NOTES: MILNOR CONJECTURE LEARNING SEMINAR

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Contents

1. Talk 1, Oct 21, Milnor's paper, Jacob

Let k be a field of characteristic not 2.

Definition 1.1. A *quadratic space* over k is a pair (V, q) where V is a finite dimensional vector space over k, and $q : V \to k$ is a non-degenerate quadratic form.

The goal is to classify quadratic spaces up to isomorphism. We denote the isomorphism class of V by $\langle V \rangle$. For an element $a \in k$, we denote the 1 dimensional quadratic space $(k, x \mapsto ax^2)$ by $\langle a \rangle$.

Proposition 1.2. *The isomorphism classes of quadratic spaces forms a commutative semi ring. The addition and the multiplication are given by*

- $\langle V \rangle + \langle W \rangle = \langle V \oplus W \rangle$, *and*
- $\langle V \rangle\langle W \rangle = \langle V \otimes W \rangle$.

The unit is $\langle 1 \rangle$.

We turn it into a ring by formally add inverses.

Definition 1.3. The Grothendieck–Witt ring $GW(k)$ of field k is the ring obtained from the semi ring of of quadratic spaces over k by formally adding inverses.

It turns out that the semi ring injects into the Grothendieck–Witt ring.

Theorem 1.4 (Witt). *If* V, V', W *are quadratic spaces with* $\langle V \rangle + \langle W \rangle = \langle V' \rangle + \langle W \rangle$, *then* $\langle V \rangle = \langle V' \rangle$.

Sketch proof. ∎

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There is a dimension functor

$$\begin{aligned}\dim : GW(k) &\longrightarrow \mathbb{Z}\\ \langle V\rangle &\longmapsto \dim_k(V).\end{aligned}$$

We denote the augmentation ideal by $I \subset GW(k)$.

Definition 1.5. We say that a quadratic space V is

- *anisotropic* if it does not contain nonzero vectors of norm zero;
- *metabolic* if $\dim(V)$ is even and there is a subspace V_0 of half the dimension with $q|_{V_0} = 0$.

Remark 1.6.

- Akshay: Why call it metabolic, not hyperbolic? Jacob: Wiki says so.
- The dimension of a subspace where q (non-degenerate) vanishes is at most $\dim(V)/2$.

Proposition 1.7. *Any quadratic space V splits as $V = n(\langle 1\rangle + \langle -1\rangle) + W$ where W is anisotropic.*

Proof. When V is not anisotropic, i.e. $\exists 0 \neq x \in V$ with $q(x) = 0$ (an isotropic vector), we can find another vector y, such that by rescaling $q(x) = q(y)$ is 0 and the bilinear form $b(x,y)$ is 1. Thus for the subspace with the basis $\{x,y\}$, the matrix for the associated bilinear form is the anti diagonal 2-by-2 matrix. One can see that this subspace equals $\langle 1\rangle + \langle -1\rangle$ by taking $x' = x + \frac{y}{2}$ and $y' = x - \frac{y}{2}$. ∎

By the Witt's cancellation thereom, the splitting is unique (on the level of isomorphism classes).

Note that the metabolic quadratic spaces form an ideal of $GW(k)$.

Definition 1.8. Define the Witt ring $W(k)$ of field k to be

$$W(k) := GW(k)/\text{the ideal of metabolic ones}.$$

Consider

$$\begin{tikzcd} & \text{metabolic quad space} \arrow[r, "\simeq"] \arrow[d, hook] & 2\mathbb{Z} \arrow[d, hook] \\ I \arrow[r, hook] & GW(k) \arrow[r, two heads] \arrow[d, two heads] & \mathbb{Z} \arrow[d] \\ & W(k) \arrow[r] & \mathbb{Z}/2 \end{tikzcd}.$$

The lower square is a pull back diagram. Thus to understand $GW(k)$, we study $W(k)$ By previous discussion on splitting, we know that as a set,

$$W(k) = \{\text{anisotropic quadratic spaces}\}.$$

By choosing orthogonal basis, we can see that $GW(k)$ is generated as an abelian group by $\langle a\rangle$ for all $a \in k^*$. Note that we have $\langle a\rangle = \langle ax^2\rangle$ for any $x \in k^*$. Thus $\langle a\rangle$ is determined by the image of a in k^*/k^{*2}. We also have $\langle ab\rangle = \langle a\rangle\langle b\rangle$.

There is another key relation.

Let $a \in k^*$ be an element that is not 0 or 1. We consider $\langle a\rangle + \langle 1-a\rangle$. This is a 2 dimensional vector space with a basis $\{x,y\}$ such that $q(x) = a, q(y) = 1-a$ and $b(x,y) = 0$. Note that $q(x+y) = 1$. To find the complement, we evaluate the quadratic form on $(1-a)x - ay$ which is orthogonal to $x+y$. We obtain that

$$\langle a\rangle + \langle 1-a\rangle = \langle 1\rangle + \langle (1-a)a\rangle.$$

It turns out these are the only relations.

Theorem 1.9. *The map $\widetilde{GW}(k) \twoheadrightarrow GW(k)$ is an isomorphism. Here $\widetilde{GW}$ is defined by the above relations.*

Proof. Suppose we have $\langle a_1 \rangle + \cdots + \langle a_n \rangle = \langle b_1 \rangle + \cdots + \langle b_n \rangle$ in $GW(k)$. We want to show the same equality holds in $\widetilde{GW}(k)$. We prove by induction on n.

