Vanishing lines and periodicities of higher real K-theories

Zhipeng Duan

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Higher real K-theories

HHR theories

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    - chromatic height $h = 2^{n-1}m$
    - good slice filtrations
    - closely related to Lubin–Tate theories/Morava E-theories

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- The Lubin–Tate theories E_h are fundamental objects in chromatic homotopy theory, and they are also equipped with group actions

Higher real K-theories

Theorem (Goerss–Hopkins–Miller, Lurie)

E_h is a commutative ($\mathbb{E}_\infty$) ring spectrum, and there is a unique $\mathbb{G}_h$-action on E_h by commutative ($\mathbb{E}_\infty$) ring maps.

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    - At prime 2, $h = 1$, and $G = C_2 \subset \mathbb{G}_1$ (formal inversion), $E_1 = KU_2^\wedge$ $\implies E_1^{hC_2} = (KU_2^\wedge)^{hC_2} = KO_2^\wedge$

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- E_h^{hG} is a very useful family of cohomology theories, because they play an important role in detecting periodic phenomena in stable homotopy

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- E_h^{hG} also detects important elements in $\pi_* S^0$:
  - $E_{p-1}^{hC_p}$ was used by Ravenel to resolve the odd primary Kervaire invariant problem ($p \geq 5$)
  - $E_4^{hC_8}$ was used by HHR to resolve the Kervaire invariant problem ($p = 2$)

Computation techniques on E_h^{hG}

- The traditional approach to compute $\pi_* E_h^{hG}$ is the homotopy fixed point spectral sequence (HFPSS)

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- With the help of the real orientation on E_h, we can transfer the computations of E_h^{hG} to the computations of $D^{-1}BP^{(\!(G)\!)}$ through the HFPSS or the slice spectral sequence (SliceSS) at least when G is a cyclic 2-group

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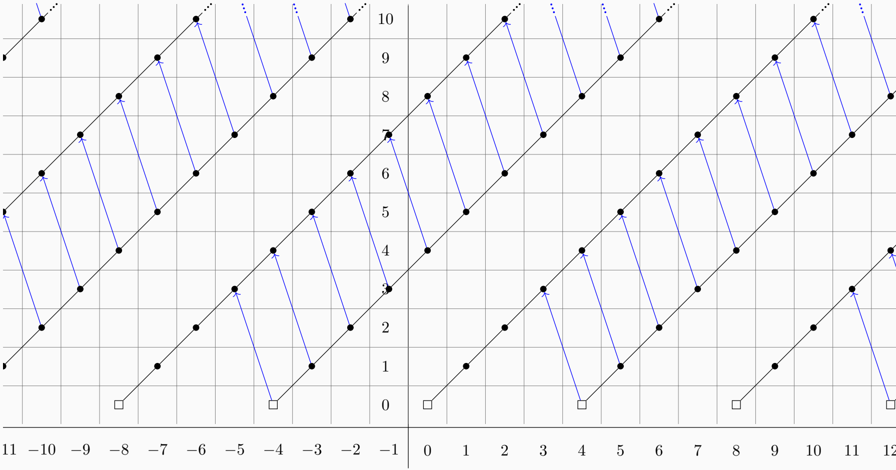
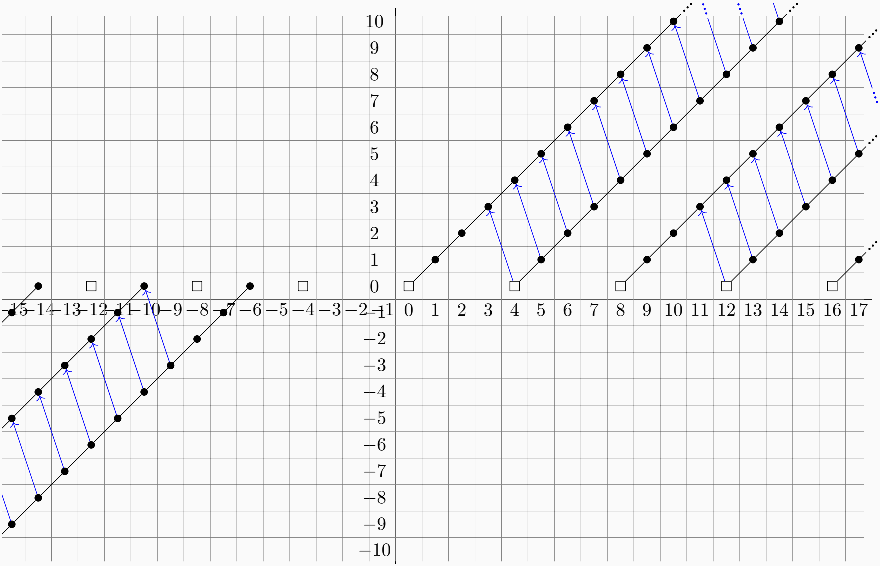
- There is a comparison map from SliceSS to HFPSS which are isomorphic under the line of slope 1

