LECTURES ON SPECTRAL ALGEBRAIC GEOMETRY BY LIN CHEN

NOTE TAKER: HANA JIA KONG

Contents

1. Notations and Conventions

$\mathbf{CAlg}^{\heartsuit}$	the category of commutative rings
$\mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}^{\heartsuit}}$	the category of locally ringed space
$\mathbf{Ab}$	the category of abelian groups
AG	algebraic geometry
POV	point of view
FOP	functor of points

2. Oct 5: Lecture 1

2.1. Motivation and overview.

2.1.1. *Why algebraic geometry (AG)?.* The first question is: why we care about AG. There are mainly two reasons.

(1) Firstly, there are intrinsic problems in algebraic geometry.
(2) What's more important, AG provides tools for problems from other subjects. The "Motto" is that, whenever you have a commutative algebra object A (possibly coming from analysis/geometry/topology), namely an object in $\mathbf{CAlg}^{\heartsuit} := \mathbf{CAlg}(\mathbf{Ab})$, you can attach an "affine scheme" $\mathrm{Spec}(A)$ to it. It allows geometric methods to study the original problem.

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2.1.2. *Some tools in AG.* There are many tools in AG, mostly due to Grothendieck.

(1) Commutative algebras.
However, there's no geometry if you use purely commutative algebra methods.
(2) Non-affine AG objects.
projective curves $\mathbb{P}^n$; elliptic curves.
(3) Relative point of view (POV): The relative POV says that you should study not the object itself, but the morphisms out of it.
Here are some examples.
   (a) For any map $X \to Y$ between AG objects X and Y, we usually view it as a family of AG objects parametrized by Y.
   (b) Moduli problems: Let (P) be a property about AG objects (again, (P) possibly originates in other fields) In lots of cases, there is a universal family $\mathcal{X}_{(\mathsf{P})} \to \mathcal{M}_{(\mathsf{P})}$ of (P)-objects.
   This turn the study of "(P) objects" into the study of "the universal morphism $\mathcal{X}_{(\mathsf{P})} \to \mathcal{M}_{(\mathsf{P})}$".

   Example 2.1. There is a "space" called the moduli stack of elliptic curves, denoted by $\mathcal{M}_{1,1}$, with a universal morphism $\mathcal{E}_{1,1} \to \mathcal{M}_{1,1}$ which classifies elliptic curves. It also has a spectral thickening $\mathcal{M}_{1,1}^{\mathsf{der}}$ where "der" means "derived". This relates elliptic cohomology, structures like orientations etc., and leads to objects like topological modular forms.

(4) There are different sheaf theories: $\mathbf{QCoh}, \mathbf{IndCoh}, \mathbf{Sh}_{\text{ét}}, \mathbf{Sh}_{\mathsf{Nis}}, \mathbf{DMod}$ where you can perform six functors yoga.

2.2. **A crash course on AG.**

2.2.1. *Affine schemes.* We have two POV for AG objects. They are:
1) geometric POV, and
2) functors of points (FOP) POV.

FOP POV

In FOP POV, we take the following definition for affine schemes.

Definition 2.2. The category of affine schemes $\mathbf{AffSch}$ is the opposite category of commutative algebras $\mathbf{CAlg}^{\heartsuit\mathsf{op}}$. Namely, we have the following correspondence:

$$\begin{array}{ccc} \mathbf{AffSch} & \mathrm{Spec}(A) & \mathrm{Spec}(B) \to \mathrm{Spec}(A) \\ \| \scriptstyle{\mathsf{def}} & \updownarrow & \updownarrow \\ \mathbf{CAlg}^{\heartsuit\mathsf{op}} & A & A \to B \end{array}$$

Geometric POV

In geometric POV, we have a more concrete definition. We define affine schemes to be some kind of locally ringed spaces, i.e. we have $\mathbf{AffSch} \subset \mathbf{Top}^{\mathsf{loc}}_{\mathbf{CAlg}^{\heartsuit}}$.

We first introduce the definition of locally ringed space.

Definition 2.3. A *ringed space* X is a pair $(|X|, \mathcal{O}_X)$ with an underlying topological space $|X|$ (also denoted by X for notational convenience) and a structure sheaf $\mathcal{O}_X$ on X.

A *locally ringed space* is a ringed space with the property that for any point $p \in X$, the stalk over it

$$\mathcal{O}_{X,p} := \lim_{p \in U} \mathcal{O}_X(U) \in \mathbf{CAlg}^{\heartsuit}$$

is local, namely, has a unique maximal ideal $m_{x,p}$.

Definition 2.4. A map between locally ringed spaces X, Y is the following data

(1) a map between the underlying spaces $f : |X| \to |Y|$, and
(2) a map of sheaves over X: $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$,
(3) with the property that it sends the maximal ideal of any stalk to the corresponding maximal ideal.

We spell out several things in the above definition:

(1) The definition of the pullback functor f^{-1} on sheaves:
$$f^{-1}\mathcal{O}_Y(U) := \lim_{f(V)\supset U} \mathcal{O}_Y(V).$$
(2) In fact, there is an adjunction $f^{-1} \dashv f_*$ where f_* denotes the pushforward functor, defined by
$$f_*\mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U)).$$
Therefore, the map in (2) is equivalently a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves over Y by adjunction.
(3) It's worth to point out that f^{-1} preserves stalk: $(f^{-1}\mathcal{O}_Y)_p \simeq \mathcal{O}_{Y,f(p)}$.
(4) The condition (3) requires the following commutative diagram for every $p \in X$:
$$\begin{tikzcd} & m_{Y,f(p)} \arrow[r, dashed] \arrow[d, hook] & m_{X,p} \arrow[d, hook] \\ \mathcal{O}_{Y,f(p)} \simeq & (f^{-1}\mathcal{O}_Y)_p \arrow[r] & \mathcal{O}_{X,p} \end{tikzcd},$$
where the bottom arrow is the induced map given by the map in (2).

Remark 2.5. Each locally ringed space is in particular a ringed space. However, the definition of a map between ringed spaces does not require (3) in Definition 2.4. As a result, the inclusion from the category of locally ringed spaces to ringed spaces is not fully faithful (viewed as ordinary categories, the inclusion is faithful.)

Example 2.6. Here is an intuitive example to keep in mind when thinking about these structure sheaves.

Let X be a topological space. Take $\mathcal{O}_X(U)$ to be the ring of regular functions on U (with the ordinary addition and multiplication). Then $\mathcal{O}_{X,p}$ is the germ of regular functions defined near p, namely the ring of all functions defined near p modulo the equivalence relation that two functions are equivalent if they coinside on a small neighboughood of p. And the maximal ideal $m_{X,p}$ is $\{\phi \in \mathcal{O}_{X,p} | \phi(p) = 0\}$.

Using the above example, the condition (3) in Definition 2.4 is a reasonable requirement: given $f : X \to Y$, a regular function ϕ on Y vanishes on $f(p)$ iff $f^{-1}\phi = \phi \circ f$ vanishes on p.

2.2.2. $\mathrm{Spec}(A)$. Let A be a commutative algebra. By FOV POV, A corresponds to a locally ringed space $\mathrm{Spec}(A)$.

Definition 2.7. The spec of A is defined to be the locally ringed space with:

- the underlying space: the space of all prime ideals of A with Zariski topology: the set $\{\mathrm{Spec}(A[f^{-1}]), f \in A^\times\}$ forms a basis of open sets.
  Exercise: Proof $\mathrm{Spec}(A[f^{-1}])$ is the space of $\{$prime ideal $p \subset A | f \notin p\}$.
- the structure sheaf: $\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A[f^{-1}])) := A[f^{-1}]$.

