Time Series Analysis

Homework of week 11-12

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10.1

(a)

$$\begin{aligned} Y_t &= e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} + Y_{t-4} \\ &= e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} + e_{t-4} - \theta_1 e_{t-5} - \theta_2 e_{t-6} + Y_{t-8} \end{aligned}$$

Thus, $\psi_0 = 1, \psi_1 = -\theta_1, \psi_2 = -\theta_2, \psi_3 = 0, \psi_4 = 1$.

(b)

$$\begin{aligned} \hat{Y}_t(1) &= Y_{t-3} - \theta_1 e_t - \theta_2 e_{t-1} = 25 - 0.5 \times 3 - (-0.25) \times 2 = 24 \\ \hat{Y}_t(2) &= Y_{t-2} - \theta_2 e_t = 20 - (-0.25) \times 3 = 20.75 \\ \hat{Y}_t(3) &= Y_{t-1} = 25 \\ \hat{Y}_t(4) &= Y_t = 40 \end{aligned}$$

(c)

$$\begin{aligned} e_t(1) &= Y_{t+1} - \hat{Y}_t(1) = e_{t+1} \\ e_t(2) &= Y_{t+2} - \hat{Y}_t(2) = e_{t+2} - \theta_1 e_{t+1} \\ e_t(3) &= Y_{t+3} - \hat{Y}_t(3) = e_{t+3} - \theta_1 e_{t+2} - \theta_2 e_{t+1} \\ e_t(4) &= Y_{t+4} - \hat{Y}_t(4) = e_{t+4} - \theta_1 e_{t+3} - \theta_2 e_{t+2} \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} Var(e_t(1)) &= 1 \\ Var(e_t(2)) &= 1 + 0.5^2 = 1.25 \\ Var(e_t(3)) &= 1 + 0.5^2 + 0.25^2 = 1.3125 \\ Var(e_t(4)) &= 1 + 0.5^2 + 0.25^2 = 1.3125 \end{aligned}$$

The 95% prediction intervals are given by

$$\begin{aligned} Y_{t+1}: &\ [24 - 1.96 * 1, 24 + 1.96 * 1] = [22.04, 25.96] \\ Y_{t+2}: &\ [20.75 - 1.96 * \sqrt{1.25}, 20.75 + 1.96 * \sqrt{1.25}] \approx [18.56, 22.94] \\ Y_{t+3}: &\ [25 - 1.96 * \sqrt{1.3125}, 25 + 1.96 * \sqrt{1.3125}] \approx [22.75, 27.25] \\ Y_{t+4}: &\ [40 - 1.96 * \sqrt{1.3125}, 40 + 1.96 * \sqrt{1.3125}] \approx [37.75, 42.25] \end{aligned}$$

10.2

(a)

$\Phi = 0.8$, the seasonal part is stationary. In the nonseasonal part, $\phi_1 = 1.6, \phi_2 = -0.7$,then $\phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1$ and $|\phi_2| < 1$, the nonseasonal part is also stationary. The model is stationary.

(b)

This is a seasonal ARIMA$(2,0,0) \times (1,0,0)_{12}$ model with $\phi_1 = 1.6, \phi_2 = -0.7, \Phi = 0.8$. It can be written as

$$\begin{aligned} Y_t &= 1.6Y_{t-1} - 0.7Y_{t-2} + 0.8(Y_{t-12} - 1.6Y_{t-13} + 0.7Y_{t-14}) + e_t \\ &= 1.6Y_{t-1} - 0.7Y_{t-2} + 0.8Y_{t-12} - 1.28Y_{t-13} + 0.56Y_{t-14} + e_t \end{aligned}$$

10.4

$$\gamma_0 = Var(Y_t) = \Phi^2 Var(Y_{t-4}) + (1+\theta^2)\sigma_e^2 = \Phi^2\gamma_0 + (1+\theta^2)\sigma_e^2$$

$\Longrightarrow$

$$\gamma_0 = \frac{1+\theta^2}{1-\Phi^2}\sigma_e^2$$

Note that $E(e_tY_t) = E(e_t\Phi Y_{t-4}) + E(e_t^2) - \theta E(e_te_{t-1}) = \sigma_e^2$. We have

$$\begin{aligned} \gamma_1 = Cov(Y_t, Y_{t-1}) &= Cov(\Phi Y_{t-4} + e_t - \theta e_{t-1}, Y_{t-1}) \\ &= \Phi Cov(Y_{t-4}, Y_{t-1}) + Cov(e_t, Y_{t-1}) - \theta E(e_{t-1}, Y_{t-1}) \\ &= \Phi\gamma_3 - \theta\sigma_e^2 \end{aligned} \tag{1}$$

and $\gamma_k = \Phi\gamma_{k-4}$ for $k > 1$. For $k = 3$, we have $\gamma_3 = \Phi\gamma_{-1} = \Phi\gamma_1$. Together with (1), we have $\gamma_1 = \frac{-\theta}{1-\Phi^2}\sigma_e^2$ and $\gamma_3 = \frac{-\theta\Phi}{1-\Phi^2}\sigma_e^2$, then $\rho_1 = \frac{-\theta}{1+\theta^2}$ and $\rho_3 = \frac{-\theta}{1+\theta^2}\Phi$. Using $\rho_k = \Phi\rho_{k-4}$, we have $\rho_{4k-1} = \rho_{4k+1} = \frac{-\theta}{1+\theta^2}\Phi^k$. Letting $k = 2$, we have $\rho_2 = \Phi\rho_{-2} = \Phi\rho_2$, which implies that $\rho_2 = 0$. Similarly, using $\rho_k = \Phi\rho_{k-4}$, we have $\rho_{4k-2} = 0$. Therefore, $\rho_{4k-1} = \rho_{4k+1} = \frac{-\theta}{1+\theta^2}\Phi^k, \rho_{4k-2} = 0, \rho_{4k} = \Phi^k,\ k = 0, 1, 2, ...$

10.5

(a)

$$(1-0.5B)(1-B^4)Y_t = (1-0.3B)e_t$$

ARIMA$(1,0,1) \times (0,1,0)_4$ with $\phi_1 = 0.5, \theta_1 = 0.3$.

