Mathematical Statistics

Assignment2

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2.1 Solution

Let $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})$ denote the pdf of $Z \sim \mathcal{N}(0,1)$ and $g(y)$ denote the pdf of $Y \sim \mathcal{X}^2(n)$.

The pdf of T is

$$\begin{aligned} F(x) = \Pr(T \le x) = \Pr(\frac{Z}{\sqrt{Y/n}} \le x) \\ &= \int \Pr(\frac{Z}{\sqrt{Y/n}} \le x | Y = y) g(y)\, dy \\ &= \int \Pr(Z \le x\sqrt{y/n}) g(y)\, dy \\ &= \int (\int_{-\infty}^{x\sqrt{y/n}} \phi(z)\, dz) g(y)\, dy \end{aligned}$$

Let $t = \frac{z}{\sqrt{y/n}}$, then $-\infty < t \le x,\ dz = \sqrt{\frac{y}{n}}\, dt$, and thus

$$\begin{aligned} F(x) &= \int_0^{\infty} (\int_{-\infty}^{x} \phi(t\sqrt{y/n})\sqrt{y/n} g(y)\, dy)\, dt \\ &= \int_{-\infty}^{x} (\int_0^{\infty} \phi(t\sqrt{y/n})\sqrt{y/n} g(y)\, dy)\, dt \quad = \int_{-\infty}^{x} f(t)\, dt \end{aligned}$$

Thus, the pdf of T is given by

$$\begin{aligned} f(t) &= \int_0^{\infty} \phi(t\sqrt{y/n})\sqrt{y/n} g(y)\, dy \\ &= \int_0^{\infty} \frac{1}{2\pi} \exp(-\frac{t^2 y}{2n}) \sqrt{y/n} \frac{(1/2)^{n/2}}{\Gamma(n/2)} y^{n/2-1} e^{-y/2}\, dy \\ &= \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)} (1 + \frac{t^2}{n})^{-\frac{n+1}{2}}, \quad -\infty < t < \infty \end{aligned}$$

When $n = 1$, $f(t) = \frac{1}{\pi(1+t^2)}$, then T obeys Cauchy distribution. Thus, there is no expectation and variance for $T \sim t(1)$.

When $n \ge 1$, we have

$$E(T) = E(E(T|Y)), \quad \text{and} \quad E(T|Y=y) = E(\frac{Z}{\sqrt{y/n}}) = \sqrt{n/y} E(Z) = 0.$$

Thus, $E(T|Y) = 0$, and so, $E(T) = E(0) = 0$.

And,

$$\begin{aligned} Var(T) &= E(Var(T|Y)) + Var(E(T|Y)) \\ &= E(Var(T|Y)) + Var(0) \\ &= E(Var(T|Y)) \end{aligned}$$

Since $Var(T|Y=y) = Var(\frac{Z}{\sqrt{y/n}}) = \frac{n}{y} Var(Z) = n/y$, we have

$$Var(T) = E(n/Y) = nE(1/Y).$$

$Y \sim \mathcal{X}^2(n)$, then the pdf of Y is given by

$$g(y) = \frac{(1/2)^{n/2}}{\Gamma(n/2)} \exp(-y/2) y^{n/2-1}, \quad 0 < y < \infty$$

Then,

$$\begin{aligned} E(1/Y) &= \int_0^\infty \frac{1}{y} g(y)\,dy = \frac{(1/2)^{n/2}}{\Gamma(n/2)} \int_0^\infty \exp(-y/2) y^{n/2-2}\,dy \\ &= \frac{(1/2)^{n/2}}{\Gamma(n/2)} 2^{n/2-1} \int_0^\infty e^{-x} x^{n/2-2}\,dx \quad (x = y/2) \\ &= \frac{1}{2\Gamma(n/2)} \Gamma(n/2-1) \\ &= \frac{1}{n-2} \quad (n > 2) \end{aligned}$$

Therefore,

$$Var(T) = nE(1/Y) = \frac{n}{n-2}, \quad n > 2.$$

In general, there is no expectation and variance if $n = 1$; and $E(T) = 0$ if $n > 1$; $Var(T) = \frac{n}{n-2}$ if $n > 2$.