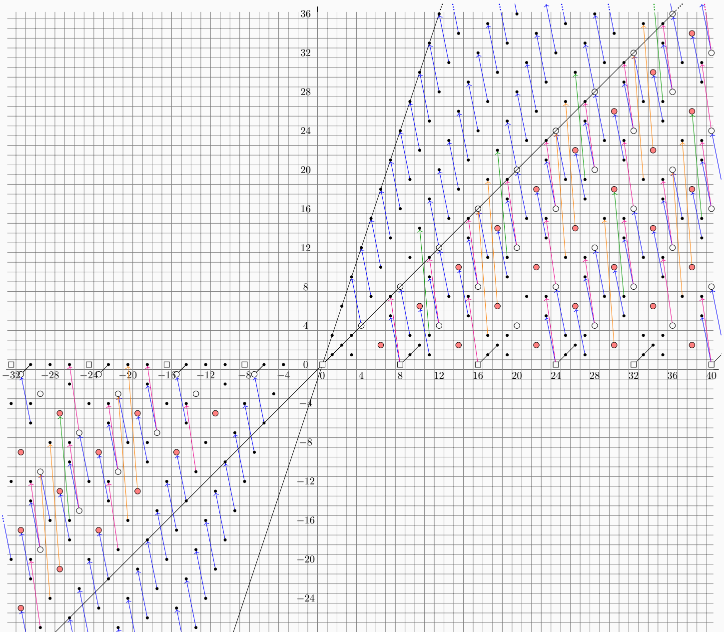
HFPSS for KO



SliceSS for $K_{\mathbb{R}}$



SliceSS for $D^{-1}BP^{((C_4))}\langle 1\rangle$



Real orientations on Lubin–Tate theories

Theorem (Hahn–Shi)

At prime $p = 2$, there is a C_2-equivariant map

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lifting the classical complex orientation $MU \to E_h$.

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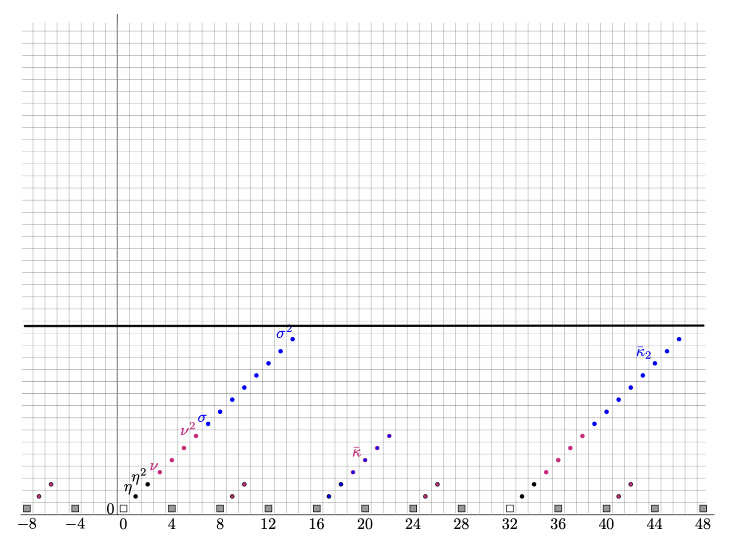
  - When $G = C_{2^n}$, this map factors through $D^{-1}BP^{(\!(G)\!)}\langle m \rangle$
    - This is a equivalence after 2-completion
    - The slices of $D^{-1}BP^{(\!(G)\!)}\langle m \rangle$ are completely known

Vanishing lines and periodicities

Examples: E_∞-page of HFPSS for $E_3^{hC_2}$



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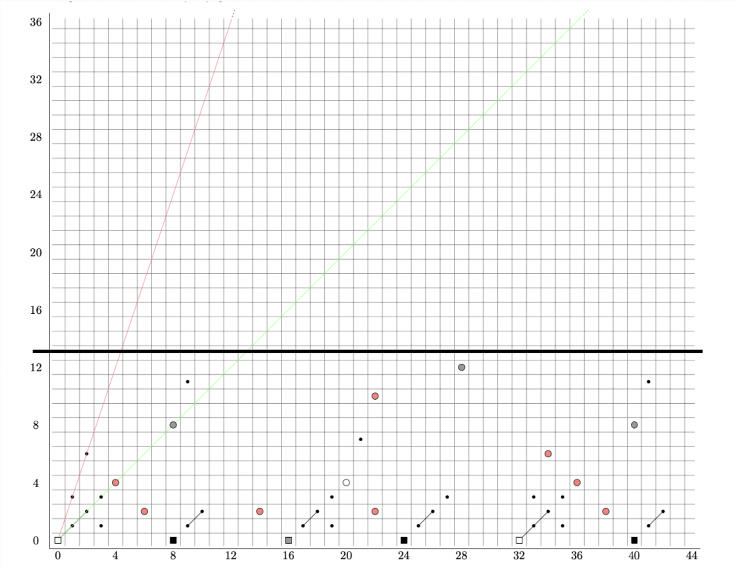

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Theorem (D.–Li–Shi)

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- Here, $N_{h,G} = N_{h,H}$, where H is a 2-Sylow subgroup of $G \cap \mathbb{S}_h$ and

$$N_{h,H} := \begin{cases} 1 & \text{if } H = e \\ 2^{h+n} - 2^n + 1 & \text{if } H = C_{2^n} \\ 2^{h+3} - 9 & \text{if } H = Q_8 \end{cases}$$

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- The filtration $N_{h,G}$ are sharp in all previously known cases, and they are very helpful in spectral sequence computations

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Applications

Orientation of bundles

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Question

When $E = E_h^{hG}$, for any vector bundle V, how many direct sums of V are E_h^{hG}-oriented?

Bundle orientation

Theorem (D.–Li–Shi)

When $p = 2$, for any finite subgroup $G < \mathbb{G}_h$ and any real vector bundle V, its d-fold direct sum is E_h^{hG}-oriented. Here, $d = 2 \cdot |K| \cdot |H|^{\frac{N_{h,H}-1}{2}}$, where $K = G \cap \mathbb{S}_h$ and H is a 2-Sylow subgroup of K.