(b)

$$\begin{aligned} (1 - B - B^{12} + B^{13})Y_t &= (1 - 0.5B - 0.5B^{12} + 0.25B^{13})e_t \\ \nabla\nabla_{12}Y_t &= (1-0.5B)(1-0.5B^{12})e_t \end{aligned}$$

ARIMA$(0,1,1) \times (0,1,1)_{12}$ with $\theta_1 = 0.5, \Theta_1 = 0.5$.

10.6

$$\gamma_0 = Var(Y_t) = \Phi^2 Var(Y_{t-12}) + (1+\theta^2)\sigma_e^2 = \Phi^2\gamma_0 + (1+\theta^2)\sigma_e^2$$

$\Longrightarrow$

$$\gamma_0 = \frac{1+\theta^2}{1-\Phi^2}\sigma_e^2$$

Note that $E(e_tY_t) = E(e_t\Phi Y_{t-12}) + E(e_t^2) - \theta E(e_te_{t-1}) = \sigma_e^2$. We have

$$\begin{aligned}
\gamma_1 = Cov(Y_t, Y_{t-1}) &= Cov(\Phi Y_{t-12} + e_t - \theta e_{t-1}, Y_{t-1}) \\
&= \Phi Cov(Y_{t-12}, Y_{t-1}) + Cov(e_t, Y_{t-1}) - \theta E(e_{t-1}, Y_{t-1}) \\
&= \Phi\gamma_{11} - \theta\sigma_e^2
\end{aligned} \tag{2}$$

and $\gamma_k = \Phi\gamma_{k-12}$ for $k > 1$. For $k = 11$, we have $\gamma_{11} = \Phi\gamma_{-1} = \Phi\gamma_1$. Together with (2), we have $\gamma_1 = \frac{-\theta}{1-\Phi^2}\sigma_e^2$ and $\gamma_{11} = \frac{-\theta\Phi}{1-\Phi^2}\sigma_e^2$, then $\rho_1 = \frac{-\theta}{1+\theta^2}$ and $\rho_{11} = \frac{-\theta}{1+\theta^2}\Phi$. Using $\rho_k = \Phi\rho_{k-12}$, we have $\rho_{12k-1} = \rho_{12k+1} = \frac{-\theta}{1+\theta^2}\Phi^k$. Therefore, $\rho_{12k-1} = \rho_{12k+1} = \frac{-\theta}{1+\theta^2}\Phi^k, \rho_{12k} = \Phi^k,\ k = 0, 1, 2, ...$

10.7

(a)

For $t = 4k + q, q = 1, 2, 3, 4$, we have

$$Y_t = e_t + Y_{t-4} = e_t + e_{t-4} + Y_{t-8} = ... = e_t + e_{t-4} + ... + e_{q+4} + Y_q = e_t + e_{t-4} + ... + e_{q+4} + e_q$$

Thus, $Var(Y_t) = (k+1)\sigma_e^2,\ t = 4k + q,\ k = 0, 1, 2, ...,\ q = 1, 2, 3, 4.$

(b)

For $t = 4k + q,\ k = 0, 1, 2, ...,\ q = 1, 2, 3, 4$ and $s = 4i + j,\ i = 0, 1, 2, ...,\ j = 1, 2, 3, 4$, we have

$$Cov(Y_t, Y_s) = Cov(e_t + e_{t-4} + ... + e_{q+4} + e_q, e_s + e_{s-4} + ... + e_{j+4} + e_j)$$

Assume $t > s$, if $q = j$, then $Cov(Y_t, Y_s) = Var(e_s + e_{s-4} + ... + e_{j+4} + e_j) = (i+1)\sigma_e^2$. If $j \neq q$, then $Cov(Y_t, Y_s) = 0$.
Therfore, for $t > s$,

$$Corr(Y_t, Y_s) = \begin{cases} \sqrt{\frac{i+1}{k+1}}, & t = 4k + q,\ s = 4i + q,\ k, i = 0, 1, 2, ...,\ q = 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