We denote the sum quadratic space by $V = kx_1 + \cdots kx_n$ where $q(x_i) = a_i$. The case $n = 1$ is trivial. For larger n, we split $V = V_- \oplus V_+$ where V_- is spanned by x_1 and V_+ is spanned by the rest.

By inductive hypothesis, it suffices to show that we can decompose it in $\widetilde{GW}(k)$ such that

$$\langle a_1 \rangle + ... + \langle a_n \rangle = \langle b_1 \rangle + \text{complement}.$$

WLOG, take $b_1 = 1$. We choose a vector of norm b_1, which under the decomposition $V = V_- \oplus V_+$ we write as $y_- + y_+$. By assumptions, we have $q(y_-) = a$ and $q(y_+) = 1 - a$. Here $\langle a \rangle = \langle a_1 \rangle$. Using the key relations, we get that $\langle a \rangle + \langle 1 - a \rangle = \langle b_1 \rangle + \langle (1 - a)a \rangle$. The result follows. ∎

This gives an upper bound on the size of $GW(k)$. Now we study the lower bounds.

Let V be a quadratic space. Define *the discriminant* of V to be the determinant of the matrix of the corresponding bilinear form. Denote this functor by disc. We have

$$\mathrm{disc}(V \oplus V') = \mathrm{disc}(V)\mathrm{disc}(V').$$

We have the following definition. The well-defined-ness is not hard to check.

Definition 1.10. The functor disc extends to a functor (*the first Stiefel–Whitney class*)

$$w_1 : GW(k) \to k^*/k^{*2}.$$

We have $w_1(\langle a \rangle) = a$. When restricted to the augmentation ideal I, it is still surjective:

$$w_1 : I \twoheadrightarrow k^*/k^{*2}.$$

Remark 1.11. A warning:
The augmentation ideal I can also be viewed as the kernel of the map

$$W(k) \to \mathbb{Z}/2.$$

Let V be a quadratic space of dimension $2d$. We have $\langle V \rangle - d\langle 1 \rangle - d\langle -1 \rangle \in I$, and

$$\mathrm{disc}(\langle V \rangle - d\langle 1 \rangle - d\langle -1 \rangle) = \mathrm{disc}(\langle V \rangle)(-1)^d.$$

Jacob calls this $\widetilde{\mathrm{disc}}$, the normalized disc.

Proposition 1.12. *The map w_1 induces an isomorphism*

$$I/I^2 \to k^*/k^{*2}.$$

Proof. A typical element in I^2 is of the form $(\langle a \rangle - 1)(\langle b \rangle - 1)$. One has

$$(\langle a \rangle - 1) + (\langle b \rangle - 1) = (\langle ab \rangle - 1) \in I/I^2.$$

∎

Definition 1.13. Let (V, q) be a quadratic space. The Clifford algebra $Cl(V)$ is defined to be the quotient of the tensor algebra on V by relations $x^2 = q(x)$.

Proposition 1.14. *If V is even dimensional, then $Cl(V)$ is a central simple algebra.*

Thus it represents an element of the Brauer group $Br(k)$ of k. Note that this only happens when it is even dimensional.

Definition 1.15. Let $br(V)$ denote the class of the Clifford algebra inside $Br(k)$.

Example 1.16. Consider the case when V is $\langle a\rangle + \langle b\rangle$. Then $Cl(V)$ is a 4 dimensional vector space over k, spanned by $1, i, j, ij = -ji$ with $i^2 = a$ and $j^2 = b$. This is anti commutative, with $(ij)^2 = -ab$. This is a generalized quaternion algebra.

In fact, $br(V)$ lives in the 2-torsion part $Br(k)[2]$. We have

$$br(V) \in Br(k)[2] = H^2(Gal(\bar{k}/k), \mathbb{Z}/2)$$

and

$$\mathrm{disc}(V) \in k^*/k^{*2} = H^1(Gal(\bar{k}/k), \mathbb{Z}/2).$$

For $a, b \in k^*/k^{*2}$, denote the corresponding cohomology class by $[a], [b]$. We have $[ab] = [a] + [b]$, and a product to get classes in degree 2: $[a][b] \in H^2(Gal(\bar{k}/k), \mathbb{Z}/2)$ given by the generalized quaternion algebra.

Take two even dimensional quadratic spaces V, V'. There is an isomorphism

$$Cl(V)\hat{\otimes}_k Cl(V') \xrightarrow{\simeq} Cl(V \oplus V')$$

Note that here we need to use the graded tensor product here. If it were $\otimes_k$ instead of $\hat{\otimes}_k$, we would have $br(V \oplus V') = br(V) + br(V')$.

We consider the actual formula:

$$br(V \oplus V') = br(V) + br(V') + \widetilde{\mathrm{disc}}(V)\widetilde{\mathrm{disc}}(V') \in H^2.$$

Thus we have

$$br(V) = br(V \oplus \langle 1\rangle \oplus \langle -1\rangle),$$

since

$$\widetilde{\mathrm{disc}}(\langle 1\rangle \oplus \langle -1\rangle) = [1] = 0 \in H^1$$

and

$$br(\langle 1\rangle \oplus \langle -1\rangle) = 0 \in H^2$$

Thus $br(V)$ depends only on the anisotropic part of V. Thus we get a map from I to H^2; it turns out that br defines a group homomorphism when restricted on I^2.