Remark 2.8. For a general open set, $\mathcal{O}_{\mathrm{Spec}(A)}(U) = \lim_{U \supset \mathrm{Spec}(A[f^{-1}])} A[f^{-1}]$. Note that not every assignment on open set basis gives a well-defined sheaf. It is actually a theorem by Serre (Serre's Lemma) that there exists such a unique sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$.

Example 2.9.

(1) Visualize the following affine schemes: $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(k)$, $\mathrm{Spec}(\mathbb{C}[t])$.

(2) Identify $\mathrm{Spec}(A\times B)$ and $\mathrm{Spec}(A\otimes B)$ where A, B are ring.
Answer: Notice that $\otimes$ is the coproduct in $\mathbf{CAlg}^\heartsuit$, and $\times$ is the product. So $\mathrm{Spec}(A\times B)\simeq \mathrm{Spec}(A)\amalg \mathrm{Spec}(B)$, and $\mathrm{Spec}(A\otimes B)\simeq \mathrm{Spec}(A)\times \mathrm{Spec}(B)$.
(3) The spectrum of power series $\mathrm{Spec}(\mathbb{C}[[t]])$: it is called an adic disk and models infinitely small disks around the original point. By definition this has two points.
(4) The spectrum of Laurent series $\mathrm{Spec}(\mathbb{C}((t)))$: it is called a punctured disk and models infinitely small disks around the original point with the original point punctured. By definition it has one point.
(5) The first order fat point $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$: it models "a point + tangent vector". By definition it has one point, and will be explained in future lectures.

2.2.3. *Schemes.*

Definition 2.10. The category of schemes $\mathbf{Sch}$ is a full subcategory of locally ringed spaces $\mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}^\heartsuit}$, with objects $(X,\mathcal{O}_X)$ that are locally equivalent to an affine scheme $\mathrm{Spec}(A)$ for some $A\in\mathbf{CAlg}^\heartsuit$m.

We have dual correspondence between geometry and algebra:

$$\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \cong \mathrm{Hom}(\mathrm{Spec}(A),\mathrm{Spec}(\mathbb{Z}[t])) \text{ ,and}$$
$$A \cong \mathrm{Hom}(\mathbb{Z}[t],A),$$

where the ccolors stand for geometric and FOP POV.

3. Oct 8: Lecture 2

Exercise: This is actually from last time. Prove the nontrivial fact that using the geometric definition, we have

$$\mathrm{Hom}(\mathrm{Spec}(A),\mathrm{Spec}(B)) \simeq \mathrm{Hom}(B,A).$$

3.1. **Schemes.** We defined schemes from last time as a full subcateogry of $\mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}^\heartsuit}$.

Remark 3.1. The inclusion $\mathbf{Sch}\to\mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}^\heartsuit}$ preserves colimits. This is also true in the spectral AG. However, note that the inclusion $\mathbf{AffSch}\hookrightarrow\mathbf{Sch}$ does not preserve colimits. For example, coproducts of affine schemes correspond to products in rings. So an infinite coproduct in $\mathbf{AffSch}$ is still affine, and thus qc and separated (see Definition 3.7). But in $\mathbf{Sch}$, the underlying space of a coproduct is the coproduct of the underlying spaces, and therefore an infinite coproduct is usually not qc.

Example 3.2. Or more precisely, Exercise:
(1) Try to define $\mathbb{P}^1$ (hint: by gluing). This is the most important non-affine example.
(2) Another important non-affine example: try to define $\mathbb{A}^2\backslash\{(0,0)\}$.
(3) A scheme X can be written as a colimit of affine schemes and open immersions. (This is true either you take colimits in $\mathbf{Sch}$ or $\mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}^\heartsuit}$ by Remark 3.1.)

3.1.1. *Aside: open and close immersions.*

Definition 3.3. A map $j: X\to Y$ of schemes is
- an open immersion if
  (1) it is open embedding on the underlying spaces, and
  (2) $j^{-1}\mathcal{O}_Y\to\mathcal{O}_X$ is an isomorphism.
- a closed immersion if
  (1) a closed embedding on the underlying spaces, and
  (2) $j^{-1}\mathcal{O}_Y\to\mathcal{O}_X$ is a surjection.

Remark 3.4. Warning: in the spectral AG, a closed immersion does not require the surjection condition b). In fact, being surjective is not very well defined in the spectral AG context. The spectral definition is as follows: $f : X \to Y$ is a closed immersion of spectral schemes if $(X, \pi_0(O_X)) \to (Y, \pi_0(O_Y))$ is a closed immersion of schemes.

Example 3.5.
- The map $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]/\epsilon^2)$ induced by $\epsilon \mapsto 0$ is a closed immersion.
- The map $\mathrm{Spec}(k[\epsilon]/\epsilon^2) \to \mathrm{Spec}(k)$ induced by the inclusion $k \hookrightarrow k[\epsilon]/\epsilon^2$ is not a closed immersion.
- The following maps are open immersions.
  1. $\mathrm{Spec}(A)[f^{-1}] \hookrightarrow \mathrm{Spec}(A)$.
  2. $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$.
  3. The map from the moduli stack of elliptic curves to the compactified moduli stack of elliptic curves $M_{1,1} \to \overline{M}_{1,1}$.

The following is technically a proposition but we take it as the definition.

Definition 3.6. A scheme X is quasi-compact and quasi-separate (qcqs) if X can be written as a finite colimit of affine schemes and open immersions.

Traditionally, the definition of qcqs is as follows:

Definition 3.7. A scheme X is
- qc if the underlying space is quasi-compact.
- qs if the diagonal map of the underlying space $X \to X \times X$ is qc (which is equivalent to that the inverse image of a qc open subspace is still qc, and again equivalent to that $U \cap V$ is qc for any qc $U, V \subset X$.
- separated if the diagonal $X \to X \times X$ is a closed immersion (equivalently, when the induced map on the underlying space is a closed immersion; this equivalence is some special property of the diagonal map).

Remark 3.8.
- It is convenient to use the finiteness proposition as the definition, since practically that's what we need and use for qcqs schemes.
- Being separated implies quasi-separated.

Exercise:
- Prove the finiteness definition of qcqs is equivalent to the traditional one.
- Construct a non-separated scheme (hint: gluing; it looks like identifying two lines except at one single point.)

3.2. FOP POV of schemes.

In the FOP POV, we first define a larger category, the category of prestacks $\mathbf{PreStk}$, using affine schemes. Then we characterize $\mathbf{Sch}$ as a full subcategory of $\mathbf{PreStk}$.

3.2.1. *Prestacks.*

Definition 3.9. The category of prestacks $\mathbf{PreStk}$ is defined to be the category of accessible functors from $\mathbf{AffSch}^{\mathrm{op}}$ to the category of not necessarily small ∞-groupoids $\hat{\mathcal{S}}$, namely:

$$\mathbf{PreStk} := \mathrm{Fun}_{\mathrm{acc}}(\mathbf{AffSch}^{\mathrm{op}}, \hat{\mathcal{S}}).$$

Remark 3.10.
1. Note that we enlarge the category of ∞-groupoids here. There would be some set theoretical problem if we did not require accessibility or took the small category).
2. This is an example of why we work ∞-categorically: Artin stacks form a 2-category; Similarly, n-Artin stacks form an $(n+1)$-category. After put toghether they form an infinity category.)

Exercise: (a very non-trivial one): There is a fully faithful functor

$$F : \mathbf{Sch} \to \mathbf{PreStk}$$
$$X \mapsto (S \mapsto \mathrm{Hom}_{\mathbf{Sch}}(S, X) \in \mathbf{Set}).$$

Remark 3.11. (1) This fully faithful embedding preserves small limits (sufficient to check: it holds for finite limit and filtered lim and inf product?). But warning: it does not preserve small colimits. Note that the Yoneda embedding is for $\mathbf{AffSch}$ but not $\mathbf{Sch}$, and $\mathbf{AffSch} \to \mathbf{Sch}$ does not preserve colimits.

