Mathematical Statistics

Assignment1

Hanbin.Liu 11912410

1.1 Solution

(a)

$$
\begin{aligned}
M_X(t) = E(e^{tX}) &= \sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k q^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}(pe^t)^k q^{n-k} \\
&= (pe^t+q)^n,
\end{aligned}
$$

where $q = 1-p$.

(b) By formula (1.34), we have

$$E(X) = M'_X(t)\Big|_{t=0} = n(pe^t+q)^{n-1}pe^t\Big|_{t=0} = np,$$

$$E(X^2) = M''_X(t)\Big|_{t=0} = npe^t((pe^t+q)^{n-1} + pe^t(n-1)(pe^t+q)^{n-2})\Big|_{t=0} = np + n(n-1)p^2.$$

Thus, $\mathrm{Var}(X) = E(X^2) - EX^2 = np + n(n-1)p^2 - n^2p^2 = np(1-p)$.

(c) If $X+Y=k<n$, then

$$
\begin{aligned}
\Pr(X+Y=k) &= \sum_{i=0}^{k}\Pr(X=i, Y=k-i) \\
&= \sum_{i=0}^{k}\Pr(X=i)\Pr(Y=k-i) \\
&= \sum_{i=0}^{k}\binom{n}{i}p^i q^{n-i}e^{-\lambda}\frac{\lambda^{k-i}}{(k-i)!}
\end{aligned}
$$

Otherwise,

$$
\begin{aligned}
\Pr(X+Y=k) &= \sum_{i=0}^{n}\Pr(X=i, Y=k-i) \\
&= \sum_{i=0}^{n}\Pr(X=i)\Pr(Y=k-i) \\
&= \sum_{i=0}^{n}\binom{n}{i}p^i q^{n-i}e^{-\lambda}\frac{\lambda^{k-i}}{(k-i)!}
\end{aligned}
$$

1.2 Solution

(a) The marginal distribution of X is given by

$$\Pr(X=1)=\sum_y \Pr(X=1,Y=y)=\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{4},$$
$$\Pr(X=2)=\sum_y \Pr(X=2,Y=y)=\frac{2}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{4},$$
$$\Pr(X=3)=\sum_y \Pr(X=3,Y=y)=\frac{3}{16}+\frac{1}{16}=\frac{1}{4},$$
$$\Pr(X=4)=\sum_y \Pr(X=4,Y=y)=\frac{4}{16}=\frac{1}{4}.$$

(b) Let Z denote $X+Y$. Then the pmf of $X+Y$ is given by

$$\Pr(Z=2)=\sum_{x+y=2}\Pr(X=x,Y=y)=\frac{1}{16},$$
$$\Pr(Z=3)=\sum_{x+y=3}\Pr(X=x,Y=y)=\frac{1}{16},$$
$$\Pr(Z=4)=\sum_{x+y=4}\Pr(X=x,Y=y)=\frac{1}{16}+\frac{2}{16}=\frac{3}{16},$$
$$\Pr(Z=5)=\sum_{x+y=5}\Pr(X=x,Y=y)=\frac{1}{16}+\frac{1}{16}=\frac{1}{8},$$
$$\Pr(Z=6)=\sum_{x+y=6}\Pr(X=x,Y=y)=\frac{1}{16}+\frac{3}{16}=\frac{1}{4},$$
$$\Pr(Z=7)=\sum_{x+y=7}\Pr(X=x,Y=y)=\frac{1}{16},$$
$$\Pr(Z=8)=\sum_{x+y=8}\Pr(X=x,Y=y)=\frac{4}{16}=\frac{1}{4}.$$

