Outline

1) Stable homotopy thry background

2) motivic theory

3) Gheorghe-Wang-Xu th'm & application

4) Generalization

joint with Bachmann, Wang, Xu

Stable homotopy groups

$\pi_n(X) = \mathrm{Map}(S^n, X) /$ homotopy eq

(Freudenthal 37)

The sequence stablizes

$\pi_n(X) \to \pi_{n+1}(X \wedge S^1) \to \pi_{n+2}(X \wedge S^2) \to \cdots$

Stable homotopy grps

$\pi_n^{st}(X) := \mathrm{colim}\ \pi_{n+k}(X \wedge S^k)$

$\pi_n^{st}(S^0)$ connects to differential geom

Milnor $\pi_n^{st}(S^0) \simeq$ cobordism class of stably framed n manifold

Spectra

"stablized" spaces : "make $\wedge S^1$ invertible"

- X space $\rightsquigarrow$ $\Sigma^\infty X$ suspension spectra

  $\pi_n^{st}(X) = \pi_n(\Sigma^\infty X)$

  π_n ← homotopy grps of spectra

  × capture the stable info

  × represent cohomology theory

- SH stable homotopy cat

  triangulated cat & t-str.

  - abstraction of str. of chain cplx & derived cat
  - use to cut obj into easier pieces

Computations in SH

want to understand $\pi_*(\mathbb{S})$

the homotopy grps of the sphere spectrum

- Adams spectral sequence

$$E_2 = \mathrm{Ext}_{A}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \overset{\text{differentials}}{\Longrightarrow} \pi_*(\mathbb{S})^{\wedge}_2$$

(A: steenrod algebra)

E_2 page : purely algebraic

differentials : contains geometric info

hard to analyze.

Motivic method

Morel
Voevodsky

Milnor
Bloch-Kato

motivic stable homotopy cat $SH(k)$
k: base field

$SH(k)$: "stable homotopy cat for Sm/k"

$$\begin{tikzcd} \text{Motivic} \arrow[r, shift right, "\text{recover}"'] & \text{Classical} \arrow[l, shift right, "\text{tools}"'] \end{tikzcd}$$

one cat $SH(k)$ for each base field k

$\mathbb{A}^1$-local, Nis-local presheaves $\in \mathrm{Fun}(Sm/k^{op}, sSet)$

obj: combine k-smooth schemes (as representable) & top spaces (as constant)

Spheres in motivic

Two circles

$S^{1,0} = \Delta^1/\partial\Delta^1$ simplicial circle

$S^{1,1} = \mathbb{A}^1 \setminus 0$ Tate circle

$S^{s,w} = (S^{1,0})^{s-w} \wedge (S^{1,1})^{w}$

$\mathbb{S} = S^{0,0} = \mathrm{Spec}(k) \amalg \mathrm{Spec}(k)$

$\pi^k_{s,w}(\mathbb{S}) = [S^{s,w}, \mathbb{S}]$ bigraded homotopy grps

$\pi^k_{n,n}(\mathbb{S}) \triangleq$ nth Milnor Witt K-theory

classical	Motivic
topological spaces	motivic spaces $\left\{ \begin{array}{l} \text{simplicial sets (top spaces} \\ \text{smooth schemes} \end{array} \right.$
S^1 : circle	$S^{1,0}$, $S^{1,1}$
spectra (invert S^1)	motivic spectra (invert $\begin{array}{l} S^{1,0} \\ S^{1,1} \end{array}$)
$\pi_n(\mathbb{S})$	$\pi_{*,*}(\mathbb{S})$
$\mathrm{Ext}_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*(\mathbb{S}_2^\wedge)$	$\mathrm{Ext}_{A_\mathbb{R}}^{***}(H\mathbb{F}_{2**}, H\mathbb{F}_{2**}) \Rightarrow \pi_{**}(\mathbb{S}_2^\wedge)$
Ex: $H\mathbb{F}_{2*} = \mathbb{F}_2$	$H\mathbb{F}_2^{\mathbb{C}}{}_{**} = \mathbb{F}_2[\tau]$ $\quad \lvert\tau\rvert = (0,-1)$