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- Kitchloo–Wilson studied $E_h^{hC_2}$-orientation when $p = 2$, our result generalizes it to larger finite subgroups
- Bhattacharya–Chatham studied $E_{k(p-1)}^{hC_p}$-orientation for odd prime p

Bundle orientation, sketch of proof

- Work on universal bundle γ over BO, our goal is to find a Thom class $u : MO[d] \to E_h^{hG}$ such that the following Thom isomorphism holds:

$$(E_h^{hG})^*(MO[d]) \simeq (E_h^{hG})^*(BO_+)[u]$$

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- If there is a Thom class $u \in H^0(G, E_h^*(MO[d]))$ which is a permanent cycle, then the following diagram

$$\begin{tikzcd}
H^*(G, E_h^*(BO_+)) \arrow[r, "\cdot u"] \arrow[d, Rightarrow] & H^*(G, E_h^*(MO[d])) \arrow[d, Rightarrow] \\
(E_h^{hG})^*(BO_+) \arrow[r] & (E_h^{hG})^*(MO[d])
\end{tikzcd}$$

  will induce an isomorphism

$$(E_h^{hG})^*(BO_+) \cdot u \cong (E_h^{hG})^*(MO[d])$$

Orientation of bundles, sketch of proof

- 2γ is E_h-oriented gives the following Thom isomorphism

$$E_h^*(MO[2]) \simeq E_h^*(BO_+)[u_2]$$

  - u_2 is the Thom class $u_2 : MO[2] \to E_h$
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  - $gu_2 : MO[2] \to E_h \xrightarrow{g} E_h$
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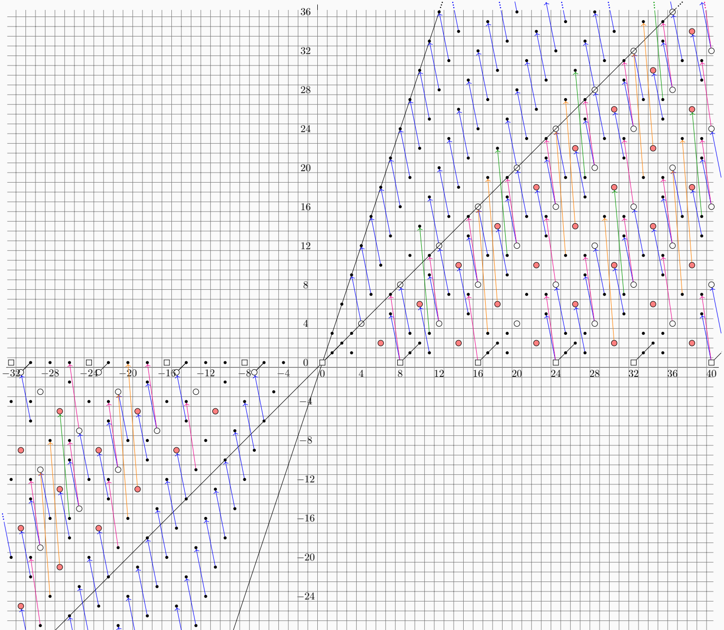
- Since each class in E_2-page is $|H|$-torsion, by the Leibniz rule and the vanishing line result, we can choose $u = u_k^n$, where $n = |H|^{\frac{N_{h,H}-1}{2}}$

SliceSS for $D^{-1}BP^{((C_4))}\langle 1\rangle$



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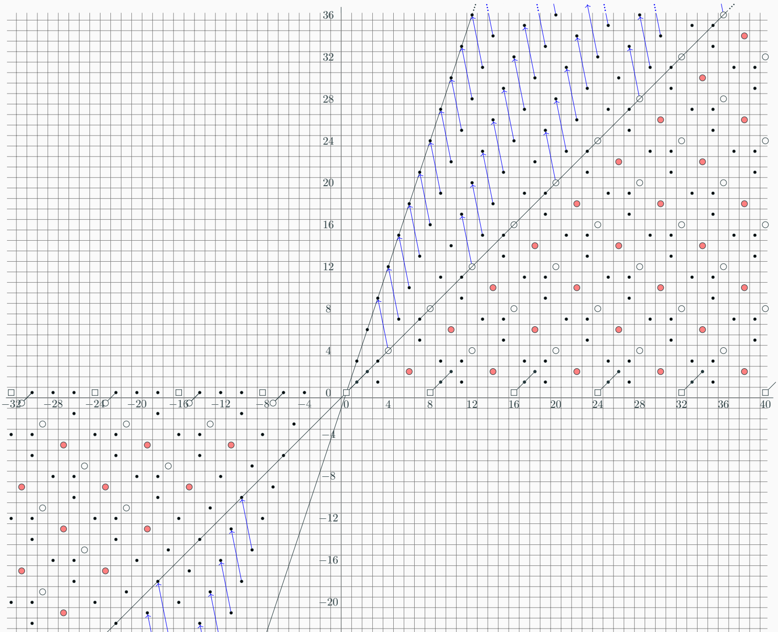