1.4 Solution

(a) The supports of X and Y are $S_X=\{x_1,x_2,x_3\}$ and $S_Y=\{y_1,y_2,y_3,y_4\}$. Then we have

$$p_1=\Pr(X=x_1)=c\frac{\Pr(X=x_1|Y=y_1)}{\Pr(Y=y_1|X=x_1)}=c\frac{a_{11}}{b_{11}}=c\frac{6}{7},$$
$$p_2=\Pr(X=x_2)=c\frac{\Pr(X=x_2|Y=y_1)}{\Pr(Y=y_1|X=x_2)}=c\frac{a_{21}}{b_{21}}=c,$$
$$p_3=\Pr(X=x_3)=c\frac{\Pr(X=x_3|Y=y_1)}{\Pr(Y=y_1|X=x_3)}=c\frac{a_{31}}{b_{31}}=c\frac{12}{7},$$

where c is a constant. Note that $p_1+p_2+p_3=1$, we obtain

$$p_1=\frac{\frac{6}{7}}{\frac{6}{7}+1+\frac{12}{7}}=\frac{6}{25},\quad p_2=\frac{1}{\frac{6}{7}+1+\frac{12}{7}}=\frac{7}{25},\quad p_3=\frac{\frac{12}{7}}{\frac{6}{7}+1+\frac{12}{7}}=\frac{12}{25}.$$

Similarly, we have

$$q_1=\frac{\frac{7}{6}}{\frac{7}{6}+\frac{4}{6}+\frac{7}{6}+\frac{7}{6}}=\frac{7}{25},\quad q_2=\frac{\frac{4}{6}}{\frac{7}{6}+\frac{4}{6}+\frac{7}{6}+\frac{7}{6}}=\frac{4}{25},$$
$$q_3=\frac{\frac{7}{6}}{\frac{7}{6}+\frac{4}{6}+\frac{7}{6}+\frac{7}{6}}=\frac{7}{25},\quad q_4=\frac{\frac{7}{6}}{\frac{7}{6}+\frac{4}{6}+\frac{7}{6}+\frac{7}{6}}=\frac{7}{25}.$$

(b) By using the formula $\Pr(X = x, Y = y) = \Pr(X = x|Y = y)\Pr(Y = y)$, we have

$$\begin{aligned}
\Pr(X = x_1, Y = y_1) &= \Pr(X = x_1|Y = y_1)\Pr(Y = y_1) = \frac{1}{7} \times \frac{7}{25} = \frac{1}{25},\\
\Pr(X = x_1, Y = y_2) &= \Pr(X = x_1|Y = y_2)\Pr(Y = y_2) = \frac{1}{4} \times \frac{4}{25} = \frac{1}{25},\\
\Pr(X = x_1, Y = y_3) &= \Pr(X = x_1|Y = y_3)\Pr(Y = y_3) = \frac{3}{7} \times \frac{7}{25} = \frac{3}{25},\\
\Pr(X = x_1, Y = y_4) &= \Pr(X = x_1|Y = y_4)\Pr(Y = y_4) = \frac{1}{7} \times \frac{7}{25} = \frac{1}{25},\\
\Pr(X = x_2, Y = y_1) &= \Pr(X = x_2|Y = y_1)\Pr(Y = y_1) = \frac{2}{7} \times \frac{7}{25} = \frac{2}{25},\\
\Pr(X = x_2, Y = y_2) &= \Pr(X = x_2|Y = y_2)\Pr(Y = y_2) = \frac{1}{2} \times \frac{4}{25} = \frac{2}{25},\\
\Pr(X = x_2, Y = y_3) &= \Pr(X = x_2|Y = y_3)\Pr(Y = y_3) = \frac{1}{7} \times \frac{7}{25} = \frac{1}{25},\\
\Pr(X = x_2, Y = y_4) &= \Pr(X = x_2|Y = y_4)\Pr(Y = y_4) = \frac{2}{7} \times \frac{7}{25} = \frac{2}{25},\\
\Pr(X = x_3, Y = y_1) &= \Pr(X = x_3|Y = y_1)\Pr(Y = y_1) = \frac{4}{7} \times \frac{7}{25} = \frac{4}{25},\\
\Pr(X = x_3, Y = y_2) &= \Pr(X = x_3|Y = y_2)\Pr(Y = y_2) = \frac{1}{4} \times \frac{4}{25} = \frac{1}{25},\\
\Pr(X = x_3, Y = y_3) &= \Pr(X = x_3|Y = y_3)\Pr(Y = y_3) = \frac{3}{7} \times \frac{7}{25} = \frac{3}{25},\\
\Pr(X = x_3, Y = y_4) &= \Pr(X = x_3|Y = y_4)\Pr(Y = y_4) = \frac{4}{7} \times \frac{7}{25} = \frac{4}{25}.
\end{aligned}$$

