Mathematical Statistics
Assignment4
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4.1 Proof

Let $W = \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}$. Note that

$$\begin{aligned} W &= \frac{\sigma_1^2}{n_1(n_1-1)} \cdot \frac{(n_1-1)}{\sigma_1^2} S_1^2 + \frac{\sigma_2^2}{n_2(n_2-1)} \cdot \frac{(n_2-1)}{\sigma_2^2} S_2^2 \\ &= aU + bV, \end{aligned}$$

where $U = \frac{(n_1-1)}{\sigma_1^2} S_1^2 \sim \chi^2(n_1-1),\ V = \frac{(n_2-1)}{\sigma_2^2} S_2^2 \sim \chi^2(n_2-1)$ and $a = \frac{\sigma_1^2}{n_1(n_1-1)},\ b = \frac{\sigma_2^2}{n_2(n_2-1)}$. We can approximate $\frac{W}{g}$ by $\chi^2(f)$, or equivalently, W can be approximated by $g\chi^2(f)$. Then $aU + bV$ and $g\chi^2(f)$ should have the same expectationa and variance, that is

$$\begin{cases} gf = a(n_1-1) + b(n_2-1) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}, \\ g^2 \cdot 2f = a^2 \cdot 2(n_1-1) + b^2 \cdot 2(n_2-1) = \frac{2}{(n_1-1)} \frac{\sigma_1^4}{n_1^2} + \frac{2}{(n_2-1)} \frac{\sigma_2^4}{n_2^2}. \end{cases}$$

Solving the equation yields that $f = (\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})^2 / (\frac{2}{(n_1-1)} \frac{\sigma_1^4}{n_1^2} + \frac{2}{(n_2-1)} \frac{\sigma_2^4}{n_2^2})$. By (4.13), we have

$$\begin{aligned} T_{\text{Welch}} &= \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{W}} \\ &= \frac{(\bar{X}_1 - \bar{X}_2 - \mu_1 + \mu_2) / \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}{\sqrt{W / (\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})}} \\ &= \frac{Z}{\sqrt{W/fg}} = \frac{Z}{\sqrt{\frac{W}{g}/f}} \\ &= \frac{Z}{\sqrt{\chi^2(f)/f}} \sim t(f), \end{aligned}$$

where $Z = (\bar{X}_1 - \bar{X}_2 - \mu_1 + \mu_2) / \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} \sim \mathbf{N}(0,1)$. Since S_i^2 is the unbiased estimator of σ_i^2, we can estimate f by

$$\nu = \frac{(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2})^2}{\frac{2}{(n_1-1)} \frac{S_1^4}{n_1^2} + \frac{2}{(n_2-1)} \frac{S_2^4}{n_2^2}} = \{ \frac{c^2}{n_1-1} + \frac{(1-c)^2}{n_2-1} \}^{-1},$$

where $c = \frac{S_1^2/n_1}{S_1^2/n_1 + S_2^2/n_2}$. Therefore, the distribution of T_{Welch} defined in (4.13) can be approximated by a t-distribution with ν degrees of freedom. □

4.2 Solution

(a) By (4.22), we have

$$\begin{aligned} 1 - \alpha &\approx \Pr\Big\{ -z_{\frac{\alpha}{2}} \le \frac{\sqrt{n}(\bar{X}_n - \lambda)}{\sigma(\lambda)} \le z_{\frac{\alpha}{2}} \Big\} \\ &= \Pr\Big\{ -1.96 \le \frac{10(6.25 - \lambda)}{\sqrt{\lambda}} \le 1.96 \Big\} \\ &= \Pr\Big\{ \Big| \frac{10(6.25 - \lambda)}{\sqrt{\lambda}} \Big| \le 1.96 \Big\}. \end{aligned}$$

Equivalently,

$$\lambda^2 - 12.5384\lambda + 39.0625 \le 0.$$

Two roots are given by $\lambda_1 = 5.7789,\ \lambda_2 = 6.7595$. Therefore, the equal-tail 95% CI for λ is given by $[5.7789, 6.7595]$.