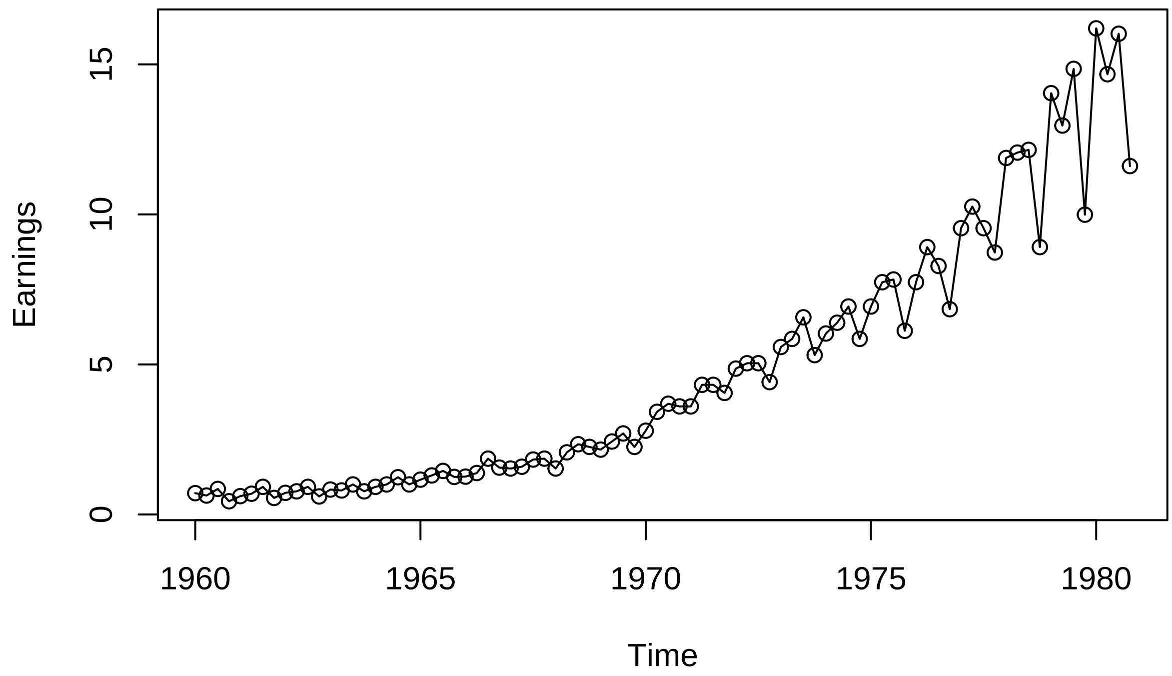
(c)

ARIMA$(0, 0, 0) \times (0, 1, 0)_4$.

