Computational Statistics

Assignment 1

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1.5

(a)

The cdf of X is

$$F(x) = \int_{-\infty}^{x} \frac{e^{-\frac{t-\mu}{\sigma}}}{\sigma(1+e^{-\frac{t-\mu}{\sigma}})}\,dt = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{e^{-y}}{(1+e^{-y})^2}\,dy \quad (y = \frac{t-\mu}{\sigma})$$
$$= -\int_{\infty}^{e^{-\frac{x-\mu}{\sigma}}} \frac{1}{(1+z)^2}\,dz \quad (z = e^{-y})$$
$$= \frac{-1}{1+z}\Big|_{e^{-\frac{x-\mu}{\sigma}}}^{\infty} = \frac{1}{1+e^{-\frac{x-\mu}{\sigma}}}.$$

Let $u = F(x)$, then

$$x = \mu - \sigma \ln \frac{1-u}{u}.$$

The algorithm is as follows:
Step 1: Draw $U = u \sim U(0,1)$;
Step 2: Return $x = \mu - \sigma \ln \frac{1-u}{u}$.

(b)

The cdf of X is

$$F(x) = \begin{cases} 0, & x \le 0 \\ \int_0^x \sigma^{-2} t e^{-\frac{t^2}{2\sigma^2}}\,dt, & x > 0 \end{cases} = \begin{cases} 0, & x \le 0 \\ \int_0^{\frac{x^2}{2\sigma^2}} e^{-y}\,dy, & x > 0 \end{cases} = \begin{cases} 0, & x \le 0 \\ 1 - e^{-\frac{x^2}{2\sigma^2}}, & x > 0 \end{cases}.$$

Let $u = F(x)$, then

$$x = \sqrt{-2\sigma^2 \ln(1-u)}.$$

The algorithm is as follows:
Step 1: Draw $U = u \sim U(0,1)$;
Step 2: Return $x = \sqrt{-2\sigma^2 \ln(1-u)}$.

(c)

The cdf of X is

$$F(x) = \int_0^x \frac{2}{a}(1 - \frac{t}{a})\,dt = \int_0^x \frac{2}{a} - \frac{2}{a^2} t\,dt = \frac{2x}{a} - \frac{x^2}{a^2}, \quad 0 \le x \le a.$$

Let $u = F(x)$, then

$$x = a - a\sqrt{1-u}.$$

The algorithm is as follows:
Step 1: Draw $U = u \sim U(0,1)$;
Step 2: Return $x = a - a\sqrt{1-u}$.

(d)

The cdf of X is

$$F(x) = \int_b^x \frac{ab^a}{t^{a+1}}\,dt = -b^a t^{-a}\Big|_b^x = 1 - (\frac{b}{x})^a, \quad x \geq b > 0.$$

Let $u = F(x)$, then

$$x = \frac{b}{(1-u)^{\frac{1}{a}}}.$$

The algorithm is as follows:
Step 1: Draw $U = u \sim U[0,1)$;
Step 2: Return $x = \frac{b}{(1-u)^{\frac{1}{a}}}$.

(e)

The cdf of X is

$$\begin{aligned} F(x) &= \int_{-\infty}^x \frac{1}{\sigma} e^{\frac{t-\mu}{\sigma}} \exp(-e^{\frac{t-\mu}{\sigma}})\,dt \\ &= \int_0^{e^{\frac{x-\mu}{\sigma}}} e^{-y}\,dy \quad (y = e^{\frac{t-\mu}{\sigma}}) \\ &= 1 - e^{-e^{\frac{x-\mu}{\sigma}}}. \end{aligned}$$

Let $u = F(x)$, then

$$x = \mu + \sigma \ln\ln \frac{1}{1-u}.$$

The algorithm is as follows:
Step 1: Draw $U = u \sim U(0,1)$;
Step 2: Return $x = \mu + \sigma \ln\ln \frac{1}{1-u}$.