We have a diagram

$$\begin{tikzcd}
I^3 \arrow[d, hook] & \\
I^2 \arrow[r, two heads, "br"] \arrow[d, hook] & Br(k)[2] = H^2(Gal, \mathbb{Z}/2) \\
I \arrow[r, two heads, "disc"] \arrow[d, hook] & k^*/k^{*2} = H^1(Gal, \mathbb{Z}/2) \\
W(k) \arrow[r, two heads, "dim"] & \mathbb{Z}/2 = H^0(Gal, \mathbb{Z}/2)
\end{tikzcd}$$

Theorem 1.17 (Merkurjev)**.** *The ideal I^3 is the kernel of the map $br : I^2 \twoheadrightarrow Br(k)[2]$.*

We have the following construction (Delzant).

Theorem 1.18. *There is a unique group homomorphism*

$$w : GW(k) \to 1 + \Pi_{n\geq 1} H^n(Gal, \mathbb{Z}/2)$$

satisfying (on generators) $w(\langle a\rangle) = 1 + [a]$.

sketch proof. ∎

We have an alternative construction.

Recall that we can compute

$$H^*(BO(n), \mathbb{Z}/2) = \mathbb{Z}/2[w_1, ...w_n]$$

where w_is are the universal Stiefel–Whitney classes.

Analogously, we can compute the étale cohomology of the algebraic stack

$$H^*_{et}(BO(n), \mathbb{Z}/2) = H^*_{et}(\mathrm{Spec}(k))[w_1, ...w_n]$$

We can think of a quadratic space V as a point in $BO(n)$ and pullback the SW classes to get $w_i(V)$ in the Galois cohomology.

Remark 1.19. Let V be a quadratic space of dim $2d$. Take $\langle V\rangle - d - d\langle -1\rangle \in I$. We have

$$br(V) = w_2(\langle V\rangle - d - d\langle -1\rangle) = w_2(V) + \text{correction terms},$$

in which the correction terms depend on dim$V$ mod 8.

Proposition 1.20. $w_3(I^3) = 0$.

Proof. We can write an element in I^n as

$$v = \Pi_1^n(1 - \langle a_i\rangle) = \Pi_{S\subset\{1,2,...n\}}(-1)^{|S|}\Pi_{i\in S}\langle a_i\rangle.$$

By the defining formula, we have

$$w(v) = \frac{\Pi_{S \text{ even}}(1 + \Sigma_{i\in S}[a_i])}{\Pi_{S \text{ odd}}(1 + \Sigma_{i\in S}[a_i])}.$$

We write it as a power series, and think of it as a formal expression $f(x_1, \ldots, x_n)$ evaluated on $(a_1, \ldots, a_n)$.

Take $a \in k^*/k^{*2}$. There are relations

$$[a][a] = [a][1/a] = [a][-1].$$

Back to the expression, we have

$$\begin{aligned} f(a_1, \ldots, a_n) &= 1 + [a_1]\cdots[a_n]g([a_1], \cdots, [a_n]]) \\ &= 1 + [a_1]\cdots[a_n]g([-1], \cdots, [-1]]). \end{aligned}$$

Take $[a_i] = [-1]$ for all i in g, we calculate that $w_i(v)$ vanishes for $0 < i < 2^{n-1}$. ■

One can ask if there is an additive function

$$c : I^n \to H^n(Gal, Z/2)$$

with $c(\Pi(1 - \langle a_i\rangle)) = [a_1]\cdots[a_n]$.

The answer is yes, when $[-1]$ is not a zero divisor in the Galois cohomology. One can take

$$c(v) = \frac{w_{2^{n-1}}(v)}{[-1]^{2^{n-1}-n}}.$$

One also need to work on well-defined-ness. For example, for $k = \mathbb{R}$, H^2 is $\mathbb{Z}/2[x]$ with a single generator, and we can take the generator to be $[-1]$.

Definition 1.21 (Milnor K-theory)**.** The Milnor K-theory $K^M_*(k)$ is defined to be the free algebra generated by classes $\{a\}$ with $a \in k^*$ such that

- $\{1\} = 0$,
- $\{ab\} = \{a\} + \{b\}$, and
- (Steinberg relation) $\{a\}\{1 - a\} = 0$ for $a \neq 0, 1$.

We have a diagram

$$\begin{tikzcd}
& \{x\} \arrow[dl, mapsto] & & K^M_*(k)/2 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \{x\} \arrow[dr, mapsto] & \\
1-\langle x\rangle & & \oplus_n I^n/I^{n+1} \arrow[rr, dashed] & & H^*(Gal,\mathbb{Z}/2) & & [x]
\end{tikzcd}$$

Conjecture 1.22 (Milnor conjecture on quadratic forms, 1970)**.** *Is α an isomorphism?*

Yes by Orlov–Vishik–Voevodsky (2000).

Conjecture 1.23 (Milnor conjecture, 1970)**.** *Is β an isomorphism?*

Yes by Voevodsky with proof sketched in 1996 and completed later.

2. Talk 2, Nov 4, Motivic cohomology, Toni

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3. Talk 3, Nov 11, Merkurjev–Suslin, Akshay

Definition 3.1. Let F be a field with $\mathrm{char}(F) \neq 2$.

- Let $K_2(F)$ denote the group $\frac{F^\times \otimes F^\times}{(x,1-x)}$.
- Define $k_2(F) := K_2(F)/2$.