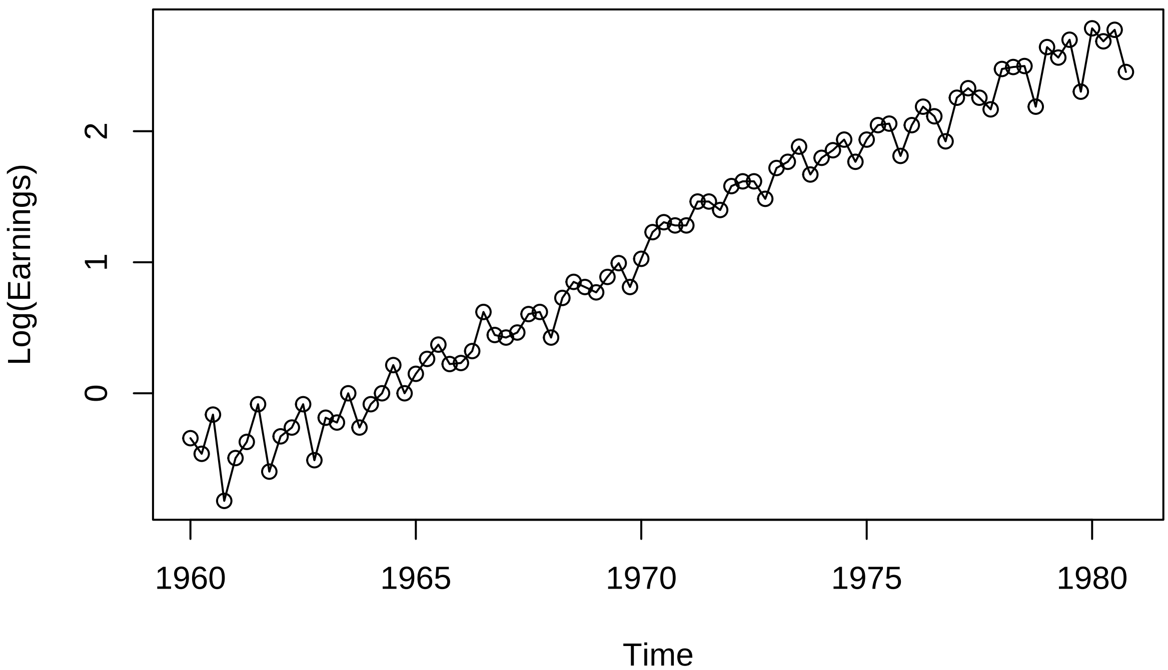
10.11

(a)

```
data(JJ)
plot(JJ,ylab='Earnings',type='o')
```



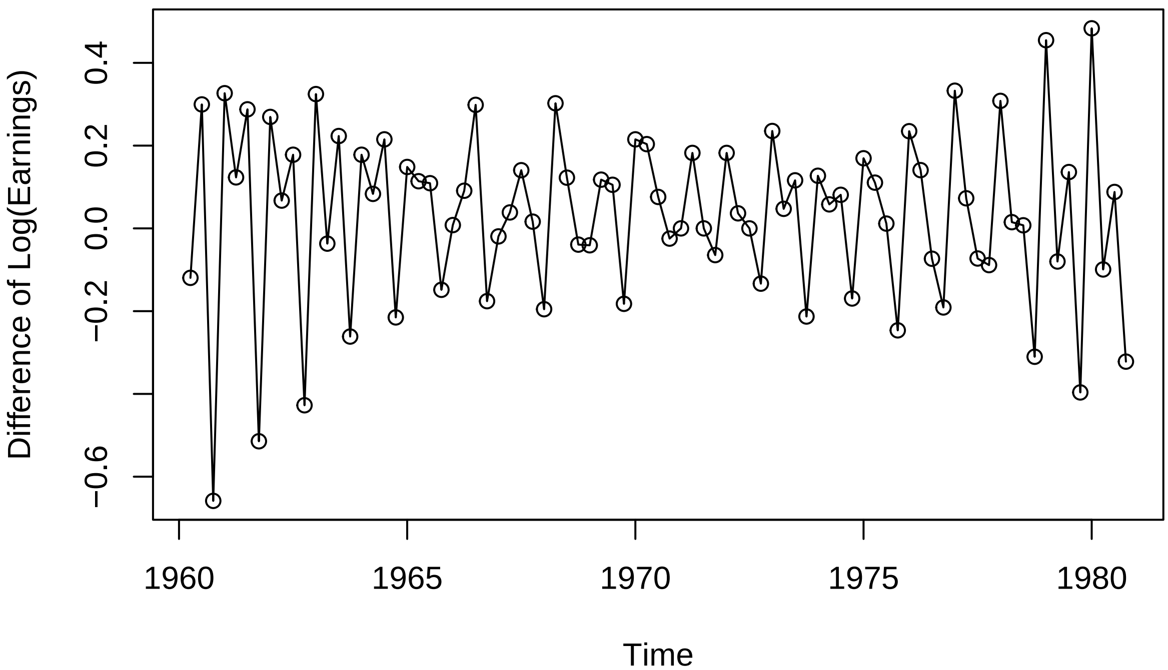

```
plot(log(JJ),ylab='Log(Earnings)',type='o')
```



In the plot at the left it is clear that at the higher values of the series there is also much more variation. The plot opf the logs at the right shows much more equal variation at all levels of the series. We use logs for all of the remaining modeling.

(b)

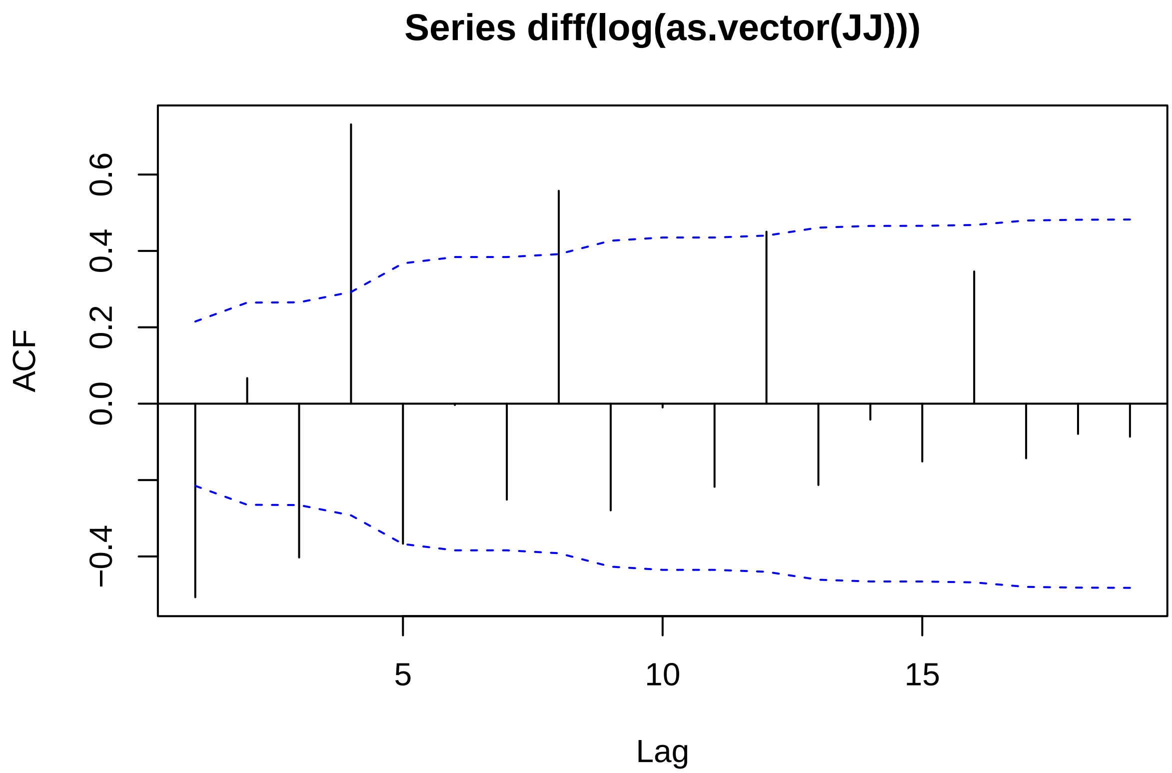
```
plot(diff(log(JJ)),ylab='Difference of Log(Earnings)',type='o')
```



We do not expect stationary series to have less variability in the middle of the series as this one does but we might entertain a stationary model and see where it leads us.