Betti Realization

Betti Realization :

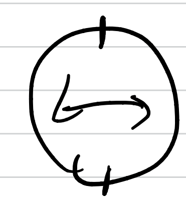
take $\mathbb{C}$ points

$$S^{1,0} \longmapsto S^1$$

$$S^{1,1} \longmapsto S^1$$

$k = \mathbb{R} \hookrightarrow \mathbb{C}$

Galois action

$\rightsquigarrow$ $S^{1,1} \mapsto S^{\sigma}$ 

$S^{1,0} \mapsto S^1$ trivial action

Advantage of using motivic

1) More classes

classical cohom of a pt $\mathbb{F}_2$

$\mathbb{C}$ cohom of a pt $\mathbb{F}_2[\tau]$

2) motivic Adams SS has a comparison map to someth purely algebraic

($\mathbb{C}$-mot)

$mASS(S^{0,0}_2/\tau) \cong$ Alg Novikov SS

Brown Petersen

$\tau : \Sigma^{0,-1} \mathbb{S}_2^\wedge \to \mathbb{S}_2^\wedge$

"Derived cat"

Thm (Gheorghe–Wang–Xu, Pstragowski)

$\mathbb{C}$-motivic

$\mathbb{S}_2^\wedge/\tau$ -Mod cell $\simeq$ Stable $(BP_*BP$-Comod$)$

$$\begin{array}{ccc} \text{mASS}(\mathbb{S}_2^\wedge/\tau) & \leftrightsquigarrow & \text{Alg Nov SS} \\ \| & & \| \\ \text{ASS for } \mathbb{S}_2^\wedge/\tau & & \text{ASS for } BP_* \\ \text{in } \mathbb{S}_2^\wedge/\tau\text{-mod} & & \text{in Stable}(BP_*BP\text{-Comod}) \end{array}$$

$$\text{mASS}(\mathbb{S}_2^\wedge) \longrightarrow \text{mASS}(\mathbb{S}_2^\wedge/\tau)$$

Use naturality

Can we generalize $\mathbb{C}$?

$\mathbb{C} \rightsquigarrow \mathbb{R}$ (Burklund – Hahn – Senger)

Thm $\overset{\mathbb{R}\text{-mot}}{\checkmark}$ $\tau : \mathbb{S}^{1} \to \mathbb{Q}_2$ an invertible object

$\mathbb{S}_2^{\wedge}/\tau\text{-Mod}_{AT} \simeq \mathrm{Stable}(\underline{BP_{2*}BP}\text{-comod})$

$\underline{(-)}$: Mackey functor $\begin{cases} C_2\text{-level} \\ e\text{-level} \end{cases}$ encode

$\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = C_2$.

AT : Artin-Tate cellular + Spec $(\mathbb{C})$

- completion
- specific field
- "cellular" part

both $\mathbb{C}$ & $\mathbb{R}$: rely on known computations

- For general base field
- For not necessarily cellular part
- For integral results

<u>Thm</u> (BKWX) char exp $k = e$

- $\mathbb{1}_{c=0}$-Mod $[1/e]$ is purely algebraic cat
- (cellular) $\mathbb{1}_{c=0}\text{-Mod}_{W\text{-cell}}[1/e] = \mathrm{Stable}(\underline{\mathcal{M}_{\mathbb{Z}*}\mathcal{M}\text{-Coll}})^{[1/e]}$

① $\mathcal{M}$: integral ver of BP

② W-cell : cells correspond to subext

③ $\mathbb{1}_{c=0}$ plays the role of $\hat{\mathbb{1}}_2/\tau$, $\hat{\mathbb{1}}_2/\bar{\tau}$

<u>Rmk</u>

1) independent of base field
2) recover $\mathbb{C}$ & $\mathbb{R}$ result
3) proof : by defining a t-str.

triangulated cat & t-str

spectra $\quad \longleftrightarrow \quad$ chain cplx

$- \wedge S^1 \quad \leftrightsquigarrow \quad$ shift $[1]$

cofiber seq = fiber seq $\leftrightsquigarrow \quad A \xrightarrow{f} B \to A[1] \oplus B$

A Δ cat is the abstraction of Δ structures.