(f)

The cdf of X is

$$\begin{aligned} F(x) &= \int_{-\infty}^x \frac{1}{\sigma} e^{-\frac{t-\mu}{\sigma}} \exp(-e^{-\frac{t-\mu}{\sigma}})\,dt \\ &= -\int_{\infty}^{e^{-\frac{x-\mu}{\sigma}}} e^{-y}\,dy \quad (y = e^{-\frac{t-\mu}{\sigma}}) \\ &= e^{-e^{-\frac{x-\mu}{\sigma}}}. \end{aligned}$$

Let $u = F(x)$, then

$$x = \mu - \sigma \ln\ln \frac{1}{u}.$$

The algorithm is as follows:
Step 1: Draw $U = u \sim U(0,1)$;
Step 2: Return $x = \mu - \sigma \ln\ln \frac{1}{u}$.

(g)

The cdf of $X_{(n)}$ is

$$F_n(x) = \Pr(X_{(n)} \le x) = \Pr(X_1 \le x, X_2 \le x, ..., X_n \le x)$$
$$= \prod_{i=1}^{n} \Pr(X_i \le x)$$
$$= \prod_{i=1}^{n} F(x) = (F(x))^n.$$

The cdf of $X_{(1)}$ is

$$F_1(x) = \Pr(X_{(1)} \le x) = 1 - \Pr(X_{(1)} > x)$$
$$= 1 - \prod_{i=1}^{n} \Pr(X_i > x)$$
$$= 1 - (1 - F(x))^n.$$

Let $u = F_n(x)$, then

$$x = F^{-1}(u^{\frac{1}{n}}).$$

The algorithm is as follows:
Step 1: Draw $U = u \sim U(0,1)$;
Step 2: Return $x_{(n)} = F^{-1}(u^{\frac{1}{n}})$.
Let $u = F_1(x)$, then

$$x = F^{-1}(1 - (1-u)^{\frac{1}{n}}).$$

The algorithm is as follows:
Step 1: Draw $U = u \sim U(0,1)$;
Step 2: Return $x_{(1)} = F^{-1}(1 - (1-u)^{\frac{1}{n}})$. □

1.7

The ratio is

$$w(x) = \frac{\phi(x)}{g(x)} = \frac{\theta_0}{\sqrt{2\pi}}(1 + e^{-\frac{x}{\theta_0}})^2 e^{\frac{x}{\theta_0} - \frac{x^2}{2}}.$$

The algorithm is as follows:
Step 1: Generate $X^{(1)}, ..., X^{(J)} \overset{iid}{\sim} g(x)$;
Step 2: Select a subset $\{X^{(k_i)}\}_{i=1}^{I}$ from $\{X^{(j)}\}_{j=1}^{J}$ via re-sampling without replacement from the discrete distribution on $\{X^{(j)}\}_{j=1}^{J}$ with probabilities $\{w_j\}_{j=1}^{J}$, where $w_j = \frac{w(X^{(j)})}{\sum_{j'=1}^{J} w(X^{(j')})}$. □

1.10

$$f(x_1, x_2, ..., x_m) = \int f(x)\, dx_{m+1} \dots dx_d = \int_{\mathbb{R}^{d-m}} \frac{\Gamma(b)}{\pi^b(1 + \sum_{i=1}^{m} x_i^2 + \sum_{j=m+1}^{d} x_j^2)^b}\, dx_{m+1} \dots dx_d$$
$$= \frac{\pi^{\frac{d-m}{2}}}{\Gamma(\frac{d-m}{2})} \int_0^{\infty} y^{\frac{d-m}{2}-1} \frac{\Gamma(b)}{\pi^b(1 + \sum_{i=1}^{m} x_i^2 + y)^b}\, dy$$
$$= \frac{\Gamma(b) \cdot \pi^{\frac{d-m}{2}}}{\Gamma(\frac{d-m}{2}) \cdot \pi^b} \int_0^{\infty} y^{\frac{d-m}{2}-1} \frac{1}{(1 + \sum_{i=1}^{m} x_i^2 + y)^b}\, dy.$$

Hence,

$$f(x_n|x_1,...,x_{n-1}) = \frac{f(x_1,...,x_n)}{f(x_1,...,x_{n-1})} = \frac{\Gamma(\frac{d-n+1}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{d-n}{2})} \cdot \frac{\int_0^\infty y^{\frac{d-n}{2}-1}\frac{1}{(1+\sum_{i=1}^n x_i^2+y)^b}\,dy}{\int_0^\infty y^{\frac{d-n+1}{2}-1}\frac{1}{(1+\sum_{i=1}^{n-1} x_i^2+y)^b}\,dy}.$$