(b) Replacing $z_{\frac{\alpha}{2}}$ and $-z_{\frac{\alpha}{2}}$ with z_{α_2} and $z_{1-\alpha+\alpha_2}$, respectively, we obtain

$$1-\alpha = \Pr\left\{ z_{1-\alpha+\alpha_2} \le \frac{62.5 - 10\lambda}{\sqrt{\lambda}} \le z_{\alpha_2} \right\}.$$

Let $a_1 = z_{1-\alpha+\alpha_2},\ a_2 = z_{\alpha_2}$. Let $t = \sqrt{\lambda}$, it then follows that

$$a_1 \le \frac{62.5 - 10t^2}{t} \le a_2.$$

Or equivalently,

$$\begin{cases} -10t^2 - a_1 t + 62.5 \ge 0, \\ -10t^2 - a_2 t + 62.5 \le 0. \end{cases}$$

Two roots for the first quadratic function are

$$t_1 = \frac{-a_1 - \sqrt{a_1^2 + 2500}}{20},\ t_2 = \frac{-a_1 + \sqrt{a_1^2 + 2500}}{20}.$$

We obtain $t_1 \le t \le t_2$. And two roots for the second quadratic function are

$$t_3 = \frac{-a_2 - \sqrt{a_2^2 + 2500}}{20},\ t_4 = \frac{-a_2 + \sqrt{a_2^2 + 2500}}{20}.$$

We obtain $t \le t_3$ or $t \ge t_4$. Note that $2500 = (-a_2 + \sqrt{a_2^2 + 2500})(a_2 + \sqrt{a_2^2 + 2500}) = (-a_1 + \sqrt{a_1^2 + 2500})(a_1 + \sqrt{a_1^2 + 2500})$. Since $\alpha = 0.05$ and $0 \le \alpha_2 \le \alpha$, we have $a_1 \le a_2$ and thus $(-a_2 + \sqrt{a_2^2 + 2500}) = 2500/(a_2 + \sqrt{a_2^2 + 2500}) < 2500/(a_1 + \sqrt{a_1^2 + 2500}) = (-a_1 + \sqrt{a_1^2 + 2500})$. Then, we have $t_3 < t_1$ and $t_4 < t_2$. Therefore, the interval of t should be $[t_4, t_2]$ and the CI for λ is given by $[t_4^2, t_2^2]$. The width of the CI is given by

$$\begin{aligned} l(\alpha_2) = t_2^2 - t_4^2 &= \frac{(a_1^2 - a_2^2) - a_1\sqrt{a_1^2 + 2500} + a_2\sqrt{a_2^2 + 2500}}{200} \\ &= \frac{(z_{0.95+\alpha_2}^2 - z_{\alpha_2}^2) - z_{0.95+\alpha_2}\sqrt{z_{0.95+\alpha_2}^2 + 2500} + z_{\alpha_2}\sqrt{z_{\alpha_2}^2 + 2500}}{200} \end{aligned}$$

Let α^* be such that

$$\alpha^* = \arg\min_{\alpha_2 \in [0,\alpha]} l(\alpha_2).$$

Therefore, the shortest $100(1-\alpha)\%$ CI for λ is given by

$$\left[\frac{z_{\alpha^*}^2 + 1250 - z_{\alpha^*}\sqrt{z_{\alpha^*}^2 + 2500}}{200}, \frac{z_{1-\alpha+\alpha^*}^2 + 1250 - z_{1-\alpha+\alpha^*}\sqrt{z_{1-\alpha+\alpha^*}^2 + 2500}}{200} \right]$$

4.3 Solution

(a) $\sigma = 3$ is known, then we have

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim \mathbf{N}(0, 1),$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$. It then follows that

$$1-\alpha = \Pr\Big\{ -z_{\frac{\alpha}{2}} \le \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \le z_{\frac{\alpha}{2}} \Big\}.$$

Plugging $\alpha = 0.1, n = 4, \sigma = 3,\ z_{0.05} = 1.645$ into the equation, we obtain

$$\begin{aligned} 0.9 &= \Pr\Big\{ -1.645 \le \frac{2((3.3-0.3-0.6-0.9)/4-\mu)}{3} \le 1.645 \Big\} \\ &= \Pr(-2.0925 \le \mu \le 2.8425). \end{aligned}$$

