K)^+ | \mathcal{F}_t] = B(t,S) \cdot E^{\mathbb{P}_S}[(B(S,T) - K)^+ | \mathcal{F}_t]$.

It is observed that after changing the probability measure, the discounter factor is avoided and the option price calculating is simplified.

An important fact:

For any forward bond price process $B(t; S, T) = \frac{B(t,S)}{B(t,T)}$, that is, the T-bond price process discounted by the S-bond price process $\frac{B(t,T)}{B(t,S)}$, the forward bond price process $B(t; S, T)$ is a martingale under the S-forward probability measure $\mathcal{P}_S \sim \mathbb{Q}$ with the defined Radon-Nikodym density process $Z_t = E_t^{\mathbb{Q}}[\frac{d\mathbb{P}_S}{dQ}] = e^{-\int_0^t r(u)du} \frac{B(t,S)}{B(0,S)}$.

Proof:

Using the Bayes' rule, the random variable $B(S; S, T)$ is $\mathcal{F}_S$-measurable, under the S-forward measure $\mathcal{P}_S \sim \mathbb{Q}$, we have $B(t, S) \cdot E_t^{\mathcal{P}_S}[\frac{B(S,T)}{B(S,S)}] = E_t^{\mathbb{Q}}[e^{-\int_t^S r(u)du} \frac{B(S,T)}{B(S,S)}] =$
$E_t^{\mathbb{Q}}[e^{-\int_t^S r(u)du} B(S, T)] = B(t, T)$.

The last equation follows the arbitrage pricing formula, any positive trading asset, its discounted share price is a martingale under the risk-neutral probability measure. so, we have $E_t^{\mathcal{P}_S}[\frac{B(S,T)}{B(S,S)}] = \frac{B(t,T)}{B(t,S)}$, or written as $E_t^{\mathcal{P}_S}[B(S; S, T)] = B(t; S, T)$.

Next, find the distribution of the forward bond price $B(t; S, T)$ under the S-forward probability measure $\mathcal{P}_S \sim \mathbb{Q}$ with the defined Radon-Nikodym density process $Z_t = E_t^{\mathbb{Q}}[\frac{d\mathbb{P}_S}{dQ}] = e^{-\int_0^t r(u)du} \frac{B(t,S)}{B(0,S)}$. So, find $dB(t; S, T)$ using Itô formula.

We have $dB(t; S, T) = \frac{B(t,T)}{B(t,S)}$, using Itô formula, we have $df(X(t)Y(t)) = f_X dX(t) + f_Y dY(t) + \frac{1}{2} f_{X,X} dX(t)dX(t) + \frac{1}{2} f_{Y,Y} dY(t)dY(t) + \frac{1}{2} f_{X,Y} dX(t)dY(t)$. After using the Itô multiplication table lemma $dW(t)dW(t) = dt$, $dtdW(t) = dW(t)dt = 0$, $dtdt = 0$, we have $df(X(t)Y(t)) = d(X(t)Y(t)) = X(t)dX(t) + Y(t)dY(t) + dX(t)dY(t)$. Simplify notation using X, Y, we have $d(XY) = XdY + YdX + dXdY$, so, $d(\frac{X}{Y}) = Xd(\frac{1}{Y}) + \frac{X}{Y} dX + dXd\frac{1}{Y}$.

Using Itô formula and Itô lemma calculate $d(\frac{1}{Y})$, $dXd\frac{1}{Y}$.
$d(\frac{1}{Y}) = f_Y dY + \frac{1}{2} f_{Y,Y}(dY)^2 = \frac{-1}{Y^2} dY + \frac{1}{Y^3}(dY)^2$,
$dXd\frac{1}{Y} = \frac{-1}{Y^2} dXdY + \frac{1}{Y^3} dX(dY)^2 = \frac{-1}{Y^2} dXdY$

Then, we have $d(\frac{X}{Y}) = X(\frac{-1}{Y^2}dY + \frac{1}{Y^3}(dY)^2) + \frac{X}{Y}dX + \frac{-1}{Y^2}dXdY$

