Statistical Linear Models

Assignment 3

Hanbin Liu 11912410

1.

Since $\mathbf{B}$ is a constant matrix, we have

$$\begin{aligned}
\frac{\partial \mathbf{H}}{\partial x} &= \mathbf{B}' \frac{\partial (\mathbf{BAB}')^{-1}\mathbf{B}}{\partial x} + \frac{\partial \mathbf{B}'}{\partial x}(\mathbf{BAB}')^{-1}\mathbf{B} \\
&= \mathbf{B}' \frac{\partial (\mathbf{BAB}')^{-1}\mathbf{B}}{\partial x} \\
&= \mathbf{B}' \Big(\frac{\partial (\mathbf{BAB}')^{-1}}{\partial x}\mathbf{B} + (\mathbf{BAB}')^{-1}\frac{\partial \mathbf{B}}{\partial x} \Big) \\
&= \mathbf{B}' \frac{\partial (\mathbf{BAB}')^{-1}}{\partial x}\mathbf{B} \\
&= \mathbf{B}' \Big(-(\mathbf{BAB}')^{-1}\frac{\partial \mathbf{BAB}'}{\partial x}(\mathbf{BAB}')^{-1} \Big)\mathbf{B} \\
&= -\mathbf{B}'(\mathbf{BAB}')^{-1}\frac{\partial \mathbf{BAB}'}{\partial x}(\mathbf{BAB}')^{-1}\mathbf{B} \\
&= -\mathbf{B}'(\mathbf{BAB}')^{-1}\mathbf{B}\frac{\partial \mathbf{A}}{\partial x}\mathbf{B}'(\mathbf{BAB}')^{-1}\mathbf{B} \\
&= -\mathbf{H}\frac{\partial \mathbf{A}}{\partial x}\mathbf{H}.
\end{aligned}$$

□

2.

(a)

Independent. Reason:

$$\mathrm{Cov}(X_2, 2X_1 - X_3) = 2\mathrm{Cov}(X_2, X_1) - \mathrm{Cov}(X_2, X_3) = 2\sigma_{21} - \sigma_{23} = 0 - 0 = 0.$$

(b)

$$\begin{pmatrix} 2X_1 - 5X_3 \\ X_1 + X_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \mathbf{AX}.$$

Since $\mathbf{X} \sim N_3(\mu, \Sigma)$, we have $\mathbf{AX} \sim N(\mathbf{A}\mu, \mathbf{A}\Sigma\mathbf{A}')$.

$$\mathbf{A}\mu = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{A}\Sigma\mathbf{A}' &= \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 & -3 \\ 0 & 9 & 0 \\ -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -5 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 25 & 0 & -16 \\ 5 & 9 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -5 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 130 & 25 \\ 25 & 14 \end{pmatrix}
\end{aligned}$$

(c)

Partition:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} Y_1 \\ X_3 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix},$$

$$\mu = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \text{where } \mu_2 = 0,$$

$$\Sigma = \begin{pmatrix} 5 & 0 & -3 \\ 0 & 9 & 0 \\ -3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \text{where } \Sigma_{22} = 2.$$

Therefore, the conditional distribution of X_3 given that $X_1 = 1$ and $X_2 = -2$ is

$$N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}).$$

And,

$$\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - \mu_1) = 0 + \begin{pmatrix} -3 & 0 \end{pmatrix} \begin{pmatrix} 1/5 & 0 \\ 0 & 1/9 \end{pmatrix} \begin{pmatrix} 1-3 \\ -2-(-2) \end{pmatrix} = \frac{6}{5}.$$

$$\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} = 2 - \begin{pmatrix} -3 & 0 \end{pmatrix} \begin{pmatrix} 1/5 & 0 \\ 0 & 1/9 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \frac{1}{5}.$$

Thus, $X_3 \mid (X_1, X_2) = (1, -2) \sim N(\frac{6}{5}, \frac{1}{5})$. □

3.

Let

$$\mathbf{J} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then,

$$U = \mathbf{Y}'(\mathbf{I} - \frac{1}{3}\mathbf{J})\mathbf{Y} = \sum_{i=1}^{3}(Y_i - \bar{Y})^2.$$

Thus,

$$E(U) = E\Big(\mathbf{Y}'(\mathbf{I} - \frac{1}{3}\mathbf{J})\mathbf{Y}\Big) = tr\Big((\mathbf{I} - \frac{1}{3}\mathbf{J})\Sigma\Big) + E(\mathbf{Y})'(\mathbf{I} - \frac{1}{3}\mathbf{J})E(\mathbf{Y}).$$