Therefore, $[-2.0925, 2.8425]$ is a 90% CI of μ.
(b) Let $\boldsymbol{\theta} = (\mu, \sigma^2)^\top$, $\mathbf{t}(\mathbf{x}) = (\bar{X}, S^2)^\top$, where $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. $\mathbf{t}(\mathbf{x})$ is jointly sufficient for θ. And

$$P = \frac{\sqrt{n}(\bar{X}-\mu)}{S} = \frac{\sqrt{n}\mathbf{e_1}^\top(\mathbf{t}(\mathbf{x})-\boldsymbol{\theta})}{\mathbf{e_2}^\top \mathbf{t}(\mathbf{x})}$$

is a function of $\mathbf{t}(\mathbf{x})$ and $\boldsymbol{\theta}$, where $\mathbf{e_1} = (1,0)^\top$ and $\mathbf{e_2} = (0,1)^\top$. Since

$$\frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t(n-1),$$

we have $P \sim t(n-1)$. That is, P is a pivot. It then follows that

$$\begin{aligned} 1-\alpha &= \Pr\Big\{ -t(\frac{\alpha}{2}, n-1) \le \frac{\sqrt{n}(\bar{X}-\mu)}{S} \le t(\frac{\alpha}{2}, n-1) \Big\} \\ &= \Pr\Big\{ \bar{X} - \frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1) \le \mu \le \bar{X} + \frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1) \Big\}. \end{aligned}$$

The CI of μ is given by

$$\Big[\bar{X} - \frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1), \bar{X} + \frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1) \Big].$$

Since $\bar{X} = 0.375,\ n = 4,\ S^2 = [(3.3-0.375)^2 + (-0.3-0.375)^2 + (-0.6-0.375)^2 + (-0.9-0.375)^2]/3 = 3.8625,\ t(0.05, 3) = 2.3534$, the CI of μ is $[-1.9376, 2.6876]$.

4.4 Solution
Let $\boldsymbol{\theta} = (\mu, \sigma^2)^\top$ and $\mathbf{t}(\mathbf{x}) = (\bar{X}, S^2)^\top$, $\mathbf{t}(\mathbf{x})$ is jointly sufficient for $\boldsymbol{\theta}$.

$$P = \frac{\sqrt{n}(\bar{X}-\mu)}{S} = \frac{\sqrt{n}\mathbf{e_1}^\top(\mathbf{t}(\mathbf{x})-\boldsymbol{\theta})}{\mathbf{e_2}^\top \mathbf{t}(\mathbf{x})}$$

is a function of both $\mathbf{t}(\mathbf{x})$ and $\boldsymbol{\theta}$. Besides,

$$P = \frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t(n-1).$$

Thus, P is a pivot.

$$\begin{aligned} 1-\alpha &= \Pr\Big\{ -t(\frac{\alpha}{2}, n-1) \le \frac{\sqrt{n}(\bar{X}-\mu)}{S} \le t(\frac{\alpha}{2}, n-1) \Big\} \\ &= \Pr\Big\{ \bar{X} - \frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1) \le \mu \le \bar{X} + \frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1) \Big\}. \end{aligned}$$

The width of a $100(1-\alpha)\%$ CI of μ is given by

$$L = \bar{X} + \frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1) - \Big(\bar{X} - \frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1)\Big) = 2\frac{S}{\sqrt{n}} t(\frac{\alpha}{2}, n-1).$$

Let $\alpha = 0.1$, then $L = 2t(0.05, n-1)S/\sqrt{n}$. And

$$\begin{aligned} 0.95 &= \Pr\Big\{ 2t(0.05, n-1)\frac{S}{\sqrt{n}} \leq \frac{\sigma}{5} \Big\} \\ &= \Pr\Big\{ \frac{(n-1)S^2}{\sigma^2} \leq \frac{n(n-1)}{100t^2(0.05, n-1)} \Big\}. \end{aligned}$$

Since $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$, the sample size n should satisfy that

$$\chi^2(0.05, n-1) = \frac{n(n-1)}{100t^2(0.05, n-1)}.$$

When $n = 309.228$, we have

$$\left| \chi^2(0.05, n-1) - \frac{n(n-1)}{100t^2(0.05, n-1)} \right| \leq 0.00002.$$

Therefore, $n = 309$.