(c)

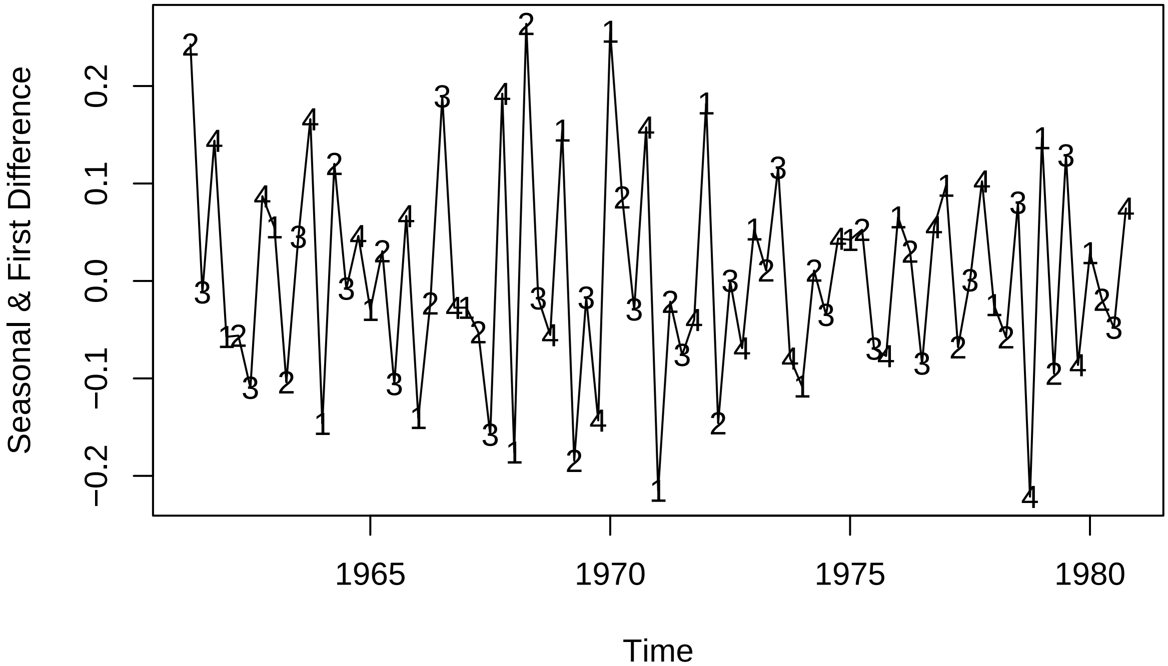
```
acf(diff(log(as.vector(JJ))),ci.type='ma')
```



In this quarterly series, the strongest autocorrelations are at the seasonal lags of 4, 8, 12, and 16. Clearly, we need to address the seasonality in this series.

(d)

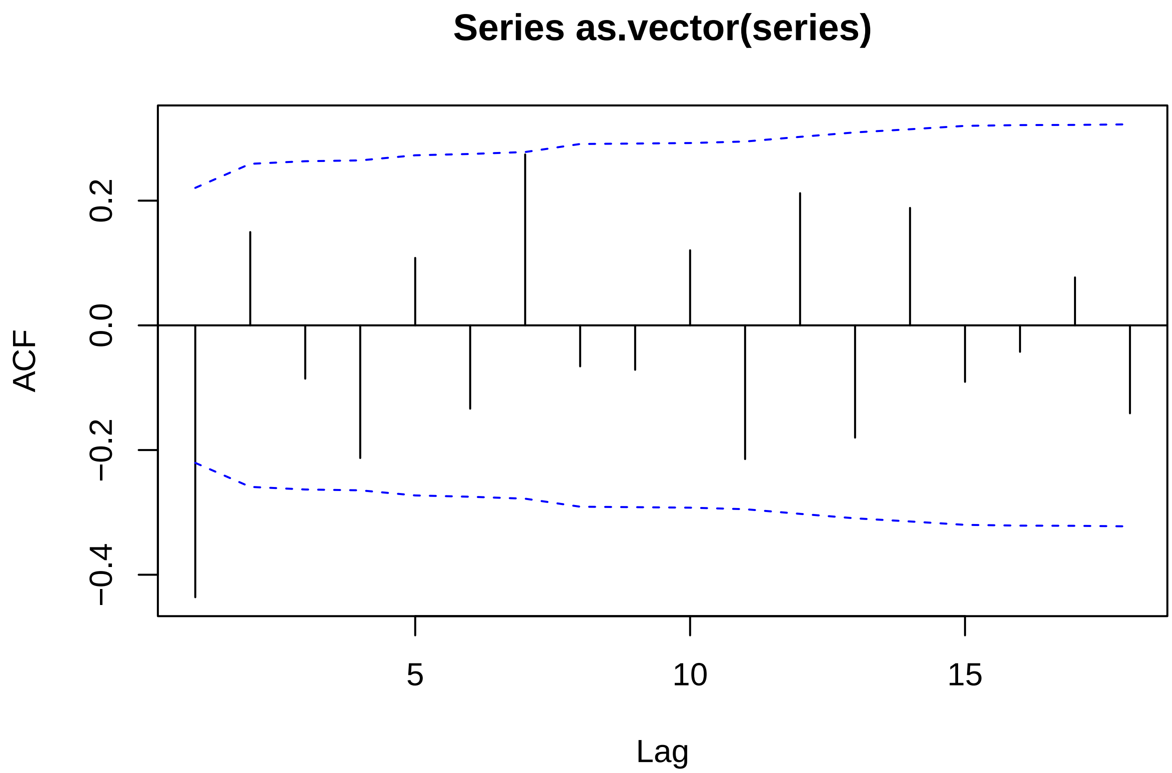
```
series=diff(diff(log(JJ),lag=4))
plot(series,ylab='Seasonal & First Difference',type='l')
points(y=series,x=time(series),pch=as.vector(season(series)))
```



The various quarters seem to be quite randomly distributed among high, middle, and low values, so that most of the seasonality is accounted for in the seasonal difference.