2.2 Solution

Let $F(x)$ be the cdf of Beta$(3, 2)$, then for $0 < x < 1$, there is

$$\begin{aligned} F(x) &= \int_0^x \frac{t^2(1-t)}{B(3,2)}\,dt \\ &= \frac{1}{B(3,2)} \int_0^x t^2 - t^3\,dt \\ &= \frac{1}{B(3,2)} (\frac{1}{3}x^3 - \frac{1}{4}x^4), \quad 0 < x < 1 \end{aligned}$$

Let $G_1(x)$ denote the cdf of $X_{(1)}$ and $G_n(x)$ denote the cdf of $X_{(n)}$. Then

$$\begin{aligned} G_n(x) &= \Pr(X_n \le x) = \Pr(max(X_1, \ldots, X_n) \le x) \\ &= \Pr(X_1 \le x)\Pr(X_2 \le x) \ldots \Pr(X_n \le x) \\ &= \prod_{i=1}^n F(x) = (F(x))^n, \quad 0 < x < 1 \end{aligned}$$

Similarly, we have

$$\begin{aligned} G_1(x) &= \Pr(X_{(1)} \le x) \\ &= 1 - \Pr(X_{(1)} > x) \\ &= 1 - \Pr(X_1 > x, \ldots, X_n > x) \\ &= 1 - (1 - F(x))^n \end{aligned}$$

Thus, the pdf of $X_{(n)}$ and $X_{(1)}$ are given by

$$
\begin{aligned}
g_n(x) = G_n^{'}(x) &= n[F(x)]^{n-1}f(x)\\
&= n[\frac{1}{B(3,2)}(\frac{1}{3}x^3 - \frac{1}{4}x^4)]^{n-1}\frac{x^2(1-x)}{B(3,2)}\\
&= 12nx^2(1-x)(4x^3-3x^4)^{n-1}, \quad 0<x<1\\
g_1(x) = G_1^{'}(x) &= n[1-F(x)]^{n-1}f(x)\\
&= 12nx^2(1-x)(1-4x^3+3x^4)^{n-1}, \quad 0<x<1
\end{aligned}
$$

2.3 Solution

(a)Note that the joint pdf of $X_{(1)}, \dots, X_{(n)}$ is

$$
g_{1,2,\dots,n}(x_1,\dots,x_n) = n!\exp(-\sum_{i=1}^{n} x_i), \quad 0 \le x_1 < x_2 < \cdots < x_n < \infty
$$

Since $Z_1 = nX_{(1)}, Z_2 = (n-1)\{X_{(2)} - X_{(1)}\}, \dots, Z_n = X_{(n)} - X_{(n-1)}$, it then follows that

$$
X_{(1)} = \frac{Z_1}{n}, X_{(2)} = \frac{Z_1}{n} + \frac{Z_2}{n-1}, ..., X_{(n)} = \frac{Z_1}{n} + \frac{Z_2}{n-1} + \cdots + Z_n.
$$

Note that the Jacobian of the transformation is $\frac{1}{n!}$, and $\sum_{i=1}^n z_i = \sum_{i=1}^n x_i$, thus the joint pdf of $Z_1, Z_2, \dots, Z_n$ is given by

$$
f_{Z_1,Z_2,\dots,Z_n}(z_1,z_2,\dots,z_n) = \exp(-\sum_{i=1}^{n} z_i), \quad 0 \le z_1, \dots, z_n < \infty
$$

Now we can find the pdf of Z_1 via the joint pdf of $Z_1, Z_2, \dots, Z_n$. That is

$$
f_{Z_1}(z_1) = \int f_{Z_1,Z_2,\dots,Z_n}(z_1,z_2,\dots,z_n)\,dz_2\,dz_3 \dots\, dz_n = e^{-z_1}, \quad 0 \le z_1 < \infty
$$

Similarly, we have $f_{Z_i}(z_i) = e^{-z_i}(i = 1, 2, \dots, n)$. Hence, $Z_1, Z_2, \dots, Z_n$ are independent and each Z_i has the exponential distribution. □

(b) By (a), we know that $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ are linear combinations of $Z_1, Z_2, \dots, Z_n$. Therefore, any linear function of $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ is also a linear function of $Z_1, Z_2, \dots, Z_n$. In other words, it can be expressed as linear functions of independent random variables. □