4.5 Solution

Let $X_i = A_i - B_i$, then $X_i \sim \mathbf{N}(\mu, \sigma)$, where $\mu = \mu_A - \mu_B$ and $\sigma = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}$. We need to construct a 95% CI for μ. By (4.6),

$$\Big[\bar{X} - t(\frac{\alpha}{2}, n-1)\frac{S}{\sqrt{n}}, \bar{X} + t(\frac{\alpha}{2}, n-1)\frac{S}{\sqrt{n}} \Big]$$

is a $100(1-\alpha)\%$ CI for μ. Next, we calculate these quantities.

$$\begin{aligned} &X_1 = 86-80 = 6, \quad X_2 = 87-79 = 8, \quad X_3 = 56-58 = -2, \quad X_4 = 93-91 = 2 \\ &X_5 = 84-77 = 7, \quad X_6 = 93-82 = 11, \quad X_7 = 75-74 = 1, \quad X_8 = 79-66 = 13 \\ &\bar{X} = (6+8-2+2+7+11+1+13)/8 = 5.75, \quad S^2 = \sum_{i=1}^{8}(X_i - \bar{X})^2/7 = 183.5/7. \end{aligned}$$

Besides, $t(\frac{\alpha}{2}, n-1) = t(0.025, 7) = 2.3646$. Therefore, a 95% CI of μ is given by $[1.4696, 10.0304]$.

4.6 Solution

(a) $\prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^n(\prod_{i=1}^n x_i)^{\theta-1} = \theta^n(\prod_{i=1}^n x_i)^{\theta-1} \times 1$. Thus, $\prod_{i=1}^n X_i$ is a sufficient statistic of θ. By (4.3), we have

$$-2\sum_{i=1}^{n} \log F(X_i;\theta) \sim \chi^2(2n).$$

$F(x) = \int f(x)\,dx = x^\theta,\ 0 < x < 1$. Thus $P = -2\sum_{i=1}^n \log F(X_i;\theta) = -2\theta\sum_{i=1}^n \log X_i = -2\theta\log(\prod_{i=1}^n X_i)$ is a pivotal quantity. It then follows that

$$\begin{aligned} 1-\alpha &= \Pr\Big\{ \chi^2(1-\frac{\alpha}{2}, 2n) \leq -2\theta\log(\prod_{i=1}^{n} X_i) \leq \chi^2(\frac{\alpha}{2}, 2n) \Big\} \\ &= \Pr\Big\{ \frac{-\chi^2(1-\frac{\alpha}{2}, 2n)}{2\log(\prod_{i=1}^n X_i)} \leq \theta \leq \frac{-\chi^2(\frac{\alpha}{2}, 2n)}{2\log(\prod_{i=1}^n X_i)} \Big\}. \quad \log(\prod_{i=1}^{n} X_i) < 0 \end{aligned}$$

The CI is given by

$$\left[\frac{-\chi^2(1-\frac{\alpha}{2},2n)}{2\log(\prod_{i=1}^n X_i)}, \frac{-\chi^2(\frac{\alpha}{2},2n)}{2\log(\prod_{i=1}^n X_i)}\right].$$

(b) Let $0 \le \alpha_2 \le \alpha$, then the $100(1-\alpha)\%$ CI for θ is given by

$$\left[\frac{-\chi^2(1-\alpha+\alpha_2,2n)}{2\log(\prod_{i=1}^n X_i)}, \frac{-\chi^2(\alpha_2,2n)}{2\log(\prod_{i=1}^n X_i)}\right].$$

The width of the CI is given by

$$\begin{aligned} l(\alpha_2) &= \frac{-\chi^2(\alpha_2,2n)}{2\log(\prod_{i=1}^n X_i)} - \frac{-\chi^2(1-\alpha+\alpha_2,2n)}{2\log(\prod_{i=1}^n X_i)} \\ &= \frac{\chi^2(1-\alpha+\alpha_2,2n)-\chi^2(\alpha_2,2n)}{2\log(\prod_{i=1}^n X_i)}. \end{aligned}$$

Let α^* be such that

$$\alpha^* = \arg\min_{\alpha_2\in[0,\alpha]} l(\alpha_2).$$

Therefore, the $100(1-\alpha)\%$ shortest CI for θ is given by

$$\left[\frac{-\chi^2(1-\alpha+\alpha^*,2n)}{2\log(\prod_{i=1}^n X_i)}, \frac{-\chi^2(\alpha^*,2n)}{2\log(\prod_{i=1}^n X_i)}\right].$$