Claim:

$$\int_0^\infty \frac{x^a}{(m+x)^b}\,dx = \frac{B(b-a-1,a+1)}{m^{b-a-1}} \quad (m>0) \tag{1}$$

Proof of (1):

$$\begin{aligned}\int_0^\infty \frac{x^a}{(m+x)^b}\,dx &= \int_0^\infty (\frac{x}{m+x})^a \cdot (\frac{m}{m+x})^{b-a} \cdot \frac{1}{m^{b-a}}\,dx \\ &= \frac{1}{m^{b-a}}\int_1^0 (1-t)^a t^{b-a}\frac{-m}{t^2}\,dt \quad (t=\frac{m}{m+x}) \\ &= \frac{1}{m^{b-a-1}}\int_0^1 t^{b-a-2}(1-t)^a\,dt \\ &= \frac{B(b-a-1,a+1)}{m^{b-a-1}}.\end{aligned}$$

By (1), we have

$$\int_0^\infty y^{\frac{d-n}{2}-1}\frac{1}{(1+\sum_{i=1}^n x_i^2+y)^b}\,dy = \frac{B(b-\frac{d-n}{2},\frac{d-n}{2})}{(1+\sum_{i=1}^n x_i^2)^{b-\frac{d-n}{2}}} = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{d-n}{2})}{(1+\sum_{i=1}^n x_i^2)^{\frac{n+1}{2}}\Gamma(\frac{d+1}{2})}.$$

Similarly,

$$\int_0^\infty y^{\frac{d-n+1}{2}-1}\frac{1}{(1+\sum_{i=1}^{n-1} x_i^2+y)^b}\,dy = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{d-n+1}{2})}{(1+\sum_{i=1}^{n-1} x_i^2)^{\frac{n}{2}}\Gamma(\frac{d+1}{2})}.$$

Therefore,

$$f(x_n|x_1,...,x_{n-1}) = \frac{\Gamma(\frac{d-n+1}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{d-n}{2})} \cdot \frac{\int_0^\infty y^{\frac{d-n}{2}-1}\frac{1}{(1+\sum_{i=1}^n x_i^2+y)^b}\,dy}{\int_0^\infty y^{\frac{d-n+1}{2}-1}\frac{1}{(1+\sum_{i=1}^{n-1} x_i^2+y)^b}\,dy} = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})}\cdot\frac{(1+\sum_{i=1}^{n-1}x_i^2)^{\frac{n}{2}}}{(1+\sum_{i=1}^{n}x_i^2)^{\frac{n+1}{2}}} \tag{2}$$

Let $m=1$ for $f(x_1,...,x_m)$, we have

$$f_1(x_1) = \frac{1}{\pi(1+x_1)^2}, \quad -\infty < x_1 < \infty,$$

which implies that $X_1 \sim$Cauchy(1). Let $y_n = x_n\sqrt{\frac{n}{1+\sum_{i=1}^{n-1}x_i^2}}$, then from (2) we have

$$f(y_n|x_1,...,x_{n-1}) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}\cdot\left(1+\frac{y_n^2}{n}\right)^{-\frac{n+1}{2}},$$

which implies that

$$X_n\sqrt{\frac{n}{1+\sum_{i=1}^{n-1}X_i^2}}\Bigg|(x_1,...,x_{n-1}) \stackrel{d}{=} T_n,$$

or

$$X_n|(X_1,...X_{n-1}) \stackrel{d}{=} T_n\sqrt{\frac{1+\sum_{i=1}^{n-1}X_i^2}{n}},$$

where $T_n \sim t(n)$.
Therefore, the algorithm is as follows:
Step 1: Draw X_1 from Cauchy(1);
Step 2: Draw $T_2 \sim t(2)$ and set $X_2 = T_2\sqrt{\frac{1+\sum_{i=1}^{1}X_i^2}{2}}$;
⋮
Step d: Draw $T_d \sim t(d)$ and set $X_d = T_d\sqrt{\frac{1+\sum_{i=1}^{d-1}X_i^2}{d}}$. □

