Computational Statistics

Assignment 4

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4.2

Using the sampling-wise IBF, we have

$$f_X(x) \propto \frac{f_{X|Y}(x|y_0)}{f_{Y|X}(y_0|x)} = \frac{\frac{1}{\sqrt{2\pi}}\exp\{-\frac{(x-y_0)^2}{2}\}}{\frac{1}{\sqrt{2\pi}}\exp\{-\frac{(y_0-x)^2}{2}\}} = 1,$$

which implies that $f_X(x) = c,\ x \in \mathcal{S}_X,\ c > 0$ is a constant. Note that the support of X is $(-\infty, \infty)$, then we have

$$1 = \int_{-\infty}^{\infty} c\, dx,$$

which is impossible. Therefore, $f_X(x)$ does not exsit. $\square$

4.3

$$\begin{cases} f_{X|Y}(x|y) = c_1^{-1}(y)\exp\left\{-\frac{\left[x-\mu_1-\rho\sigma_1\sigma_2^{-1}(y-\mu_2)\right]^2}{2\sigma_1^2(1-\rho^2)}\right\},\ a_1 \le x \le b_1 \\ f_{Y|X}(y|x) = c_2^{-1}(x)\exp\left\{-\frac{\left[y-\mu_2-\rho\sigma_2\sigma_1^{-1}(x-\mu_1)\right]^2}{2\sigma_2^2(1-\rho^2)}\right\},\ a_2 \le y \le b_2 \end{cases}$$

where

$$\begin{aligned} c_1(y) &= \int_{a_1}^{b_1} \exp\left\{-\frac{\left[x-\mu_1-\rho\sigma_1\sigma_2^{-1}(y-\mu_2)\right]^2}{2\sigma_1^2(1-\rho^2)}\right\} dx \\ &= \int_{a_1^*}^{b_1^*} e^{-\frac{z^2}{2}}\sqrt{\sigma_1^2(1-\rho^2)}\, dz \\ &= \sqrt{2\pi\sigma_1^2(1-\rho^2)}\left[\Phi(b_1^*) - \Phi(a_1^*)\right] \end{aligned}$$

and

$$a_1^* = \frac{a_1-\mu_1-\rho\sigma_1\sigma_2^{-1}(y-\mu_2)}{\sqrt{\sigma_1^2(1-\rho^2)}}, \quad b_1^* = \frac{b_1-\mu_1-\rho\sigma_1\sigma_2^{-1}(y-\mu_2)}{\sqrt{\sigma_1^2(1-\rho^2)}}.$$

Similarly,

$$\begin{aligned} c_2(x) &= \int_{a_2}^{b_2} \exp\left\{-\frac{\left[y-\mu_2-\rho\sigma_2\sigma_1^{-1}(x-\mu_1)\right]^2}{2\sigma_2^2(1-\rho^2)}\right\} dy \\ &= \int_{a_2^*}^{b_2^*} e^{-\frac{z^2}{2}}\sqrt{\sigma_2^2(1-\rho^2)}\, dz \\ &= \sqrt{2\pi\sigma_2^2(1-\rho^2)}\left[\Phi(b_2^*) - \Phi(a_2^*)\right] \end{aligned}$$

and

$$a_2^* = \frac{a_2 - \mu_2 - \rho\sigma_2\sigma_1^{-1}(x-\mu_1)}{\sqrt{\sigma_2^2(1-\rho^2)}}, \quad b_2^* = \frac{b_2 - \mu_2 - \rho\sigma_2\sigma_1^{-1}(x-\mu_1)}{\sqrt{\sigma_2^2(1-\rho^2)}}.$$

Then, we have

$$\begin{aligned}
\frac{f_{Y|X}(y|x)}{f_{X|Y}(x|y)} &= \frac{c_2^{-1}(x)\exp\left\{-\frac{\left[y-\mu_2-\rho\sigma_2\sigma_1^{-1}(x-\mu_1)\right]^2}{2\sigma_2^2(1-\rho^2)}\right\}}{c_1^{-1}(y)\exp\left\{-\frac{\left[x-\mu_1-\rho\sigma_1\sigma_2^{-1}(y-\mu_2)\right]^2}{2\sigma_1^2(1-\rho^2)}\right\}} \\
&= \frac{c_1(y)}{c_2(x)}\exp\left\{\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\int_{a_2}^{b_2} \frac{f_{Y|X}(y|x)}{f_{X|Y}(x|y)}\,dy &= c_2^{-1}(x)\int_{a_2}^{b_2} c_1(y)\exp\left\{\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}dy \\
&= c_2^{-1}(x)\exp\left\{\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}\int_{a_2}^{b_2} c_1(y)\exp\left\{-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}dy \\
&\propto c_2^{-1}(x)\exp\left\{\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}
\end{aligned}$$

Using the point-wise IBF, we have

$$\begin{aligned}
f_X(x) &\propto \left\{\int_{a_2}^{b_2} \frac{f_{Y|X}(y|x)}{f_{X|Y}(x|y)}\right\}^{-1} \\
&\propto c_2(x)\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\} \\
&\propto \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}[\Phi(b_2^*) - \Phi(a_2^*)], a_1 \le x \le b_1
\end{aligned}$$