Here, $d(\frac{X}{Y}) = d(\frac{B(t,T)}{B(t,S)})$ Substitute
$dX = dB(t,T) = r(t)B(t,T)dt + \Sigma(t,T)B(t,T)d\widetilde{W}(t)$,
$dY = dB(t,S) = r(t)B(t,S)dt + \Sigma(t,S)B(t,S)d\widetilde{W}(t)$,
$dXdY = dB(t,T)dB(t,S) = \Sigma(t,T)\Sigma(t,S)dt$,
$(dY)^2 = dB(t,S)dB(t,S) = \Sigma^2(t,S)dt$, simplify the equation,

we have $d(\frac{B(t,T)}{B(t,S)}) =$
$\frac{B(t,T)}{B(t,S)}\left(\Sigma^2(t,S)dt - \Sigma(t,T)\Sigma(t,S)dt + (\Sigma(t,T) - \Sigma(t,S))d\widetilde{W}(t)\right) =$
$\frac{B(t,T)}{B(t,S)}\left((\Sigma(t,T) - \Sigma(t,S))(d\widetilde{W}(t) - \Sigma(t,S)dt)\right)$

Under the equivalent probability measure $\mathcal{P}_S \sim \mathbb{Q}$ with the defined Radon-Nikodym density process $Z_t = E_t^{\mathbb{Q}}[\frac{d\mathbb{P}_S}{dQ}] = e^{-\int_0^t r(u)du}\frac{B(t,S)}{B(0,S)}$, the Itô process $(d\widetilde{W}(t) - \Sigma(t,S)dt)$ is a Brownian motion, denoted by $d\widetilde{W}^S(t)$.

Therefore, under the S-forward measure $\mathcal{P}_S \sim \mathbb{Q}$, the dynamics of the forward bond price process $B(t; S, T) = \frac{B(t,T)}{B(t,S)}$ is modelled by $dB(t; S, T) = ((\Sigma(t, T) - \Sigma(t, S)))\, B(t; S, T) d\widetilde{W}^S(t)$, where $\widetilde{W}^S(t)$ is a Brownian motion under the S-forward measure $\mathcal{P}_S \sim \mathbb{Q}$.

Solving the SDE can be analogously with solving the SDE $dS(t) = \sigma(t)S(t)dW(t)$. This SDE is solved as follows, 1st, find $dln(S(t))$ using the Itô formula and Itô lemma multiplication table. 2nd, Substitute $\frac{dS(t)}{S(t)}$ into $dln(S(t))$. 3rd, Integrate both sides of the obtained equation and find the formula of $S(t)$.

Using the Itô formula, we have $dln(S(t)) = df(S(t)) = f_X dS(t) + \frac{1}{2} f_{X,X}(dS(t))^2 = \frac{1}{S(t)} dS(t) - \frac{1}{2} \frac{-1}{S(t)^2}(dS(t))^2 = \frac{dS(t)}{S(t)} - \frac{1}{2}\left(\frac{dS(t)}{S(t)}\right)^2$. Using the Itô lemma multiplication table, we have $(dS(t))^2 = \sigma^2(t)S^2(t)(dW(t))^2 = \sigma^2(t)S^2(t)dt$, that is $(\frac{dS(t)}{S(t)})^2 = \sigma^2(t)dt$.

Substitute $\frac{dS(t)}{S(t)} = \sigma(t)dW(t)$, $(\frac{dS(t)}{S(t)})^2 = \sigma^2(t)dt$ into $dln(S(t)) = \frac{dS(t)}{S(t)} - \frac{1}{2}\left(\frac{dS(t)}{S(t)}\right)^2$, we have $dln(S(t)) = -\frac{1}{2}\sigma^2(t)dt + \sigma(t)dW(t)$. It can be seen that after applying the function $ln(x)$ to the asset process $S(t)$, the drift of price process has drawn down by $\frac{1}{2}\sigma^2(t)$.

Next, integrate both sides of the equation from time t to time T $\int_t^T dln(S(t)) = \int_t^T (-\frac{1}{2}\sigma^2(t))dt + \int_t^T \sigma(t)dW(t)$ and solve for $S(T)$ with a given value $S(t)$. We have $lnS(T) - lnS(t) = \int_t^T (-\frac{1}{2}\sigma^2(u))du + \int_t^T \sigma(u)dW(u) = \int_t^T (-\frac{1}{2}\sigma^2(u))du + \int_t^T \sigma(u)du \cdot \mathcal{N}(0,1)$. Then, we have $\frac{S(T)}{S(t)} = e^{\int_t^T \sigma(u)dW(u) - \frac{1}{2}\int_t^T \sigma^2(u)du} = e^{\int_t^T \sigma(u)du \cdot \mathcal{N}(0,1) - \frac{1}{2}\int_t^T \sigma^2(u)du}$, the second equation is also written as $\frac{S(T)}{S(t)} = e^{\int_t^T \sigma(u)du \cdot Y - \frac{1}{2}\int_t^T \sigma^2(u)du}$, with $Y \sim \mathcal{N}(0,1)$