(2) $\mathbf{PreStk}$ contains all small colimits and limits. This follow directly from the fact that it is a presheaf category.

3.2.2. $\mathbf{Sch}$ *as a subcategory of* $\mathbf{PreStk}$.

The idea of characterizing $\mathbf{Sch}$ as a subcategory of $\mathbf{PreStk}$ is to find some conditions such that objects satisfying these conditions are exactly schemes. Among these conditions, the most important condition is "descent".

We have the following fact.

Example 3.12. A scheme X, viewed as an object in $\mathbf{PreStk}$, satisfies descent condition under the Zariski Grothendieck topology on $\mathbf{AffSch}$.

We spell out the terminologies in this example: Zariski Grothendieck topology, and descent.

Definition 3.13. The Zariski Grothendieck topology on $\mathbf{AffSch}$ is defined as follows: a morphism $S \to T$ is a covering if

- it is a surjection, and
- each connected component S_i of S openly embedded into T, namely, $S_i \hookrightarrow S \to T$ is an open immersion.

Remark 3.14. The way to think about this (not as rigorous):

(1) we have $S = \amalg_i S_i$, and
(2) each S_i is an open subscheme of T and all together they cover T.

Definition 3.15. Let X be a product preserving presheaf on a category $\mathcal{C}$ equipped with a Grothendieck topology. We say X satisfies descent condition, if for any covering $S \to T \in \mathcal{C}_1$, the map

$$X(T) \to \mathrm{Tot}\, X(S^\bullet) := \mathrm{Tot}[X(S) \rightrightarrows X(S \times_T S) \substack{\rightarrow\\\rightarrow\\\rightarrow} X(S \times_T S \times_T S) \cdots]$$

is an equivalence.

Remark 3.16. This happens in ∞-categorical context. In the ordinary category context, the above diagram is truncated at the second stage, namely, the right hand side is replaced with the equalizer of $X(S) \rightrightarrows X(S \times_T S)$.

Remark 3.17. This is a simplified version of the most common definition of the descent condition.

The definition of Grothendieck topology requires not only the data of covering maps, but also what it means to be a cover. When the category $\mathcal{C}$ has coproducts, it is usually required that the covering maps $\{S_i \to T\}_{i \in I}$ forms a cover iff $\coprod S_i \to T$ forms a cover. For example, this is satisfied in $\mathbf{AffSch}$ under Zariski topology. This allows us to simplify the statement of descent condition to Definition 3.15 under the extra assumption that the presheaf is product preserving.

Convension: Let (P) denote any Grothendieck topology. If a presheaf satisfies (P) decent, we sometimes say it is a sheaf in (P)-topology.

Definition 3.18. The category of schemes $\mathbf{Sch}$ is the full subcategory of $\mathbf{PreStk}$ containing objects that satisfying:

- Zariski descent, and
- some other things that are not as essential.

Exercise: Verify that schemes defined in the geometric POV satisfy Zariski descent.

Example 3.19. Prestacks that are not schemes:

(1) Consider the formal disk: $\mathrm{Spf}(k[[t]]) := \mathrm{colim}\,\mathrm{Spec}(k[t]/t^n)$. Here spef functor $\mathrm{Spf}(-)$ is the topological version of $\mathrm{Spec}(-)$, taken over "topological" rings.
Assume this is a scheme. Then the colimit exists in $\mathbf{Sch}$. Therefore $\mathrm{Spf}(k[[t]]) := \mathrm{colim}\,\mathrm{Spec}(k[t]/t^n) = \mathrm{Spec}(k[[t]])$. For notational convenience, we denote this object by S. We have $S(S) := \mathrm{Hom}_{\mathbf{Sch}}(S, S) = \mathrm{colim}\,\mathrm{Spec}(k[t]/t^n)(S)$. However, the LHS contains id which can not factor through $\mathrm{Spec}(k[t]/t^n)$ for any n. Contradiction.
Here we use the facts that $\mathbf{Sch} \hookrightarrow \mathbf{PreStk}$ is f.f., and that in this case, the colimit in $\mathbf{Sch}$ is $\mathrm{Spec}(k[[t]])$ as if it is taken in $\mathbf{AffSch}$.

(2) Let G be a group scheme over $\mathrm{Spec}(R)$, i.e. it comes with a multiplication map $G \times G \to G$ and a unit map $\eta : \mathrm{Spec}(R) \to G$ fitting in the expected commutativity diagrams. We define

$$\mathbb{B}G := \mathrm{colim}[\mathrm{Spec}(R) \leftleftarrows G \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} G \times G \cdots].$$

Alternatively, it can be defined as

$$\mathbb{B}G(S) := \mathbb{B}(G(S)),$$

where $G(S) := \mathrm{Hom}_{\mathbf{Sch}}(G, S)$ is a group with structures induced by structures on G, and $\mathbf{B}(G(S))$ is the associated ∞-groupoid. Therefore it is not a scheme. (Also, it does not satisfy fpqc descent which is stronger than Zariski descent and satisfied by all schemes.)

3.2.3. *Stacks.*

Definition 3.20. The fppf topology on $\mathbf{AffSch}$ is defined as follows: a map $S = \mathrm{Spec}(A) \to T = \mathrm{Spec}(B)$ is a covering if the corresponding ring map $B \to A$ is faithfully flat and finitely presented.

Definition 3.21. The category of (fppf) stacks $\mathbf{Stk}$ is a full subcategory of $\mathbf{PreStk}$ with objects satisfying fppf descent.

Remark 3.22. There are definitions of stacks under different topologies. Usually people mean fppf stacks when the topology is not specified.

4. Oct 11: Lecture 3

4.1. More Grothendieck topologies on AffSch.

There are different Grothendieck topologies on $\mathbf{AffSch}$:

$$\text{Zariski} \subset \text{étale} \subset \text{smooth} \subset \text{fppf} \subset \text{fpqc}.$$

Definition 4.1. A morphism $p : X \to S \in \mathbf{Sch}$ is *formally smooth* if for any $A \in \mathbf{CAlg}^{\heartsuit}$ and ideal I with $I^2 = 0$, there is a lifting in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}.$$

Exercise: The map $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ with $I^2 = 0$ is a homeomorphism of the topological spaces, for example $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]/\epsilon)$.

Example 4.2. To better understand the definition of being smooth, we introduce the following example. Let A be the ring of dual numbers $k[\epsilon]/\epsilon^2$, and let I be $\langle\epsilon\rangle$. Consider the following lifting diagram.

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "x"] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{Spec}(k[\epsilon]/\epsilon^2) \arrow[r, "(s,\eta)"] \arrow[ur, dashed] & S \end{tikzcd}.$$

The top horizontal arrow is equivalent to picking a point $x \in X$, and the bottom horizontal arrow is equivalent to picking a point $s \in S$ as well as a tangent vector η at s. By commutativity of the square, we have $p(x) = s$, namely x is a lift of s. And the lifting map is asking for a lift of η at x.

Briefly, being formally smooth says the map is locally a surjection between spaces with a tangent. This is analogues to submersion between manifolds.

Definition 4.3. A morphism $p : X \to S \in \mathbf{Sch}$ is *formally étale* if it is smooth and each required lifting is unique.

In other words, being formally étale means the map is locally an isomorphism between spaces with a tangent.

Definition 4.4. A morphism $p : X \to S \in \mathbf{Sch}$ is smooth (resp. étale) if it is formally smooth (resp. étale) and p is finitely presented.

Definition 4.5. A morphism $p : X \to S \in \mathbf{Sch}$ is finitely presented if it is

(1) quasi-compact, and
(2) locally of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ when $A \to B$ is finitely presented.

Remark 4.6. In $\mathbf{AffSch}$, being quasi-compact is automatic. Therefore a finitely presented morphism of affine schemes only requires the corresponding ring map is finitely presented.