Similarly, the marginal pdf of Y is given by

$$f_Y(y) \propto \exp\left\{-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}[\Phi(b_1^*) - \Phi(a_1^*)], a_2 \le y \le b_2$$

□

4.4

(a)

We have

$$\frac{f_{Y|X}(y|x)}{f_{X|Y}(x|y)} = \frac{xe^{-xy}/(1-e^{-bx})}{ye^{-xy}/(1-e^{-by})} = \frac{x(1-e^{-by})}{y(1-e^{-bx})},$$

so that

$$\int_{\mathcal{S}_Y} \frac{f_{Y|X}(y|x)}{f_{X|Y}(x|y)}\,dy = \int_0^b \frac{x(1-e^{-by})}{y(1-e^{-bx})}\,dy = \frac{x}{1-e^{-bx}}\int_0^b \frac{1-e^{-by}}{y},$$

Using the point-wise IBF, we have

$$f_X(x) \propto \left\{\frac{f_{Y|X}(y|x)}{f_{X|Y}(x|y)}\right\}^{-1} \propto \frac{1-e^{-bx}}{x},\ 0 \le x < b,$$

Let $h(x)$ denote $\frac{1-e^{-bx}}{x}$, $0 \le x < b$. It suffices to show that $\int_0^b h(x)\,dx < +\infty$. Using $e^a \ge a+1$ for any real number a, we have

$$h(x) = \frac{1-e^{-bx}}{x} \le \frac{bx}{x} = b.$$

Thus, $\int_0^b h(x)\,dx \le \int_0^b b\,dx = b^2 < +\infty$. Therefore, $f_X(x)$ exists and the pdf is given by

$$f_X(x) = \frac{1-e^{-bx}}{cx},\ 0 \le x < b,$$

where $c = \int_0^b \frac{1-e^{-bx}}{x}\,dx$.

(b)

If $b = +\infty$, then

$$f_X(x) \propto \frac{1}{x},\ 0 \le x < +\infty$$

However,

$$\int_0^\infty \frac{1}{x}\,dx = +\infty,$$

which implies that $f_X(x)$ does not exist. □

4.5

(a)

Using IBF formula, we have

$$\Pr(X = x_1) \propto \frac{\Pr(X = x_1|Y = y_2)}{\Pr(Y = y_2|X = x_1)} = \frac{a_{12}}{b_{12}} = \frac{1/2}{2/3} = \frac{3}{4}$$
$$\Pr(X = x_2) \propto \frac{\Pr(X = x_2|Y = y_2)}{\Pr(Y = y_2|X = x_2)} = \frac{a_{22}}{b_{22}} = \frac{1/2}{2/5} = \frac{5}{4}$$

and thus

$$\Pr(X = x_1) = \frac{3/4}{3/4 + 5/4} = \frac{3}{8}$$
$$\Pr(X = x_2) = \frac{5/4}{3/4 + 5/4} = \frac{5}{8}$$

Similarly, we have

$$\Pr(Y = y_1) = \frac{1}{2}, \quad \Pr(Y = y_2) = \frac{1}{2}$$

(b)

Using $\Pr(X = x_i, Y = y_j) = \Pr(X = x_i|Y = y_i)\Pr(Y = y_j)$, the joint distribution of (X, Y) is given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{8} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} \end{pmatrix},$$

where the (i, j) element of $\mathbf{P}$ is $p_{ij} = \Pr(X = x_i, Y = y_j)$. □

4.6

(a)

Note that $z_i = y_i,\ i = r+1, ..., m$, the complete-data likelihood is given by

$$L(\theta|Y_{\text{obs}}, z) = \prod_{i=1}^{m} \theta e^{-\theta y_i} = \theta^m \exp\Big\{ -\theta(\sum_{i=1}^{r} y_i + \mathbf{1}^\top z)\Big\}$$

(b)

The complete-data posterior distribution is given by

$$\begin{aligned} p(\theta|Y_{\text{obs}}, z) &\propto L(\theta|Y_{\text{obs}}, z) \times \text{Gamma}(\theta|\alpha_0, \beta_0) \\ &\propto \theta^m \exp\Big\{ -\theta(\sum_{i=1}^{r} y_i + \mathbf{1}^\top z)\Big\} \times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} e^{-\beta_0\theta}\theta^{\alpha_0 - 1} \\ &\propto \theta^{m+\alpha_0-1} \exp\Big\{ -\theta(y^* + \mathbf{1}^\top z + \beta_0)\Big\}, \end{aligned}$$

where $y^* = \sum_{i=1}^{r} y_i$. Hence, $\theta|(Y_{\text{obs}}, z) \sim \text{Gamma}(m + \alpha_0, y^* + \mathbf{1}^\top z + \beta_0)$. Since $f(y_i|\theta) = \theta e^{-\theta y_i},\ i = r+1, ..., m$, we have

$$f(z_i|Y_{\text{obs}}, \theta) = f(z_i|c_i, \theta) = \frac{\theta e^{-\theta z_i} I(z_i > c_i)}{\int_{c_i}^{\infty} \theta e^{-\theta z_i}\, dz_i} = \theta e^{-\theta(z_i - c_i)} I(z_i > c_i).$$

