Mathematical Statistics
Assignment3

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3.1 Solution

The pdf of X_i is $f(x_i) = \frac{1}{\theta_2-\theta_1}\mathbb{1}_{[\theta_1,\theta_2]}(x_i)$. Then the likelihood function is given by

$$L(\theta_1,\theta_2) = \begin{cases} \frac{1}{(\theta_2-\theta_1)^n} & \theta_1 \le x_{(1)}, \theta_2 \ge x_{(n)}, \\ 0 & otherwise. \end{cases}$$

Then, $\max L(\theta) = \frac{1}{(x_{(n)}-x_{(1)})^n}$ and thus

$$\hat{\theta_1} = X_{(1)}, \quad \hat{\theta_2} = X_{(n)}.$$

3.2 Solution

(a) The likelihood function of μ_1 is $L(\mu_1) = \prod_{i=1}^{n_1} \frac{1}{\sigma_1\sqrt{2\pi}} \exp{(-\frac{(x_i-\mu_1)^2}{2\sigma_1^2})}$, so that

$$l(\mu_1) = c_1 - c_2\sum_{i=1}^{n}(x_i-\mu_1)^2,$$

where c_1, c_2 are two constants. Solving $0 = l'(\mu_1)$ yields that $\mu_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$. Therefore,

$$\hat{\mu}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i,$$

where $X_i \sim \mathcal{N}(\mu_1, \sigma_1^2)$. Similarly, we have $\hat{\mu}_2 = \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i$, where $Y_i \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Then,

$$\hat{\theta} = \hat{\mu}_1 - \hat{\mu}_2 = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i - \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i.$$

(b) The variance of $\hat{\theta}$ is $Var(\hat{\theta}) = Var(\hat{\mu}_1)+Var(\hat{\mu}_2) = \frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}$. Let $f(x) = \frac{\sigma_1^2}{x}+\frac{\sigma_2^2}{n-x}, 0 < x < n$. $f'(x) = -\frac{\sigma_1^2}{x^2}+\frac{\sigma_2^2}{(n-x)^2}$. Letting $0 = f'(x)$, we have $x = \frac{\sigma_1}{\sigma_1+\sigma_2}n$. Therefore

$$n_1 = \lfloor\frac{\sigma_1}{\sigma_1+\sigma_2}n\rfloor, \quad n_2 = n - \lfloor\frac{\sigma_1}{\sigma_1+\sigma_2}n\rfloor,$$

or

$$n_1 = \lceil\frac{\sigma_1}{\sigma_1+\sigma_2}n\rceil, \quad n_2 = n - \lceil\frac{\sigma_1}{\sigma_1+\sigma_2}n\rceil.$$

3.4 Solution

Note that

$$\begin{aligned} a &= \frac{1}{4}(\mu_1+\mu_2+\mu_3+\mu_4), \\ b &= \frac{1}{4}(\mu_1+\mu_2-\mu_3-\mu_4), \\ c &= \frac{1}{4}(\mu_1+\mu_3-\mu_2-\mu_4). \end{aligned}$$

The likelihood function is

$$L(\mu_1,\mu_2,\mu_3,\sigma^2) = \prod_{i=1}^{4}\prod_{j=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}\exp{(-\frac{(x_{ij}-\mu_i)^2}{2\sigma^2})},$$

so that

$$l(\mu_1,\mu_2,\mu_3,\sigma^2) = c + \sum_{i=1}^{4}[-\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_{ij}-\mu_i)^2],$$

where c is a constant. Solving $0 = \frac{\partial l}{\partial \mu_1}$ yields that $\mu_1 = \frac{1}{n}\sum_{j=1}^n x_{1j}$. Then

$$\hat{\mu}_1 = \frac{1}{n}\sum_{j=1}^n X_{1j}.$$

Similarly, we have

$$\hat{\mu}_2 = \frac{1}{n}\sum_{j=1}^n X_{2j}, \quad \hat{\mu}_3 = \frac{1}{n}\sum_{j=1}^n X_{3j}, \quad \hat{\mu}_4 = \frac{1}{n}\sum_{j=1}^n X_{4j}.$$

And,

$$\frac{\partial l}{\partial \sigma^2} = -\frac{2n}{\sigma^2} + \frac{1}{2}\sum_{i=1}^4\sum_{j=1}^n (x_{ij}-\mu_i)^2\frac{1}{\sigma^4} = 0.$$

Then,

$$\hat{\sigma}^2 = \frac{1}{4n}\sum_{i=1}^4\sum_{j=1}^n (X_{ij}-\hat{\mu}_i)^2.$$