1.11

Let $X = \sum_{i=1}^d Z_i$, then we have the following transformation:

$$
\begin{aligned}
z_i &= y_i x, \quad i = 1, ..., d-1, \\
z_d &= y_d x = (1 - \sum_{i=1}^{d-1} y_i)x.
\end{aligned}
$$

The Jacobian determinant is

$$
\begin{aligned}
J(z \to y_{-d}, x) &= \frac{\partial(z_1, ..., z_{d-1}, z_d)}{\partial(y_1, ..., y_{d-1}, x)} \\
&= \begin{vmatrix} x & 0 & ... & 0 & y_1 \\ 0 & x & ... & 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & x & y_{d-1} \\ -x & -x & ... & -x & y_d \end{vmatrix} \\
&= \begin{vmatrix} x & & & & \\ & x & & & \\ & & \ddots & & \\ & & & x & \\ & & & & \sum_{i=1}^d y_i \end{vmatrix} \\
&= x^{d-1}.
\end{aligned}
$$

Therefore, we obtain

$$
f(y_1, ..., y_{d-1}, x) = \prod_{i=1}^{d} f_{Z_i}(z_i) \cdot |x^{d-1}| = x^{d-1} e^{-x}, \quad x > 0.
$$

The joint distribution of $(y_1, ..., y_{d-1})$ is

$$
f(y_1, ..., y_{d-1}) = \int_0^\infty x^{d-1} e^{-x}\, dx = \Gamma(d) = (d-1)! \cdot I(y_{-d} \in \mathbb{V}_{d-1}),
$$

where

$$
\mathbb{V}_{d-1} = \{(y_1, ..., y_{d-1}) : y_i > 0, i = 1, ..., d-1, \sum_{i=1}^{d-1} y_i \le 1\}.
$$

In other words, $y_{-d} \sim U(\mathbb{V}_{d-1})$ and the volume of $\mathbb{V}_{d-1}$ is $\frac{1}{(d-1)!}$. Equivalently, we can write $y \sim U(\mathbb{T}_d)$, where

$$
\mathbb{T}_d = \{(y_1, ..., y_d) : y_i > 0, i = 1, ..., d, \sum_{i=1}^{d} y_i = 1\}.
$$

□

1.12

(a)

If $y = 0$, then

$$
\Pr(ZX = y) = \Pr(ZX = 0) = \Pr(Z = 0) + \Pr(Z = 1, X = 0) = \phi + (1-\phi)e^{-\lambda}.
$$

If $y > 0$, then

$$\Pr(ZX = y) = \Pr(Z = 1, X = y) = \Pr(Z = 1) \cdot \Pr(X = y) = (1-\phi)\frac{e^{-\lambda}\lambda^y}{y!}, \quad y = 1, ..., \infty.$$

Therefore,

$$\Pr(ZX = y) = \begin{cases} \phi + (1-\phi)e^{-\lambda}, & y = 0 \\ (1-\phi)\frac{e^{-\lambda}\lambda^y}{y!}, & y > 0. \end{cases}$$

$Y \stackrel{d}{=} ZX$, the algorithm is as follows:
Step 1: Draw Z from Bernoulli$(1-\phi)$ and independently draw X from Poisson(λ);
Step 2: Return $Y = ZX$.

(b)

The pmf of Y is

$$\begin{aligned} f_Y(y) &= \int f_W(w) f_{Y|W}(y|w)\, dw \\ &= \Pr(W = 0) f_{Y|W}(y|0) + \Pr(W = 1) f_{Y|W}(y|1) \\ &= \phi \frac{e^{-0}0^y}{y!} + (1-\phi)\frac{e^{-\lambda}\lambda^y}{y!} \\ &= \phi \cdot I(y = 0) + (1-\phi)\frac{e^{-\lambda}\lambda^y}{y!} \cdot I(y \geq 0) \\ &= \left\{\phi + (1-\phi)e^{-\lambda}\right\} I(y = 0) + \left\{(1-\phi)\frac{e^{-\lambda}\lambda^y}{y!}\right\} I(y > 0). \end{aligned}$$

□

1.13

(a)