2.4 Solution

(a)The pdf of X_i is $\frac{x_i^{a_i-1}e^{-x_i}}{\Gamma(a_i)}, x_i \ge 0$. Since X_i are mutually independent, the joint pdf of $(X_1, \dots, X_n)$ is $\prod_{i=1}^n \frac{x_i^{a_i-1}e^{-x_i}}{\Gamma(a_i)}, x_i \ge 0$

Let $X = \sum_{i=1}^n X_i$, then $X_i = Y_i X$ and

$$
\frac{\partial(X_1,\dots,X_{n-1},X_n)}{\partial(Y_1,\dots,Y_{n-1},X)} = \begin{vmatrix} X & & & Y_1 \\ & X & & Y_2 \\ & & \dots & \dots \\ -X & -X & \dots \quad -X & 1-Y_1-\dots Y_{n-1} \end{vmatrix} = X^{n-1}
$$

So $f_{Y_1,\ldots,Y_{n-1},X}(y_1,\ldots,y_{n-1},x) = \left[\prod_{i=1}^n \frac{x_i^{a_i-1}e^{-x_i}}{\Gamma(a_i)}\right] \cdot x^{n-1}, x \geq 0$ and thus

$$\begin{aligned}
f_{Y_1,\ldots,Y_{n-1}}(y_1,\ldots,y_{n-1}) &= \int_0^\infty f_{Y_1,\ldots,Y_{n-1},X}(y_1,\ldots,y_{n-1},x)\,dx \\
&= \int_0^\infty [\prod_{i=1}^{n-1} \frac{y_i^{a_i-1}}{\Gamma(a_i)}] \cdot \frac{(1-\sum_{i=1}^{n-1} y_i)^{a_n-1}}{\Gamma(a_n)} x^{\sum_{i=1}^n a_n - 1} e^{-x}\,dx \\
&= [\prod_{i=1}^{n-1} \frac{y_i^{a_i-1}}{\Gamma(a_i)}] \cdot \frac{(1-\sum_{i=1}^{n-1} y_i)^{a_n-1}}{\Gamma(a_n)} \cdot \Gamma(\sum_{i=1}^n a_i) \\
&= \frac{1}{\mathrm{B}(a)} \prod_{i=1}^n y_i^{a_i-1} \qquad y_1,\ldots,y_{n-1} \geq 0, y_1+\cdots+y_{n-1} \leq 1, y_n = 1 - \sum_{i=1}^{n-1} y_i
\end{aligned}$$

where $\mathrm{B}(a) = \frac{\prod_{i=1}^n \Gamma(a_i)}{\Gamma(\sum_{i=1}^n a_i)}, \quad a = (a_1,\ldots,a_n)$.

(b) Since $\{X_i\}_{i=1}^n \overset{ind}{\sim} \Gamma(a_i, 1)$, the mgf of X_i is given by

$$M_{X_i}(t) = (\frac{1}{1-t})^{a_i}$$

we have then

$$M_{X_1+X_2+\cdots+X_n}(t) = \prod_{i=1}^n M_{X_i}(t) = (\frac{1}{1-t})^{\sum_{i=1}^n a_i}$$

Thus, $X_1+X_2+\cdots+X_n \sim \Gamma(\sum_{i=1}^n a_i, 1)$, the pdf of $X_1+X_2+\ldots X_n$ is given by

$$f_{X_1+X_2+\cdots+X_n}(x) = \frac{e^{-x}x^{\sum_{i=1}^n a_i - 1}}{\Gamma(\sum_{i=1}^n a_i)}, \quad x > 0$$

2.5 Solution

Let $Z = XY$, thus the cdf of Z is given by

$$\begin{aligned}
F_Z(z) = \Pr(Z \leq z) &= \Pr(XY \leq z) \\
&= \int_0^1 \Pr(Xy \leq z | Y = y) f_Y(y)\,dy \\
&= \int_0^1 F_X(z/y) f_Y(y)\,dy
\end{aligned}$$