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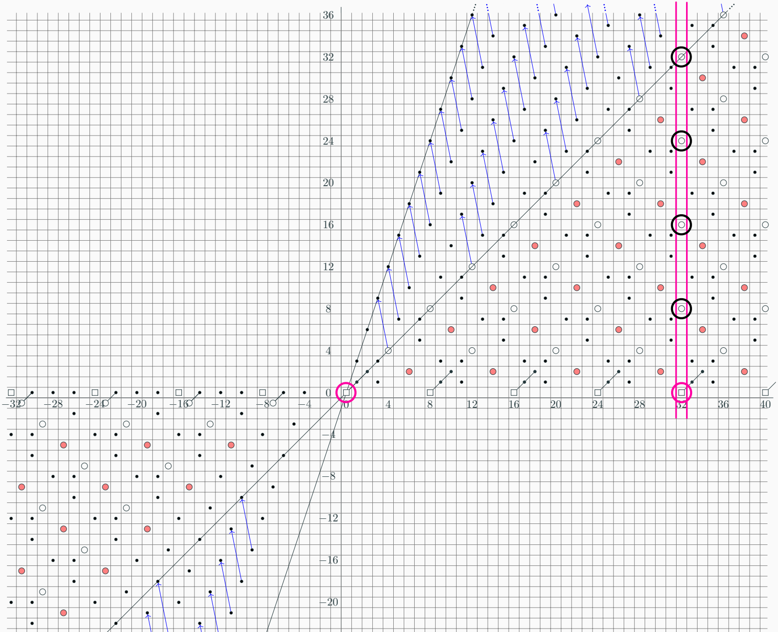

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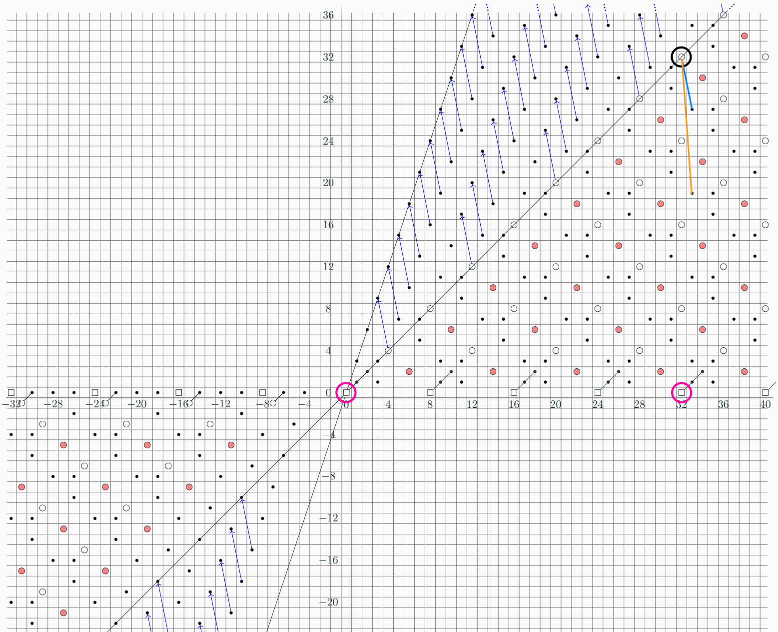

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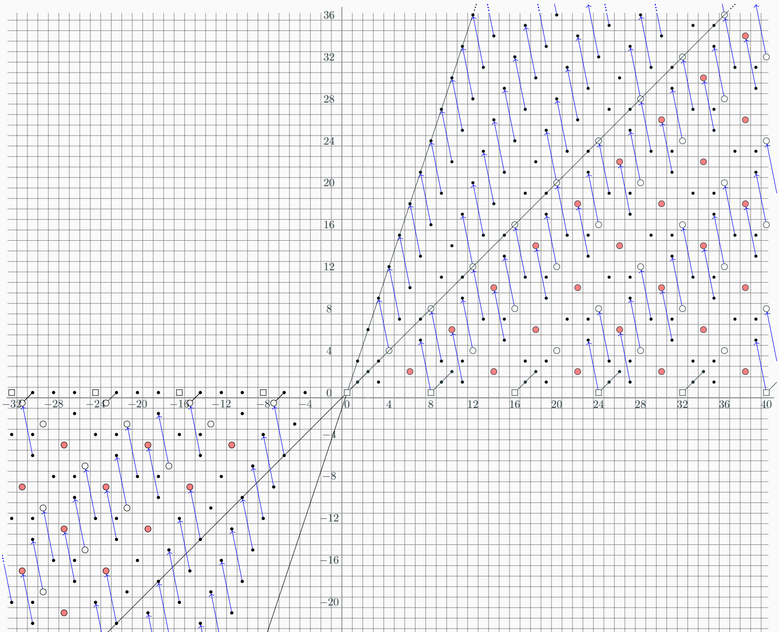

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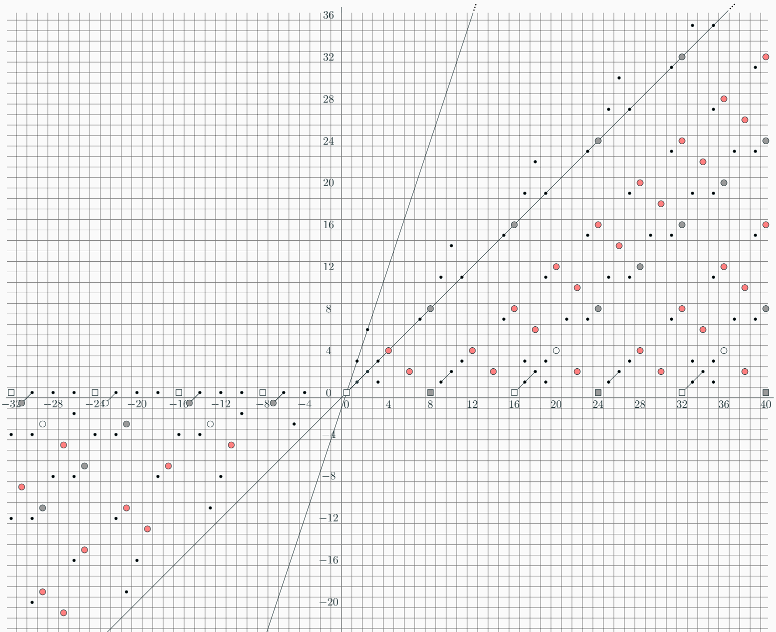