4.7 Solution

(a) $\prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \theta e^{-\theta x_i} = \theta^n e^{-\theta\sum_{i=1}^n x_i} \times 1$, then $\sum_{i=1}^n X_i = n\bar{X}$ is sufficient for θ. We have $X_i \sim \text{Exponential}(\theta)$, $n\bar{X} \sim \text{Gamma}(n,\theta)$, and

$$2\theta n\bar{X} \sim \text{Gamma}(\frac{2n}{2},\frac{1}{2}) = \chi^2(2n).$$

Thus, $P = 2\theta n\bar{X}$ is a pivot. The $100(1-\alpha)\%$ CI of θ can be constructed as

$$\begin{aligned} 1-\alpha &= \Pr\left\{\chi^2(1-\frac{\alpha}{2},2n) \le 2\theta n\bar{X} \le \chi^2(\frac{\alpha}{2},2n)\right\} \\ &= \Pr\left\{\frac{\chi^2(1-\frac{\alpha}{2},2n)}{2n\bar{X}} \le \theta \le \frac{\chi^2(\frac{\alpha}{2},2n)}{2n\bar{X}}\right\}. \end{aligned}$$

Let $\alpha = 0.05$ and plug into the values, the 95% equal-tail CI of θ is given by $[0.00871, 0.03101]$.

(b) By (a), $P = 2\theta n\bar{X}$ is a pivot, and

$$\begin{aligned} 1-\alpha &= \Pr\left\{\chi^2(1-\frac{\alpha}{2},2n) \le 2\theta n\bar{X} \le \chi^2(\frac{\alpha}{2},2n)\right\} \\ &= \Pr\left\{\frac{2n\bar{X}}{\chi^2(\frac{\alpha}{2},2n)} \le \frac{1}{\theta} \le \frac{2n\bar{X}}{\chi^2(1-\frac{\alpha}{2},2n)}\right\}. \end{aligned}$$

Plugging into the values, then the 95% equal-tail CI of $1/\theta$ is given by $[32.2429, 114.8723]$.

4.8 Solution

(a) Let $Y = \min(X, \mu^2/X)$, the support is $(0,\mu]$. The cdf is given by

$$\begin{aligned} \Pr(Y \le y) &= \Pr(X \le y) + \Pr(\frac{\mu^2}{X} \le y, X \ge \mu) \\ &= \Pr(X \le y) + 1 - \Pr(X \le \frac{\mu^2}{y}), \end{aligned}$$

which leads to $f_Y(y) = f_X(y) + (\mu^2/y^2)f_X(\mu^2/y)$. Note that

$$\frac{\lambda(\frac{\mu^2}{X}-\mu)^2}{\mu^2\frac{\mu^2}{X}} = \frac{\lambda(\mu^2-\mu X)^2}{\mu^4 X} = \frac{\lambda(\mu-X)^2}{\mu^2 X} = \frac{\lambda(X-\mu)^2}{\mu^2 X}.$$

If we let $Z = \frac{\lambda(Y-\mu)^2}{\mu^2 Y}$, then $Z = \frac{\lambda(Y-\mu)^2}{\mu^2 Y} = \frac{\lambda(X-\mu)^2}{\mu^2 X}$. And

$$\begin{aligned} f_Y(y) &= f_X(y) + (\mu^2/y^2)f_X(\mu^2/y) \\ &= \sqrt{\frac{\lambda}{2\pi}}e^{-\frac{z}{2}}[y^{-\frac{3}{2}} + \frac{\mu^2}{y^2}(\frac{\mu^2}{y})^{-\frac{3}{2}}] \\ &= \sqrt{\frac{\lambda}{2\pi}}e^{-\frac{z}{2}}(y^{-\frac{3}{2}} + \mu^{-1}y^{-\frac{1}{2}}). \end{aligned}$$

Denote $Z = \frac{\lambda(Y-\mu)^2}{\mu^2 Y} = H(Y)$, then $H'(y) = \frac{\lambda(y^2-\mu^2)}{\mu^2 y^2}$, we have

$$\begin{aligned} f_Z(z) &= f_Y(y)\cdot|\frac{1}{H'(y)}| \\ &= \sqrt{\frac{\lambda}{2\pi}}e^{-\frac{z}{2}}(y^{-\frac{3}{2}} + \mu^{-1}y^{-\frac{1}{2}})\cdot\frac{\mu^2 y^2}{\lambda(\mu^2-y^2)} \\ &= \sqrt{\frac{1}{2\pi}}e^{-\frac{z}{2}}\frac{\mu y^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}(\mu-y)} \\ &= \sqrt{\frac{1}{2\pi}}e^{-\frac{z}{2}}z^{-\frac{1}{2}}. \end{aligned}$$