For $x, y \in F^\times$ we have $(x,y) \in K_2(F)$ and $(x,y)(y,x) = 1$.
Denote the 2-torsion part in Brauer group by $br_2(F)$.
There is a map

$$k_2(F) \to br_2(F)$$

that sends (x,y) to the quaternion algebra $F\langle i,j\rangle/(ij = -ji, i^2 = x, j^2 = y)$.
One can check that $(x, 1-x)$ under this map is sent to $M_2(F)$, the 2 by 2 matrix.

Theorem 3.2 (Merkurjev)**.** *The map described above induces an isomorphism*

$$k_2(F) \simeq br_2(F).$$

Idea of the proof:

- First prove for simple classes of fields, like finite fields, global fields, purely transcendental extension of those + one more class of fields. The proofs are computations by hand using results about K-theories.
- The nice part of the proof is to reduce the general case to the special cases.

3.1. **Special cases.**

3.1.1. *For finite fields.* We have $k_2(F) \simeq br_2(F) \simeq 0$.

3.1.2. *For number fields.* One can compute everything by hand.

Definition 3.3 (cyclic algebras)**.** Assume there is a cyclic extension L/F with Galois group $\mathbb{Z}/n = \langle\sigma\rangle$. Given an element $z \in F^\times$, we can attach to this a central simple algebra (cyclic algebra), denoted by $[L,z] = \langle L, \tau, \tau\lambda\tau^{-1} = \sigma(\lambda), \lambda \in L, \tau^n = z\rangle = L \oplus L\tau \oplus ... \oplus L\tau^{n-1}$.

Example 3.4. The cyclic algebra $[F(\sqrt{x}), y]$ is isomorphic to the quaternion algebra described above.

Remark 3.5. $[L, z]$ only depends on $z \in F^\times/NL^\times$. In fact, this construction realizes the identification.

$$\begin{array}{ccccc} F^\times/NL^\times & \xrightarrow{\simeq} & H^2(Gal(F/L), L^\times) & \xrightarrow{\sim} & \ker(Br(F) \to Br(L)) \\ z & \longmapsto & & & [L, z] \end{array}$$

Here the map $Br(F) \to Br(L)$ is given by the assignment $D \mapsto D \otimes_F L$. One can compute $F^\times/NL^\times \simeq H^2(Gal(L/F), L^\times)$.

Remark 3.6. This also gives a proof that $(x, 1-x)$ is trivial in $br_2(F)$. Here we denote by $(x, 1-x)$ the image of it under the map $k_2(F) \to br_2(F)$. Note that it is the cyclic algebra $[F(\sqrt{x}), 1-x]$. It follows by the fact that $z = (1-x)$ is a norm from $F(\sqrt{x})$.

We abuse notation and sometimes denote by (x, y) the image of it under the map $k_2(F) \to br_2(F)$.

3.1.3. *For global fields.*

Theorem 3.7 (Tate). *$k_2F \to br_2F$ is an isomorphism for global fields.*

Facts:

- If F is a local fields (not $\mathbb{C}$), then $br_2(F) \simeq \mathbb{Z}/2$. E.g. $br_2\mathbb{R} = \{\mathbb{R}, \mathbb{H} = [\mathbb{C}, -1]\}$.
- For any quadratic extension E/F, the map $br_2F \to br_2E$ is zero.
- For F a global field, the map $br_2F \to \oplus_v br_2F_v$ is injective.

Lemma 3.8. *For any F, $x, y \in F^\times$, $(x, y) = 0$ in $br_2(F) \implies (x, y) = 0$ in $k_2(F)$.*

Proof. We have equivalent statements:
$(x, y) = 0$ in $br_2(F)$. $\iff$ y is a norm from $F(\sqrt{x})$. $\iff$ $y = a^2 - b^2x$.
Thus we need to show $(x, a^2 - b^2x) = 0$ in $k_2(F)$. Up to a square, it follows from the triviality of the element $(\frac{b^2}{a^2}x, 1 - \frac{b^2}{a^2}x)$. ∎

proof of Tate's theorem with global field F.

- Step 1: $(x_1, y_1) = (x_2, y_2)$ in $br_2 \implies (x_1, y_1) = (x_2, y_2)$ in k_2.
  Proof of 1: Want to find an element $z \in F^\times$ such that
  $$(x_1, y_1) = (x_1, z) = (z, y_2) = (x_2, y_2)$$
  in br_2. Then by Lemma 3.8 we are done.
  Here is how to find z.
  We consider $D = F \oplus Fi \oplus Fj \oplus Fk = F \oplus Fi' \oplus Fj' \oplus Fk'$ where i, j, k, i', j', k' are determined as follows:
  1. $i^2 = x_1$, $j^2 = y_1$, $i'^2 = x_2$, $j'^2 = y_2$.
  2. Let D^0 denote the trace zero sub. We have $i, j' \in D^0$. Let k'' span orthogonal complement of i, j', with respect to norm in D^0.
  3. Take $z = k''^2$.

  This proof does not use the fact that F is a global field and is true for any fields. The next one uses.
- Step 2: $k_2(F) \to br_2(F)$ is an isomorphism.
  1. Surjectivity is not hard: every element of br_2 is a cyclic algebra $[L, \alpha]$ for a quadratic extension L/K. Find L and work in $L \otimes_K K_v$.
  2. Injectivity is hard part. Start with $(x_1, y_1) + ... + (x_n, y_n) \in ker(k_2 \to br_2)$. Enough to find for any x_1, x_2, y_1, y_2, there are x_3, y_3 such that
  $$(x_1, y_1) + (x_2, y_2) = (x_3, y_3) \in k_2.$$
  Then we can recursively reduce the number of pairs.