1) translation functor Σ
2) distinguished triangles

+ Axioms

A t-str. : abstraction of str. of connective (coconnective) subcats

$(D_{\geq 0}, D_{\leq 0})$ two subcat of Δ cat D + axioms

$\cdots \hookrightarrow \Sigma D_{\geq 0} \hookrightarrow D_{\geq 0} \hookrightarrow \Sigma^{-1} D_{\geq 0} \hookrightarrow \cdots$ gives a filtration

Postnikov t-structure on SH

$$SH_{\geq 0} := \{X \mid \pi_* X \text{ concentrated in } \geq 0\}$$
$$SH_{\leq 0} := \{X \mid \pi_* X \text{ concentrated in } \leq 0\}$$

$$\begin{tikzcd} \arrow[r] & X_{\geq 3} \arrow[r] \arrow[d] & X_{\geq 2} \arrow[r] \arrow[d] & X_{\geq 1} \arrow[r] \arrow[d] & X_{\geq 0} \arrow[d] \\ & X_{=3} & X_{=2} & X_{=1} & X_{=0} \end{tikzcd}$$

$$\pi_* X_{\geq n} = \begin{cases} \pi_* X & * \geq n \\ 0 & * < n \end{cases} \qquad \pi_* X_{=n} = \begin{cases} \pi_n X & * = n \\ 0 & \text{o.w.} \end{cases}$$

$\Sigma^{-n} X_n$ lies in the heart of the t-str.

$$SH^{\heartsuit} := SH_{\geq 0} \cap SH_{\leq 0} \simeq \mathbb{Z}\text{-Mod}$$

In general: use t-str. to cut things into easier pieces.

$\mathbb{1}_{c=0}$: the 0th part of the t-str we defined

<u>Def</u> Chow t-str.

$SH(k)_{\geq 0}$: generated under colim/ext by

$Th_X(\xi)$ $X \in Sm_k$ smooth proper

ξ virtual bundle

<u>Ex</u> $X = \mathrm{Spec}(k)$ $\xi = \mathbb{A}^1$ dim n trivial bundle

$Th_X(\xi) = \mathbb{A}^1 / (\mathbb{A}^1 - \{0\}) = S^{2,1}$

$\Rightarrow$ if $X \in SH(k)_{c<0}$ then $\pi_{s,w}(X)$ is concentrated in deg $\underset{\text{Chow deg}}{\underset{\shortparallel}{s-2w}} < 0$

Proof sketch

Prop (BKWX) $X \in SH(k)^{c\heartsuit}$

$MGL_{**}X$ concentrated in Chow deg 0

MGL: motivic alg cobordism spec

① mANSS: use $MGL_{**}X$ to compute $\pi_{**}X$

② use $\pi_{**}(-\wedge \text{Thom})$ to detect equivalences.

$\Rightarrow$ $-\wedge MGL$ detect equivalences (for bounded obj)

$\Rightarrow$ use Monadicity theorem to show the equivalence (on bounded part)

Extend the application

$$\begin{tikzcd}
 & \downarrow & \\
 & \mathbb{1}_{c\geq 2} \arrow[r] \arrow[d] & \mathbb{1}_{c=2} \\
 & \mathbb{1}_{c\geq 1} \arrow[r] \arrow[d] & \mathbb{1}_{c=1} \\
\mathbb{1} = & \mathbb{1}_{c\geq 0} \arrow[r] & \mathbb{1}_{c=0}
\end{tikzcd}$$

tower associated to t-str.

know E_2

know E_2 & diff

use naturality to compute differentials.