Analogously, from the SDE
$dB(t;S,T) = (\Sigma(t,T) - \Sigma(t,S))\, B(t;S,T) d\widetilde{W}^S(t)$, with the given value $B(t;S,T)$, solve for $B(S;S,T)$. we have
$lnB(S;S,T) - lnB(t;S,T) =$
$-\frac{1}{2}\int_t^S (\Sigma(u,T) - \Sigma(u,S))^2\, du + \int_t^S (\Sigma(u,T) - \Sigma(u,S))\, dW(u)$,
then, we have $B(S;S,T) =$
$B(t;S,T) \cdot e^{\int_t^S (\Sigma(u,T) - \Sigma(u,S)) dW(u) - \frac{1}{2}\int_t^S (\Sigma(u,T) - \Sigma(u,S))^2 du} =$
$B(t;S,T) \cdot e^{\pm\int_t^S (\Sigma(u,T) - \Sigma(u,S)) du \cdot Y - \frac{1}{2}\int_t^S (\Sigma(u,T) - \Sigma(u,S))^2 du}$, with
$Y \sim \mathcal{N}(0,1)$.

Denote the volatility of the forward bond process by
$\xi(t,S) = \int_t^S (\Sigma(u,T) - \Sigma(u,S))^2\, du$, we have
$B(S;S,T) = B(t;S,T) e^{\pm\sqrt{\xi(t,S)} \cdot Y - \frac{1}{2}\xi(t,S)}$, with $Y \sim \mathcal{N}(0,1)$.

Back to the European call option on T-bond, with the payoff $(B(S,T)-K)^+$ at time S, in the S-forward probability measure $\mathbb{P}_S \sim \mathbb{Q}$ with the defined Radon-Nikodym density process, the option value at time t is
$V(t) = B(t,S) \cdot E^{\mathbb{P}_S}[(B(S,T)-K)^+|\mathcal{F}_t] = B(t,S) \cdot E^{\mathbb{P}_S}[(\frac{B(S,T)}{B(S,S)}-K)^+|\mathcal{F}_t] = B(t,S) \cdot E^{\mathbb{P}_S}[(B(S;S,T)-K)^+|\mathcal{F}_t]$

Now, calculate $E^{\mathbb{P}_S}[(B(S;S,T)-K)^+|\mathcal{F}_t]$. 1st, using independence lemma to change the conditional expectation to expectation. From the dynamic of the forward bond process $B(t;S,T)$ in the S-forward measure, $lnB(S;S,T) - B(t;S,T) = \int_t^S (\Sigma(u,T)-\Sigma(u,S))\, d\widetilde{W}^S(u) - \frac{1}{2}\int_t^S (\Sigma(u,T)-\Sigma(u,S))^2\, du$ contain an Itô integral which is independent of the available information $\mathcal{F}_t$. If given the initial value $B(t;S,T)$, $B(S;S,T)$ is independent of the filtration $\mathcal{F}_t$, and calculating the conditional expectation is simplified to calculating the expectation. Therefore, fix the value of $B(t;S,T)$, we calculate $E^{\mathbb{P}_S}[(B(S;S,T)-K)^+]$.

Now, calculate $E^{\mathbb{P}_S}[(B(S;S,T)-K)^+]$.

Substitute $B(S;S,T) = B(t;S,T)e^{\pm\sqrt{\xi(t,S)}\cdot Y - \frac{1}{2}\xi(t,S)}$, with $Y \sim \mathcal{N}(0,1)$ and calculate the expectation $\int_{-\infty}^{\infty}(B(t;S,T)e^{\pm\sqrt{\xi(t,S)}\cdot y - \frac{1}{2}\xi(t,S)} - K)^+ \cdot e^{\frac{-1}{2}y^2}\frac{1}{\sqrt{2\pi}}dy$.

1st, remove the $+$ sign, by letting $B(S;S,T) > K$, we have
$B(t;S,T)e^{\pm\sqrt{\xi(t,S)}\cdot y - \frac{1}{2}\xi(t,S)} > K$
$\leftrightarrow \frac{B(t;S,T)}{K} > e^{\mp\sqrt{\xi(t,S)}\cdot y + \frac{1}{2}\xi(t,S)}$
$\leftrightarrow \frac{1}{2}\xi(t,S) \mp \sqrt{\xi(t,S)}\cdot y < ln\frac{B(t;S,T)}{K}$, here, we would like the upper bound of y, so we choose
$\frac{1}{2}\xi(t,S) + \sqrt{\xi(t,S)}\cdot y < ln\frac{B(t;S,T)}{K}$, then, we have
$y < \frac{1}{\sqrt{\xi(t,S)}}\left(ln\frac{B(t;S,T)}{K} - \frac{1}{2}\xi(t,S)\right)$. Let
$d_- = \frac{1}{\sqrt{\xi(t,S)}}\left(ln\frac{B(t;S,T)}{K} - \frac{1}{2}\xi(t,S)\right)$