To do this, we find $a, b, c \in F^\times$ such that $(x_1, y_1) = (b, a)$, and $(x_2, y_2) = (c, a)$ in br_2 by choosing a such that $F(\sqrt{a})$ kills $(x_1, y_1), (x_2, y_2)$. By Lemma 3.8, the same equations hold in k_2: $(x_1, y_1) + (x_2, y_2) = (bc, a)$.

■

Facts we are gonna use:

- (Bass–Tate) There is a norm map $k_2(F(\sqrt{a})) \to k_2F$ compatible with maps to br_2. This is characterized by
$$\left(x \in F^\times, y \in F(\sqrt{a})^\times\right) \mapsto (x, Ny).$$
- Consider $\alpha_F : k_2F \to br_2F$. If α_F is an iso, then the same holds true for a purely transcendental field $F(x)$, i.e. $\alpha_{F(X)}$ is an iso.
  (a variation) Let Y be the conic with $ax^2 + by^2 = 1$. If α_F and $\alpha_{F(\sqrt{a})}$ are ismorphisms, then $\alpha_{F(Y)}$ and $\alpha_{F(Y)(\sqrt{a})}$.
- Specialization. Let Y be a smooth variety over F, p closed point.
  There is a specialization map
$$\begin{aligned} sp : K_2F(Y) &\to K_2F(p), \\ (f, g) &\mapsto (f(p), g(p)) \end{aligned}$$
  where f, g are regular at p.
  Side: If R is a discrete evaluation ring with quotient field Q and residue field κ. We want to study
$$\begin{aligned} K_2F(Q) &\to K_2F(\kappa), \\ (q_1, q_2) &\mapsto (\frac{\bar{q_1}}{\pi^{\nu(q_1)}}, \frac{\bar{q_2}}{\pi^{\nu(q_2)}}) \end{aligned}$$
  where pi is from a regular sequence at p.

Let F be a field and let $a \in F^\times$ be a nonsquare. There is a norm map $k_2F(\sqrt{a}) \xrightarrow{N} k_2(F)$, and the composite $k_2F \to k_2F(\sqrt{a}) \xrightarrow{N} k_2(F)$ is squaring.

Lemma 3.9 (Merkujev). *The sequence*
$$F^*/2 \to k_2F \to k_2F(\sqrt{a}) \xrightarrow{N} k_2(F)$$
is exact. Here the first map is given by $x \mapsto (a, x)$.

Intuition. Consider the diagram:

$$\begin{tikzcd} F^*/2 \arrow[r] \arrow[d, equal] & k_2F \arrow[r] \arrow[d, "\alpha_F"] & k_2F(\sqrt{a}) \arrow[r, "N"] \arrow[d, "\alpha_{F(\sqrt{a})}"] & k_2F \arrow[d, "\alpha_F"] \\ F^*/2 \arrow[r, "\beta"] & br_2F \arrow[r] & br_2F(\sqrt{a}) \arrow[r, "N"] & br_2F \end{tikzcd} \quad . \tag{3.10}$$

The bottom line is the exact sequence in Galois cohomology for F associated to
$$0 \to \mathbb{Z}/2 \to ind_{G_{F\sqrt{a}}}^{G_F} \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0.$$

■

Proof of the theorem using the Lemma. 1. Injectivity: Suppose $\Sigma(x_i, y_i) \in k_2F$ vanishes under α_F. Pass to $F(\sqrt{x_1})$. We use induction.
$$\begin{aligned} &(x_2, y_2) + \cdots + (x_n, y_n) = 0 \in k_2F(\sqrt{x_1}) \\ \implies &(x_2, y_2) + \cdots + (x_n, y_n) = (x_1, a) \in k_2F \text{ for some } a \\ \implies &\Sigma(x_i, y_i) = (x_1, b) \in k_2F \text{ for some } a. \end{aligned}$$

2. Surjectivity: $D \in br_2(F)$.

- If D is split by a quadratic extension L/F, then $D = [L, z]$.
- If D is split by Galois extension L/K of order 2^n, then proceed by induction on n. Consider Diagram 3.10. Choose $F(\sqrt{a}) \subset L$. By induction, $D \otimes_F F(\sqrt{a}) \in Im(\alpha_{F(\sqrt{a})})$. By diagram chasing and injectivity of α_F, we know that $N(D \otimes_F F(\sqrt{a})) = 0$ in k_2F. Thus it lifts to k_2F and it maps to D up to an element in $\beta(F^*/2)$. Therefore, this is contained in $Im(\alpha_F)$.
- In the general case, where D is split by some Galois extension L/F, we consider $L \supset L_1 \supset F$ where L/L_1 is 2 Sylow and F/L_1 is odd. Then we use answer for L_1 and norm on L_1/F.

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This leaves us to prove Lemma 3.9. We need the following fact which characterizes when a product is trivial.

Lemma 3.11. *The element $\Pi^n(a_i, b_i)$ is trivial in k_2F, iff (possibly after increasing n with $b_{n+1} \ldots = 1$) for each nonempty subset $S \subset \{1, 2, \ldots N\}$, there are $c_S, d_S \in F$ such that $b_i = \Pi_{i \in S}(c_s^2 - a_S d_S^2)$ in F, $a_S = \Pi_{i \in S} a_i$.*

Proof of Lemma 3.9. We show the exactness at $k_2F(\sqrt{d})$ of

$$F^*/2 \to k_2F \to k_2F(\sqrt{d}) \xrightarrow{N} k_2(F).$$

The rest is similar. Here we change a to d to avoid confusing notations.

