Statistical Linear Models

Assignment 2

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1.

Let

$$\mathbf{X} = \begin{pmatrix} -\mathbf{B}^{-1} & \mathbf{Q} \\ \mathbf{P} & \mathbf{A} \end{pmatrix} = \begin{pmatrix} -\mathbf{B}^{-1} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{BQ} \\ \mathbf{P} & \mathbf{A} \end{pmatrix} \hat{=} \mathbf{CD}$$

Then,

$$\det(\mathbf{X}) = \det(\mathbf{C}) \det(\mathbf{D}) = \det(-\mathbf{B}^{-1}) \cdot \det(\mathbf{A} + \mathbf{PBQ}) \neq 0,$$

since $\mathbf{B}^{-1}$ and $\mathbf{A} + \mathbf{PBQ}$ are non-singular. Then $\mathbf{X}$ is non-singular. Note that $\mathbf{X}$ is also a block matrix. Using the formula learned in class, we have

$$\begin{aligned}
(\mathbf{A} - \mathbf{P}(-\mathbf{B}^{-1})^{-1}\mathbf{Q})^{-1} &= \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{P}(-\mathbf{B}^{-1} - \mathbf{Q}\mathbf{A}^{-1}\mathbf{P})^{-1}\mathbf{Q}\mathbf{A}^{-1} \\
&= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{P}(\mathbf{B}^{-1} + \mathbf{Q}\mathbf{A}^{-1}\mathbf{P})^{-1}\mathbf{Q}\mathbf{A}^{-1} \\
&= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{P}\mathbf{B}\mathbf{B}^{-1}(\mathbf{B}^{-1} + \mathbf{Q}\mathbf{A}^{-1}\mathbf{P})^{-1}\mathbf{B}^{-1}\mathbf{B}\mathbf{Q}\mathbf{A}^{-1} \\
&= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{P}\mathbf{B}\Big[\mathbf{B}(\mathbf{B}^{-1} + \mathbf{Q}\mathbf{A}^{-1}\mathbf{P})\mathbf{B}\Big]^{-1}\mathbf{B}\mathbf{Q}\mathbf{A}^{-1} \\
&= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{P}\mathbf{B}(\mathbf{B} + \mathbf{B}\mathbf{Q}\mathbf{A}^{-1}\mathbf{P}\mathbf{B})^{-1}\mathbf{B}\mathbf{Q}\mathbf{A}^{-1}
\end{aligned}$$

That is,

$$(\mathbf{A} + \mathbf{PBQ})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{P}\mathbf{B}(\mathbf{B} + \mathbf{B}\mathbf{Q}\mathbf{A}^{-1}\mathbf{P}\mathbf{B})^{-1}\mathbf{B}\mathbf{Q}\mathbf{A}^{-1}.$$

$\square$

2.

(a)

Let

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

where $\mathbf{A}_{11} = 4$. Then, a generalized inverse for $\mathbf{A}$ is given by

$$\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Obviously, it is symmetric.

(b)

Similarly, let

$$\boldsymbol{A} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix},$$

where $\mathbf{B}_{12} = 2$. Then, a generalized inverse for $\mathbf{A}$ is given by

$$\begin{pmatrix} \mathbf{0} & \mathbf{B}_{21}^{-1} \\ 0 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Obviously, it is nonsymmetric. $\square$

3.

Lemma If $\mathbf{X}'\mathbf{X}\mathbf{A} = \mathbf{X}'\mathbf{X}\mathbf{B}$, then $\mathbf{X}\mathbf{A} = \mathbf{X}\mathbf{B}$.
Proof of lemma

$$\begin{aligned} \mathbf{X}'\mathbf{X}\mathbf{A} = \mathbf{X}'\mathbf{X}\mathbf{B} &\Rightarrow \mathbf{X}'\mathbf{X}\mathbf{A} - \mathbf{X}'\mathbf{X}\mathbf{B} = 0 \\ &\Rightarrow \mathbf{X}'\mathbf{X}(\mathbf{A} - \mathbf{B}) = 0 \\ &\Rightarrow (\mathbf{A} - \mathbf{B})'\mathbf{X}'\mathbf{X}(\mathbf{A} - \mathbf{B}) = 0 \\ &\Rightarrow [\mathbf{X}(\mathbf{A} - \mathbf{B})]'[\mathbf{X}(\mathbf{A} - \mathbf{B})] = 0 \\ &\Rightarrow \mathbf{X}(\mathbf{A} - \mathbf{B}) = 0 \\ &\Rightarrow \mathbf{X}\mathbf{A} = \mathbf{X}\mathbf{B}. \end{aligned}$$

For $(\mathbf{X}'\mathbf{X})^{-}$ is a generalized inverse for $\mathbf{X}'\mathbf{X}$, there is

$$\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X} = \mathbf{X}\mathbf{X}' = \mathbf{X}'\mathbf{X}\mathbf{I}.$$

By the lemma, we have

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X} = \mathbf{X}\mathbf{I} = \mathbf{X},$$

which implies that $(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$ is a generalized inverse of $\mathbf{X}$. $\square$