And

$$\mathbf{J}\Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

$$E(\mathbf{Y})'E(\mathbf{Y}) = 2^2 + 3^2 + 4^2 = 29,$$

$$E(\mathbf{Y})'\mathbf{J}E(\mathbf{Y}) = (2 \quad 3 \quad 4)\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}2\\3\\4\end{pmatrix} = 81.$$

Therefore,

$$\begin{aligned}E(U) &= tr\Big((\mathbf{I} - \frac{1}{3}\mathbf{J})\Sigma\Big) + E(\mathbf{Y})'(\mathbf{I} - \frac{1}{3}\mathbf{J})E(\mathbf{Y})\\ &= tr(\Sigma) - \frac{1}{3}tr(\mathbf{J}\Sigma) + E(\mathbf{Y})'E(\mathbf{Y}) - \frac{1}{3}E(\mathbf{Y})'\mathbf{J}E(\mathbf{Y})\\ &= 6 - \frac{1}{3}\cdot 6 + 29 - \frac{1}{3}\cdot 81\\ &= 6.\end{aligned}$$

□

4.

Note that

$$U = \sum_{i<j}(Y_i - Y_j)^2 = \sum_{i>j}(Y_i - Y_j)^2 = \sum_{i\geq j}(Y_i - Y_j)^2.$$

Thus,

$$\begin{aligned}2U &= \sum_{i<j}(Y_i - Y_j)^2 + \sum_{i\geq j}(Y_i - Y_j)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(Y_i - Y_j)^2\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(Y_i^2 + Y_j^2 - 2Y_iY_j)\\ &= \sum_{i=1}^{n}\Big[nY_i^2 + \sum_{j=1}^{n}Y_j^2 - 2Y_i\sum_{j=1}^{n}Y_j\Big]\\ &= n\sum_{i=1}^{n}Y_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}Y_j^2 - 2\sum_{i=1}^{n}Y_i\sum_{j=1}^{n}Y_j\\ &= n\sum_{i=1}^{n}Y_i^2 + n\sum_{j=1}^{n}Y_j^2 - 2(\sum_{i=1}^{n}Y_i)^2\\ &= 2n\sum_{i=1}^{n}Y_i^2 - 2(\sum_{i=1}^{n}Y_i)^2,\end{aligned}$$

which implies that $U = n\sum_{i=1}^{n}Y_i^2 - (\sum_{i=1}^{n}Y_i)^2$. Let $\mathbf{A} = n\mathbf{I} - \mathbf{J}$, then we have

$$\begin{aligned}\mathbf{Y'AY} &= \mathbf{Y}'(n\mathbf{I} - \mathbf{J})\mathbf{Y}\\ &= n\mathbf{Y'IY} - \mathbf{Y'JY}\\ &= n\mathbf{Y'Y} - (Y_1, ..., Y_n)\begin{pmatrix}1 & ... & 1\\ \vdots & \ddots & \vdots\\ 1 & ... & 1\end{pmatrix}\begin{pmatrix}Y_1\\ \vdots\\ Y_n\end{pmatrix}\\ &= n\sum_{i=1}^{n}Y_i^2 - \left(\textstyle\sum_{i=1}^{n}Y_i, ..., \sum_{i=1}^{n}Y_i\right)\begin{pmatrix}Y_1\\ \vdots\\ Y_n\end{pmatrix}\\ &= n\sum_{i=1}^{n}Y_i^2 - (\sum_{i=1}^{n}Y_i)(Y_1 + ... + Y_n)\\ &= n\sum_{i=1}^{n}Y_i^2 - (\sum_{i=1}^{n}Y_i)^2 = U.\end{aligned}$$

Therefore,

$$\begin{aligned}
E(U) &= E\Big(\mathbf{Y}'(n\mathbf{I}-\mathbf{J})\mathbf{Y}\Big) \\
&= tr\Big((n\mathbf{I}-\mathbf{J})\mathrm{Cov}(\mathbf{Y})\Big) + E(\mathbf{Y})'(n\mathbf{I}-\mathbf{J})E(\mathbf{Y}) \\
&= ntr\Big(\mathrm{Cov}(\mathbf{Y})\Big) - tr\Big(\mathbf{J}\mathrm{Cov}(\mathbf{Y})\Big) + nE(\mathbf{Y})'E(\mathbf{Y}) - E(\mathbf{Y})'\mathbf{J}E(\mathbf{Y}) \\
&= n^2\sigma^2 - n\sigma^2 + n^2\mu^2 - n^2\mu^2 \\
&= (n^2-n)\sigma^2.
\end{aligned}$$

Clearly, $k = \frac{1}{n^2-n}$. □