Then, we have

$$
\begin{aligned}
& E^{\mathbb{P}_S}[(B(S;S,T)-K)^+] \\
= \quad & \int_{-\infty}^{\infty}(B(t;S,T)e^{\pm\sqrt{\xi(t,S)}\cdot y-\frac{1}{2}\xi(t,S)}-K)^+\cdot e^{\frac{-1}{2}y^2}\frac{1}{\sqrt{2\pi}}dy \\
= \quad & \int_{-\infty}^{d_-}B(t;S,T)e^{-\sqrt{\xi(t,S)}\cdot y-\frac{1}{2}\xi(t,S)}\cdot e^{\frac{-1}{2}y^2}\frac{1}{\sqrt{2\pi}}dy \\
& -\int_{-\infty}^{d_-}K\cdot e^{\frac{-1}{2}y^2}\frac{1}{\sqrt{2\pi}}dy \\
= \quad & \int_{-\infty}^{d_-}B(t;S,T)e^{-\frac{-1}{2}(2\sqrt{\xi(t,S)}\cdot y+\xi(t,S)+y^2)}\cdot e^{\frac{-1}{2}y^2}\frac{1}{\sqrt{2\pi}}dy \\
& -\int_{-\infty}^{d_-}K\cdot e^{\frac{-1}{2}y^2}\frac{1}{\sqrt{2\pi}}dy
\end{aligned}
$$

$$= \int_{-\infty}^{d_-} B(t;S,T)e^{-\frac{1}{2}(y+\sqrt{\xi(t,S)})^2} \cdot \frac{1}{\sqrt{2\pi}} dy - K \cdot \mathcal{N}(d_-)$$

Next, using change of variable, let $z = y + \sqrt{\xi(t,S)}$, then, $dz = dy$. If $y = d_-$, then, $z = d_- + \sqrt{\xi(t,S)}$. Denoted by d_+, let $d_+ = d_- + \sqrt{\xi(t,S)}$. If $y = -\infty$, then, $z = -\infty$. Then, we have

$$\begin{aligned} & E^{\mathbb{P}_S}[(B(S;S,T) - K)^+] \\ = & \int_{-\infty}^{d_+} B(t;S,T)e^{-\frac{1}{2}z^2} \cdot \frac{1}{\sqrt{2\pi}} dz - K \cdot \mathcal{N}(d_-) \\ = & B(t;S,T) \cdot \mathcal{N}(d_+) - K \cdot \mathcal{N}(d_-) \end{aligned}$$

with
$d_+ = d_- + \sqrt{\xi(t,S)} = \frac{1}{\sqrt{\xi(t,S)}} ln \frac{B(t;S,T)}{K} - \frac{1}{2}\sqrt{\xi(t,S)} + \sqrt{\xi(t,S)} = \frac{1}{\sqrt{\xi(t,S)}} \left(ln \frac{B(t;S,T)}{K} + \frac{1}{2}\xi(t,S) \right)$

Therefore, the European call option on T-bond, with the payoff $(B(S,T)-K)^+$ at time S, in the S-forward probability measure $\mathbb{P}_S \sim \mathbb{Q}$ with the defined Radon-Nikodym density process, with given value $B(t;S,T)$ the option value at time t is
$V(t) = B(t,S) \cdot E^{\mathbb{P}_S}[(B(S;S,T)-K)^+|\mathcal{F}_t] =$
$B(t,S) \cdot E^{\mathbb{P}_S}[(B(t;S,T)e^{\pm\sqrt{\xi(t,S)}\cdot y - \frac{1}{2}\xi(t,S)} - K)^+] =$
$B(t,S) \cdot [B(t;S,T) \cdot \mathcal{N}(d_+) - K \cdot \mathcal{N}(d_-)] =$
$B(t,T) \cdot \mathcal{N}(d_+) - B(t,S) \cdot K \cdot \mathcal{N}(d_-)$, with
$d_+ = \frac{1}{\sqrt{\xi(t,S)}}\Big(ln\frac{B(t;S,T)}{K} + \frac{1}{2}\xi(t,S)\Big)$,
$d_- = \frac{1}{\sqrt{\xi(t,S)}}\Big(ln\frac{B(t;S,T)}{K} - \frac{1}{2}\xi(t,S)\Big)$. This is the analytical result for the price of European call option on T-bond, with the time S payoff claim $(B(S,T)-K)^+$.

In another way,