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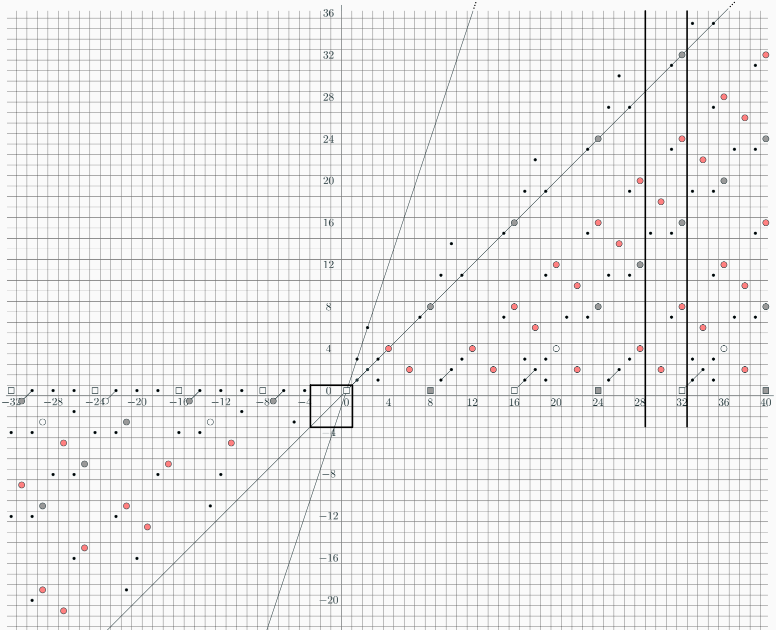

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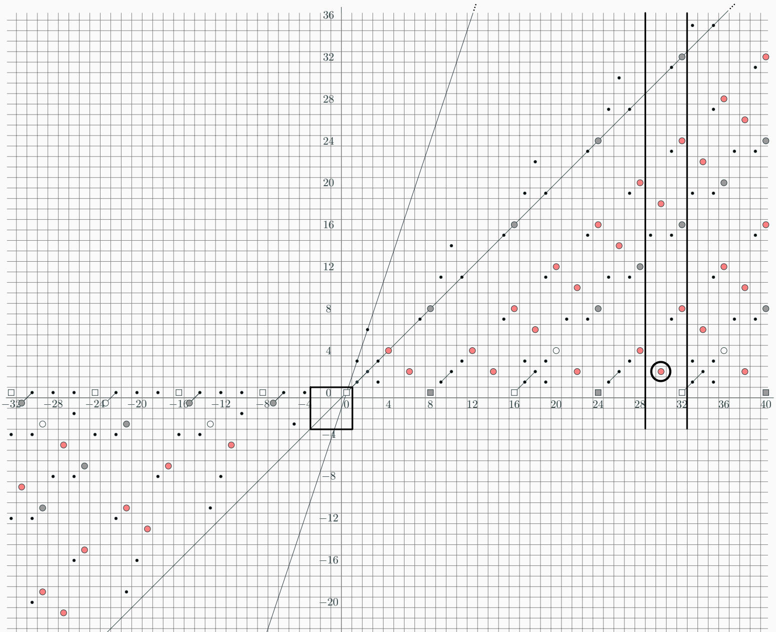

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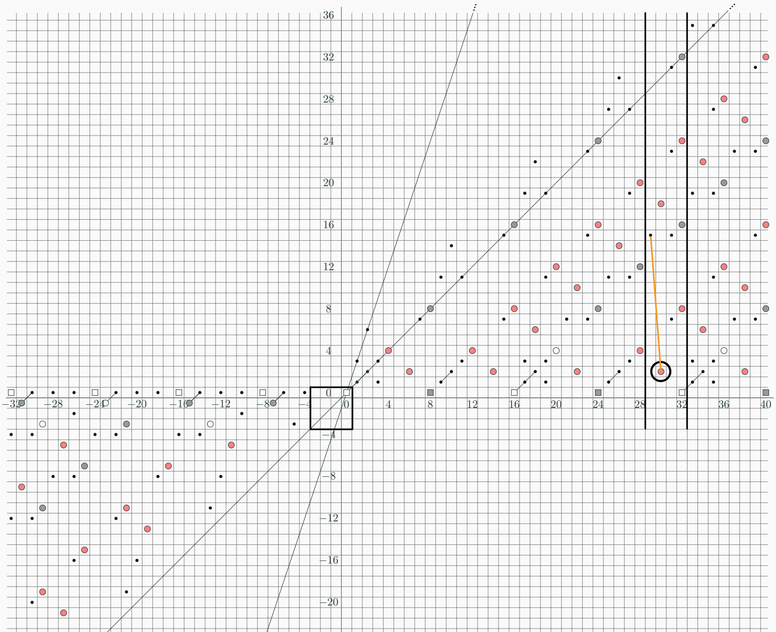