Therefore,

$$\begin{aligned}
\hat{a} &= \frac{1}{4}(\hat{\mu}_1+\hat{\mu}_2+\hat{\mu}_3+\hat{\mu}_4) = \frac{1}{4n}\sum_{i=1}^4\sum_{j=1}^n X_{ij},\\
\hat{b} &= \frac{1}{4}(\hat{\mu}_1+\hat{\mu}_2-\hat{\mu}_3-\hat{\mu}_4) = \frac{1}{4n}(\sum_{j=1}^n X_{1j}+\sum_{j=1}^n X_{2j}-\sum_{j=1}^n X_{3j}-\sum_{j=1}^n X_{4j}),\\
\hat{c} &= \frac{1}{4}(\hat{\mu}_1+\hat{\mu}_3-\hat{\mu}_2-\hat{\mu}_4) = \frac{1}{4n}(\sum_{j=1}^n X_{1j}+\sum_{j=1}^n X_{3j}-\sum_{j=1}^n X_{2j}-\sum_{j=1}^n X_{4j}),\\
\hat{\sigma}^2 &= \frac{1}{4n}\sum_{i=1}^4\sum_{j=1}^n (X_{ij}-\frac{1}{n}\sum_{j=1}^n X_{ij})^2.
\end{aligned}$$

3.6 Solution

(a) The joint pdf is

$$\prod_{i=1}^n f(x_i;\theta) = \begin{cases} \exp\left(-\sum_{i=1}^n (x_i-\theta)\right) & x_i \geq \theta, \\ 0 & otherwise. \end{cases}$$

Then, $L(\theta) = \exp\{n\theta - \sum_{i=1}^n x_i\}$, $\theta \leq x_{(1)}$. $L(\theta)$ reaches its maximum iff $\theta = x_{(1)}$. Therefore, the MLE of θ is $\hat{\theta} = X_{(1)}$.

(b)

$$E(X) = \int_\theta^\infty x e^{-(x-\theta)}\,dx = \theta + 1.$$

Thus,

$$\theta + 1 = E(X) = \frac{1}{n}\sum_{i=1}^n X_i.$$

The moment estimator of θ is

$$\hat{\theta}^M = \frac{1}{n}\sum_{i=1}^n X_i - 1.$$

(c)

$$f(x_1,...,x_n,\theta) = \prod_{i=1}^{n} f(x_i;\theta) \times \pi(\theta)$$
$$= \exp\{(n-1)\theta - \sum_{i=1}^{n} x_i\} \cdot \mathbb{1}_{(0,\infty)}(\theta) \cdot \prod_{i=1}^{n} \mathbb{1}_{(\theta,\infty)}(x_i).$$

Then,

$$p(\theta|x) \propto e^{(n-1)\theta} \cdot \mathbb{1}_{(0,x_{(1)})}(\theta).$$

i.e. $p(\theta|x) = c \cdot e^{(n-1)\theta} \cdot \mathbb{1}_{(0,x_{(1)})}(\theta)$. Solving $1 = \int_{-\infty}^{\infty} c \cdot e^{(n-1)\theta} \cdot \mathbb{1}_{(0,x_{(1)})}(\theta)\, d\theta$ yields that

$$c = \frac{n-1}{e^{(n-1)x_{(1)}} - 1}.$$

Therefore,

$$E(\theta|x) = \int_0^{x_{(1)}} \theta \cdot \frac{n-1}{e^{(n-1)x_{(1)}} - 1} \cdot e^{(n-1)\theta}\, d\theta$$
$$= \frac{[(n-1)x_{(1)} - 1]e^{(n-1)x_{(1)}} + 1}{(n-1)(e^{(n-1)x_{(1)}} - 1)}$$

is the Bayesian estimate of θ, and

$$\frac{[(n-1)X_{(1)} - 1]e^{(n-1)X_{(1)}} + 1}{(n-1)(e^{(n-1)X_{(1)}} - 1)}$$

is the Bayesian estimator of θ.

3.7 Solution

(a)

$$E(X) = \theta, \quad E(t_1(X)) = E(X) = \theta, \quad E(t_2(X)) = E(\frac{1}{2}) = \frac{1}{2}.$$

Therefore, $t_1(X)$ is unbiased, $t_2(X)$ is unbiased iff $\theta = \frac{1}{2}$.