Thus,

$$\begin{aligned}
f_Z(z) &= \int_0^1 f_X(z/y) f_Y(y) \frac{1}{y}\,dy \\
&= \frac{1}{\Gamma(p)B(q,p-q)} \int_0^1 e^{-z/y}(z/y)^{p-1} y^{q-1}(1-y)^{p-q-1}\frac{1}{y}\,dy \\
&= \frac{1}{\Gamma(q)\Gamma(p-q)} z^{q-1} \int_z^\infty e^{-t} t^{p-q-1} (\frac{t-z}{t})^{p-q-1}\,dt \quad (t = z/y) \\
&= \frac{z^{q-1}}{\Gamma(q)\Gamma(p-q)} \int_z^\infty e^{-t}(t-z)^{p-q-1}\,dt \quad (u = t - z) \\
&= \frac{z^{q-1}}{\Gamma(q)\Gamma(p-q)} \int_0^\infty e^{-z} e^{-u} u^{p-q-1}\,du \\
&= \frac{z^{q-1}e^{-z}}{\Gamma(q)\Gamma(p-q)} \Gamma(p-q) \\
&= \frac{z^{q-1}e^{-z}}{\Gamma(q)}
\end{aligned}$$

Thus, $XY \sim \Gamma(q,1)$.

2.6 Solution

If $ZX = \vec{0}$, then the probability is

$$\phi + (1-\phi)\prod_{i=1}^{m} e^{-\lambda_i} = \phi + (1-\phi)e^{-\sum_{i=1}^{m}\lambda_i}$$

If $ZX \neq \vec{0}$, then $Z = 1, Y_i = X_i$ are mutually independent, and thus the pmf is given by

$$p(y_1, y_2, \ldots, y_m) = (1-\phi)\prod_{i=1}^{m}\frac{e^{-\lambda_i}\lambda_i^{y_i}}{y_i!}$$

In general, the pmf is

$$p(y_1, \ldots, y_m) = \begin{cases} \phi + (1-\phi)e^{-\sum_{i=1}^{m}\lambda_i}, & y_1 = \cdots = y_m = 0 \\ (1-\phi)\prod_{i=1}^{m}\frac{e^{-\lambda_i}\lambda_i^{y_i}}{y_i!}, & \text{otherwise} \end{cases}$$

2.7 Solution

(a) Consider the mgf of X_1 and X_2. $M_{X_1}(t) = M_{X_2}(t) = \exp(\frac{\sigma^2 t^2}{2})$. Since X_1 and X_2 are independent, we have

$$M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t) = \exp(\sigma^2 t^2)$$
$$M_{X_1-X_2}(t) = M_{X_1}(t)M_{-X_2}(t) = \exp(\sigma^2 t^2)$$

Thus, $X_1 + X_2 \sim \mathcal{N}(0, 2\sigma^2)$, $X_1 - X_2 \sim \mathcal{N}(0, 2\sigma^2)$, and

$$\begin{aligned} Cov(X_1+X_2, X_1-X_2) &= E(X_1^2 - X_2^2) - E(X_1+X_2)E(X_1-X_2) \\ &= E(X_1^2) - E(X_2^2) - 0 \\ &= 0 \end{aligned}$$

Hence, $X_1 + X_2$ and $X_1 + X_2$ are unrelatted. Since they are normal $r.v.s$, $X_1 + X_2$ and $X_1 - X_2$ are independent. Let $U = (\frac{X_1-X_2}{\sqrt{2}\sigma})^2$, $V = (\frac{X_1+X_2}{\sqrt{2}\sigma})^2$, then U and V are independent, and $U \sim \mathcal{X}^2(1)$, $V \sim \mathcal{X}^2(1)$. Let $W = \frac{(X_1-X_2)^2}{(X_1+X_2)^2}$, then $W = \frac{U}{V}$, thus $W \sim \mathcal{F}(1,1)$. Therefore, the pdf of $W \sim \mathcal{F}(1,1)$ is given by

$$\begin{aligned} f(w) &= \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)} w^{1/2-1}(1+w)^{-1} \\ &= \frac{1}{\pi\sqrt{w}(1+w)}, \quad w > 0 \end{aligned}$$

(b)

$$\begin{aligned} 0.1 &= \Pr(\frac{(X_1+X_2)^2}{(X_1+X_2)^2 + (X_1-X_2)^2} > k) \\ &= \Pr(1/(1 + (\frac{X_1-X_2}{X_1+X_2})^2) > k) \\ &= \Pr((\frac{X_1-X_2}{X_1+X_2})^2 < \frac{1}{k} - 1) \end{aligned}$$