Take $P = \Pi(x_i, u_i + v_i\sqrt{d}) \in k_2F(\sqrt{d})$, with traivial norm $\Pi(x_i, u_i^2 - dv_i^2) \in k_2F$. We want to show P is from k_2F.

By Lemma 3.11, we can find c_S, d_S, with $a_S = \Pi_{i \in S} x_i$ such that

$$u_i^2 - dv_i^2 = \Pi_{i \in S}(c_s^2 - a_S d_S^2).$$

This defiens a variety Y in the affine space with variables u_i, v_i, a_i, c_S, d_S defined over $F_0(d) = F_1$. Here F_0 is the prime field of F_1.

The previous facts show that $\alpha_{F_1(Y)}$ and $\alpha_{F_1(Y)(\sqrt{d})}$ are isomorphisms. This implies that the lemma is true for $F_1(Y)$.

Now we think of P as an element living in the function field, denoted by $\tilde{P} = \Pi(\alpha_i, u_i + v_i\sqrt{d}) \in k_2F(Y)(\sqrt{d})$ with $N_{F_1(Y)}^{F_1(Y)(\sqrt{d})} \tilde{P}$ trivial.

The result follows by considering

$$\begin{array}{ccc}
\tilde{\bar{P}} \in k_2F_1(Y) & \longmapsto & \tilde{P} \in k_2F_1(Y)(\sqrt{d}) \\
\downarrow sp & & \downarrow sp \\
\bar{P} \in k_2F & \longmapsto & P \in k_2F(\sqrt{d})
\end{array}.$$

■

4. Talk 4, Nov 18, Beilinson–Lichtenbaum conjecture, Peter

4.1. **Big picture.** We have a family of conjectures.

- BL(n): weight n Beilinson–Lichtenbaum conjecture.
- BK(n): Bloch–Kato conjecture.
- H90(n): Hilbert's Theorem 90.

Remark 4.1.
- The latter two are about fields, and $BL(n)$ is more general about motivic theory and smooth schemes.
- It will be clear that BL(n) $\Longrightarrow$ BK(n) $\Longrightarrow$ H90(n).
- The key input to Voevodsky's proof is that all implications are reversible.

Proof outline: use induction on n.

(1) First show the $n = 1$ case is true.

(2) Assume BL(n-1), then do many hard things.
(3) get seemingly weaker H90(n).

Remark 4.2. A historical remark: given the existence of $\mathbb{Z}(n)$, the equivalence of the 3 conjectures was known to Voevodsky–Suslin in 1994.

4.2. **A reminder of BK(n).** Recall the norm-residue map. Take the Kummer sequence

$$1 \to \mu_l \to \mathbb{G}_m \xrightarrow{(-)^l} \mathbb{G}_m \to 1.$$

Taking etale cohomology, we get a long exact sequence.

$$\to H^0_{et}(k, \mathbb{G}_m) \xrightarrow{l} H^0_{et}(k, \mathbb{G}_m) \xrightarrow{\partial} H^1_{et}(k, \mu_l) \to 0.$$

Here we have $H^0_{et}(k, \mathbb{G}_m) = k^\times$. By the LES, ∂ gives an isomorphism

$$\partial : k^\times / l \xrightarrow{\sim} H^1_{et}(k, \mathbb{G}_m).$$

We have the map

$$k^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} k^\times \xrightarrow{\partial^{\otimes q}} H^1_{et}(k, \mu_l) \xrightarrow{\cup} H^q_{et}(k, \mu_l^{\otimes q}).$$

Conjecture 4.3 (BK(n)). *$K^M_n(k)/l \to H^n_{et}(k, \mu_l^{\otimes n})$ is an isomorphism when $l \neq$ $\mathrm{char(k)}$.*

4.3. **Steinberg relation (under Piotr's request).**

Proposition 4.4 (Projection formula). *Let E/F be a finite separable extension. Pick $a \in F^\times, b \in E^\times$. Then we have*

$$\mathrm{tr}_{E/F}(a \cup b) = a \cup \mathrm{Nm}_{E/F}(b).$$

Proposition 4.5. *If m is an integer invertible in F then*

$$F^\times \otimes_{\mathbb{Z}} F^\times \to H^2_{et}(F, \mu_m^\otimes 2)$$

factor through $K^M_2(F)/m$.

Proof. Choose any $a \in F \backslash \{0, 1\}$. We want to show $a \cup (1-a)$ vanishes mod m. The trick is to factor the separable polynomial (separable because a is not zero and n is invertible in the field).

Assume we have

$$t^m - a = \Pi_i f_i \in F[t]$$

with f_i irreducible.