Let $W_i = Z_i - c_i$, then $f(w_i|Y_{\text{obs}}, \theta) = f(w_i|Y_{\text{obs}}, \theta) = \theta e^{-\theta w_i},\ w_i > 0$. i.e., $W_i|(c_i, \theta) \sim \text{Exponential}(\theta) = \text{Gamma}(1, \theta)$. Therefore,

$$\mathbf{1}^\top z - c_{\cdot}|(Y_{\text{obs}}, \theta) = \sum_{i=r+1}^{m} (z_i - c_i)|(Y_{\text{obs}}, \theta) = \sum_{i=r+1}^{m} W_i|(Y_{\text{obs}}, \theta) \sim \text{Gamma}(m - r, \theta).$$

(c)

The Gibbs sampler is as follows.
i) Given $\theta^{(t)}$, draw $W^{(t)}$ from $\text{Gamma}(m - r, \theta^{(t)})$.
ii) Let $\mathbf{1}^\top Z^{(t)} = W^{(t)} + c_{\cdot}$.
iii) Draw $\theta^{(t+1)}$ from $\text{Gamma}(m + \alpha_0, y^* + \mathbf{1}^\top Z^{(t)} + \beta_0)$.
iv) Repeat this process until convergence. □

4.7

(a)

The joint distribution is given by

$$\begin{aligned} f(Y_{\text{obs}}, \lambda, \beta) &= f(Y_{\text{obs}}|\lambda, \beta) f(\lambda|\beta) f(\beta) \\ &= \Big\{ \prod_{i=1}^{m} e^{-\lambda_i t_i} \frac{(\lambda_i t_i)^{N_i}}{N_i!} \Big\} \times \Big\{ \prod_{i=1}^{m} \frac{\beta^{\alpha_0}}{\Gamma(\alpha_0)} \lambda_i^{\alpha_0 - 1} e^{-\beta\lambda_i} \Big\} \times \frac{b_0^{a_0}}{\Gamma(a_0)} \beta^{a_0 - 1} e^{-b_0\beta} \end{aligned}$$

Hence, we have

$$f(\lambda|Y_{\text{obs}},\beta) \propto f(Y_{\text{obs}},\lambda,\beta) \propto \prod_{i=1}^{m} \lambda_i^{N_i+\alpha_0-1} e^{-\lambda_i(t_i+\beta)},$$

which implies that

$$f(\lambda|Y_{\text{obs}},\beta) = \prod_{i=1}^{m} \text{Gamma}(\lambda_i|N_i+\alpha_0, t_i+\beta)$$

and

$$f(\beta|Y_{\text{obs}},\lambda) \propto f(Y_{\text{obs}},\lambda,\beta) \propto \beta^{a_0+m\alpha_0-1} e^{-\beta(b_0+\sum_{i=1}^{m}\lambda_i)},$$

which implies that

$$f(\beta|Y_{\text{obs}},\lambda) = \text{Gamma}(\beta|a_0+m\alpha_0, b_0+\sum_{i=1}^{m}\lambda_i)$$