By (a), we know that $\frac{(X_1-X_2)^2}{(X_1+X_2)^2} \sim \mathcal{F}(1,1)$, And we have $\Pr(\mathcal{F}(1,1) < 0.0251) = 0.1$. Therefore,

$$\frac{1}{k} - 1 = 0.0251$$
$$k = \frac{10000}{10251}$$

2.8 Proof

Let $U = 2X$, $V = 2Y$, then

$$f_U(u) = f_X(\frac{u}{2})\frac{1}{2} = \frac{1}{2}e^{-u/2}, \quad u > 0$$
$$f_V(v) = f_Y(\frac{v}{2})\frac{1}{2} = \frac{1}{2}e^{-v/2}, \quad v > 0$$

Thus,

$$U \sim \Gamma(1, \frac{1}{2}) = \mathcal{X}^2(2), \quad V \sim \Gamma(1, \frac{1}{2}) = \mathcal{X}^2(2)$$

Therefore,

$$W = \frac{U/2}{V/2} \sim \mathcal{F}(2,2)$$

i.e.,

$$\frac{X}{Y} \sim \mathcal{F}(2,2)$$

2.9 Solution

(a) The cdf of X is given by

$$\begin{aligned} F_X(x) &= \Pr(X \le x) = \Pr(max(aW, -bW) \le x) \\ &= \Pr(aW \le x, -bW \le x) \\ &= \Pr(-x/b \le W \le x/a) \\ &= F_W(x/a) - F_W(-x/b) \end{aligned}$$

Thus, the pdf of X is given by

$$\begin{aligned} f_X(x) &= f_W(x/a)(1/a) - f_W(-x/b)(-1/b) \\ &= \big[\frac{1}{a\sigma\sqrt{2\pi}}e^{-(x/a-u)^2/(2\sigma^2)} + \frac{1}{b\sigma\sqrt{2\pi}}e^{-(x/b+u)^2/(2\sigma^2)}\big] \cdot \mathbb{1}_{(0,\infty)}(x) \end{aligned}$$

Then the cdf is

$$\begin{aligned} F_X(x) &= \Pr(-x/b \le W \le x/a) \\ &= \Pr(\frac{-x/b - \mu}{\sigma} \le \frac{W-\mu}{\sigma} \le \frac{x/a - \mu}{\sigma}) \\ &= \big[\Phi(\frac{x - a\mu}{a\sigma}) - \Phi(\frac{-x - b\mu}{b\sigma})\big] \end{aligned}$$

(b) If $x = 0$, then $0 = max(aW, -bW)$, $a > 0, b > 0$. Thus, the conditional pdf of $W|(X = x)$ is

$$\Pr(W = 0 | X = 0) = 1.$$

i.e., $W|(X=x) \sim \text{Degenerate}(0)$.

If $x>0$, then $x = max(aW, -bW)$, $W = \frac{x}{a}$ or $\frac{-x}{b}$. Note that

$$\begin{aligned}
\frac{\Pr(W=\frac{x}{a}|X=x)}{\Pr(W=\frac{-x}{b}|X=x)} &= \lim_{\epsilon\to 0^+} \frac{\Pr(\frac{x}{a}-\epsilon \le W \le \frac{x}{a}+\epsilon)}{\Pr(\frac{-x}{b}-\epsilon \le W \le \frac{-x}{b}+\epsilon)} = \frac{f_W(\frac{x}{a})}{f_W(\frac{-x}{b})} \\
&= \exp\{\frac{(x+b\mu)^2}{2b^2\sigma^2} - \frac{(x-a\mu)^2}{2a^2\sigma^2}\}
\end{aligned}$$