The pdf of X is given by

$$f_X(x) = (1-\phi) f_{X_2}(x) + \phi f_{X_1}(x).$$

Let

$$Y = ZX_2 + (1-Z)X_1,$$

where $Z \sim$Bernoulli$(1-\phi)$ and $Z \perp\!\!\!\perp X_1$, $Z \perp\!\!\!\perp X_2$. The cdf of Y is given by

$$\begin{aligned} \Pr(Y \leq x) &= \Pr(ZX_2 + (1-Z)X - 1 \leq x) \\ &= \sum_{z=0}^{1} \Pr(ZX_2 + (1-Z)X_1 \leq x \big| Z = z) \Pr(Z = z) \\ &= (1-\phi)\Pr(X_2 \leq x) + \phi \Pr(X_1 \leq x), \end{aligned}$$

so that the pdf of Y is

$$f_Y(x) = (1-\phi) f_{X_2}(x) + \phi f_{X_1}(x) = f_X(x).$$

Thus $Y \stackrel{d}{=} X$ is an SR of X.

(b)-1

Similarly, an SR of X is

$$X \stackrel{d}{=} Z_1 X_1 + ... + Z_n X_n,$$

where $\mathbb{z} = (Z_1, ..., Z_n) \sim$ Multinomial$(1; \phi_1, ..., \phi_n)$, and $\mathbb{z} \perp\!\!\!\perp \{X_1, ..., X_n\}$. The pdf of X is

$$f_X(x) = \phi_1 f_{X_1}(x) + \phi_2 f_{X_2}(x) + ... + \phi_n f_{X_n}(x) = \sum_{i=1}^{n} \phi_i f_{X_i}(x).$$

(b)-2

Let $Z \sim$ FDiscrete$(\{m\}, \{p_m\})$ with $\Pr(X = m) = p_m = \phi_m, m = 1, ..., n$. Let

$$Y = \left[\frac{1}{(-1)^{n-1}(n-1)!}\prod_{i=2}^{n}(Z-i)\right]X_1 + \left[\frac{1}{(-1)^{n-2}(n-2)!}(Z-1)\prod_{i=3}^{n}(Z-i)\right]X_2 + ... + \left[\frac{1}{(n-1)!}\prod_{i=1}^{n-1}(Z-i)\right]X_n$$
$$= \sum_{k=1}^{n}\left[\frac{1}{(-1)^{n-k}(k-1)!(n-k)!}\prod_{i=1, i\neq k}^{n}(Z-k)\right]X_k.$$

Then,

$$\Pr(Y = X_i) = \Pr(Z = k) = \phi_k.$$

Therefore, Y is an SR of X. The pdf of X is

$$f_X(x) = f_Y(x) = \phi_1 f_{X_1}(x) + \phi_2 f_{X_2}(x) + ... + \phi_n f_{X_n}(x) = \sum_{i=1}^{n} \phi_i f_{X_i}(x).$$

$\square$

1.14

Method 1

Let $X_1 \sim f_{X_1}(x)$,

$$f_{X_1}(x) = \frac{5}{2}(x-1)^4, \quad 0 \leq x \leq 2.$$

Let $X_2 \sim f_{X_2}(x)$,

$$f_{X_2}(x) = \frac{1}{2}, \quad 0 \leq x \leq 2.$$

Then,

$$Z \stackrel{d}{=} ZX_1 + (1-Z)X_2 \hat{=} Y,$$

where $Z \sim$Bernoulli$(\frac{1}{6})$, and $Z \perp\!\!\!\perp X_1$, $Z \perp\!\!\!\perp X_2$, since

$$\begin{aligned} f_Y(x) &= \frac{1}{6} f_{X_1}(x) + \frac{5}{6} f_{X_2}(x) \\ &= \frac{1}{6} \cdot \frac{5}{2}(x-1)^4 + \frac{5}{6} \cdot \frac{1}{2} \\ &= \frac{5}{12}\{1 + (x-1)^4\}. \end{aligned}$$

The cdf of X_1 is given by

$$F_{X_1}(x) = \int_0^x \frac{5}{2}(t-1)^4\,dt = \frac{1}{2}(x-1)^5 + \frac{1}{2}, \quad 0 \le x \le 2.$$

Let $u = F_{X_1}(x)$, then

$$x = 1 + (2u-1)^{\frac{1}{5}}.$$

Therefore, the algorithm is as follows:
Step 1: Draw $Z = z$ from Bernoulli($\frac{1}{6}$) and independently draw $U_1 = u_1, U_2 = u_2 \overset{iid}{\sim} U(0,1)$;
Step 2: Let $x_1 = (2u_1 - 1)^{\frac{1}{5}} + 1$ and $x_2 = 2u_2$;
Step 2: Return $x = zx_1 + (1-z)x_2$.