(e)

```
acf(as.vector(series),ci.type='ma')
```



They only significant autocorrelations are at lags 1 and 7. Lag 4 (the quarterly lag) is nearly significant.

(f)

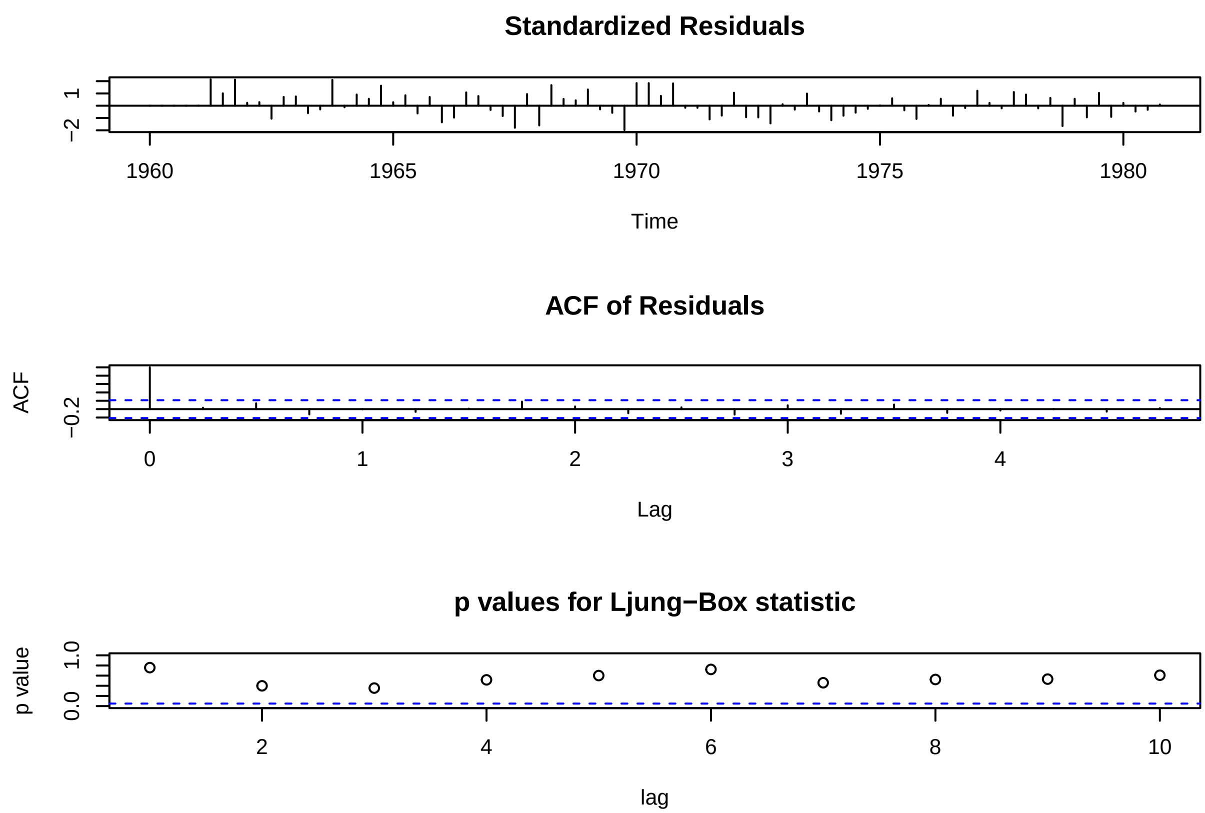
```
model=arima(log(JJ),order=c(0,1,1),seasonal=list(order=c(0,1,1),period=4))
model
```

```
##
## Call:
## arima(x = log(JJ), order = c(0, 1, 1), seasonal = list(order = c(0, 1, 1), period = 4))
##
## Coefficients:
##           ma1     sma1
##       -0.6809  -0.3146
## s.e.   0.0982   0.1070
##
## sigma^2 estimated as 0.007931:  log likelihood = 78.38,  aic = -152.75
```

Both the seasonal and nonseasonal ma parameters are significant in this model.

(g)