Let $k = \exp\{\frac{(x+b\mu)^2}{2b^2\sigma^2} - \frac{(x-a\mu)^2}{2a^2\sigma^2}\}$, since

$$\Pr(W=\frac{x}{a}|X=x) + \Pr(W=\frac{-x}{b}|X=x) = 1,$$

we have

$$\begin{aligned}
\Pr(W=\frac{x}{a}|X=x) &= \frac{k}{k+1} = \frac{\exp\{\frac{(x+b\mu)^2}{2b^2\sigma^2}\}}{\exp\{\frac{(x+b\mu)^2}{2b^2\sigma^2}\} + \exp\{\frac{(x-a\mu)^2}{2a^2\sigma^2}\}} \\
\Pr(W=\frac{-x}{b}|X=x) &= \frac{1}{k+1} = \frac{\exp\{\frac{(x-a\mu)^2}{2a^2\sigma^2}\}}{\exp\{\frac{(x+b\mu)^2}{2b^2\sigma^2}\} + \exp\{\frac{(x-a\mu)^2}{2a^2\sigma^2}\}}
\end{aligned}$$

2.10 Proof

(a) $\quad Y \overset{d}{=} X + Z$,

$$E(Y) = E(X) + E(Z), \quad Var(Y) = Var(X) + Var(Z).$$

And,

$$\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x \frac{1}{e^\lambda - 1}\frac{\lambda^x}{x!} \\
&= \frac{\lambda}{e^\lambda - 1}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} \\
&= \frac{\lambda}{e^\lambda - 1}\sum_{x-1=0}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} \\
&= \frac{\lambda}{1-e^{-\lambda}} \\
E(X^2) &= \sum_{x=1}^{\infty} x^2 \frac{1}{e^\lambda - 1}\frac{\lambda^x}{x!} \\
&= \frac{1}{e^\lambda - 1}\sum_{x=1}^{\infty}(x^2-x)\frac{\lambda^x}{x!} + \frac{1}{e^\lambda - 1}\sum_{x=1}^{\infty} x\frac{\lambda^x}{x!} \\
&= \frac{1}{e^\lambda - 1}\sum_{x-2=0}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} + E(X) \\
&= \frac{\lambda^2}{1-e^{-\lambda}} + \frac{\lambda}{1-e^{-\lambda}} \\
&= \frac{\lambda^2+\lambda}{1-e^{-\lambda}}
\end{aligned}$$

Thus,

$$E(Y) = E(X) + E(Z) = \frac{\lambda}{1-e^{-\lambda}} + \rho\lambda$$

$$\begin{aligned}
Var(Y) &= Var(X) + Var(Z) \\
&= E(X^2) - (EX)^2 + \rho\lambda \\
&= \frac{\lambda^2+\lambda}{1-e^{-\lambda}} - (\frac{\lambda}{1-e^{-\lambda}})^2 + \rho\lambda \\
&= E(Y) - e^{\lambda}(\frac{\lambda}{1-e^{\lambda}})^2.
\end{aligned}$$

□

(b) The pdf of X and Z are

$$\begin{aligned}
p_X(x) &= \frac{1}{e^{\lambda}-1}\frac{\lambda^x}{x!}, \quad x = 1, 2, \ldots \\
p_Z(z) &= e^{-\rho\lambda}\frac{(\rho\lambda)^z}{z!}, \quad z = 0, 1, \ldots
\end{aligned}$$

Thus the pmf of Y is given by

$$\begin{aligned}
\Pr(Y = y) &= \Pr(X + Z = y) \\
&= \sum_{k=1}^{y} \Pr(X = k, Z = y - k) \\
&= \sum_{k=1}^{y} \frac{1}{e^{\lambda}-1}\frac{\lambda^k}{k!}e^{-\rho\lambda}\frac{(\rho\lambda)^{y-k}}{(y-k)!} \\
&= \frac{1}{e^{\rho\lambda}(e^{\lambda}-1)}[\frac{1}{y!}\sum_{k=0}^{y}\frac{y!}{k!(y-k)!}\lambda^k(\rho\lambda)^{y-k} - \frac{(\rho\lambda)^y}{y!}] \\
&= \frac{1}{e^{\rho\lambda}(e^{\lambda}-1)}[\frac{(\lambda+\rho\lambda)^y}{y!} - \frac{(\rho\lambda)^y}{y!}] \\
&= \frac{[(1+\rho)^y - \rho^y]\lambda^y}{e^{\rho\lambda}(e^{\lambda}-1)}, \quad y = 1, 2, \ldots
\end{aligned}$$