(b)

```
n <- 10000
alpha0 <- 1
a0 <- 1
b0 <- 1


N <- c(5, 1, 5, 14, 3, 19, 1, 1, 4, 22)
t <-
  c(94.32, 15.72, 62.88, 125.76, 5.24, 31.44, 1.05, 1.05, 2.10, 10.48)


lambda <- rep(0.5, 10)
beta <- 1
mat <- matrix(ncol = 11, nrow = n)
mat[1, ] <- c(lambda, beta)

set.seed(1234)
for (m in 1:n) {
  for (i in 1:10) {
    lambda[i] <- rgamma(1, N[i] + alpha0, t[i] + beta)
  }
  beta <- rgamma(1, a0 + 10 * alpha0, b0 + sum(lambda))
  mat[m, ] <- c(lambda, beta)
}

mat[9950:10000, ]
```

```
##             [,1]        [,2]       [,3]       [,4]      [,5]      [,6]
##  [1,] 0.07353450 0.204061422 0.08033234 0.10324629 0.3088557 0.6229854
##  [2,] 0.06250969 0.042101485 0.02280456 0.12874712 0.1219917 0.4736009
##  [3,] 0.05583002 0.119255149 0.05297725 0.11706280 0.7479253 0.6764637
##  [4,] 0.05175539 0.016131383 0.11999412 0.07677465 0.8595706 0.6214857
##  [5,] 0.09215644 0.365901586 0.14161898 0.09013144 0.7909859 0.4210601
##  [6,] 0.05197123 0.154719335 0.15329792 0.08422580 0.8935561 0.7403831
##  [7,] 0.03686762 0.053005555 0.16835965 0.10344207 0.8878001 0.5882680
```

```
##  [8,] 0.07151591 0.195008649 0.17501919 0.09526962 0.8951012 0.8287128
##  [9,] 0.04457825 0.159333465 0.17904099 0.18441480 0.2615333 0.4120538
## [10,] 0.06980651 0.103392310 0.09012791 0.09693604 0.5113643 0.4518033
## [11,] 0.03847968 0.175497067 0.15289325 0.12495565 0.4117295 0.5533086
## [12,] 0.09631558 0.079293837 0.11099987 0.07422503 0.5409589 0.6876709
## [13,] 0.03384858 0.070455430 0.08811005 0.09473536 0.2171642 0.3765553
## [14,] 0.11411685 0.070359543 0.12373297 0.08121332 0.7324899 0.5866853
## [15,] 0.06602495 0.237328421 0.09125351 0.16042984 0.4073881 0.9448174
## [16,] 0.02461771 0.013618902 0.05546255 0.17535493 0.5512339 0.4549193
## [17,] 0.05420008 0.008948322 0.11204881 0.11631005 0.4105170 0.7041836
## [18,] 0.06201393 0.088543966 0.06740920 0.11411157 0.5049797 0.5974267
## [19,] 0.02382143 0.031851636 0.05179560 0.08057501 0.3568728 0.4794808
## [20,] 0.02361553 0.218376041 0.06971431 0.11595372 0.1921277 0.7213908
## [21,] 0.04665014 0.098423211 0.06554434 0.04340644 0.8413010 0.8621217
## [22,] 0.08382762 0.127360329 0.05517627 0.11415512 0.8161970 0.5307615
## [23,] 0.04458077 0.170486554 0.08143647 0.14831515 0.5584280 0.4951978
## [24,] 0.07483877 0.137554103 0.04348644 0.09623333 0.3756058 0.7187787
## [25,] 0.08055714 0.126328257 0.08413724 0.14582109 0.4228057 0.8587946
## [26,] 0.06201184 0.049265537 0.16567759 0.13250429 0.4672531 0.7291321
## [27,] 0.01600213 0.171627438 0.10361476 0.13499201 1.1388651 0.6149570
## [28,] 0.07940879 0.148904816 0.04860803 0.10851522 0.8344174 0.7058548
## [29,] 0.02937354 0.054429140 0.11581703 0.09173315 0.3580453 0.5725521
## [30,] 0.06333030 0.023197904 0.06943298 0.14166052 0.3957409 0.4017863
## [31,] 0.03552508 0.163939330 0.09078111 0.12819752 0.3055134 0.4329759
## [32,] 0.03526407 0.130009932 0.11981758 0.17103935 0.4167168 0.5188151
## [33,] 0.07109764 0.067962051 0.09395807 0.18203698 0.4878714 0.4511628
## [34,] 0.03014774 0.311428262 0.08839979 0.09052131 0.2989881 0.4400832
## [35,] 0.07512313 0.054659997 0.09152545 0.08697698 0.7002864 0.7535099
## [36,] 0.04740664 0.057241719 0.12268902 0.11800182 1.0349720 0.8015847
## [37,] 0.03526749 0.141333033 0.07031148 0.13956200 0.6508357 0.5664373
## [38,] 0.08268006 0.185190074 0.07562829 0.11483262 0.7805492 0.4440239
## [39,] 0.04291070 0.159341703 0.08172650 0.11011985 0.6988133 0.5625658
## [40,] 0.02274903 0.021183055 0.03286396 0.12387412 1.0108868 0.5525899
## [41,] 0.06242373 0.161397594 0.14572966 0.15146172 0.3456309 0.6182911
## [42,] 0.09672091 0.040304163 0.05766923 0.06963775 0.5632016 0.5097371
## [43,] 0.02995400 0.244315830 0.07906379 0.11916496 0.1893704 0.5633115
## [44,] 0.03845610 0.065904964 0.08312125 0.08306419 0.8055137 0.6416601
## [45,] 0.03717797 0.024099161 0.09185350 0.11831032 0.4456933 0.7196971
## [46,] 0.05110784 0.128072434 0.09064961 0.10875420 0.4842019 0.5484174
## [47,] 0.09098341 0.408915570 0.08673648 0.10245073 0.4746848 0.6846793
## [48,] 0.07279499 0.012958065 0.06983403 0.14091658 0.4917587 0.6281145
## [49,] 0.08740037 0.205245122 0.06429123 0.08337870 1.1658233 0.4577323
## [50,] 0.05911178 0.152358593 0.10601436 0.11765373 0.7169412 0.4940003
## [51,] 0.10241402 0.284293592 0.03578838 0.09736803 0.5410463 0.4594338
##             [,7]       [,8]      [,9]     [,10]     [,11]
##  [1,] 0.44367007 0.23521602 1.1659114 1.8625372 1.4008468
##  [2,] 0.78055038 0.21915021 1.9230526 2.0891963 1.7220121
##  [3,] 1.10962181 0.77345129 1.9734764 1.5394219 1.0433180
##  [4,] 0.75638298 2.26965988 0.8440189 2.0435933 1.2548160
##  [5,] 1.08714055 0.20091071 1.2057002 1.9559775 0.7539630
##  [6,] 0.23871022 1.15043634 2.1385155 1.6496689 0.5990290
##  [7,] 0.88413157 1.33999005 1.3053475 2.2446705 1.3024771
##  [8,] 0.18507139 0.22124888 0.9724763 1.4981529 0.8438646
##  [9,] 0.57099847 0.22833781 1.5650608 1.3182066 1.1753095
```

```
## [10,] 0.48932720 1.99220151 2.7878254 1.7045623 1.3036165
## [11,] 1.29971511 1.08657596 1.2533575 2.3563062 1.1074541
## [12,] 0.74879676 0.97173508 0.7546605 2.1159708 1.3247095
## [13,] 0.13500113 1.20838025 0.6887984 2.4378675 1.8401039
## [14,] 0.72708871 0.48295004 1.5999461 1.9595881 1.2526975
## [15,] 0.35427546 0.39568653 0.8297624 1.1604681 2.7414193
## [16,] 0.07925373 0.09594436 0.9444956 1.6534806 3.1100852
## [17,] 0.12147218 0.34597532 1.0822213 2.1482934 1.0649145
## [18,] 1.04743002 1.41812431 1.3264357 1.9539501 1.5219845
## [19,] 0.68753530 0.28019222 0.8724809 0.9282799 1.1812343
## [20,] 0.53279956 0.82614576 1.7491816 2.1941156 0.7127947
## [21,] 0.11285204 1.97382989 2.3647147 1.8361515 1.9701015
## [22,] 0.77472796 0.07791089 0.6665495 1.9879047 1.7814194
## [23,] 0.96440234 0.28543228 1.3237845 1.7836505 2.4808648
## [24,] 0.81576289 0.86366655 1.3530226 1.9840024 1.7997002
## [25,] 1.24008826 0.76637017 0.6391311 2.3323098 1.7151060
## [26,] 1.42181283 0.79628416 1.2742690 1.9007296 0.9674347
## [27,] 0.76697294 0.28513121 1.2113016 1.2533803 1.6505060
## [28,] 0.38338238 0.17304414 1.0601634 1.8500526 1.6241074
## [29,] 0.43050032 0.56834223 1.4542986 2.0478710 1.6700162
## [30,] 0.16665606 0.79911063 2.1107564 1.6885779 1.7017635
## [31,] 0.68622227 0.40616373 1.9325028 2.3395780 1.8492246
## [32,] 1.62221105 0.62068977 1.6631903 2.5796786 1.1558432
## [33,] 1.91501307 1.04278116 2.3232304 1.2028849 1.1308806
## [34,] 1.36805611 0.62294129 1.3873342 1.8101432 0.8493279
## [35,] 0.58533296 0.30344892 1.1593658 1.9888323 1.5822664
## [36,] 0.60960689 0.69343612 2.0439546 2.0757781 1.3546107
## [37,] 0.72210767 1.77456141 1.0398454 2.8351209 1.6306238
## [38,] 0.23136611 1.03294610 1.2800730 1.5962522 1.1047720
## [39,] 1.08246563 0.91885271 1.5815164 1.9512997 1.9645827
## [40,] 0.23095440 0.65450250 1.2058357 1.2880476 1.5201854
## [41,] 1.00674241 0.58852354 0.7844482 1.8059324 1.3204243
## [42,] 0.87281151 2.17025014 0.8319322 2.3391956 1.2438915
## [43,] 1.33393175 0.28768658 1.0532093 1.7630403 1.4223975
## [44,] 1.34647801 0.79988859 1.8975095 1.6991916 1.7168140
## [45,] 1.03531089 0.31854121 1.6272563 1.4952581 1.4941085
## [46,] 1.46110284 1.07433918 0.8009939 1.6671874 1.3945678
## [47,] 0.68415727 1.26832740 1.3368372 1.9691353 2.3968087
## [48,] 0.82946478 0.50150704 0.8136017 1.8930207 1.1862767
## [49,] 0.32014863 1.20787442 1.6558498 1.9727852 0.8489814
## [50,] 0.90978087 0.49972842 2.0150072 2.5421071 0.6882440
## [51,] 0.44084822 2.19997079 1.6619009 1.8316924 1.5463806
```