Therefore, $Z \sim \chi^2(1)$. □

(b)

$$\begin{aligned} \prod_{i=1}^n f(x_i) &= \prod_{i=1}^n \sqrt{\frac{\lambda}{2\pi}}x_i^{-\frac{3}{2}}\exp\{\frac{-\lambda}{2\mu^2 x_i}(x_i-\mu)^2\} \\ &= (\frac{\lambda}{2\pi})^{\frac{n}{2}}\exp\{-\frac{\lambda}{2\mu^2}\sum_{i=1}^n\frac{(x_i-\mu)^2}{x_i}\}\times\prod_{i=1}^n x_i^{-\frac{3}{2}} \\ &= (\frac{\lambda}{2\pi})^{\frac{n}{2}}\exp\{-\frac{\lambda}{2\mu^2}\sum_{i=1}^n(x_i-2\mu+\frac{\mu^2}{x_i})\}\times\prod_{i=1}^n x_i^{-\frac{3}{2}} \\ &= (\frac{\lambda}{2\pi})^{\frac{n}{2}}\exp\{-\frac{\lambda}{2\mu^2}(\sum_{i=1}^n x_i + \mu^2\sum_{i=1}^n x_i^{-1} - 2n\mu)\}\times\prod_{i=1}^n x_i^{-\frac{3}{2}} \end{aligned}$$

Hence, $(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^{-1})$ are jointly sufficient statistics of (μ,λ). □

(c) $\lambda = \lambda_0$, then

$$\begin{aligned} \prod_{i=1}^n f(x_i) &= (\frac{\lambda_0}{2\pi})^{\frac{n}{2}}\exp\{-\frac{\lambda_0}{2\mu^2}(\sum_{i=1}^n x_i - 2n\mu) - \frac{\lambda_0}{2}\sum_{i=1}^n x_i^{-1}\}\times\prod_{i=1}^n x_i^{-\frac{3}{2}} \\ &= (\frac{\lambda_0}{2\pi})^{\frac{n}{2}}\exp\{-\frac{\lambda_0}{2\mu^2}(\sum_{i=1}^n x_i - 2n\mu)\}\times e^{-\frac{\lambda_0}{2}\sum_{i=1}^n x_i^{-1}}\prod_{i=1}^n x_i^{-\frac{3}{2}} \end{aligned}$$

Thus, $\sum_{i=1}^n x_i$ is sufficient for μ. □

(d) By Q3.18(c), $W = \sum_{i=1}^n X_i \sim \mathrm{IG}(n\mu, n\mu^3/\lambda)$ with $\mu' = n\mu,\ \lambda' = n^2\lambda$. Then by (a), we have

$$\frac{n^2\lambda(W-n\mu)^2}{(n\mu)^2 W} \sim \chi^2(1),$$

which is a pivot since $W = \sum_{i=1}^n X_i$ is sufficient for μ.
Let $\alpha = 0.05$, then the equal-tail 95% CI can be constructed as

$$0.95 = \Pr(\chi^2(0.975, 1) \le \frac{n^2\lambda(W - n\mu)^2}{(n\mu)^2 W} \le \chi^2(0.025, 1)).$$

Note that

$$\begin{aligned} \frac{n^2\lambda(W-n\mu)^2}{(n\mu)^2W} &= \frac{\lambda(W-n\mu)^2}{\mu^2W} = \frac{\lambda(W^2-2n\mu W+n^2\mu^2)}{\mu^2W} \\ &= \frac{\lambda(W^2-2n\mu W)}{\mu^2W} + \frac{\lambda n^2\mu^2}{\mu^2W} \\ &= \frac{\lambda(W-2n\mu)}{\mu^2} + \frac{\lambda n^2}{W}. \end{aligned}$$

Let $a = \chi^2(0.975, 1) - \frac{\lambda n^2}{W}$, $b = \chi^2(0.025, 1) - \frac{\lambda n^2}{W}$. Then we obtain

$$\begin{aligned} 0.95 &= \Pr\left\{\chi^2(0.975,1) \le \frac{\lambda(W-2n\mu)}{\mu^2} + \frac{\lambda n^2}{W} \le \chi^2(0.025,1)\right\} \\ &= \Pr\left\{a \le \frac{\lambda(W-2n\mu)}{\mu^2} \le b\right\}. \end{aligned}$$

