Time Series Analysis

Homework of week 13

Hanbin Liu 11912410

12.3

Since

$$\eta_t = r_t^2 - \sigma_{t|t-1}^2, \quad r_t = \sigma_{t|t-1}\epsilon_t,$$

where $\{\epsilon_t\}$ is a sequence of i.i.d random variables with zero mean and unit variance and ϵ_t is independet of $r_{t-j}, j = 1, 2, ...$, we have

$$\begin{aligned}
E(\eta_t) &= E(\sigma_{t|t-1}^2(\epsilon_t^2 - 1)) \\
&= E(E(\sigma_{t|t-1}^2(\epsilon_t^2 - 1|r_{t-j}, j = 1, 2, ...)) \\
&= E(\sigma_{t|t-1}^2 E((\epsilon_t^2 - 1)|r_{t-j}, j = 1, 2, ...)) \\
&= E(\sigma_{t|t-1}^2 \times 0) \\
&= 0
\end{aligned}$$

$\forall k > 0$,

$$\begin{aligned}
E(\eta_t\eta_{t-k}) &= E(\sigma_{t|t-1}^2(\epsilon_t^2 - 1)(r_{t-k}^2 - \sigma_{t-k|t-k-1}^2)) \\
&= E(E(\sigma_{t|t-1}^2(\epsilon_t^2 - 1)(r_{t-k}^2 - \sigma_{t-k|t-k-1}^2)|r_{t-j}, j = 1, 2, ...)) \\
&= E(\sigma_{t|t-1}^2(r_{t-k}^2 - \sigma_{t-k|t-k-1}^2)E((\epsilon_t^2 - 1)|r_{t-j}, j = 1, 2, ...)) \\
&= E(\sigma_{t|t-1}^2(r_{t-k}^2 - \sigma_{t-k|t-k-1}^2) \times 0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
E(\eta_t r_{t-k}^2) &= E(\eta_t\eta_{t-k}) + E(\eta_t\sigma_{t-k|t-k-1}^2) \\
&= 0 + E(\sigma_{t|t-1}^2(\epsilon_t^2 - 1)(\sigma_{t-k|t-k-1}^2)) \\
&= E(E(\sigma_{t|t-1}^2(\epsilon_t^2 - 1)(\sigma_{t-k|t-k-1}^2)|r_{t-j}, j = 1, 2, ...)) \\
&= E(\sigma_{t|t-1}^2\sigma_{t-k|t-k-1}^2 E((\epsilon_t^2 - 1)|r_{t-j}, j = 1, 2, ...)) \\
&= E(\sigma_{t|t-1}^2\sigma_{t-k|t-k-1}^2 \times 0) \\
&= 0
\end{aligned}$$

Therefore,

$$Cov(\eta_t, \eta_{t-k}) = E(\eta_t\eta_{t-k}) - E(\eta_t)E(\eta_{t-k}) = 0$$

and

$$Cov(\eta_t, r_{t-k}^2) = E(\eta_t r_{t-k}^2) - E(\eta_t)E(r_{t-k}^2) = 0,$$

which implies that $\{\eta_t\}$ is a serially uncorrelated sequence and η_t is uncorrelated with past squared returns.

12.4

Substituting $\sigma^2_{t|t-1} = r^2_t - \eta_t$ into Equation (12.2.2), we have

$$r^2_t - \eta_t = \omega + \alpha r^2_{t-1},$$

which implies

$$r^2_t = \omega + \alpha r^2_{t-1} + \eta_t.$$

12.5

By Equation (12.2.2), we have

$$\sigma^2_{t|t-1} = \omega + \alpha r^2_{t-1}.$$

Squaring both sides of the equation, we have

$$\sigma^4_{t|t-1} = \omega^2 + 2\omega\alpha r^2_{t-1} + \alpha^2 r^4_{t-1}.$$

Let τ denote $E(\sigma^4_{t|t-1})$ and σ^2 denote $E(r^2_{t-1})$. Equation (12.2.7) tells us that $E(r^4_t) = 3\tau$. Therefore, taking expectation on both sides of the equation above yields that

$$\tau = \omega^2 + 2\omega\alpha\sigma^2 + \alpha^2 3\tau.$$