1.5 Solution

(a) Since m is the unique median of the distribution of X, we have

$$\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx = \frac{1}{2}.$$

It then follows that

$$
\begin{aligned}
\text{RHS} &= \int_{-\infty}^{\infty} |x-m| f(x)\,dx + 2\int_{m}^{b} (b-x) f(x)\,dx \\
&= \int_{-\infty}^{m} (m-x) f(x)\,dx + \int_{m}^{\infty} (x-m) f(x)\,dx + \int_{m}^{b} -x f(x)\,dx + \int_{b}^{m} x f(x)\,dx + 2\int_{m}^{b} b f(x)\,dx \\
&= \int_{-\infty}^{b} -x f(x)\,dx + \int_{b}^{\infty} x f(x)\,dx + m\int_{-\infty}^{m} f(x)\,dx - m\int_{m}^{\infty} f(x)\,dx + 2\int_{m}^{b} b f(x)\,dx \\
&= \int_{-\infty}^{b} -x f(x)\,dx + \int_{b}^{\infty} x f(x)\,dx + \frac{m}{2} - \frac{m}{2} + [\int_{m}^{-\infty} b f(x)\,dx + \int_{-\infty}^{b} b f(x)\,dx] + \int_{m}^{b} b f(x)\,dx \\
&= \int_{-\infty}^{b} (b-x) f(x)\,dx + \int_{b}^{\infty} x f(x)\,dx + \int_{m}^{-\infty} b f(x)\,dx + \int_{m}^{b} b f(x)\,dx \\
&= \int_{-\infty}^{b} (b-x) f(x)\,dx + \int_{b}^{\infty} x f(x)\,dx - \frac{b}{2} + [\int_{m}^{\infty} b f(x)\,dx + \int_{\infty}^{b} b f(x)\,dx] \\
&= \int_{-\infty}^{b} (b-x) f(x)\,dx + \int_{b}^{\infty} (x-b) f(x)\,dx - \frac{b}{2} + \frac{b}{2} \\
&= \int_{-\infty}^{\infty} |x-b| f(x)\,dx \\
&= \text{LHS}.
\end{aligned}
$$

(b) If $b = m$, then $E(|X-b|)$ is minimized. Otherwise, assume that $b > m$, then

$$\int_{m}^{b} (b-x) f(x)\,dx > 0$$

since $(b-x)f(x) \geq 0$ for any $x \in (m, b)$. Similarly the integration is also positive if $b < m$. And the integration is 0 iff $b = m$. By (a), we know that $E(|X-b|)$ is minimized when $b = m$.

1.6 Solution

(a)

$$\Pr(\frac{1}{4} < X < \frac{5}{8}) = F(\frac{5}{8}) - F(\frac{1}{2}) = \frac{19}{32}.$$

(b) Taking the derivative of $F(x)$ yeilds the pdf of X:

$$f(x) = \begin{cases} 4x, & 0 \leq x < \frac{1}{2}, \\ -4x+4, & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let's find out $E(X)$ and $E(X^2)$ first.

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\frac{1}{2}} 4x^2\,dx + \int_{\frac{1}{2}}^{1} -4x^2 + 4x\,dx = \frac{1}{2}. \\
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{0}^{\frac{1}{2}} 4x^3\,dx + \int_{\frac{1}{2}}^{1} -4x^3 + 4x^2\,dx = \frac{7}{24}.
\end{aligned}
$$

Therefore, $\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{24}$.