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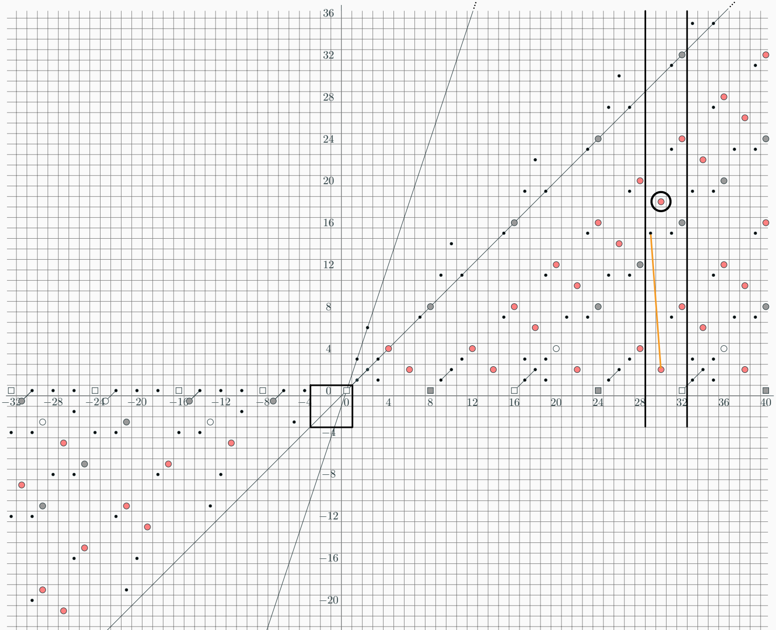

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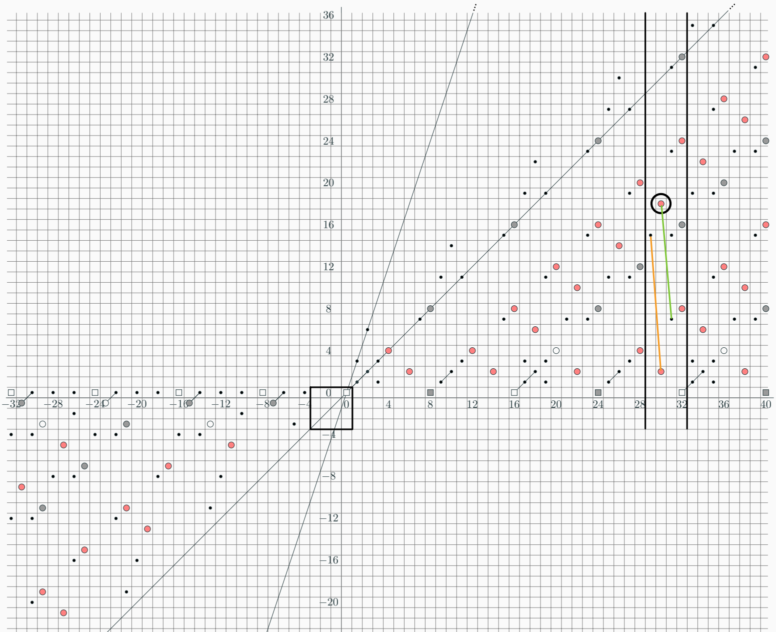

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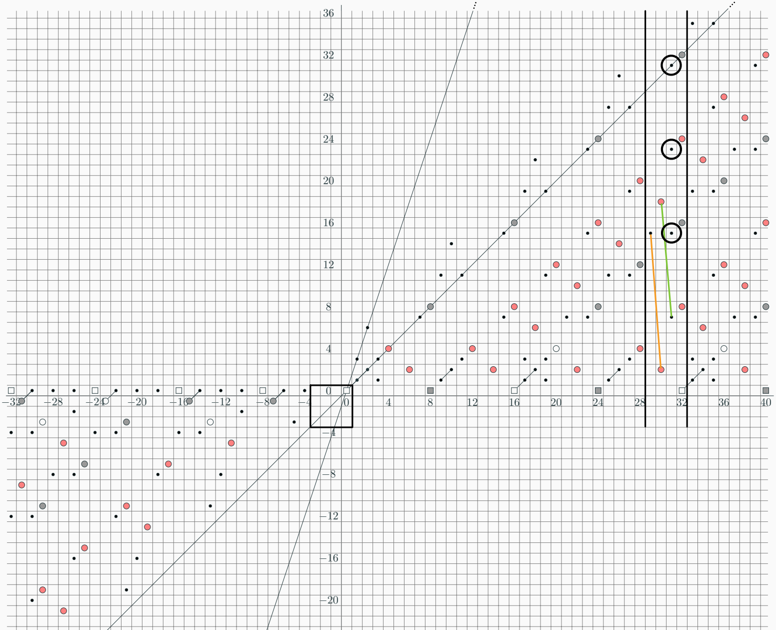