Exercise:

(1) Show $\mathrm{Spec}(R)[t] \to \mathrm{Spec}(R)$ is smooth.
(2) Show $\mathbb{A}^1 \xrightarrow{Sq} \mathbb{A}^1 : t \mapsto t^2$ is smooth but not étale.
(3) Show $\mathbb{A}^1\backslash 0 \xrightarrow{Sq} \mathbb{A}^1\backslash 0 : t \mapsto t^2$ is smooth and étale.

Definition 4.7. A *sieve* $\mathcal{C} \subset \mathbf{AffSch}_{/S}$ is a covering sieve under topology (P) on $S \in \mathbf{AffSch}$ if there exist finitely many objects $\{T_i \to S\}_{i\in I}$ in $\mathcal{C}$ such that

- $\{T_i \to S\}_{i\in I}$ is jointly surjective, and
- each $T_i \to S$ is a (A), where the correspondence between (A) and (P) are as in the following table.

A	open embedding	étale	smooth	flat and fp	flat
P	Zariski	étale	smooth	fppf	fpqc

Remark 4.8. We have inclusions of topologies, the latter the finer (more opens).

Zariski ⊂ étale ⊂ smooth ⊂ fppf ⊂ fpqc.

The inclusion "smooth ⊂ fppf" is non-obvious.

Recall the definition of (abstract, or fppf) stacks (See Definition 3.21) .

Definition 4.9. An (fppf, or abstract) stack is a fppf-sheaf on $\mathbf{AffSch}$.

Remark 4.10. The fppf descent is for defining abstract stacks. An algebraic stack is an abstract stack satisfying some other nice properties. In literature, people require different nice properties in definitions of algebraic stacks. The actual definitions are for building theories; practically, you take the essential properties as definitions.

For example, we have a sequence of inclusions of categories of different algebraic stacks.

$$\mathbf{Sch} \subset \text{Algebraic spaces} \subset \text{Deligne-Mumford stack} \subset 1\text{-Artin stack} \subset 2\text{-Artin stack} \subset \cdots$$

In fact, it not obvious that schemes are stacks. In other words, $\mathrm{Hom}(-, X)$ satisfies not only Zariski descent condition, but also fppf descent condition. This is due to Grothendieck in [FGA].

Remark 4.11. Scheme is some sort of locally ringed spaces. And algebraic space is some sort of locally ringed topos, so are DM stacks. Starting from Artin stacks, things are mostly defined using FOP POV; it's harder to make definitions using geometric POV.

Exercise: Read the proof of the following statement: any étale sheaf is a smooth sheaf.

4.2. Quasi-coherent sheaves.

Definition 4.12. Define the category of quasi-coherent sheaves and its heart to be

$$\mathbf{QCoh}(\mathrm{Spec}(A)) := A\text{-}\mathbf{Mod} \simeq D(A\text{-}\mathbf{Mod})^{\heartsuit}, \text{ and}$$
$$\mathbf{QCoh}(\mathrm{Spec}(A))^{\heartsuit} := A\text{-}\mathbf{Mod}^{\heartsuit}.$$

Remark 4.13. It's okay to to take either A-$\mathbf{Mod}(\mathbf{Sp})$ or A-$\mathbf{Mod}(\mathbf{Ab})$. Since A is an abelian group, A-$\mathbf{Mod}(\mathbf{Sp})$ and A-$\mathbf{Mod}(\mathbf{Ab})$ agrees.

Definition 4.14. Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ (equivalently, $f : A \to B$) be a map between affine schemes.

$$f^* : \mathbf{QCoh}(\mathrm{Spec}(A)) = A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod} = \mathbf{QCoh}(\mathrm{Spec}(B))$$
$$M \mapsto B \otimes_A M.$$

There is a adjunction $f^* \dashv f_*$ where f_* is the functor:

$$f_* : A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod} : M \mapsto M.$$

Now we have a functor $\mathbf{QCoh}$ such that

$$\mathbf{QCoh} : \mathbf{AffSch}^{\mathrm{op}} \to \mathbf{Pr}^{\mathrm{L,st}}$$
$$\mathrm{Spec}(A) \mapsto A\text{-}\mathbf{Mod}$$
$$f \mapsto f^*$$

where the target is the category of presentable stable categories with colimit-preserving functors.

Definition 4.15. Define the functor $\mathbf{QCoh} : \mathbf{PreStk} \to \mathbf{Pr}^{\mathrm{L,st}}$ to be

$$\begin{tikzcd} \mathbf{AffSch}^{\mathrm{op}} \arrow[r, "\mathbf{QCoh}"] \arrow[d] & \mathbf{Pr}^{\mathrm{L,st}} \\ \mathbf{PreStk}^{\mathrm{op}} \arrow[ur, dashed, "\mathrm{RKE}"'] & \end{tikzcd}.$$

Where the dashed arrow is given by right Kan extension (RKE) which we abuse notation and denote again by $\mathbf{QCoh}$.

Remark 4.16. A quasi-coherent sheaf $\mathcal{F}$ on Y is equivalent to the following data:

- $\mathcal{F}_S \in \mathbf{QCoh}(S)$ for any $\mathbf{AffSch} \ni S \to Y$, and

- for any morphism of affine schemes over Y

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & T \arrow[dl] \\ Y & \end{tikzcd},$$

  an equivalence $f^*\mathcal{F}_T \xrightarrow{\sim} \mathcal{F}_S$, and
- higher compatibilities.

Remark 4.17. Since there is high conherence, it's hard to avoid ∞-categories if you want to define $\mathbf{QCoh}(Y)$ even only on Artin stacks.

Theorem 4.18 (Grothendieck)**.** *The category of quasi-coherent sheaves* $\mathbf{QCoh}$ *satisfies fpqc decsent, namely the functor* $\mathbf{QCoh} : \mathbf{AffSch}^{op} \to \mathbf{Pr}^{L,st}$ *viewed as a presheaf on* $\mathbf{AffSch}$ *is a fpqc sheaf.*

Remark 4.19. The formula for RKE involves all objects in $\mathbf{AffSch}$ and is thus incomputable. However, with Theorem 4.18, we can use fpqc cover and descent condition to compute $\mathbf{QCoh}$, namely, $\mathbf{QCoh}(X) = \mathrm{Tot}[\mathbf{QCoh}(Y) \rightrightarrows \mathbf{QCoh}(Y \times_X Y) \cdots]$ where $Y \to X$ is a fpqc cover.

Theorem-Definition 4.20. *Let* $\mathcal{C}$ *be a category with a Grothendieck topology. The Grothendieck topology on* $\mathcal{C}$ *can be canonically extended to a Grothendieck topology on* $P(\mathcal{C})$.

The induced topology is defined as follows.

A collection of morphisms $J = \{Y_i \to Y\}_i \subset P(C)_{/Y}$ *form a covering family on* $Y \in P(\mathcal{C})$ *if it is a representable covering family, namely, under base change, the image of* $J \times_{P(\mathcal{C})_{/S}} \mathcal{C}_{/S} \to \mathcal{C}_{/S}$ *is a covering family for any* $S \in \mathcal{C}$. *Here the pullback is taken along the inclusion and the map* $J \to P(\mathcal{C})_{/S} : (Y_i \to Y) \mapsto (S \times_Y Y_i \to S)$ *(so the image contains morphisms of the form* $S \times_Y Y_i \to S$ *with the domain in* $\mathcal{C}$.*)*

Remark 4.21. The above definition uses covering families and Grothendieck pretopology. In the language of covering sieves, the induced Grothendieck topology on $P(\mathcal{C})$ can be defined as follows.

A sieve $J = \{Y_i \to Y\} \subset P(\mathcal{C})_{/Y}$ is a covering sieve on $Y \in P(\mathcal{C})$ if for any $\mathcal{C} \ni S \to Y$, the sieve generated by the image of $J \times_{P(\mathcal{C})_{/S}} \mathcal{C}_{/S} \to \mathcal{C}_{/S}$ (i.e. the smallest sieve containing the image) is a covering sieve on S.