1.7 Solution

(a) Let a denote $t_1X_1 + t_2X_2 + \dots + t_dX_d$, then we have

$$\begin{aligned} E(e^a) &= E(1 + a + \frac{a^2}{2!} + \dots + \frac{a^n}{n!} + \dots) \\ &= E(1) + E(a) + E(\frac{a^2}{2!}) + \dots + E(\frac{a^n}{n!}) + \dots \\ &= E(1) + E(t_1X_1 + t_2X_2 + \dots + t_dX_d) + \frac{1}{2!}E(t_1X_1 + t_2X_2 + \dots + t_dX_d)^2 + \dots \end{aligned}$$

Thus, the partial derivative of the joint mgf $w.r.t.$ t_i is

$$\frac{dE(e^a)}{dt_i} = E(X_i) + \sum_{j=1}^{d} E(X_jX_i)t_j + \dots + \frac{1}{(n-1)!}E[(t_1X_1 + t_2X_2 + \dots + t_dX_d)^{n-1}X_i] + \dots$$

Letting $t_1 = \dots = t_d = 0$ yeilds that

$$\left.\frac{dM_X(t)}{dt_i}\right|_{t=0} = E(X_i),$$

where $X = (X_1, ..., X_d)^T,\ t = (t_1, ..., t_d)^T$.

(b) By(a), the second derivative of the joint mgf $w.r.t.$ t_i and t_j is

$$\frac{d^2E(e^a)}{dt_jt_i} = E(X_iX_j) + \dots + \frac{1}{(n-2)!}E[(t_1X_1 + \dots t_dX_d)^{n-2}X_iX_j] + \dots$$

Letting $t_1 = \dots = t_d = 0$ yeilds that

$$\left.\frac{d^2M_X(t)}{dt_it_j}\right|_{t=0} = E(X_iX_j),$$

where $X = (X_1, ..., X_d)^T,\ t = (t_1, ..., t_d)^T$.

(c)

$$\begin{aligned} M_{(X,Y)}(t_1, t_2) = E(e^{t_1X+t_2Y}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x, y)\, dx\, dy \\ &= \int_0^{\infty} e^{(t_1-1)x}\, dx \int_0^{\infty} e^{(t_2-1)y}\, dy \\ &= \frac{1}{1-t_1}\frac{1}{1-t_2} \quad (t_1 < 1, t_2 < 1) \\ &= \frac{1}{(1-t_1)(1-t_2)}. \end{aligned}$$

Thus,

$$\begin{aligned} E(X) &= \left.\frac{dM_{(X,Y)}(t_1, t_2)}{dt_1}\right|_{t1=t2=0} = \left.\frac{1}{(1-t_1)^2(1-t_2)}\right|_{t_1=t_2=0} = 1, \\ E(Y) &= \left.\frac{dM_{(X,Y)}(t_1, t_2)}{dt_2}\right|_{t1=t2=0} = \left.\frac{1}{(1-t_1)(1-t_2)^2}\right|_{t_1=t_2=0} = 1, \\ E(XY) &= \left.\frac{d^2M_{(X,Y)}(t_1, t_2)}{dt_1t_2}\right|_{t1=t2=0} = \left.\frac{1}{(1-t_1)^2(1-t_2)^2}\right|_{t_1=t_2=0} = 1, \\ \mathrm{Cov}(X, Y) &= E(XY) - E(X)E(Y) = 1 - 1 = 0. \end{aligned}$$

1.8 Solution

(a)

$$1=\sum_{x=1}^{\infty}\Pr(X=x)=ce^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{x!}=ce^{-\lambda}(e^{\lambda}-1)\Rightarrow c=\frac{1}{1-e^{-\lambda}}.$$