That is

$$\begin{cases} b\mu^2 + 2n\lambda\mu - \lambda W \ge 0, \\ a\mu^2 + 2n\lambda\mu - \lambda W \le 0. \end{cases}$$

Two roots for the first quadratic function are

$$\mu_1 = \frac{-n\lambda - \sqrt{n^2\lambda^2 + b\lambda W}}{b}, \quad \mu_2 = \frac{-n\lambda + \sqrt{n^2\lambda^2 + b\lambda W}}{b}.$$

Two roots for the second quadratic function are

$$\mu_3 = \frac{-n\lambda - \sqrt{n^2\lambda^2 + a\lambda W}}{a}, \quad \mu_4 = \frac{-n\lambda + \sqrt{n^2\lambda^2 + a\lambda W}}{a}.$$

Since $\mu > 0$, the lower confidence bound of the CI of μ should be positive. There are 5 cases according to different a and b.
Case 1: $0 < a < b$, the 95% equal-tail CI of μ is given by

$$[\mu_2, \mu_4] = \left[\frac{-n\lambda + \sqrt{n^2\lambda^2 + b\lambda W}}{b}, \frac{-n\lambda + \sqrt{n^2\lambda^2 + a\lambda W}}{a}\right].$$

Case 2: $a < 0 < b$, the 95% equal-tail CI of μ is given by

$$[\max(\mu_2, \mu_3), \infty) = \left[\max(\frac{-n\lambda + \sqrt{n^2\lambda^2 + b\lambda W}}{b}, \frac{-n\lambda - \sqrt{n^2\lambda^2 + a\lambda W}}{a}), \infty\right).$$

Case 3: $a < b < 0$, the 95% equal-tail CI of μ is given by

$$[\mu_3, \mu_1] = \left[\frac{-n\lambda - \sqrt{n^2\lambda^2 + a\lambda W}}{a}, \frac{-n\lambda - \sqrt{n^2\lambda^2 + b\lambda W}}{b}\right].$$

Case 4: $0 = a < b$, the 95% equal-tail CI of μ is given by

$$[\mu_2, W/2n] = \left[\frac{-n\lambda + \sqrt{n^2\lambda^2 + b\lambda W}}{b}, \frac{W}{2n}\right].$$

Case 5: $a < b = 0$, the 95% equal-tail CI of μ is given by

$$[\max(\mu_3, W/2n), \infty) = \left[\max(\frac{-n\lambda - \sqrt{n^2\lambda^2 + a\lambda W}}{a}, \frac{W}{2n}), \infty\right).$$

4.9 Solution

(a) If $0 \le x \le m$,

$$\begin{aligned}\Pr(X = x) &= \sum_{k=0}^{x} \Pr(X_1 = k, X_2 = x - k) \\ &= \sum_{k=1}^{x} \binom{m}{k} p^k (1-p)^{m-k} \cdot \frac{\lambda^{x-k} e^{-\lambda}}{(x-k)!}.\end{aligned}$$

If $x > m$,

$$\begin{aligned}\Pr(X = x) &= \sum_{k=0}^{m} \Pr(X_1 = k, X_2 = x - k) \\ &= \sum_{k=0}^{m} \binom{m}{k} p^k (1-p)^{m-k} \cdot \frac{\lambda^{x-k} e^{-\lambda}}{(x-k)!}.\end{aligned}$$

Therefore, the pmf of X is

$$\Pr(X = x) = \sum_{k=0}^{\min(m,x)} \binom{m}{k} p^k (1-p)^{m-k} \frac{\lambda^{x-k} e^{-\lambda}}{(x-k)!}$$

for $x = 0, 1, ..., \infty$. □

(b) Since X_1 and X_2 are independent, we have

$$\begin{aligned}&E(X) = E(X_1) + E(X_2) = mp + \lambda = \mu, \\ &Var(X) = Var(X_1) + Var(X_2) = mp(1-p) + \lambda = \mu - mp^2.\end{aligned}$$