Note that the image of the base change is not necessarily a sieve; it may not be closed under precomposition.

Corollary 4.22. *Using the induced topology on* $\mathbf{PreStk}$, $\mathbf{QCoh} : \mathbf{PreStk}^{op} \to \mathbf{Pr}^{L,st}$ *satisfies fpqc descent.*

5. Oct 15: lecture 4

What is SAG (some words from Jacob):

- SAG is AG plus $\mathbb{E}_\infty$-rings.
- SAG is AG plus ∞-categories.
- SAG is AG plus deformation theory

Notation:

- Denote by $(\mathbf{Sp}, S, \otimes)$ the symmetric monoidal stable presentable category of spectra. This category is equipped with a t-structure with $\mathbf{Sp}^\heartsuit = \mathbf{Ab}$, but note that $\mathbf{Sp} \neq D(\mathbf{Ab})$.
- Denote by $\mathbf{CAlg}$ the category of $\mathbb{E}_\infty$-algebras in the category of $\mathbf{Sp}$. We have that

$$\mathbf{CAlg}^\heartsuit := \mathbf{CAlg} \times_{\mathbf{Sp}} \mathbf{Sp}^\heartsuit,$$

namely the category of ordinary rings is the category of $\mathbb{E}_\infty$-algebras in the category of $\mathbf{Sp}^\heartsuit$.

- Denote by $\mathbf{CAlg}^{\mathsf{cn}} := \mathbf{CAlg}^{\leq 0} = \mathbf{CAlg}_{\geq 0}$ the connective part. Note that we use the supscript for the cohomological convention and the subscript for the homological convention.

5.1. FOP POV.

Definition 5.1. Define the category of spectral affine schemes to be the oppsite category

$$\begin{array}{c} \mathbf{AffSpSch} := (\mathbf{CAlg}^{\mathsf{cn}})^{\mathsf{op}} \\ \mathrm{Spec}(A) \leftarrow\!\shortmid A \end{array},$$

and define the connective analog to be

$$\mathbf{AffSpSch}^{\mathsf{nc}} := (\mathbf{CAlg})^{\mathsf{op}}.$$

Here "nc" is for "non-connective".

Example 5.2. The element $\mathrm{Spec}(S)$ is the final object.

5.2. Geometric POV.

Definition 5.3. Denote by $\mathbf{Top}_{\mathbf{CAlg}}$ the category of $\mathbf{CAlg}$-valued ringed spaces (spectral ringed space). An object is a pair $(X, \mathcal{O}_X)$ where X is in $\mathbf{Top}$ and $\mathcal{O}_X$ is a $\mathbf{CAlg}$-valued sheaf, namely $\mathcal{O}_X \in \mathbf{Shv}_{\mathbf{CAlg}}(X)$.

Define the spectral locally ringed space to be the fiber product

$$\mathbf{Top}^{\mathsf{loc}}_{\mathbf{CAlg}} := \mathbf{Top}_{\mathbf{CAlg}} \times_{\mathbf{Top}_{\mathbf{CAlg}^\heartsuit}} \mathbf{Top}^{\mathsf{loc}}_{\mathbf{CAlg}^\heartsuit}.$$

Note that by definition, a spectral ringed space $(X, \mathcal{O}_X)$ is local if $(X, \pi_0\mathcal{O}_X)$ is local.

Remark 5.4. For $\mathcal{F} \in \mathbf{Shv}_{\mathbf{Sp}}(X)$, we denote by $\pi_n\mathcal{F} \in \mathbf{Shv}_{\mathbf{Ab}}(X)$ the homotopy π_n sheaf. It is the sheafification of the presheaf $U \mapsto \pi_n(\mathcal{F}(U))$.

Warning:

- Note that for $\mathcal{F} \in \mathbf{Shv}_{\mathbf{Sp}}(X)$, the functor $U \mapsto \pi_n(\mathcal{F}(U))$ is just a presheaf. You need to sheafify to get the corresponding homotopy sheaf.
- On the other hand, note that the functor $\mathbf{Psh}_{\mathbf{Ab}}(X) \to \mathbf{Psh}_{\mathbf{Sp}}$ induced by $\mathbf{Ab} \hookrightarrow \mathbf{Sp}$ does not send sheaves to sheaves. Because the inclusion of the heart does not preserve limits.

The above warnings lead to the following example.

Example 5.5. For $\mathcal{F} \in \mathbf{Shv}_{\mathbf{Sp}}(X)$, the homotopy sheaf $\pi_n\mathcal{F}$ can vanish even if $\pi_n(\mathcal{F}(U)) \in \mathbf{Ab}$ does not vanish for some U, and vice versa.

Remark 5.6. We can restrict to $\mathbb{E}_\infty$-algebras. If $\mathcal{F} \in \mathbf{Shv}_{\mathbf{Sp}}(X)$ takes value in $\mathbf{CAlg}$, the homotopy sheaf $\pi_0\mathcal{F} \in \mathbf{Shv}_{\mathbf{Ab}}(X)$ can be lifted to an object in the $\mathbf{Shv}_{\mathbf{CAlg}^\heartsuit}(X)$.

Definition 5.7. Define *the category of connective spectral sheaves* $\mathbf{Shv}^{\mathsf{cn}}_{\mathbf{CAlg}}$ to be the full subcategory of $\mathbf{Shv}_{\mathbf{CAlg}}$ containing the connective objects, namely objects $\mathcal{O}$ with property $\pi_n\mathcal{O} = 0$ for any $n < 0$.

Definition 5.8. Define *the category of connective ringed spaces* $\mathbf{Top}^{\mathsf{cn}}_{\mathbf{CAlg}}$ to be the full subcategory of $\mathbf{Top}_{\mathbf{CAlg}}$ containing ringed spaces with connective structure sheaves.

Similarly, we can define *the category of connective locally ring spaces* $\mathbf{Top}^{\mathsf{cn,loc}}_{\mathbf{CAlg}}$ to be the full subcategory of $\mathbf{Top}^{\mathsf{loc}}_{\mathbf{CAlg}}$ containing locally ringed spaces that are also connective.

Warning, or rather non-warning: There is an equivalence

$$\mathbf{Shv}^{\mathrm{cn}}_{\mathbf{CAlg}}(X) \xrightarrow{\simeq} \mathbf{Shv}_{\mathbf{CAlg}^{\mathrm{cn}}}(X)$$

given by $\mathcal{F} \mapsto (U \mapsto \tau_{\geq 0}(\mathcal{F}(U)))$. This is saying every connective sheaf can be viewed as a sheaf valued in connective $\mathbb{E}_\infty$-objects. Note that we need to take a $\tau_{\geq 0}$ truncation here.

Connectivity is not defined for presheaves.

We can deduce the equivalence for spectral (locally) ringed spaces, namely, we have that

$$\mathbf{Top}^{\mathrm{cn}}_{\mathbf{CAlg}} \xrightarrow{\simeq} \mathbf{Top}_{\mathbf{CAlg}^{\mathrm{cn}}}$$

as well as the locally ringed space version, with the caveat that the underlying presheaf are changed.

Notation: We write A_0 for $\pi_0 A \in \mathbf{Ab}$ for $A \in \mathbf{Sp}$.

We need the following theorem for the later construction.