□

4.8

```r
exsercise4.8 <- function(theta0) {
  y <- c(14, 0, 1, 5)
  a <- b <- 1
  q <- rep(0, y[1] + 1)
```

```r
  for (k in 1:length(q)) {
    zk <- k - 1
    q[k] <- dbinom(zk, y[1], theta0 / (theta0 + 2)) /
      dbeta(theta0, a + y[4] + zk, b + y[2] + y[3])
  }

  p <- q / sum(q)
  qp <- round(data.frame(q = q, p = p), 8)
  return(qp)
}
```

```r
## \theta_0 = 0.8
exsercise4.8(0.8)
```

```
##             q          p
## 1  0.00326948 0.00938290
## 2  0.01716475 0.04926022
## 3  0.04338868 0.12451890
## 4  0.06942189 0.19923024
## 5  0.07809963 0.22413401
## 6  0.06508303 0.18677835
## 7  0.04130269 0.11853241
## 8  0.02022989 0.05805669
## 9  0.00767050 0.02201316
## 10 0.00223723 0.00642051
## 11 0.00049351 0.00141629
## 12 0.00007976 0.00022890
## 13 0.00000892 0.00002560
## 14 0.00000062 0.00000177
## 15 0.00000002 0.00000006
```

```r
## \theta_0 = 0.5
exsercise4.8(0.5)
```

```
##             q          p
## 1  0.06701785 0.00938290
## 2  0.35184372 0.04926022
## 3  0.88938274 0.12451890
## 4  1.42301238 0.19923024
## 5  1.60088893 0.22413401
## 6  1.33407411 0.18677835
## 7  0.84662395 0.11853241
## 8  0.41467296 0.05805669
## 9  0.15723016 0.02201316
## 10 0.04585880 0.00642051
## 11 0.01011591 0.00141629
## 12 0.00163489 0.00022890
## 13 0.00018285 0.00002560
## 14 0.00001266 0.00000177
## 15 0.00000041 0.00000006
```

Based on the results, $\{q_k(0.8)\}_{k=1}^{15}$ are different from $\{q_k(0.5)\}_{k=1}^{15}$, but $\{p_k(0.8)\}_{k=1}^{15}$ are the same with $\{p_k(0.5)\}_{k=1}^{15}$, which implies that $\{p_k(\theta_0)\}_{k=1}^{15}$ do not depend on the value of θ_0.