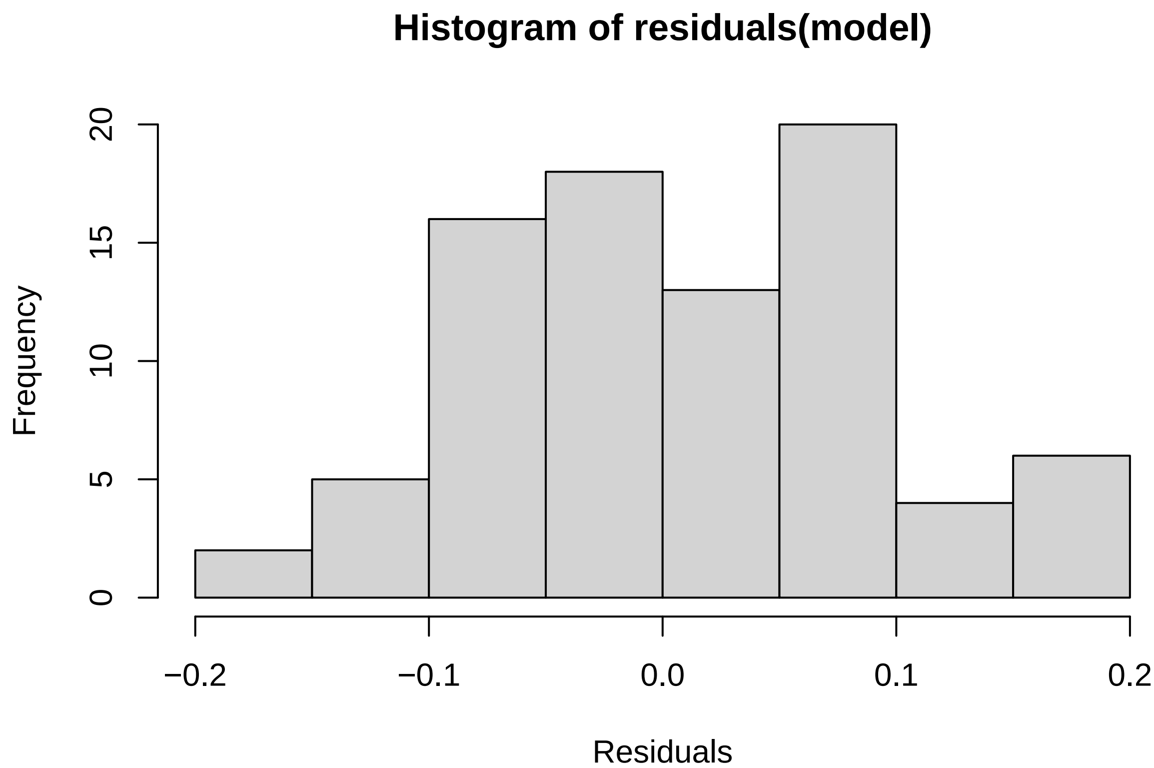
```
tsdiag(model)
```



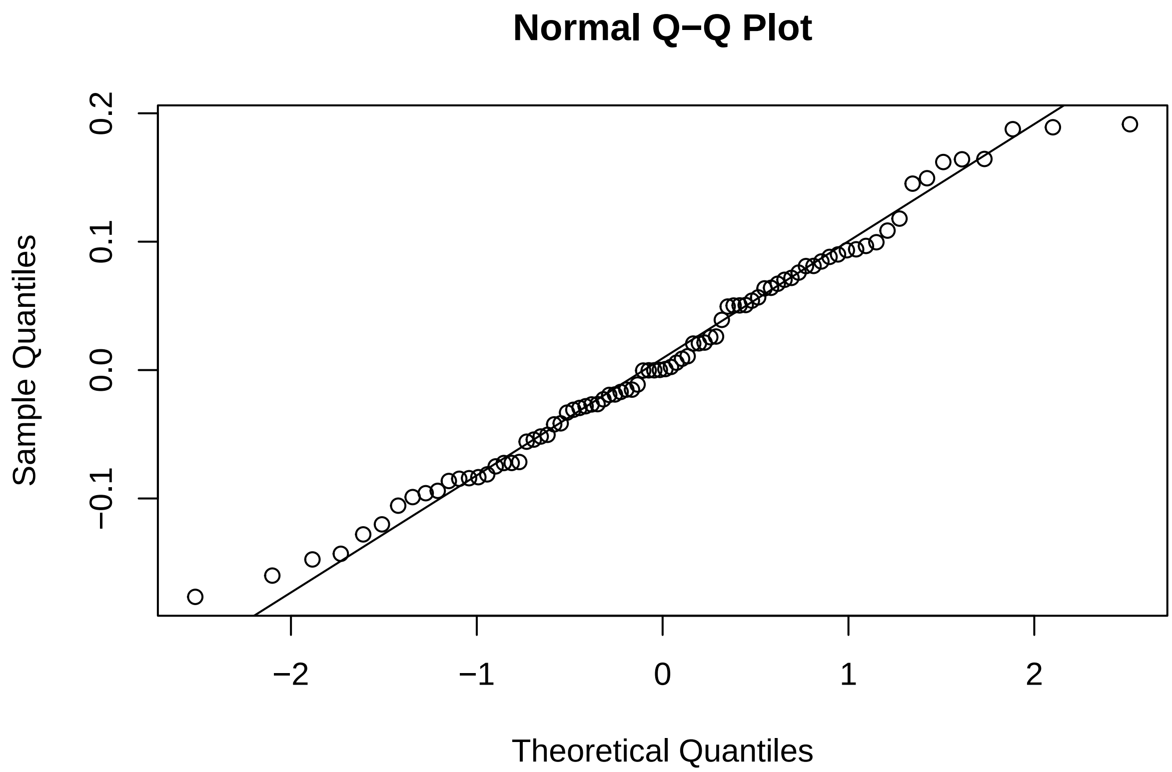
These diagnostic plots do not show any inadequacies with the model. No outliers are detected and there is little autocorrelation in the residuals.