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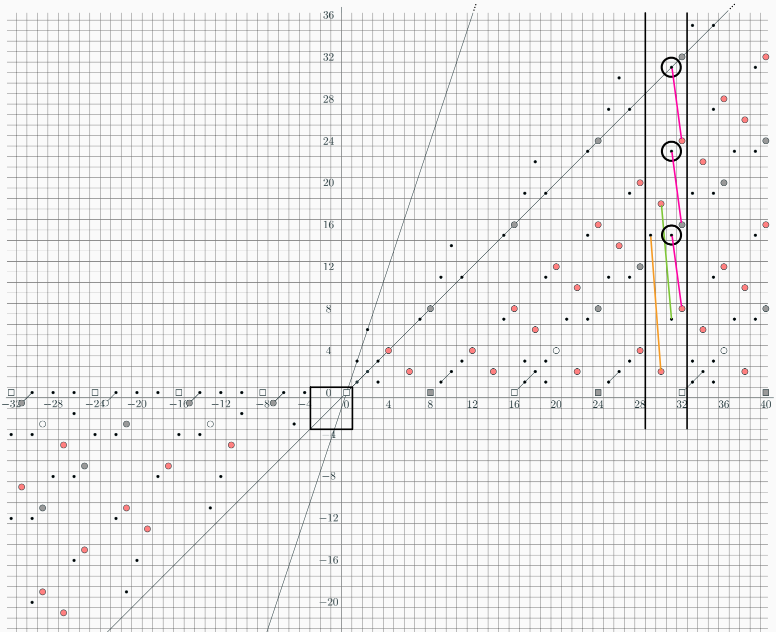

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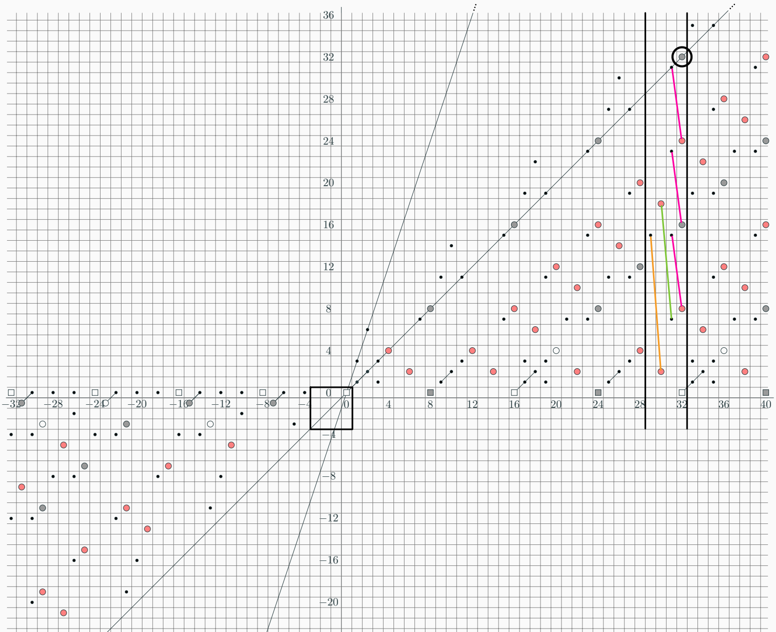

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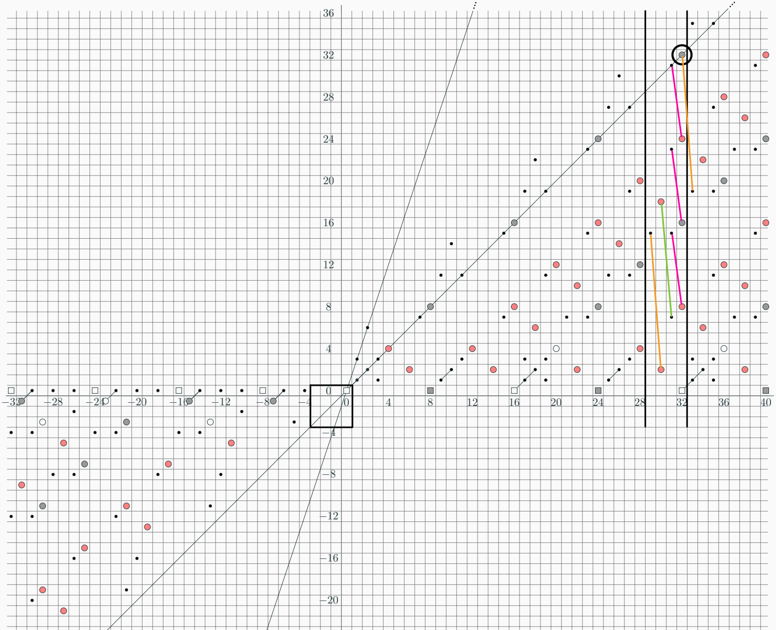
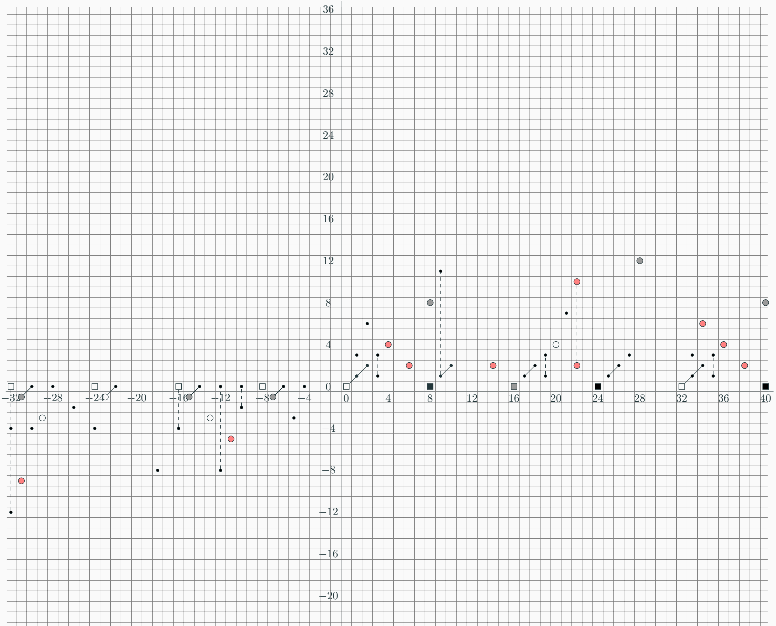