$\square$

4.9

The P-step The complete-data likelihood function of (ϕ, λ) is

$$L(\phi, \lambda | Y_{\text{com}}) = \left\{ \prod_{i=1}^{n} (1-\phi)^{z_i} \phi^{1-z_i} \right\} \times \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}.$$

The joint prior distribution of (ϕ, λ) is

$$\pi(\phi, \lambda) = \frac{\phi^{a_0-1}(1-\phi)^{b_0-1}}{B(a_0, b_0)} \times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \lambda^{\alpha_0-1} e^{-\beta_0\lambda}$$

so that the joint posterior distribution of (ϕ, λ) is given by

$$\begin{aligned} p(\phi, \lambda | Y_{\text{com}}) &\propto L(\phi, \lambda | Y_{\text{com}}) \times \pi(\phi, \lambda) \\ &\propto \phi^{n+a_0-1-\sum_{i=1}^n z_i} (1-\phi)^{b_0-1+\sum_{i=1}^n z_i} \lambda^{\alpha_0-1+\sum_{i=1}^n x_i} e^{-(n+\beta_0)\lambda}, \end{aligned}$$

which implies that

$$\begin{aligned} p(\phi, \lambda | Y_{\text{com}}) &= p(\phi | Y_{\text{com}}) \times p(\lambda | Y_{\text{com}}) \\ &= \text{Beta}\Big(\phi \Big| n + a_0 - \sum_{i=1}^n z_i, b_0 + \sum_{i=1}^n z_i\Big) \times \text{Gamma}\Big(\lambda \Big| \alpha_0 + \sum_{i=1}^n x_i, n + \beta_0\Big) \end{aligned}$$

The I-step By exsercise 2.21, we have

$$Z_i | (Y_i = y_i, \phi, \lambda) \sim \begin{cases} \text{Bernoulli}(p_0), & y_i = 0, \\ \text{Degenerate}(1), & y_i > 0, \end{cases}$$

and

$$\begin{cases} X_i | (Y_i = 0, \phi, \lambda) \sim \text{ZIP}(p_0, \lambda) \\ X_i | (Y_i = y_i > 0, \phi, \lambda) \sim \text{Degenerate}(y_i), \end{cases}$$

where

$$p_0 = \frac{1-\phi}{1-\phi+\phi e^{\lambda}}.$$

$\square$

4.10

(a)

By example 2.6, the conditional predictive distribution is

$$Z | (Y_{\text{obs}}, \theta) \sim \text{Binomial}\Big(y_1, \frac{\theta}{\theta+2}\Big)$$

and the complete-data likelihood function of θ is

$$L(\theta | Y_{\text{obs}}, Z) \propto \theta^{z+y_4}(1-\theta)^{y_2+y_3}.$$

Since the prior distribution of θ is Beta(a_0, b_0), the complete-data posterior distribution of θ is given by

$$p(\theta|Y_{\text{obs}}, Z) = \text{Beta}(z + y_4 + a_0, y_2 + y_3 + b_0).$$

Hence, the mode $\tilde{\theta}$ is given by

$$\tilde{\theta} = \frac{z + y_4 + a_0 - 1}{z + y_2 + y_3 + y_4 + a_0 + b_0 - 2}$$

Note that $E(Z|Y_{\text{obs}}, \theta^{(t)}) = y_1\theta^{(t)}/(y_1\theta^{(t)} + 2)$, then the EM algorithm is given by

$$\theta^{(t+1)} = \frac{y_1\theta^{(t)}/(y_1\theta^{(t)} + 2) + y_4 + a_0 - 1}{y_1\theta^{(t)}/(y_1\theta^{(t)} + 2) + y_2 + y_3 + y_4 + a_0 + b_0 - 2}$$

By R code, the value of mode $\tilde{\theta}$ is 0.6268215.

R code:

```
exsercise4.10.a <- function(theta0, iteNum) {
  y <- c(125, 18, 20, 34)
  a0 <- b0 <- 1
  theta <- rep(0, iteNum + 1)
  theta[1] <- theta0
  for (t in 1:iteNum) {
    theta[t + 1] <- (y[1] * theta[t] / (theta[t] + 2) + y[4] + a0 - 1) /
      (y[1] * theta[t] / (theta[t] + 2) + sum(y) - y[1] + a0 + b0 - 2)
  }
  return(theta)
}

exsercise4.10.a(0.1, 10)[-1]
```

```
##  [1] 0.5125229 0.6102501 0.6245940 0.6265252 0.6267822 0.6268163 0.6268208
##  [8] 0.6268214 0.6268215 0.6268215
```

```
exsercise4.10.a(0.3, 10)[-1]
```

```
##  [1] 0.5696701 0.6188996 0.6257635 0.6266809 0.6268028 0.6268190 0.6268212
##  [8] 0.6268215 0.6268215 0.6268215
```

```
exsercise4.10.a(0.6, 10)[-1]
```

```
##  [1] 0.6231884 0.6263378 0.6267573 0.6268130 0.6268204 0.6268213 0.6268215
##  [8] 0.6268215 0.6268215 0.6268215
```

```
exsercise4.10.a(0.9, 10)[-1]
```

```
##  [1] 0.6570184 0.6307438 0.6273408 0.6268904 0.6268306 0.6268227 0.6268217
##  [8] 0.6268215 0.6268215 0.6268215
```