Theorem 5.9 (HA, 7.5.0.6)**.** *Let $A \in \mathbf{CAlg}$ be an $\mathbb{E}_\infty$-algebra and let $f \in A_0$ be a non zero-divisor. There exists $\mathbb{E}_\infty$-algebra $A[f^{-1}]$ such that $\pi_*(A[f^{-1}]) \simeq \pi_*(A)[f^{-1}]$.*

Remark 5.10. This is the universal object with the property that $\mathrm{Map}_{\mathbf{CAlg}}(A[f^{-1}], B)$ is the fiber product

$$\mathrm{Map}_{\mathbf{CAlg}}(A, B) \times_{\mathrm{Map}_{\mathbf{CAlg}^\heartsuit}(A_0, B_0)} \mathrm{Map}_{\mathbf{CAlg}^\heartsuit}(A_0[f^{-1}], B_0).$$

Now we introduce our construction of spectral spectra.

Construction: For $A \in \mathbf{CAlg}$, we can construct a spectral locally ringed space $\mathrm{Spec}(A) = (\mathrm{Spec}(A_0), \mathcal{O}_A)$ with $\mathcal{O}_A(\mathrm{Spec}(A_0[f^{-1}])) := A[f^{-1}]$.

Theorem 5.11 (SAG, 1.1.5.5)**.** *The functor*

$$\begin{aligned} \mathbf{CAlg}^{op} &\to \mathbf{Top}^{loc}_{\mathbf{CAlg}} \\ A &\mapsto \mathrm{Spec}(A) \end{aligned}$$

is fully faithful.

By Theorem 5.11, we can view $\mathbf{AffSpSch}^{\mathrm{nc}}$ as a full subcategory of $\mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}}$.

We also have the connective version of this:

$$\begin{tikzcd} \mathbf{AffSpSch} \arrow[r, hook, "\mathrm{f.f}"] \arrow[d, hook] & \mathbf{Top}^{\mathrm{cn,loc}}_{\mathbf{CAlg}} \arrow[d, hook] \\ \mathbf{AffSpSch}^{\mathrm{nc}} \arrow[r, hook, "\mathrm{f.f}"] & \mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}} \end{tikzcd}$$

Remark 5.12. In fact, we have a stronger statement.

$$\mathrm{Map}_{\mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}}}((X, \mathcal{O}_X), \mathrm{Spec}(A)) = \mathrm{Map}_{\mathbf{CAlg}}(A, \mathcal{O}_X(X)).$$

Namely, we can replace the codomain with an arbitrary ringed space $(X, \mathcal{O}_X)$ instead of $\mathrm{Spec}(B)$. This is not true if we replace the domain.

Exercise: Suppose we have $A \in \mathbf{CAlg}^\heartsuit$ and $\mathrm{Spec}(A) \in \mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}}$. Show that we have

$$\pi_n(\Gamma(U, \mathcal{O}_A)) = H^{-n}(U, \pi_0 \mathcal{O}_A).$$

In SAG, the spectral structure sheaf captures all structures of the cohomology of $\pi_0 \mathcal{O}_A$.

6. Oct 22: Lecture 5

Last time we defined affine spectral schemes. Today we will define general spectral schemes and spectral quasi-coherent sheaves, and introduce the base change theorem.

6.1. **spectral schemes.**

Definition 6.1. A spectral locally ringed space $(X, \mathcal{O}_X) \in \mathbf{Top}^{\mathsf{loc}}_{\mathbf{CAlg}}$ is a *non-connective spectral scheme* if locally it is of the form $\operatorname{Spec}(A)$ where $A \in \mathbf{CAlg}$.

The *category of non-connective spectral schemes*, denoted by $\mathbf{SpSch}^{\mathsf{nc}}$, is defined to be the full subcategory of $\mathbf{Top}^{\mathsf{loc}}_{\mathbf{CAlg}}$ containing non-connective spectral schemes.

We also have the connective version.

Definition 6.2. A spectral locally ringed space $(X, \mathcal{O}_X) \in \mathbf{Top}^{\mathsf{loc}}_{\mathbf{CAlg}}$ is a *spectral scheme* if locally it is of the form $\operatorname{Spec}(A)$ where $A \in \mathbf{CAlg}^{\mathsf{cn}}$.

The *category of spectral schemes*, denoted by $\mathbf{SpSch}$, is defined to be the full subcategory of $\mathbf{Top}^{\mathsf{cn,loc}}_{\mathbf{CAlg}}$ containing spectral schemes.

Remark 6.3. Note that the above definition is not the same as the original definition in SAG, but is equivalent by [SAG, 1.1.6.2].

The following gives a characterization of spectral schemes.

Proposition 6.4. *A spectral ringed space $(X, \mathcal{O}_X) \in \mathbf{Top}_{\mathbf{CAlg}}$ is a non-connective spectral scheme if and only if it satisfies that*

- $(X, \pi_0 \mathcal{O}_X) \in \mathbf{Top}^{loc}_{\mathbf{CAlg}^\heartsuit}$ *is a scheme, and*
- $\pi_n \mathcal{O}_X \in \pi_0 \mathcal{O}_X$-$\mathbf{Mod}$ *is quasi-coherent over* $(X, \pi_0 \mathcal{O}_X)$, *and*
- $\mathcal{O}_X$ *is hypercomplete, namely, it satisfies descent condition for hyper covers.*

Remark 6.5. For now we skip the definition of hypercompeleteness or hyper covers. This is some condition you need to consider when dealing with infinite dimensional objects. When a spectral scheme X has underlying topological space of finite homotopy dimensional, every scheme over X is hypercompelte.

Analogously the category of spectral schemes can also be embedded into spectral stacks.

Theorem 6.6. *The functor*

$$\begin{aligned} \mathbf{SpSch}^{nc} &\to \mathbf{SpPreStk}^{nc} := P(\widehat{\mathbf{CAlg}^{op}}) \\ X &\mapsto (A \mapsto \operatorname{Map}_{\mathbf{SpSch}^{nc}}(\operatorname{Spec}(A), X)) \end{aligned}$$

is fully faithful.

Here $\mathbf{SpPreStk}^{\mathsf{nc}}$ denotes the category of non-connective prestacks, and the notation $P(\widehat{\mathbf{CAlg}^{\mathsf{op}}})$ is defined to be $\operatorname{Fun}_{\mathsf{acc}}(\mathbf{CAlg}, \hat{\mathcal{S}})$ where $\widehat{(-)}$ means including not necessarily small categories, for set theoretical reasons.

The similar construction works for the connective version, and is also fully faithful. We use $\mathbf{SpPreStk} := P(\widehat{\mathbf{CAlg}^{\mathsf{cn,op}}})$ for the connective prestacks.

Remark 6.7. The two fully faithful embeddings form a diagram

$$\begin{tikzcd} \mathbf{SpSch} \arrow[r, hook, "f.f."] \arrow[d, hook] & \mathbf{SpPreStk} = P(\widehat{\mathbf{CAlg}^{\mathsf{cn,op}}}) \arrow[d, dashed, "\subset"] \\ \mathbf{SpSch}^{\mathsf{nc}} \arrow[r, hook, "f.f."] & \mathbf{SpPreStk}^{\mathsf{nc}} = P(\widehat{\mathbf{CAlg}^{\mathsf{op}}}) \end{tikzcd}$$

The right vertical arrow can be filled in, constructed as a left Kan extension. It is also fully faithful.

Theorem 6.8 (SAG, 1.6.3.1)**.** *Any object in $\mathbf{SpSch}^{nc}$ viewed as a spectral prestack is a fpqc sheaf on $\mathbf{SpAffSch}^{nc}$, i.e., it satisfies descent condition under the fpqc Grothendieck topology.*

As in the ordinary AG case (see Definition 4.7) the fpqc topology on $\mathbf{SpAffSch}^{\mathsf{nc}}$ is defined analogously, with the condition $T_i \to S$ being flat replaced by spectral flatness.

Spectral flatness in $\mathbf{CAlg} = (\mathbf{SpAffSch}^{\mathsf{nc}})^{\mathsf{op}}$ is characterized by the following, which we take as the definition.