(b) The MSE of $t_1(X)$ is given by

$$E(X-\theta)^2 = Var(X) = \theta - \theta^2.$$

The MSE of $t_2(X)$ is given by

$$E(\frac{1}{2} - \theta)^2 = \theta^2 - \theta + \frac{1}{4}.$$

The difference of the MSE of $t_2(X)$ and $t_1(X)$ is

$$\phi(\theta) = \theta^2 - \theta + \frac{1}{4} - (\theta - \theta^2) = 2\theta^2 - 2\theta + \frac{1}{4}$$

Two zero points of this function are $\theta_1 = \frac{2-\sqrt{2}}{4}$, $\theta_2 = \frac{2+\sqrt{2}}{4}$. Hence,

$$\begin{cases} \text{MSE}(t_1(X)) > \text{MSE}(t_2(X)) & \text{if } \frac{2-\sqrt{2}}{4} < \theta < \frac{2+\sqrt{2}}{4}, \\ \text{MSE}(t_1(X)) \le \text{MSE}(t_2(X)) & \text{if } 0 < \theta \le \frac{2-\sqrt{2}}{4} \text{ or } \frac{2+\sqrt{2}}{4} \le \theta < 1. \end{cases}$$

3.10 Solution

(a)

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp(-\frac{(x-\mu_0)^2}{2\theta}), \quad \theta > 0.$$
$$L(\theta) = (\frac{1}{\sqrt{2\pi\theta}})^n \exp(-\frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\mu_0)^2), \quad \theta > 0.$$
$$l(\theta) = -\frac{n}{2}\log(2\pi\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\mu_0)^2, \quad \theta > 0.$$

Solving $0 = l'(\theta)$ yields that

$$\theta = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2.$$

Then,

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_0)^2.$$

(b) $X_1, ..., X_n \overset{iid}{\sim} \mathcal{N}(\mu_0, \theta)$, thus $\frac{(X_i-\mu_0)^2}{\theta} \sim \mathcal{X}^2(1)$. Let $Y_i = \frac{(X_i-\mu_0)^2}{\theta}$, then $\hat{\theta} = \frac{\theta}{n}\sum_{i=1}^{n} Y_i$, and $\sum_{i=1}^{n} Y_i \sim \mathcal{X}^2(n)$. Hence

$$E(\hat{\theta}) = \frac{\theta}{n}E(\sum_{i=1}^{n} Y_i) = \frac{\theta}{n}\cdot n = \theta,$$
$$Var(\hat{\theta}) = \frac{\theta^2}{n^2}Var(\sum_{i=1}^{n} Y_i) = \frac{\theta^2}{n^2}\cdot 2n = \frac{2\theta^2}{n}.$$

Therefore, by central limit theorem, we have

$$\frac{\hat{\theta}-\theta}{\sqrt{\frac{2\theta^2}{n}}} \xrightarrow{L} Z \text{ as } n \to \infty,$$

where $Z \sim \mathcal{N}(0, 1)$. Then

$$\sqrt{n}(\hat{\theta}-\theta) = \sqrt{2}\theta \cdot \frac{\hat{\theta}-\theta}{\sqrt{\frac{2\theta^2}{n}}} \xrightarrow{L} Z_1 \text{ as } n \to \infty,$$

where $Z_1 \sim \mathcal{N}(0, 2\theta^2)$. i.e. the asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ is $\mathcal{N}(0, 2\theta^2)$.

3.13 Solution

(a)

$$\begin{aligned} f(x_1, ..., x_n; \theta) &= \prod_{i=1}^{n} e^{-(x_i-\theta)} \cdot \mathbb{1}_{(\theta,\infty)}(x_i) \\ &= e^{n\theta - \sum_{i=1}^{n} x_i} \cdot \mathbb{1}_{(\theta,\infty)}(x_{(1)}) \\ &= e^{n\theta}\mathbb{1}_{(\theta,\infty)}(x_{(1)}). \end{aligned}$$

Therefore, $Y_1 = \min(X_1, ..., X_n)$ is sufficient for θ. The pdf of Y_1 is $f_{Y_1}(y) = nf(y)[1-F(y)]^{n-1}$. Since the cdf of X is

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} e^{-(t-\theta)} \cdot \mathbb{1}_{(\theta,\infty)}(t)\, dt \\ &= (1 - e^{\theta - x})\mathbb{1}_{(\theta,\infty)}(x). \end{aligned}$$

Then,

$$\begin{aligned} f_{Y_1}(y) &= ne^{-(y-\theta)}\mathbb{1}_{(\theta,\infty)}(y)[e^{\theta-y}\mathbb{1}_{(\theta,\infty)}(y)]^{n-1} \\ &= ne^{n(\theta-y)}\mathbb{1}_{(\theta,\infty)}(y). \end{aligned}$$