Normality:

```
hist(residuals(model),xlab='Residuals')
```




```
qqnorm(residuals(model)); qqline(residuals(model))
```



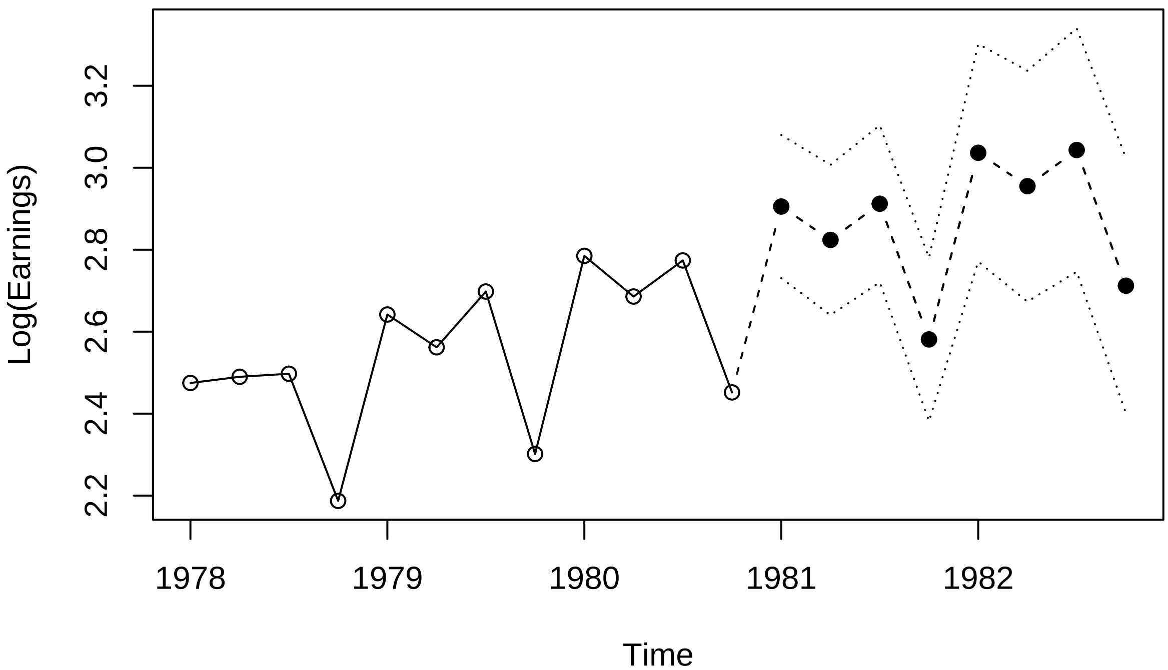

```
shapiro.test(residuals(model))
```

```
##
##  Shapiro-Wilk normality test
##
## data:  residuals(model)
## W = 0.98583, p-value = 0.489
```

Normality of the error terms looks like a very good assumption.

(h)

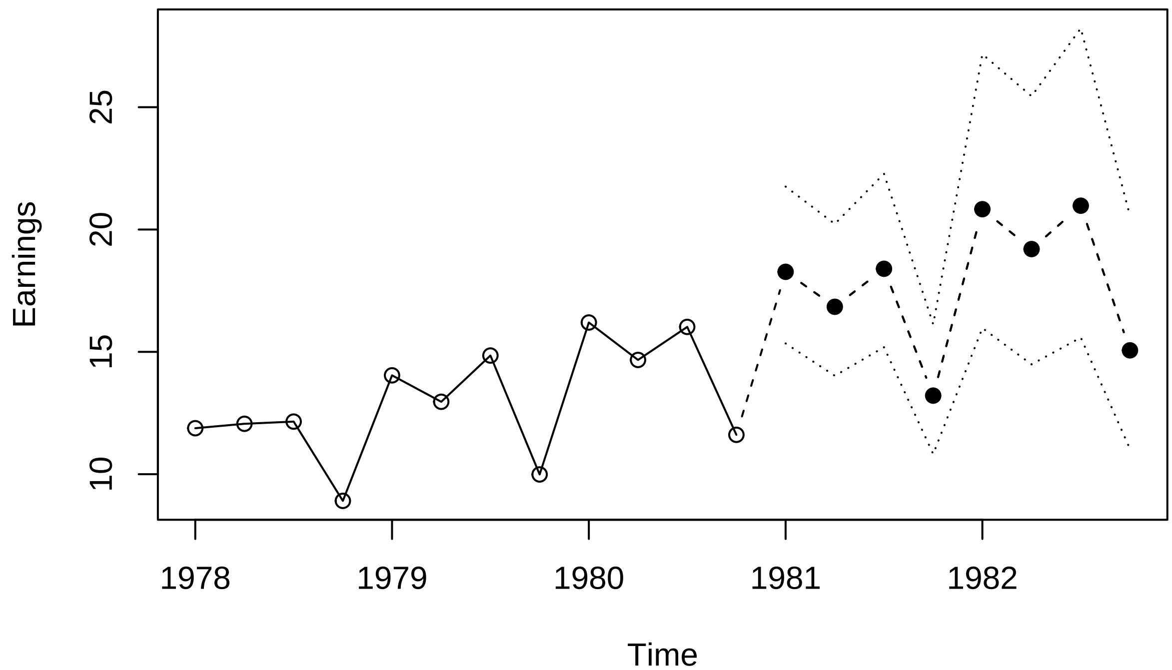
```
plot(model,n1=c(1978,1),n.ahead=8,pch=19,ylab='Log(Earnings)')
```



The forecasts follow the general pattern of seasonality and "trend" in the earnings series and the forecast limits give a good indication of the confidence in these forecasts.

Lastly, we display the forecasts in orginal terms.

```
plot(model,n1=c(1978,1),n.ahead=8,pch=19,ylab='Earnings',transform=exp)
```



In original terms, the uncertainty in the forecasts is easier to understand.