Extension issues



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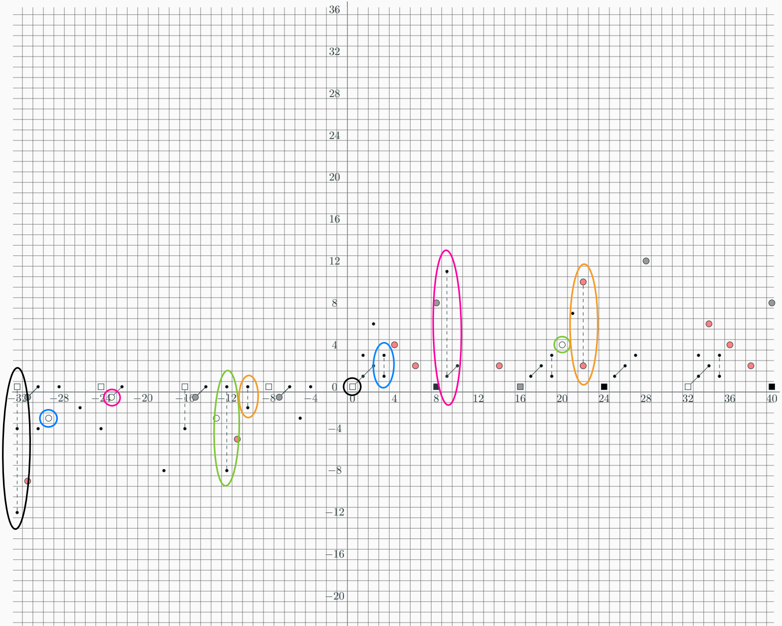
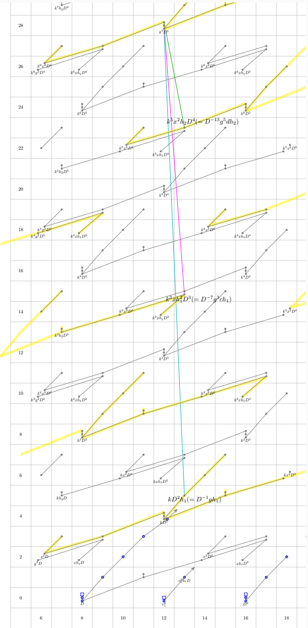


HFPSS for $E_2^{hG_{24}}$



Ideas of proofs

Tate construction

Definition

Given a G-spectrum X, we have the following so-called Tate construction of X

$$X^{tG} := \mathrm{cofib}(X_{hG} \to X^{hG}).$$

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- When $X = HM$: the Eilenberg-Maclane spectrum, which represents the singular cohomology with coefficient M
  - $\pi_*(HM_{hG}) = H_*(G, M)$
  - $\pi_*(HM^{hG}) = H^{-*}(G, M)$
  - $\pi_*(HM^{tG}) = \hat{H}^{-*}(G, M)$
  - $\pi_0(HM_{hG}) \to \pi_0(HM^{hG})$ is just the norm map $H_0(G, M) \xrightarrow{N} H^0(G, M)$ in group cohomology.

Tate spectral sequence

Theorem

There is a spectral sequence to compute $\pi_* X^{tG}$ called the Tate spectral sequence

$$\hat{H}^s(G, \pi_t X) \Rightarrow \pi_{t-s} X^{tG}.$$

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- This is a whole plane spectral sequence
    - There is a natural map from the HFPSS to the TateSS which induces an one–one correspondence of classes and differentials beyond the filtration 0
    - If there is a d_r-differential hitting the unity 1 in TateSS(E_h^{hG}), then there is a strong vanishing line at filtration r in HFPSS(E_h^{hG})

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- The unity 1 is killed by a $d_{2^{h+1}-1}$-differential in C_2-TateSS for E_h by Hahn–Shi's computation
  - The Tate spectrum $E_h^{tC_2}$ is contractible
  - There is a strong vanishing line at filtration $2^{h+1}-1$ in the HFPSS for $E_h^{hC_2}$

TateSS for $D^{-1}BP^{((C_4))}\langle 1\rangle$

HHR norm functor and differentials

Theorem (HHR)

For a spectral sequence with a norm structure, if there is a differential $d_r(x) = y$ on H-level, then there is a predicted differential on G-level

$$d_{|G/H|(r-1)+2}(N_H^G(x)a_{\bar{\rho}}) = N_H^G(y)$$

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With the same conditions as above, we have the following predicted differential on G-level:

$$d_r(N_H^G(x)) = \mathrm{tr}_H^G(y \prod_{[g]\in T} N_{H\cap H^g}^H(x^g))$$

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- When G is abelian, $d_r(N_H^G(x)) = \mathrm{tr}_H^G(yx^{|G/H|-1})$

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- Apply the map from HFPSS to TateSS to deduce the vanishing line
- As for the general case, we just take the transfer map from the 2-Sylow subgroup to $G \cap \mathbb{S}_h$

Thank you!