(b)

$$E(X)=\sum_{x=1}^{\infty}x\Pr(X=x)=\frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=1}^{\infty}x\frac{\lambda^x}{x!}\overset{t=x-1}{=}\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\sum_{t=0}^{\infty}\frac{\lambda^t}{t!}=\frac{\lambda}{1-e^{-\lambda}},$$

$$\begin{aligned}E(X^2)=\sum_{x=1}^{\infty}x^2\Pr(X=x)&=\frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=1}^{\infty}x^2\frac{\lambda^x}{x!}\\&=\frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=1}^{\infty}x(x-1)\frac{\lambda^x}{x!}+\frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=1}^{\infty}x\frac{\lambda^x}{x!}\\&=\frac{\lambda^2e^{-\lambda}}{1-e^{-\lambda}}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\ (m=x-2)\\&=\frac{\lambda^2}{1-e^{-\lambda}}+\frac{\lambda}{1-e^{-\lambda}}\\&=\frac{\lambda^2+\lambda}{1-e^{-\lambda}},\end{aligned}$$

$$\mathrm{Var}(X)=E(X^2)-(E(X))^2=\frac{\lambda-(\lambda^2+\lambda)e^{-\lambda}}{(1-e^{-\lambda})^2}.$$

(c)

$$M_X(t)=E(e^{tX})=\frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=1}^{\infty}e^{tx}\frac{\lambda^x}{x!}=\frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=1}^{\infty}\frac{(\lambda e^t)^x}{x!}=\frac{e^{-\lambda}}{1-e^{-\lambda}}(e^{\lambda e^t}-1)=\frac{e^{\lambda e^t}-1}{e^{\lambda}-1}.$$

(d) The support of $X+Y$ is $\{2,3,...\}$.

$$\begin{aligned}\Pr(X_1+X_2=n)&=\sum_{k=1}^{n-1}\Pr(X_1=k,X_2=n-k)\\&=\sum_{k=1}^{n-1}\Pr(X_1=k)\Pr(X_2=n-k)\\&=\sum_{k=1}^{n-1}\frac{1}{e^{\lambda_1}-1}\frac{\lambda_1^k}{k!}\cdot\frac{1}{e^{\lambda_2}-1}\frac{\lambda_2^{n-k}}{(n-k)!}\\&=\frac{1}{(e^{\lambda_1}-1)(e^{\lambda_2}-1)}[\sum_{k=0}^{n}\frac{\lambda_1^k}{k!}\cdot\frac{\lambda_2^{n-k}}{(n-k)!}-\frac{\lambda_1^n}{n!}-\frac{\lambda_1^n}{n!}]\\&=\frac{1}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)}[\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k\lambda_2^{n-k}-\lambda_1^n-\lambda_2^n]\\&=\frac{(\lambda_1+\lambda_2)^n-\lambda_1^n-\lambda_2^n}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)},\quad n=2,3,...\end{aligned}$$

(e) The pmf is given by

$$\begin{aligned}
\Pr(X_1 = y | X_1 + X_2 = x) &= \frac{\Pr(X_1 = y, X_1 + X_2 = x)}{\Pr(X_1 + X_2 = x)} \\
&= \frac{\Pr(X_1 = y)\Pr(X_2 = x - y)}{\Pr(X_1 + X_2 = x)} \\
&= \frac{\frac{\lambda_1^y}{y!(e^{\lambda_1}-1)} \cdot \frac{\lambda_2^{x-y}}{(x-y)!(e^{\lambda_2}-1)}}{\frac{(\lambda_1+\lambda_2)^x - \lambda_1^x - \lambda_2^x}{x!(e^{\lambda_1}-1)(e^{\lambda_2}-1)}} \\
&= \binom{x}{y} \frac{\lambda_1^y \lambda_2^{x-y}}{(\lambda_1 + \lambda_2)^x - \lambda_1^x - \lambda_2^x},
\end{aligned}$$

where $1 \le y \le x - 1$, y is an integer.