(b)

By (a), the expression of $h_1(\cdot)$ is

$$\begin{aligned} h_1(\theta) &= \frac{y_1\theta/(y_1\theta+2)+y_4+a_0-1}{y_1\theta/(y_1\theta+2)+y_2+y_3+y_4+a_0+b_0-2} \\ &= 1-\frac{y_2+y_3+b_0-1}{y_1\theta/(\theta+2)+y_2+y_3+y_4+a_0+b_0-2} \\ &= 1-\frac{(y_2+y_3+b_0-1)(\theta+2)}{y_1\theta+(y_2+y_3+y_4+a_0+b_0-2)(\theta+2)} \end{aligned}$$

so that the derivative is given by

$$h_1'(\theta) = -\frac{(y_2+y_3+b_0-1)\Big[y_1\theta+(N-y_1)(\theta+2)\Big]-N(y_2+y_3+b_0-1)(\theta+2)}{\Big[y_1\theta+(N-y_1)(\theta+2)\Big]^2},$$

where $N = y_1+y_2+y_3+y_4+a_0+b_0-2$. Since $(y_1,y_2,y_3,y_4)^\top = (125,18,20,34)^\top$ and $a_0 = b_0 = 1$, we have

$$h_1'(\theta) = \frac{9500}{(197\theta+144)^2} = \frac{9500}{(197\times 0.6268215+144)^2} = 0.1327787.$$

(c)

Given $\theta = \theta^{(t)}$, the I-step is to draw $Z^{(t)}$ from $\text{Binomal}(y_1, \theta^{(t)}/(\theta^{(t)}+2))$. Given $Z = Z^{(t)}$, the P-step is to draw $\theta^{(t+1)}$ from $\text{Beta}(Z^{(t)}+y_4+a_0, y_2+y_3+b_0)$.

R code:

```r
z <- rep(0, 100001)
theta <- rep(0, 100001)
z[1] <- 1 # z^{(0)}
theta[1] <- 0.5 # \theta^{(0)}

for (i in 1:100001) {
  z[i] <- rbinom(1, 125, theta[i] / (theta[i] + 2))
  theta[i + 1] <- rbeta(1, z[i] + 35, 39)
}

theta.half <- theta[50002:100001]
```

(d)

Similar to (a), the conditional predictive distribution is

$$Z|(Y_{\text{obs}},\theta) \sim \text{Binomial}\Big(y_1, \frac{3\theta}{\theta+2}\Big)$$

and the complete-data likelihood function of θ is

$$L(\theta|Y_{\text{obs}}, Z) \propto \theta^{z+y_4}(1-\theta)^{y_1-z+y_2+y_3}.$$

Since the prior distribution of θ is Beta(a_0, b_0), the complete-data posterior distribution of θ is given by

$$p(\theta|Y_{\text{obs}}, Z) = \text{Beta}(z + y_4 + a_0, y_1 - z + y_2 + y_3 + b_0).$$

Hence, the mode $\tilde{\theta}$ is given by

$$\tilde{\theta} = \frac{z + y_4 + a_0 - 1}{y_1 + y_2 + y_3 + y_4 + a_0 + b_0 - 2}$$

Note that $E(Z|Y_{\text{obs}}, \theta^{(t)}) = 3y_1\theta^{(t)}/(y_1\theta^{(t)} + 2)$, then the EM algorithm is given by

$$\theta^{(t+1)} = \frac{3y_1\theta^{(t)}/(y_1\theta^{(t)} + 2) + y_4 + a_0 - 1}{y_1 + y_2 + y_3 + y_4 + a_0 + b_0 - 2}$$

By R code, the value of mode $\tilde{\theta}$ is 0.6268215.