Now we look for a field E to apply the projection formula. Take $F_i := F(\text{root of } f_i)$. Setting $t = 1$, we have

$$1 - a = \Pi_i \mathrm{Nm}_{F_i/F}(1 - x_i).$$

Now we can use the formula to compute the cup product.

$$\begin{aligned} a \cup (1-a) &= \Sigma_i a \cup \mathrm{Nm}_{F_i/F}(1 - x_i) \\ &= \Sigma_i \mathrm{tr}_{F_i/F}(a \cup (1 - x_i)) \\ &= m \Sigma_i \mathrm{tr}_{F_i/F}(x_i \cup (1 - x_i)) = 0 \mod m. \end{aligned}$$

The last equality holds by $x_i^m = a$. ∎

4.4. **Recap of Toni's talk.** In the second half, Toni introduced the motivic complexes. Let k be a field. We have an additive category $\mathbf{Cor}_k$ of finite correspondences. We have a faithful functor $\mathbf{Sm}_k \to \mathbf{Cor}_k$ with assignment

$$(X \xrightarrow{f} Y) \mapsto (X \leftarrow \text{ graph } f \to Y).$$

Definition 4.6. A presheaf with transfers is an additive functor

$$F : \mathbf{Cor}_k^{op} \to \mathbf{Ab}.$$

We have a restriction functor

$$\mathbf{PSh}^{\mathrm{tr}}(k) \xrightarrow{rest} \mathbf{PSh}(\mathbf{Sm}_k).$$

Definition 4.7. Let τ be a topology on $\mathbf{Sm}_k$. A presheaf with transfers F is a τ-sheaf if $F|_{\mathbf{Sm}_k}$ is a τ-sheaf.

Toni constructed in his talk that given $A \in \mathbf{Ab}, q \geq 0$, we can get a cochain complex $A(q)$ of etale sheaves with transfers. It is concentrated in cohomological degrees $\leq q$.

Definition 4.8 (Motivic cohomology)**.** Let X be a smooth scheme over k. We have motivic cohomology

$$H^p(X, A(q)) := H^p_{Zar}(X, A(q)|_{X_{Zar}}),$$

and etale motivic cohomology

$$H^p_{et}(X, A(q)) := H^p_{et}(X, A(q)|_{X_{et}}).$$

Remark 4.9. The proof of BK(n) interprets one side as etale motivic cohomology, and the other side as Zariski motivic cohomology.

We have a pushforward functor

$$\pi_* : \mathbf{Sh}_{et}(\mathbf{Sm}_k) \to \mathbf{Sh}_{Zar}(\mathbf{Sm}_k)$$

and the pullback functor π^* is the etale sheafification.

Applying the unit $\mathrm{id} \to R\pi_*\pi^*$ to $A(q)$, we get

$$\begin{tikzcd} A(q) \arrow[r] \arrow[dr, dashed] & R\pi_* A(q) \\ & \tau^{\leq q} R\pi_* A(q) \arrow[u] \end{tikzcd}.$$

4.5. **Properties/Axioms.** We will use the following properties for

(1) The complex $\mathbb{Z}(0) \simeq \mathbb{Z}$ is the constant sheaf. And we have $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$.
(2) The complex $A(q)$ is in degree $\leq q$.
(3) A vanishing property following from (2): if $p > q + dim(X)$, then we have
$$H^p(X, A(q)) = 0.$$
(4) If $n \geq 2$ is invertible in k, there is an exact triangle
$$\mathbb{Z}(q) \xrightarrow{n} \mathbb{Z}(q) \to \mathbb{Z}/n(q).$$
(5) There is a natural isomorphism
$$K^M_q(K) \xrightarrow{\sim} H^q(k, \mathbb{Z}(q)).$$
(6) If char$(k) \neq l$, then we have
$$K^M_q(k)/l \xrightarrow{\sim} H^q(k, \mathbb{Z}/l(q)).$$
This relates the left hand side of BK(n) to motivic cohomology.

(7) If $n \geq 2$ is invertible in k,
$$\phi : \mu_n \xrightarrow{\sim \text{ qis of etale sheaves}} \mathbb{Z}/n(1)$$
This gives
$$\mu_n^{\otimes q} \xrightarrow{\simeq} \mathbb{Z}/n(q)$$
This relates the right hand side of BK(n) to motivic cohomology.

(8) The equivalences in $(6,7)$ is compatible with the norm-residue map
$$\begin{array}{ccc} K_q^M(k)/l & \xrightarrow{\sim} & H^q(k, \mathbb{Z}/l(q)) \\ \downarrow & & \downarrow \\ H_{et}^q(k, \mu_l^{\otimes q}) & \xrightarrow{\sim} & H_{et}^q(k, \mathbb{Z}/l(q)) \end{array}$$

(9) Localization: Assume X/k is smooth. We have
$$H_{et}^p(X, \mathbb{Z}(q)) \otimes \mathbb{Q} \xrightarrow{\sim} H_{et}^p(X; \mathbb{Q}(q)).$$
This is also true when $\mathbb{Q}$ is replaced by $\mathbb{Z}_{(l)}$.

(10) Rational isomorphism: Assume X/k is smooth. We have
$$H^p(X; \mathbb{Q}(q)) \xrightarrow{\sim} H_{et}^p(X; \mathbb{Q}(q)).$$

Remark 4.10. By (7), change of topology gives a map
$$\mathbb{Z}/l(q) \to R\pi_* \mu_l^{\otimes q}.$$
Since $\mathbb{Z}/l(q)$ is in degree $\leq q$, it factors
$$\mathbb{Z}/l(q) \to \tau^{\leq q} R\pi_* \mu_l^{\otimes q}.$$

4.6. **Conjecture BL(n).**

Conjecture 4.11 (BL(n)). *Let l be a prime. For every field k with char$(k) \neq l$, the change of topology map*
$$\mathbb{Z}/l(q) \to \tau^{\leq q} R\pi_* \mu_l^{\otimes q}$$
is a quasi-isomorphism.

By the reinterpretation, this implies BK(n).