Note that

$$E(h(Y_1)) = \int_{\theta}^{\infty} h(y)ne^{n(\theta-y)}\, dy,$$

and $E(h(Y_1)) = 0$ for all $-\infty < \theta < \infty$ if and only if

$$\int_{\theta}^{\infty} h(y)e^{-ny}\, dy = 0 \quad \text{for all } -\infty < \theta < \infty.$$

Differentiating both sides of this identity with respect to θ produces

$$h(\theta)e^{-n\theta} = 0,$$

which in turn implies $h(\theta) = 0$ for all $-\infty < \theta < \infty$. Therefore, Y_1 is a complete sufficient statistic for θ. $\square$

(b) Suppose the function is $g(y)$, then

$$E(g(Y)) = \int_\theta^\infty g(y)ne^{n(\theta-y)}\,dy = ne^{n\theta}\int_\theta^\infty g(y)e^{-ny}\,dy = \theta.$$

Let $\phi(x) = g(x)e^{-nx}$, $\Phi(x) = \int \phi(x)\,dx$, then

$$\int_\theta^\infty \phi(x)\,dx = \frac{\theta}{ne^{n\theta}} = -\frac{x}{ne^{nx}}\Big|_\theta^\infty.$$

Therefore,

$$\Phi(x) = -\frac{x}{ne^{nx}}$$

and thus

$$\begin{aligned}\phi(x) &= \Phi'(x) = (x - \frac{1}{n})e^{-nx},\\ g(y) &= \frac{\phi(y)}{e^{-ny}} = y - \frac{1}{n}.\end{aligned}$$

Hence, the function of Y_1 is $g(Y_1) = Y_1 - \frac{1}{n}$.

3.17 Solution

(a)

$$E(X) = \sum_{k=1}^\infty kp(1-p)^{k-1} = p\sum_{k=1}^\infty(1-p)^{k-1}.$$

Since $\frac{x}{1-x} = \sum_{k=1}^\infty x^k$, we have

$$\frac{1}{(1-x)^2} = (\frac{x}{1-x})' = (\sum_{k=1}^\infty x^k)' = \sum_{k=1}^\infty kx^{k-1}.$$

Hence, $E(X) = p\cdot\frac{1}{[1-(1-p)]^2} = \frac{1}{p}$. Then,

$$\begin{aligned}\frac{1}{p} &= E(X) = \frac{1}{n}\sum_{i=1}^n X_i\\ \hat{p}^M &= \frac{n}{\sum_{i=1}^n X_i}.\end{aligned}$$

(b) Likelihood and log-likelihood functions :

$$\begin{aligned}L(p) &= \prod_{i=1}^n \Pr(X_i = k_i) = p^n\prod_{i=1}^n(1-p)^{k_i-1}\\ l(p) &= n\log p + \sum_{i=1}^n(k_i-1)\log(1-p)\end{aligned}$$

Solving $l'(p) = 0$ yields that $p = \frac{n}{\sum_{i=1}^n k_i}$. Thus the MLE of p is $\hat{p} = n/(\sum_{i=1}^n X_i)$.

(c)

$$f(k_1,...,k_n,p) = \prod_{i=1}^n \Pr(X_i = k_i)\times f(p)$$

where $f(p) = 1,\ p \in [0,1]$. Then

$$f(k_1, ..., k_n, p) = p^n \prod_{i=1}^{n} (1-p)^{k_i - 1}.$$

$$f(p|k_1, ..., k_n) \propto f(k_1, ..., k_n, p) = p^n \prod_{i=1}^{n} (1-p)^{k_i - 1}.$$

Then,

$$f(p|k_1, ..., k_n) = c \cdot p^n \prod_{i=1}^{n} (1-p)^{k_i - 1}.$$

Solving $1 = \int_0^1 f(p|k_1, ..., k_n)\, dp$ yields that $c = \frac{1}{B(n+1, (\sum_{i=1}^n k_i) - n + 1)}$. Therefore, the posterior distribution of p is Beta$(n+1, (\sum_{i=1}^n k_i) - n + 1)$. And thus

$$E(p|k_1, ..., k_n) = \frac{n+1}{n+1+(\sum_{i=1}^n k_i) - n + 1} = \frac{n+1}{(\sum_{i=1}^n k_i) + 2},$$

and $(n+1)/((\sum_{i=1}^n X_i) + 2)$ is the Bayesian estimator of p.