R code:

```
exsercise4.10.d <- function(theta0, iteNum) {
  y <- c(125, 18, 20, 34)
  a0 <- b0 <- 1
  theta <- rep(0, iteNum + 1)
  theta[1] <- theta0
  for (t in 1:iteNum) {
    theta[t + 1] <-
      (3 * y[1] * theta[t] / (theta[t] + 2) + y[4] + a0 - 1) /
      (sum(y) + a0 + b0 - 2)
  }
  return(theta)
}

exsercise4.10.d(0.1, 30)[-1]
```

```
##  [1] 0.2632342 0.3939890 0.4858647 0.5446402 0.5800145 0.6005277 0.6121675
##  [8] 0.6186909 0.6223216 0.6243345 0.6254480 0.6260633 0.6264030 0.6265906
## [15] 0.6266941 0.6267512 0.6267827 0.6268001 0.6268097 0.6268150 0.6268179
## [22] 0.6268195 0.6268204 0.6268209 0.6268212 0.6268213 0.6268214 0.6268214
## [29] 0.6268215 0.6268215
```

```
exsercise4.10.d(0.3, 30)[-1]
```

```
##  [1] 0.4208784 0.5035284 0.5554457 0.5863408 0.6041371 0.6141966 0.6198222
##  [8] 0.6229494 0.6246820 0.6256401 0.6261694 0.6264616 0.6266229 0.6267119
## [15] 0.6267610 0.6267881 0.6268031 0.6268113 0.6268159 0.6268184 0.6268198
## [22] 0.6268206 0.6268210 0.6268212 0.6268213 0.6268214 0.6268214 0.6268215
## [29] 0.6268215 0.6268215
```

```
exsercise4.10.d(0.6, 30)[-1]
```

```
##  [1] 0.6118704 0.6185251 0.6222296 0.6242835 0.6254198 0.6260477 0.6263945
##  [8] 0.6265858 0.6266915 0.6267498 0.6267819 0.6267997 0.6268094 0.6268148
```

```
## [15] 0.6268178 0.6268195 0.6268204 0.6268209 0.6268212 0.6268213 0.6268214
## [22] 0.6268214 0.6268215 0.6268215 0.6268215 0.6268215 0.6268215 0.6268215
## [29] 0.6268215 0.6268215
```

```
exsercise4.10.d(0.9, 30)[-1]
```

```
##  [1] 0.7633468 0.6984263 0.6652803 0.6477346 0.6382689 0.6331101 0.6302829
##  [8] 0.6287288 0.6278730 0.6274014 0.6271414 0.6269980 0.6269189 0.6268752
## [15] 0.6268511 0.6268379 0.6268305 0.6268265 0.6268242 0.6268230 0.6268223
## [22] 0.6268220 0.6268218 0.6268216 0.6268216 0.6268215 0.6268215 0.6268215
## [29] 0.6268215 0.6268215
```

(e)

By (d), the expression of $h_2(\cdot)$ is

$$\begin{aligned} h_2(\theta) &= \frac{3y_1\theta/(y_1\theta+2)+y_4+a_0-1}{y_1+y_2+y_3+y_4+a_0+b_0-2} \\ &= \frac{3y_1\theta/(\theta+2)+y_4}{N} \end{aligned}$$

so that the derivative is given by

$$h_2'(\theta) = \frac{6y_1}{N(\theta+2)^2},$$

where $N = y_1+y_2+y_3+y_4+a_0+b_0-2$. Since $(y_1, y_2, y_3, y_4)^\top = (125, 18, 20, 34)^\top$ and $a_0 = b_0 = 1$, we have

$$h_2'(\theta) = \frac{6 \times 125}{197 \times (0.6268215+2)^2} = 0.5517393.$$

□

4.11

The P-step By exserciese2.22(b), the complete-data likelihood function is

$$L(\lambda|Y_{\text{com}}) = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!},$$

where $y_i = z_i x_i$. Since the prior distribution of λ is Gamma(α_0, β_0), the complete-data posterior distribution of λ is given by

$$\begin{aligned} p(\lambda|Y_{\text{com}}) &\propto L(\lambda|Y_{\text{com}}) \times \text{Gamma}(\lambda|\alpha_0, \beta_0) \\ &\propto \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} e^{-\beta_0\lambda} \lambda^{\alpha_0-1} \\ &\propto e^{-(n+\beta_0)\lambda} \lambda^{\sum_{i=1}^{n} y_i + \alpha_0 - 1} \end{aligned}$$

which implies that

$$p(\lambda|Y_{\text{com}}) = \text{Gamma}(\sum_{i=1}^{n} y_i + \alpha_0, n+\beta_0)$$

The I-step By exserciese2.22(b), we have

$$Z_i|(X_i, \lambda) = Z_i|\lambda \sim \text{Bernoulli}(1-e^{-\lambda})$$

Therefore, we have

$$\text{I step}: \quad \text{Draw } z^{(t)} \sim \text{Bernoulli}(1 - e^{-\lambda^{(t)}})$$

$$\text{P step}: \quad \text{Draw } \lambda^{(t+1)} \sim \text{Gamma}(\sum_{i=1}^{n} y_i^{(t)} + \alpha_0, n + \beta_0)$$

where $y_i^{(t)} = z_i^{(t)} x_i$. $\square$

Additional Problem

Question Let $X \sim N(\mu, \sigma^2)$. The prior distribution of μ is $N(0, \sigma^2)$. Derive the complete-data posterior distribution of μ. (σ^2 is known)

Solution The complete-data likelihood function of μ is

$$L(\mu|X, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\mu^2}{2\sigma^2}} = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2-2x\mu+2\mu^2}{2\sigma^2}}$$

so that the posterior distribution of μ is given by

$$p(\mu|X, \sigma^2) \propto L(\mu|X, \sigma^2) \propto \exp\Big\{-\frac{\mu^2 - x\mu}{\sigma^2}\Big\} \propto \exp\Big\{-\frac{(\mu - \frac{1}{2}x)^2}{2(\frac{\sqrt{2}}{2}\sigma)^2}\Big\}$$

which implies that

$$\mu|(X, \sigma^2) \sim N(\frac{x}{2}, \frac{\sigma^2}{2})$$

$\square$