Definition 6.9. A map of $\mathbb{E}_\infty$-rings $f : A \to B \in \mathbf{CAlg}$ is flat iff

- $\pi_0 f : A_0 \to B_0$ is flat, and
- $A_n \otimes_{A_0} B_0 \to B_n$ is an isomorphism.

Remark 6.10. Definition 6.9 says that flatness is essentially characterized by π_*.

Here is another way to think of spectral flatness. Roughly speaking, a map $f : A \to B \in \mathbf{CAlg}$ being flat is equivalent to that the functor $B \otimes_A (-) : \mathbf{Mod}_A \to \mathbf{Mod}_B$ is t-exact with respect to the canonical t-structures. However, this is hard to make precise when A or B are not connective: in that case, the canonical t-structure on the module category is not well-defined.

6.2. **Quasi-coherent sheaves in SAG.**

Definition 6.11. Define the functor $\mathbf{QCoh}$ to be the following:

$$\begin{aligned}\mathbf{QCoh} : (\mathbf{AffSpSch}^{\mathsf{nc}})^{\mathsf{op}} &\to \mathbf{Pr}^{\mathsf{L,st}}\\ \mathrm{Spec}(A) &\mapsto \mathbf{Mod}_A\\ (A \to B) &\mapsto (B \otimes_A (-) : \mathbf{Mod}_A \to \mathbf{Mod}_B).\end{aligned}$$

Definition 6.12. We extend the define of $\mathbf{QCoh}$ to $\mathbf{SpPreStk}^{\mathsf{nc}}$ by a right Kan extension, but still denote it by $\mathbf{QCoh}$.

$$\begin{tikzcd}(\mathbf{AffSpSch}^{\mathsf{nc}})^{\mathsf{op}} \arrow[r, "\mathbf{QCoh}"] \arrow[d] & \mathbf{Cat}^{\mathsf{st}} \\ (\mathbf{SpPreStk}^{\mathsf{nc}})^{\mathsf{op}} \arrow[ur, dashed, "{\mathsf{RKE},\ \mathbf{QCoh}}"'] & \end{tikzcd}.$$

Remark 6.13.

- Knowing $\mathcal{F} \in \mathbf{QCoh}(X)$ is equivalent to knowing $\mathcal{F}|_S \in \mathbf{QCoh}(S)$ for all $S = \mathrm{Spec}(A) \to X$ together with all higher coherence.
- In the case of $\mathbf{SpSch}$, we have a more explicit description of $\mathbf{QCoh}$ like in the ordinary AG case. For a $X \in \mathbf{SpSch}^{\mathsf{nc}}$, $\mathbf{QCoh}(X)$ is a full subcategory of $\mathcal{O}_X$-$\mathbf{Mod}(\mathbf{Shv}_{\mathsf{Sp}}(X))$ containing objects $\mathcal{F}$ such that for any $\mathrm{Spec}(A) \to X$, $\mathcal{F}|_{\mathrm{Spec}(A)} \simeq \tilde{M}$ for some A-module M.
- Note that here the target is $\mathbf{Cat}^{\mathsf{st}}$, due to some set theoretical issue. Under a mild assumption on the size of X, $\mathbf{QCoh}(X)$ is presentable.

  We record this mild assumption:

  There is a localization functor

  $$L : \mathbf{SpPreStk}^{\mathsf{nc}} \to \mathbf{Shv}_{\mathsf{fpqc}}(\widehat{\mathbf{CAlg}}^{\mathsf{op}}).$$

  Suppose that $L(X)$ belongs to the smallest full subcategory of $\mathbf{Shv}_{\mathsf{fpqc}}(\widehat{\mathbf{CAlg}}^{\mathsf{op}})$ which is closed under small colimits and contains the essential image of the Yoneda embedding. Then the $\mathbf{QCoh}(X)$ is presentable. For more details, see DAGVIII 2.7.17(3).

  Most of time we will assume $\mathbf{QCoh}(X) \in \mathbf{Pr}^{\mathsf{L,st}}$.

Theorem 6.14. $\mathbf{QCoh}$ *satisfies fpqc descent (on* $\mathbf{AffSchSch}$ *or* $\mathbf{SpPreStk}$ *or the non-connective versions).*

6.2.1. *t-structure on $\mathbf{QCoh}(X)$.*

Definition 6.15. Let $f : X \to Y$ be a map of spectral prestacks. The pullback functor, usually denoted by f^*, is defined to be the functor $\mathbf{QCoh}(f)$. Namely, f^* is defined to be the induced morphism in the following diagram.

$$\mathbf{QCoh} : \mathbf{SpPreStk}^{\mathrm{op}} \to \mathbf{Cat}^{\mathrm{st}}$$
$$(f : X \to Y) \mapsto (f^* : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)).$$

For $X \in \mathbf{SpPreStk}$, $\mathbf{QCoh}(X)$ has a unique t-structure such that $\mathcal{F} \in \mathbf{QCoh}(X)_{\geq 0}$ iff for any $S = \mathrm{Spec}(A) \to X$, $\mathcal{F}|_S \in \mathbf{QCoh}(S)_{\geq 0} = \mathbf{Mod}_{A\geq 0}$.

In other words, under this t-structure, pullback functors are right t-exact, i.e. preserve non-negative part.

This t-structure has the following properties.

- It is right complete, namely there are no infinitely negative things (homological convention).
- When X is in $\mathbf{SpSch}$, $\mathbf{QCoh}(X)$ is also left complete.
- It is compatible with filtered colimits, i.e. filtered colimit functor is t-exact. Note that formally taking formal colimits is only left t-exact.

Exercise: Compute $k \otimes_{k[t]} k$ where t has degree $1, -1, 2, -2$. You will see the tensor product of bounded things are not necesssarily bounded.

6.2.2. *Pushforward functor.*

Definition 6.16. A morphism $f : X \to Y \in \mathbf{SpPreStk}^{\mathrm{nc}}$ is *spectrally schematic* if for all $\mathrm{Spec}(A) \to Y$, $X \times_Y \mathrm{Spec}(A)$ is a spectral scheme.

Further more, it is *qcqs spectrally schematic* if $X \times_Y \mathrm{Spec}(A)$ is qcqs.

Remark 6.17.

- Note the definition of qcqs only involves the underlying topological space.
- This condition is stable under composition.
- This condition is stable under base change, i.e. in the following pullback diagram, if f is spectrally shematic, then f' is.

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}.$$

The following result shows the importance of this notion.

Proposition 6.18. *If a morphism between non-connective spectral prestalk $f : X \to Y$ is qcqs spectral schematic, then $f^* : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$ has a colimit preserving right adjoint f_*.*

Remark 6.19.

- Note that f^* always has a right adjoint, but it is not necessarily colimit preserving, without the qcqs spectral schematic condition.
- Since f_* is colimit preserving, it has a right adjoint. But the right adjoint of f_* is usually not colimit preserving. When we replace $\mathbf{QCoh}$ by $\mathbf{IndCoh}$, then the right adjoint of f_* is colimit preserving under the condition that f is proper and almost of finite presentation.

Proposition 6.20. *Let $f : X \to Y$ be a morphism between spectral schemes. The following diagram*

$$\begin{tikzcd} \mathbf{QCoh}(X) \arrow[r, "f_*"] \arrow[d] & \mathbf{QCoh}(Y) \arrow[d] \\ \mathcal{O}_X\text{-}\mathbf{Mod}(\mathbf{Shv}_{\mathbf{Sp}}(X)) \arrow[r, "f_*"] & \mathcal{O}_Y\text{-}\mathbf{Mod}(\mathbf{Shv}_{\mathbf{Sp}}(Y)) \end{tikzcd}$$

is commutative only when f is qcqs spectrally schematic. Here the horizontal arrows are the right adjoints of the pullback functors f^*.