Method 2

Let $g(x) = \frac{1}{2}, \quad 0 \le x \le 2$. Then

$$\frac{f(x)}{g(x)} = \frac{5}{6}(1 + (x-1)^4), \quad 0 \le x \le 2.$$

Hence,

$$c = \underset{0\le x\le 2}{max} \frac{f(x)}{g(x)} = \frac{5}{3},$$

and

$$\frac{f(x)}{cg(x)} = \frac{1+(x-1)^4}{2}.$$

Therefore, the algorithm is as follows:
Step 1: Draw $U \sim U[0,1]$ and independently draw $Y \sim U[0,2]$;
Step 2: If $U \le \frac{1+(Y-1)^4}{2}$, set $X = Y$; Otherwise, go to Step 1.
The acceptance probability is $\frac{1}{c} = \frac{3}{5}$. $\square$

1.15

Let $W = \frac{1}{Y}$, $Z = XY$, then

$$\frac{\partial(x,y)}{\partial(w,z)} = \begin{vmatrix} z & w \\ -\frac{1}{w^2} & 0 \end{vmatrix} = \frac{1}{w}.$$

The joint pdf of (W,Z) is given by

$$f_{(W,Z)}(w,z) = f_{(X,Y)}(x,y) \cdot |\frac{1}{w}|.$$

Since $f_X(x) = n\int_1^\infty y^{-n}e^{-xy}\,dy,\ x > 0$, it then follows that

$$f_{(X,Y)}(x,y) = ny^{-n}e^{-xy}, \quad x > 0, y > 1.$$

Thus,

$$f_{(W,Z)}(w,z) = nw^n e^{-z} \cdot \frac{1}{w} = nw^{n-1} \cdot e^{-z} \hat{=} f_W(w) \cdot f_Z(z), \quad z > 0, 0 < w < 1,$$

where $f_W(w) = nw^{n-1},\ 0 < w < 1$ and $f_Z(z) = e^{-z},\ z > 0$. Thus $W \perp\!\!\!\perp Z$.
Therefore, the algorithm is as follows:
Step 1: Draw W from Beta$(n,1)$ and independently draw Z from Exponential(1);
Step 2: Return $X = WZ$. $\square$

1.17

Question: Use the method of mixture representation to generate a r.v. from the following pdf

$$f_X(x) = \int_0^\infty \frac{1}{y} e^{-\sqrt{2\pi}y - \frac{x^2}{2y^2}}\, dy, \quad -\infty < x < \infty.$$

Solution:
Since

$$f_X(x) = \int_0^\infty \frac{1}{y} e^{-\sqrt{2\pi}y - \frac{x^2}{2y^2}}\, dy, \quad -\infty < x < \infty,$$

it then follows that

$$f_{(X,Y)}(x,y) = \frac{1}{y} e^{-\sqrt{2\pi}y - \frac{x^2}{2y^2}}, \quad -\infty < x < \infty, y > 0.$$

Let $X = WZ, Y = W$, then

$$\frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w.$$

Thus,

$$\begin{aligned} f_{(W,Z)}(w,z) &= f_{(X,Y)}(x,y) \cdot |w| \\ &= \frac{1}{w} e^{-\sqrt{2\pi}w - \frac{z^2}{2}} \cdot w \\ &= e^{-\sqrt{2\pi}w} \cdot e^{-\frac{z^2}{2}}, \quad w > 0, -\infty < z < \infty. \end{aligned}$$

Hence,

$$f_{(W,Z)}(w,z) = \sqrt{2\pi} e^{-\sqrt{2\pi}w} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \hat{=} f_W(w) \cdot f_Z(z),$$

where $f_W(w) = \sqrt{2\pi} e^{-\sqrt{2\pi}w},\ w > 0$ and $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},\ -\infty < z < \infty$.
Therefore, the algorithm is as follows:
Step 1: Draw W from Exponential($\sqrt{2\pi}$) and independently draw Z from standard normal distribution;
Step 2: Return $X = WZ$. $\square$