

# Statistical Linear Models

## Assignment 3

Hanbin Liu 11912410

1.

Since  $\mathbf{B}$  is a constant matrix, we have

$$\begin{aligned}
 \frac{\partial \mathbf{H}}{\partial x} &= \mathbf{B}' \frac{\partial (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \mathbf{B}}{\partial x} + \frac{\partial \mathbf{B}'}{\partial x} (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \mathbf{B} \\
 &= \mathbf{B}' \frac{\partial (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \mathbf{B}}{\partial x} \\
 &= \mathbf{B}' \left( \frac{\partial (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1}}{\partial x} \mathbf{B} + (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \frac{\partial \mathbf{B}}{\partial x} \right) \\
 &= \mathbf{B}' \frac{\partial (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1}}{\partial x} \mathbf{B} \\
 &= \mathbf{B}' \left( -(\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \frac{\partial \mathbf{B} \mathbf{A} \mathbf{B}'}{\partial x} (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \right) \mathbf{B} \\
 &= -\mathbf{B}' (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \frac{\partial \mathbf{B} \mathbf{A} \mathbf{B}'}{\partial x} (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \mathbf{B} \\
 &= -\mathbf{B}' (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \mathbf{B} \frac{\partial \mathbf{A}}{\partial x} \mathbf{B}' (\mathbf{B} \mathbf{A} \mathbf{B}')^{-1} \mathbf{B} \\
 &= -\mathbf{H} \frac{\partial \mathbf{A}}{\partial x} \mathbf{H}.
 \end{aligned}$$

□

2.

(a)

Independent. Reason:

$$\text{Cov}(X_2, 2X_1 - X_3) = 2\text{Cov}(X_2, X_1) - \text{Cov}(X_2, X_3) = 2\sigma_{21} - \sigma_{23} = 0 - 0 = 0.$$

(b)

$$\begin{pmatrix} 2X_1 - 5X_3 \\ X_1 + X_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \mathbf{A}\mathbf{X}.$$

Since  $\mathbf{X} \sim N_3(\mu, \Sigma)$ , we have  $\mathbf{A}\mathbf{X} \sim N(\mathbf{A}\mu, \mathbf{A}\Sigma\mathbf{A}')$ .

$$\mathbf{A}\mu = \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{A}\Sigma\mathbf{A}' &= \begin{pmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 & -3 \\ 0 & 9 & 0 \\ -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -5 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 25 & 0 & -16 \\ 5 & 9 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -5 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 130 & 25 \\ 25 & 14 \end{pmatrix}
\end{aligned}$$

(c)

Partition:

$$\begin{aligned}
\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} &= \begin{pmatrix} Y_1 \\ X_3 \\ Y_2 \end{pmatrix}, \\
\mu &= \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \text{where } \mu_2 = 0, \\
\Sigma &= \begin{pmatrix} 5 & 0 & -3 \\ 0 & 9 & 0 \\ -3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \text{where } \Sigma_{22} = 2.
\end{aligned}$$

Therefore, the conditional distribution of  $X_3$  given that  $X_1 = 1$  and  $X_2 = -2$  is

$$N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}).$$

And,

$$\begin{aligned}
\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - \mu_1) &= 0 + (-3 \ 0) \begin{pmatrix} 1/5 & 0 \\ 0 & 1/9 \end{pmatrix} \begin{pmatrix} 1-3 \\ -2-(-2) \end{pmatrix} = \frac{6}{5}. \\
\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} &= 2 - (-3 \ 0) \begin{pmatrix} 1/5 & 0 \\ 0 & 1/9 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \frac{1}{5}.
\end{aligned}$$

Thus,  $X_3 \mid (X_1, X_2) = (1, -2) \sim N(\frac{6}{5}, \frac{1}{5})$ . □

### 3.

Let

$$\mathbf{J} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then,

$$U = \mathbf{Y}'(\mathbf{I} - \frac{1}{3}\mathbf{J})\mathbf{Y} = \sum_{i=1}^3 (Y_i - \bar{Y})^2.$$

Thus,

$$E(U) = E\left(\mathbf{Y}'(\mathbf{I} - \frac{1}{3}\mathbf{J})\mathbf{Y}\right) = \text{tr}\left((\mathbf{I} - \frac{1}{3}\mathbf{J})\Sigma\right) + E(\mathbf{Y})'(\mathbf{I} - \frac{1}{3}\mathbf{J})E(\mathbf{Y}).$$