1.10 Solution

(a) Since X and Y are independent, we have

$$M_W(t) = M_{3X}(t) \cdot M_{2Y}(t) = M_X(3t) \cdot M_Y(2t) = \exp(-2t + \frac{25}{2}t^2).$$

This shows that $W \sim N(-2, 25)$. Now we have

$$\begin{aligned}
\Pr(-12 < W < 3) &= \Pr(\frac{-12 - (-2)}{5} < \frac{W - (-2)}{5} < \frac{3 - (-2)}{5}) \\
&= \Phi(1) - \Phi(-2) \\
&= \Phi(1) + \Phi(2) - 1
\end{aligned}$$

(b) Since $W \sim N(-2, 25)$, we have

$$E(W) = -2, \mathrm{Var}(W) = 25.$$

Therefore,

$$E(W^2) = \mathrm{Var}(W) + (EW)^2 = 25 + (-2)^2 = 29.$$

1.11 Solution

(a)

$$E(Y) = \int_{-\infty}^{\infty} |x| f(x)\, dx = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}}\, dx \overset{t=x^2/2}{=} \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-t}\, dt = \frac{2}{\sqrt{2\pi}}.$$

$$E(Y^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = E(X^2) = \mathrm{Var}(X) + (EX)^2 = 1 + 0 = 1.$$

$$\mathrm{Var}(Y) = E(Y^2) - (EY)^2 = 1 - \frac{4}{2\pi} = 1 - \frac{2}{\pi}.$$

(b) The pmf is given by

$$F_Y(y) = \Pr(Y \le y) = \Pr(|X| \le y) = \Pr(-y \le X \le y) = \int_{-y}^{y} f(x)\, dx = \frac{2}{\sqrt{2\pi}} \int_0^{y} e^{-\frac{x^2}{2}}\, dx$$

Thus,

$$F_Y(y) = \begin{cases} 0, & y \le 0, \\ \frac{2}{\sqrt{2\pi}} \int_0^y e^{-\frac{x^2}{2}}, & y > 0. \end{cases}$$

And the pdf is

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

1.12 Proof

Let Y denote $F(X)$, then the set of all possible values of Y is $[0, 1]$.

If $y < 0$, then

$$F_Y(y) = \Pr(Y \le y) = 0.$$

If $y \in [0, 1]$, then

$$F_Y(y) = \Pr(Y \le y) = \Pr(F(X) \le y) = \Pr(X \le F^{-1}(y)) = F(F^{-1}(y)) = y.$$

If $y > 1$, then

$$F_Y(y) = \Pr(Y \le y) = 1.$$

Hence, the pmf of Y is same as the pmf of U. i.e. $F(X) \overset{d}{=} U$. □

1.13 Proof

Define the random variable:

$$I = \begin{cases} 1, & X > \lambda\mu, \\ 0, & X \le \lambda\mu. \end{cases}$$

Then,

$$E(I^2) = \Pr(X > \lambda\mu),$$

$$\begin{aligned} E(IX) &= \sum_{x > \lambda\mu} x\Pr(X = x) \\ &= \sum_x x\Pr(X = x) - \sum_{0 < x \le \lambda\mu} x\Pr(X = x) \\ &\ge \mu - \sum_{0 < x \le \lambda\mu} \lambda\mu\Pr(X = x) \\ &\ge \mu - \lambda\mu \sum_x \Pr(X = x) \\ &= (1 - \lambda)\mu. \end{aligned}$$

By Chauchy-Schwarz inequality, we obtain that

$$E(I^2)E(X^2) \ge (E(IX))^2.$$

i.e.

$$\Pr(X > \lambda\mu)E(X^2) \geq (1-\lambda)^2\mu^2,$$

which completes the proof for discrete case. Subsitituting the sum of x by the integral yeilds the proof for continuous case. $\square$