□

(c) Let $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$, then by the central limit theorem, an approximate $100(1-\alpha)\%$ CI of μ can be constructed as

$$\begin{aligned}1 - \alpha &= \Pr\left\{ -z_{\frac{\alpha}{2}} \le \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu - mp^2}} \le z_{\frac{\alpha}{2}} \right\} \\ &= \Pr\left\{ \frac{n(\bar{X}_n - \mu)^2}{\mu - mp^2} \le z^2_{\frac{\alpha}{2}} \right\} \\ &= \Pr\left\{ n\mu^2 - (2n\bar{X}_n + z^2_{\frac{\alpha}{2}})\mu + n\bar{X}_n^2 + mp^2 z^2_{\frac{\alpha}{2}} \le 0 \right\}.\end{aligned}$$

Therefore, an approximate $100(1-\alpha)\%$ CI of the μ is given by

$$\frac{2n\bar{X}_n + z^2_{\frac{\alpha}{2}} \mp \sqrt{(2n\bar{X}_n + z^2_{\frac{\alpha}{2}})^2 - 4n(n\bar{X}_n^2 + mp^2 z^2_{\frac{\alpha}{2}})}}{2n}$$

$$= \left[\bar{X}_n + \frac{z^2_{\frac{\alpha}{2}} - z_{\frac{\alpha}{2}}\sqrt{z^2_{\frac{\alpha}{2}} + 4n\bar{X}_n - 4nmp^2}}{2n}, \bar{X}_n + \frac{z^2_{\frac{\alpha}{2}} + z_{\frac{\alpha}{2}}\sqrt{z^2_{\frac{\alpha}{2}} + 4n\bar{X}_n - 4nmp^2}}{2n}\right].$$

4.10 Solution

(a) The cdf is given by

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} \frac{1}{\sigma_0} e^{-\frac{t-\mu}{\sigma_0}} \exp(e^{-\frac{t-\mu}{\sigma_0}})\, dt \\
&= \int_{-\infty}^{\frac{x-\mu}{\sigma_0}} e^{-y} \exp(e^{-y})\, dy \quad (y = \frac{t-\mu}{\sigma_0}) \\
&= \int_{\infty}^{e^{\frac{\mu-x}{\sigma_0}}} -e^{-z}\, dz \quad (z = e^{-y}) \\
&= \exp(-e^{\frac{\mu-x}{\sigma_0}}).
\end{aligned}
$$

(b) Based on (4.3),

$$
-2\sum_{i=1}^{n} \log F(X_i) = 2\sum_{i=1}^{n} e^{\frac{\mu-X_i}{\sigma_0}} \sim \chi^2(2n)
$$

is a pivot. Then the $100(1-\alpha)\%$ CI for μ can be constructed as

$$
\begin{aligned}
1-\alpha &= \Pr\Big\{\chi^2(1-\frac{\alpha}{2}, 2n) \le 2\sum_{i=1}^{n} e^{\frac{\mu-X_i}{\sigma_0}} \le \chi^2(\frac{\alpha}{2}, 2n)\Big\} \\
&= \Pr\Big\{e^{\frac{-\mu}{\sigma_0}}\chi^2(1-\frac{\alpha}{2}, 2n) \le 2\sum_{i=1}^{n} e^{\frac{-X_i}{\sigma_0}} \le e^{\frac{-\mu}{\sigma_0}}\chi^2(\frac{\alpha}{2}, 2n)\Big\} \\
&= \Pr\Big\{-\sigma_0\Big[\log 2 + \log\sum_{i=1}^{n} e^{\frac{-X_i}{\sigma_0}} - \log\chi^2(1-\frac{\alpha}{2}, 2n)\Big] \le \mu \\
&\qquad \le -\sigma_0\Big[\log 2 + \log\sum_{i=1}^{n} e^{\frac{-X_i}{\sigma_0}} - \log\chi^2(\frac{\alpha}{2}, 2n)\Big]\Big\}.
\end{aligned}
$$

Therefore, the $100(1-\alpha)\%$ equal-tail CI of μ is given by

$$
\Big[-\sigma_0\Big(\log 2 + \log\sum_{i=1}^{n} e^{\frac{-X_i}{\sigma_0}} - \log\chi^2(1-\frac{\alpha}{2}, 2n)\Big), -\sigma_0\Big(\log 2 + \log\sum_{i=1}^{n} e^{\frac{-X_i}{\sigma_0}} - \log\chi^2(\frac{\alpha}{2}, 2n)\Big)\Big].
$$