Theorem 6.21 (Base-change isomorphism)**.** *For a pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "\psi"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "\varphi"] & Y \end{tikzcd}$$

where f is qcqs spectrally schematic, $\varphi^* \circ f_* \to (f')_* \circ \psi^*$ is an equivalence.

Remark 6.22. There's a naturally functor form $\mathbf{AffSch} \to \mathbf{AffSpSch}$, but this does not preserve limits. Therefore, a pullback square in ordinary AG is not necessarily a pullback square in spectral AG.

As an example, we calculate $A = k \otimes_B k$ where B is $k[\epsilon]/\epsilon^2$.

- The spectral case: We take spec and use duality. The spec of A is the fiber product $\mathrm{Spec}(k) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(k)$. This is a "derived" fat point: it's underlying topological space is just a point, and the structure sheaf has $\pi_i \mathcal{O}_X = k, i \geq 0$ (when char(k) is not 2).

  The homotopy of the structure sheaf can be read off by the algebraic computation. We calculate using the derived tensor, and will obtain that $k \otimes_S k = k[\alpha, \beta]$ where α, β are in degree -1 and -2 respectively. Since we are working in graded commutative context, $\alpha^2 = 0$ (when char(k) is not 2), and thus the $k[\alpha, \beta]_i = k$ for any $i \leq 0$.
- The ordinary case: $k \otimes_S k = k$.

Question:

Is there a general way to compute the $\Omega X := pt \times_X pt$?

Answer: Not in a obvious way. Note that $pt \times_X pt$ only depends on local information, i.e. if both map $pt \to X$ factors through $U \subset X$, then $pt \times_X pt$ is equivalent to $pt \times_U pt$. On the other hand, taking loop in topological spaces requires global information of X.

Theorem 6.23. *If X is an ordinary scheme, X can be viewed as a spectral scheme (here we need to take some sheafification as mentioned before; equivalently, the spectral scheme has the structure sheaf such that π_i vanish for all $i \neq 0$). We have that* $\mathbf{QCoh}(X) = D(\mathbf{QCoh}(X)^{\heartsuit})$.

There are truncation functors such that the following diagram commutes.

$$\begin{tikzcd} \mathbf{SpSch} \arrow[r, "\tau_{\leq 0}"] \arrow[d, hook] & \mathbf{Sch} \arrow[d, hook] \\ \mathbf{Top}^{\mathsf{loc,cn}}_{\mathbf{CAlg}} \arrow[r, "\tau_{\leq 0}"] & \mathbf{Top}^{\mathsf{loc}}_{\mathbf{CAlg}^{\heartsuit}} \end{tikzcd} .$$

The two truncation functors both have left adjoints which also make the square commutes (with horizontal arrow inverted).

Exercise:

Consider a chain complex $A = \cdots A_{-1} \leftarrow A_0 \leftarrow A_1 \leftarrow \cdots$.

Then $\mathrm{Spec}(A)$ has the underlying topological space as $\mathrm{Spec}(H_0A)$. Its structure sheaf has that $\pi_n\mathcal{O}_A = H^{-n}(A)$.

Exercise: Tale $B = k[t]$ with $|t| = 1$. By definition, we have $\mathbf{QCoh}(\mathrm{Spec}(B)) = \mathbf{Mod}_B$, and $\mathbf{QCoh}(\mathrm{Spec}(B))^{\heartsuit} = \mathbf{Mod}_B^{\heartsuit} = \mathbf{Mod}_{B_0}^{\heartsuit} = \mathbf{Mod}_k^{\heartsuit}$. However, $D(\mathbf{Mod}_B^{\heartsuit}) \neq \mathbf{Mod}_B$. This can be seen via the example: $\mathrm{Map}_{D(\mathbf{Mod}_B^{\heartsuit})}(k, k[1]) \neq \mathrm{Map}_{\mathbf{Mod}_B}(k, k[1])$.

Exercise: Consider the scheme $X := \mathbb{A}^2\backslash(0,0)$ and the map $p : X \to \mathrm{Spec}(S)$.

(1) Calculate $p_*\mathcal{O}_X \in \mathbf{Sp}$. Here p_* is also called the global section functor. Hint:
- Use $\pi_{-n}(p_*\mathcal{O}_X) = H^n(X, \mathcal{O}_X)$.
- Form a fpqc Cech nerve

$$Y_\bullet = \cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} Y^1 \rightrightarrows Y^0 \xrightarrow{f^0} X.$$

  of affine schemes. For any $\mathcal{F} \in \mathbf{QCoh}(X)$, we have $\mathcal{F} = \mathrm{Tot}(f^0_*(f^0)^*\mathcal{F} \rightrightarrows f^1_*(f^1)^*\mathcal{F} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots)$, where f^1 is the map from $Y^1 \to X$.
- Use the Cech nerve and descent to calculate the homotopy groups. In particular, for an affine scheme $Y = \mathrm{Spec}(A)$ over $\mathrm{Spec}(S)$, $p_*\mathcal{O}_Y = HA$ since it only has a non-zero $\pi_0 = A$.
- More hint: We use the cover $\{\mathrm{Spec}(\mathbb{C}[x, y^{\pm}]), \mathrm{Spec}(\mathbb{C}[x^{\pm}, y])\}$ to form the Cech nerve. The homotopy can be calculated by taking the homology of the following chain

$$\mathbb{C}[x, y^{\pm}] \oplus \mathbb{C}[x^{\pm}, y] \to \mathbb{C}[x^{\pm}, y^{\pm}].$$

(2) The spectrum $p_*\mathcal{O}_X$ is a $\mathbb{E}_\infty$-ring, with the ring structure inherited from X. Show the ring structure. Hint:
- For a proof, notice that p^* is symmetric monoidal, and p_* is right lax symmetric monoidal thus sending algebra to algebra.
- From previous computations, $\pi_*\mathcal{O}_X$ has only non-trivial π_0 and π_{-1}.

(3) Check that $p_* : \mathbf{QCoh}(X) \to \mathbf{Sp}$ factor through $\mathbf{Mod}_{p_*\mathcal{O}_X}$. Hint: this is formal.

(4) Show that the map $p_* : \mathbf{QCoh}(X) \to \mathbf{Mod}_{p_*\mathcal{O}_X}$ constructed in the previous question is an equivalence.

(5) Visualize $\mathrm{Spec}(p_*\mathcal{O}_X)$. Answer: The underlying space is $\mathbb{A}^2$; its π_0 is as calculated, and its π_{-1} is a sheaf supported at a 0.

(6) Use the equivalence in Remark 5.12

$$\mathrm{Map}_{\mathbf{Top}^{\mathrm{loc}}_{\mathbf{CAlg}}}((X, \mathcal{O}_X), \mathrm{Spec}(A)) = \mathrm{Map}_{\mathbf{CAlg}}(A, \mathcal{O}_X(X))$$

to construct a map $\psi : X \to \mathrm{Spec}(p_*\mathcal{O}_X)$.

(7) Show that ψ^* is an equivalence.

Remark 6.24. In the previous exercise, we showed an example of an open embedding $\psi : X \to \mathrm{Spec}(p_*\mathcal{O}_X)$ which induces equivalence on $\mathbf{QCoh}$. This is not the case in the ordinary AG: when you have an open embedding, the corresponding $\mathbf{QCoh}$ are never the same. Roughly, in this example, "the fat point at the origin cancels itself."

In ordinary AG, a scheme A is affine if $\mathbf{QCoh}(A) = \mathbf{Mod}_A$. But in spectral AG, this can happen even when the scheme is not affine or even for some stacks (modulo the definitions of spectral stacks and the corresponding $\mathbf{QCoh}$). Lurie named spectral objects with property $\mathbf{QCoh}(A) = \mathbf{Mod}_A$ "co-affine".

Example: the spectral scheme $\mathbb{B}\mathbb{G}_a$ is coaffine over a characteristic 0 field.

References