Remark 4.12. Akhil asks Beilinson why he made this conjecture; Beilinson responded that he observed they satisfy the projective bundle formula.

Remark 4.13. To prove BK(1), we need H90: $0 = H_{et}^1(k, \mathbb{G}_m) \simeq H_{et}^2(k, \mathbb{Z}(1))$. We only need that the l-torsion subgroup is zero.

Lemma 4.14. *If $p > q$ then the following are true.*

(1) *$H_{et}^p(k, \mathbb{Z}(q))$ is torsion.*
(2) *The l-torsion subgroup of $H_{et}^p(k, \mathbb{Z}(q))$ is $H_{et}^p(k, \mathbb{Z}_{(l)}(q))$.*

Proof. 1) Consider the LES associated to the triangle
$$\mathbb{Z}(q) \to \mathbb{Q}(q) \to \mathbb{Q}/\mathbb{Z}(q).$$
We have
$$\cdots \to H_{et}^{p-1}(k, \mathbb{Q}/\mathbb{Z}(q)) \xrightarrow{\partial} H_{et}^p(k, \mathbb{Z}(q)) \to H_{et}^p(k, \mathbb{Q}(q)) \xrightarrow{\sim \text{ by (10)}} H^p(k, \mathbb{Q}(q)) = 0.$$
The group $H_{et}^{p-1}(\mathbb{Q}/\mathbb{Z}(q))$ is torsion, and the middle map is zero. Thus the boundary map is surjective. The result follows.
2) The l-torsion subgroup is $H_{et}^p(k, \mathbb{Z}(q)) \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)}$. By (9), this is the same as
$$H_{et}^p(k, \mathbb{Z}_{(l)}(q)).$$
∎

Conjecture 4.15 (H90(n)).

$$H_{et}^{n+1}(k, \mathbb{Z}_{(l)}(n)) = 0.$$

4.7. **Why BK(n) implies H90(n).**

Proposition 4.16. *BK(n) $\implies$ H90(n).*

Proof. Consider the exact triangle

$$\mathbb{Z}(n) \xrightarrow{l} \mathbb{Z}(n) \to \mathbb{Z}/l(n) \simeq_{et} \mu_l^{\otimes n}.$$

To simplify, write $H_{et}^p(A(q))$ for $H_{et}^p(k, A(q))$.
Consider the following diagram.

$$\begin{array}{ccccccccc} \cdots \longrightarrow H^n(\mathbb{Z}(n)) & \xrightarrow{l} & H^n(\mathbb{Z}(n)) & \longrightarrow & H^n(\mathbb{Z}/l(n)) & \longrightarrow & H^{n+1}(\mathbb{Z}(n)) & \overset{\text{by (3)}}{=} & 0 \\ \downarrow & & \downarrow & & \downarrow f & & \downarrow & & \\ \cdots \longrightarrow H_{et}^n(\mathbb{Z}(n)) & \xrightarrow{l} & H_{et}^n(\mathbb{Z}(n)) & \xrightarrow{g} & H_{et}^n(\mu_l^{\otimes n}) & \longrightarrow & H_{et}^{n+1}(\mathbb{Z}(n)) & & \end{array}.$$

If f is surjective, then g is surjective. Therefore

$$\begin{aligned} &\ker(H_{et}^{n+1}(\mathbb{Z}(n)) \xrightarrow{l} H_{et}^{n+1}(\mathbb{Z}(n))) \text{ is zero} \\ \iff &H_{et}^{n+1}(\mathbb{Z}(n)) \text{ has no } l\text{-torsion} \\ \iff &H_{et}^{n+1}(\mathbb{Z}_{(l)}(n)) \text{ is zero} = \text{H90(n)}. \end{aligned}$$

∎

4.8. **Equivalence of three conjectures.** Very rough ideas:

(1) Show BL(n) $\Longrightarrow$ BL(n-1), BK(n) $\Longrightarrow$ BK(n-1), and H90(n) $\Longrightarrow$ H90(n-1).
(2) use a dimension shifting argument involving some auxiliary cohomology theory of $\partial\Delta^n$, and use (1) to show by induction BK(n) implies BL(n).

We show (1): use localization and specialization.

Proposition 4.17. *For any p, q, there are split exact sequences as follows:*

$$0 \to H_{et}^p(k, A(q)) \to H_{et}^p(k(t), A(q)) \xrightarrow{\partial} \oplus_{\text{closed } x \in \mathbb{A}^1} H_{et}^{p-1}(k(x), A(q-1)) \to 0.$$

.

Corollary 4.18. (1) *If H90(n) holds for $k(t)$, then it holds for k.*
(1") H90(n) implies H90(n-1).
(2) *If $K_n^M(k(t))/l \to H_{et}^n(k(t), \mu_l^{\otimes n})$ is an surjective, so is*

$$K_{n-1}^M(k(t))/l \to H_{et}^{n-1}(k(t), \mu_l^{\otimes n}).$$

(2") BK(n) implies BK(n-1).

Proof. 1) apply the localization sequence with respect to $A = \mathbb{Z}_{(l)}$.
2) Take $A = \mathbb{Z}/l$.

$$\begin{array}{ccc} K_n^M(k(t))/l & \xrightarrow{\partial} & K_{n-1}^M(k)/l \\ \downarrow & & \downarrow \\ H_{et}^n(k(t), \mu_l^{\otimes n}) & \xrightarrow{\partial} & H_{et}^{n-1}(k, \mu_l^{\otimes n-1}) \end{array}.$$

∎