3.19 Proof

(a) $X \sim$ Poisson(θ), then

$$f(x; \theta) = e^{-\theta} \frac{\theta^x}{x!} = e^{-\theta} \cdot \frac{1}{x!} \cdot \exp[\ln \theta \cdot x], \quad x = 0, 1, ...$$

□

(b) $Y \sim$ Exponential(θ), then

$$f(y; \theta) = \theta e^{-\theta y} = \theta \cdot 1 \cdot \exp[-\theta \cdot y], \quad y \geq 0$$

□

(c)

$$\begin{aligned}
\prod_{i=1}^{n} f(x_i; \theta) &= \prod_{i=1}^{n} a(\theta) b(x_i) \exp[c(\theta) d(x_i)] \\
&= a^n(\theta) \prod_{i=1}^{n} b(x_i) \cdot \exp[c(\theta) \sum_{i=1}^{n} d(x_i)] \\
&= a^n(\theta) \cdot \exp[c(\theta) \sum_{i=1}^{n} d(x_i)] \times \prod_{i=1}^{n} b(x_i)
\end{aligned}$$

Therefore, $\sum_{i=1}^n d(X_i)$ is a sufficient statistics of θ. □

3.20 Solution

(a) $Y = X_1^2$, then

$$f_Y(y) = f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} = \frac{\sqrt{y}}{\sigma^2} e^{-\frac{y}{2\sigma^2}} \frac{1}{2\sqrt{y}} = \frac{1}{2\sigma^2} e^{-\frac{1}{2\sigma^2} y}.$$

Hence, $X_1^2 \sim$ Exponential(β) with $\beta = \frac{1}{2\sigma^2}$. □

(b) Note that

$$\begin{aligned}
\log f(x; \sigma) &= \log x - 2 \log \sigma - \frac{x^2}{2\sigma^2}, \\
\frac{d \log f(x; \sigma)}{d\sigma} &= -\frac{2}{\sigma} + \frac{x^2}{\sigma^3}, \\
\frac{d^2 \log f(x; \sigma)}{d\sigma^2} &= \frac{2}{\sigma^2} - \frac{3x^2}{\sigma^4}.
\end{aligned}$$

Then,

$$\begin{aligned}
I(\sigma) &= E\{-\frac{d^2 \log f(X;\sigma)}{d\sigma^2}\} \\
&= E(\frac{3X^2}{\sigma^4} - \frac{2}{\sigma^2}) \\
&= \frac{4}{\sigma^2},
\end{aligned}$$

so that $I_n(\sigma) = nI(\sigma) = \frac{4n}{\sigma^2}$, and the C-R lower bound of σ is $\frac{1}{I_n(\sigma)} = \frac{\sigma^2}{4n}$.
(c)
"Method 1" Let $\theta = \sigma^2$, then

$$\begin{aligned}
\log f(x;\theta) &= \log x - \log\theta - \frac{x^2}{2\theta}, \\
\frac{d \log f(x;\theta)}{d\theta} &= -\frac{1}{\theta} + \frac{x^2}{2\theta^2}, \\
\frac{d^2 \log f(x;\theta)}{d\theta^2} &= \frac{1}{\theta^2} - \frac{x^2}{\theta^3}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I(\theta) &= E\{-\frac{d^2 \log f(X;\theta)}{d\theta^2}\} \\
&= E(\frac{X^2}{\theta^3} - \frac{1}{\theta^2}) \\
&= \frac{1}{\theta^2},
\end{aligned}$$

so that $I_n(\theta) = nI(\theta) = \frac{n}{\theta^2}$, and the C-R lower bound of σ^2 is $\frac{1}{I_n(\sigma^2)} = \frac{\sigma^4}{n}$.
"Method 2" Let $g(\sigma)$ be a function of σ, then we have

$$\frac{d \log f(x;\sigma)}{d\sigma} = \frac{d \log f(x;\sigma)}{dg(\sigma)} \cdot \frac{dg(\sigma)}{d\sigma}.$$

Then

$$\begin{aligned}
E(\frac{d \log f(x;\sigma)}{d\sigma})^2 &= E(\frac{d \log f(x;\sigma)}{dg(\sigma)} \cdot \frac{dg(\sigma)}{d\sigma})^2 \\
&= [g'(\sigma)]^2 E(\frac{d \log f(x;\sigma)}{dg(\sigma)})^2.
\end{aligned}$$

i.e.

$$I(\sigma) = [g'(\sigma)]^2 I(g(\sigma)).$$

Therefore, $I_n(\sigma) = [g'(\sigma)]^2 I_n(g(\sigma))$, and hence

$$\frac{1}{I_n(g(\sigma))} = [g'(\sigma)]^2 \frac{1}{I_n(\sigma)}.$$

Particularly, let $g(\sigma) = 2\sigma$, we obtain

$$\frac{1}{I_n(\sigma^2)} = (2\sigma)^2 \cdot \frac{1}{I_n(\sigma)} = 4\sigma^2 \cdot \frac{\sigma^2}{4n} = \frac{\sigma^4}{n}.$$