And

$$\begin{aligned}
\mathbf{J}\Sigma &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \\
E(\mathbf{Y})'E(\mathbf{Y}) &= 2^2 + 3^2 + 4^2 = 29,
\end{aligned}$$

$$E(\mathbf{Y})' \mathbf{J} E(\mathbf{Y}) = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 81.$$

Therefore,

$$\begin{aligned} E(U) &= \text{tr}\left(\left(\mathbf{I} - \frac{1}{3}\mathbf{J}\right)\Sigma\right) + E(\mathbf{Y})'\left(\mathbf{I} - \frac{1}{3}\mathbf{J}\right)E(\mathbf{Y}) \\ &= \text{tr}(\Sigma) - \frac{1}{3}\text{tr}(\mathbf{J}\Sigma) + E(\mathbf{Y})'E(\mathbf{Y}) - \frac{1}{3}E(\mathbf{Y})'\mathbf{J}E(\mathbf{Y}) \\ &= 6 - \frac{1}{3} \cdot 6 + 29 - \frac{1}{3} \cdot 81 \\ &= 6. \end{aligned}$$

□

#### 4.

Note that

$$U = \sum_{i < j} (Y_i - Y_j)^2 = \sum_{i > j} (Y_i - Y_j)^2 = \sum_{i \geq j} (Y_i - Y_j)^2.$$

Thus,

$$\begin{aligned} 2U &= \sum_{i < j} (Y_i - Y_j)^2 + \sum_{i \geq j} (Y_i - Y_j)^2 = \sum_{i=1}^n \sum_{j=1}^n (Y_i - Y_j)^2 \\ &= \sum_{i=1}^n \sum_{j=1}^n (Y_i^2 + Y_j^2 - 2Y_i Y_j) \\ &= \sum_{i=1}^n \left[ nY_i^2 + \sum_{j=1}^n Y_j^2 - 2Y_i \sum_{j=1}^n Y_j \right] \\ &= n \sum_{i=1}^n Y_i^2 + \sum_{i=1}^n \sum_{j=1}^n Y_j^2 - 2 \sum_{i=1}^n Y_i \sum_{j=1}^n Y_j \\ &= n \sum_{i=1}^n Y_i^2 + n \sum_{j=1}^n Y_j^2 - 2 \left( \sum_{i=1}^n Y_i \right)^2 \\ &= 2n \sum_{i=1}^n Y_i^2 - 2 \left( \sum_{i=1}^n Y_i \right)^2, \end{aligned}$$

which implies that  $U = n \sum_{i=1}^n Y_i^2 - \left( \sum_{i=1}^n Y_i \right)^2$ . Let  $\mathbf{A} = n\mathbf{I} - \mathbf{J}$ , then we have

$$\begin{aligned} \mathbf{Y}' \mathbf{A} \mathbf{Y} &= \mathbf{Y}' (n\mathbf{I} - \mathbf{J}) \mathbf{Y} \\ &= n\mathbf{Y}' \mathbf{I} \mathbf{Y} - \mathbf{Y}' \mathbf{J} \mathbf{Y} \\ &= n\mathbf{Y}' \mathbf{Y} - (Y_1, \dots, Y_n) \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \\ &= n \sum_{i=1}^n Y_i^2 - \left( \sum_{i=1}^n Y_i, \dots, \sum_{i=1}^n Y_i \right) \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \\ &= n \sum_{i=1}^n Y_i^2 - \left( \sum_{i=1}^n Y_i \right) (Y_1 + \dots + Y_n) \\ &= n \sum_{i=1}^n Y_i^2 - \left( \sum_{i=1}^n Y_i \right)^2 = U. \end{aligned}$$

Therefore,

$$\begin{aligned}
E(U) &= E\left(\mathbf{Y}'(n\mathbf{I} - \mathbf{J})\mathbf{Y}\right) \\
&= \text{tr}\left((n\mathbf{I} - \mathbf{J})\text{Cov}(\mathbf{Y})\right) + E(\mathbf{Y})'(n\mathbf{I} - \mathbf{J})E(\mathbf{Y}) \\
&= n\text{tr}\left(\text{Cov}(\mathbf{Y})\right) - \text{tr}\left(\mathbf{J}\text{Cov}(\mathbf{Y})\right) + nE(\mathbf{Y})'E(\mathbf{Y}) - E(\mathbf{Y})'\mathbf{J}E(\mathbf{Y}) \\
&= n^2\sigma^2 - n\sigma^2 + n^2\mu^2 - n^2\mu^2 \\
&= (n^2 - n)\sigma^2.
\end{aligned}$$

Clearly,  $k = \frac{1}{n^2 - n}$ .

□