Assignments: Chapter two (P20-24)

Part I Theoretical problems

2.7 Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k.

(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary by finding the mean and autocovariance function for $\{W_t\}$.

(b) Show that $U_t = \nabla^2 Y_t = \nabla[Y_t - Y_{t-1}] = Y_t - 2Y_{t-1} + Y_{t-2}$ is stationary. (You need not find the mean and autocovariance function for $\{U_t\}$.)

2.9 Suppose $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with autocovariance function γ_k and β_0 and β_1 are constants.

(a) Show that $\{Y_t\}$ is not stationary but that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary.

(b) In general, show that if $Y_t = \mu_t + X_t$, where $\{X_t\}$ is a zero-mean stationary series and μ_t is a polynomial in t of degree d, then $\nabla^m Y_t = \nabla(\nabla^{m-1} Y_t)$ is stationary for $m \geq d$ and nonstationary for $0 \leq m < d$.

2.13 Let $Y_t = e_t - \theta(e_{t-1})^2$. For this exercise, assume that the white noise series is normally distributed.

(a) Find the autocorrelation function for $\{Y_t\}$.

(b) Is $\{Y_t\}$ stationary?

2.20 Consider the standard random walk model where $Y_t = Y_{t-1} + e_t$ with $Y_1 = e_1$.

(a) Use the representation of Y_t above to show that $\mu_t = \mu_{t-1}$ for $t > 1$ with initial condition $\mu_1 = E(e_1) = 0$. Hence show that $\mu_t = 0$ for all t.

(b) Similarly, show that $Var(Y_t) = Var(Y_{t-1}) + \sigma_e^2$ for $t > 1$ with $Var(Y_1) = \sigma_e^2$ and hence $Var(Y_t) = t\sigma_e^2$.

(c) For $0 \leq t \leq s$, use $Y_s = Y_t + e_{t+1} + e_{t+2} + \cdots + e_s$ to show that $Cov(Y_t, Y_s) = Var(Y_t)$ and, hence, that $Cov(Y_t, Y_s) = \min(t, s)\sigma_e^2$.

Part II Computation problems by R coding

§ 2.4 Estimation of Correlation/相关性估计(reading materials)

Assuming that the time series $\{x_t\}$, $t = 1, \ldots, n$ is stationary

- Definition 1: **the sample autocovariance** is defined by

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h} - \bar{x})(x_t - \bar{x}). \tag{1.1}$$

  - $\hat{\gamma}(h) = \hat{\gamma}(-h)$ for $h = 0, 1, \ldots, n-1$.

- Definition 2: **the sample autocorrelation function** is defined as

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}. \tag{1.2}$$

  - $\hat{\rho}(h) = \hat{\rho}(-h)$ for $h = 0, 1, \ldots, n-1$.

Exercise 4. To give an example of the procedure for calculating numerically the autocovariance functions, we consider a artificial set of data generated by tossing a fair coin, letting $x_t = 1$ when a head is obtained and $x_t = -1$ when a tail is obtained. Construct y_t as

$$y_t = 5 + X_t - 0.7X_{t-1} \tag{1.3}$$

with $x_0 = -1$. We obtain the following table

Table 1.1. Sample Realization of the Contrived Series y_t

t	1	2	3	4	5	6	7	8	9	10
Coin	H	H	T	H	T	T	T	H	T	H
x_t	1	1	-1	1	-1	-1	-1	1	-1	1
y_t	6.7	5.3	3.3	6.7	3.3	4.7	4.7	6.7	3.3	6.7
$y_t - \bar{y}$	1.56	.16	-1.84	1.56	-1.84	$-.44$	$-.44$	1.56	-1.84	1.56

with $\bar{y}_t = 5.14$.

The sample autocorrelation for the series y_t can be calculated using (1.1) and (1.2) for $h = 0, 1, 2, \ldots$. For example, when $h = 3$, the autocorrelation becomes the ratio of

$$\begin{aligned}\widehat{\gamma}_y(3) &= \tfrac{1}{10}\sum_{t=1}^{7}(y_{t+3}-\bar{y})(y_t-\bar{y}) \\ &= \tfrac{1}{10}\Big[(1.56)(1.56)+(-1.84)(.16)+(-.44)(-1.84)+(-.44)(1.56) \\ &\quad +(1.56)(-1.84)+(-1.84)(-.44)+(1.56)(-.44)\Big] = -.048\end{aligned}$$

to

$$\widehat{\gamma}_y(0) = \tfrac{1}{10}[(1.56)^2+(.16)^2+\cdots+(1.56)^2] = 2.030$$

so that

$$\widehat{\rho}_y(3) = \frac{-.048}{2.030} = -.024.$$

The theoretical ACF can be obtained from the model

$$y_t = 5 + X_t - 0.7X_{t-1} \tag{1.4}$$

with $x_0 = -1$ using the fact that the mean of x_t is zero and the variance of x_t is one. It can be shown that

$$\rho_y(1) = \frac{-0.7}{1+(0.7)^2} = -0.47$$

and $\rho_y(h) = 0$ for $|h| > 1$. Table 1.2 compares the theoretical ACF with sample ACFs for a realization where $n = 10$ and another realization where $n = 100$.

Table 1.2. Theoretical and Sample ACFs for $n = 10$ and $n = 100$

		$n = 10$	$n = 100$
h	$\rho_y(h)$	$\widehat{\rho}_y(h)$	$\widehat{\rho}_y(h)$
0	1.00	1.00	1.00
±1	$-.47$	$-.55$	$-.45$
±2	.00	.17	$-.12$
±3	.00	$-.02$	.14
±4	.00	.15	.01
±5	.00	$-.46$	$-.01$

Assignment 2.21 (Computational problem):

a) Please write your R codes to realize the above computation of $\hat{\rho}_y(h)$ for n=10, n=100, n=1000 in Exercise 4 and report your results using a table similar to Table 1.2, where h=1, 2, 3, 4, 5, 6, 7, 8.

b) Compare the results you obtain with the results by running R function: acf(y, lag.max=8).

Notes: R function acf(y, lag.max=8) will return the acfs at different lags up to lag.max.

Submission deadline: Tuesday (00:00, 14, Sep)