□

2.11 Solution

(a) If $y \geq 0$, then

$$\begin{aligned}
\Pr(Y = y) &= \Pr(X_2 - X_1 = y) \\
&= \sum_{k=0}^{\infty} \Pr(X_1 = k, X_2 = y + k) \\
&= \sum_{k=0}^{\infty} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{y+k}}{(y+k)!} \\
&= e^{-(\lambda_1+\lambda_2)}\lambda_2^y\sum_{k=0}^{\infty}\frac{(\lambda_1\lambda_2)^k}{k!(y+k)!}
\end{aligned}$$

If $y < 0$, then

$$\begin{aligned}
\Pr(Y = y) &= \Pr(X_2 - X_1 = y) \\
&= \sum_{k=0}^{\infty} \Pr(X_2 = k, X_1 = k - y) \\
&= \sum_{k=0}^{\infty} e^{-\lambda_2} \frac{\lambda_2^k}{k!} e^{-\lambda_1} \frac{\lambda_1^{k-y}}{(k-y)!} \\
&= e^{-(\lambda_1+\lambda_2)} \lambda_1^{-y} \sum_{k=0}^{\infty} \frac{(\lambda_1\lambda_2)^k}{k!(k-y)!}
\end{aligned}$$

Thus,

$$\Pr(Y = y) = \begin{cases} e^{-(\lambda_1+\lambda_2)} \lambda_1^{-y} \sum_{k=0}^{\infty} \frac{(\lambda_1\lambda_2)^k}{k!(k-y)!}, & y < 0, \\ e^{-(\lambda_1+\lambda_2)} \lambda_2^{y} \sum_{k=0}^{\infty} \frac{(\lambda_1\lambda_2)^k}{k!(k+y)!}, & y \geq 0. \end{cases}$$

(b)

$$E(Y) = E(X_2 - X_1) = E(X_2) - E(X_1) = \lambda_2 - \lambda_1$$

$$Var(Y) = Var(X_2 - X_1) = Var(X_2) - Var(X_1) = \lambda_2 - \lambda_1.$$

2.12 Solution

(a)The joint pdf of X_1 and X_2 is given by

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}, \quad x_1 \geq 0, x_2 \geq 0$$

Let

$$\begin{cases} y_1 = x_1 + x_2 \\ y_2 = \frac{x_1}{x_2} \end{cases}$$

Then,

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{y_2}{y_2+1} & \frac{y_1}{(y_2+1)^2} \\ \frac{1}{y_2+1} & \frac{-y_1}{(y_2+1)^2} \end{vmatrix} = -\frac{y_1}{(y_2+1)^2}$$

Thus, the joint pdf of Y_1 and Y_2 is given by

$$\begin{aligned}
f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(\frac{y_1 y_2}{y_2+1}, \frac{y_1}{y_2+1}) \frac{y_1}{(y_2+1)^2} \\
&= \lambda^2 \exp\left(-\lambda(\frac{y_1 y_2}{y_2+1} + \frac{y_1}{y_2+1})\right) \frac{y_1}{(y_2+1)^2} \\
&= \lambda^2 e^{-\lambda y_1} \frac{y_1}{(y_2+1)^2}, \quad y_1 \geq 0, y_2 \geq 0
\end{aligned}$$

(b) The marginal distribution of Y_1 is given by

$$\begin{aligned}
f_{Y_1}(y_1) &= \int_0^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2 \\
&= \int_0^{\infty} \lambda^2 e^{-\lambda y_1} \frac{y_1}{(y_2+1)^2}\, dy_2 \\
&= \lambda^2 e^{-\lambda y_1} y_1 \int_0^{\infty} \frac{1}{(y_2+1)^2}\, dy_2 \\
&= \lambda^2 y_1 e^{-\lambda y_1}, \quad y_1 \geq 0
\end{aligned}$$

(c) The marginal distribution of Y_2 is given by

$$\begin{aligned}
f_{Y_2}(y_2) &= \int_0^\infty f_{Y_1,Y_2}(y_1, y_2)\, dy_1 \\
&= \int_0^\infty \lambda^2 e^{-\lambda y_1} \frac{y_1}{(y_2+1)^2}\, dy_1 \\
&= \lambda^2 \frac{1}{(y_2+1)^2} \int_0^\infty e^{-\lambda y_1} y_1\, dy_1 \\
&= \frac{1}{(y_2+1)^2}, \quad y_2 \geq 0
\end{aligned}$$