Discrete Mathematics

Function

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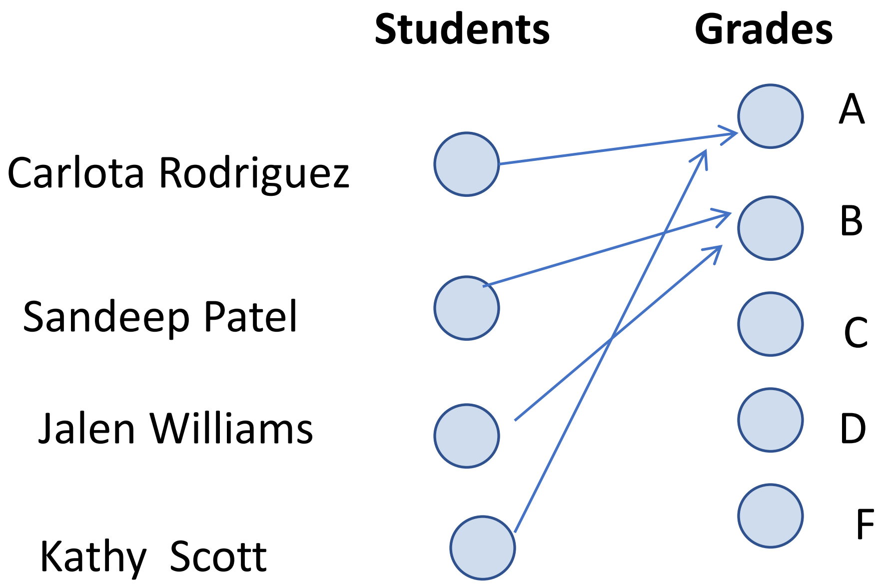
Functions

- Let A and B be nonempty sets.
- A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- Functions are sometimes called *mappings* or *transformations*.



Functions

- A function f: $A \to B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

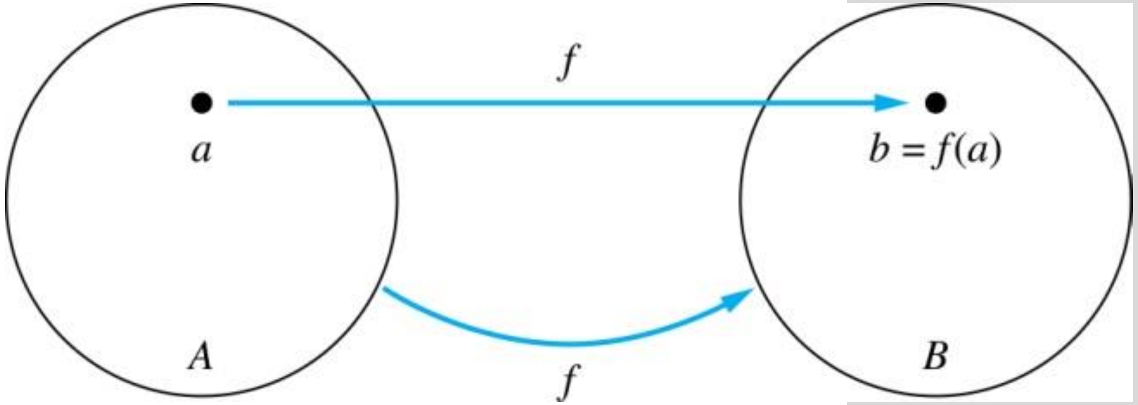
$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2)] \rightarrow y_1 = y_2]$$

Functions

Given a function $f: A \to B$:

- We say *f maps A* to *B or*

  f is a *mapping* from *A* to *B.*
- *A* is called the *domain* of *f.*
- *B* is called the *codomain* of *f.*
- If $f(a) = b$,
  - then *b* is called the *image* of *a* under *f.*
  - *a* is called the *preimage* of *b.*



Questions

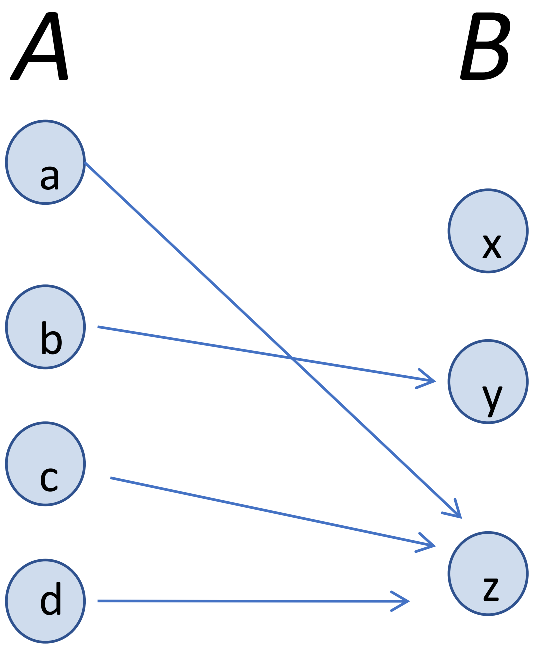
f(a) = ? z

The image of d is ? z

The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b



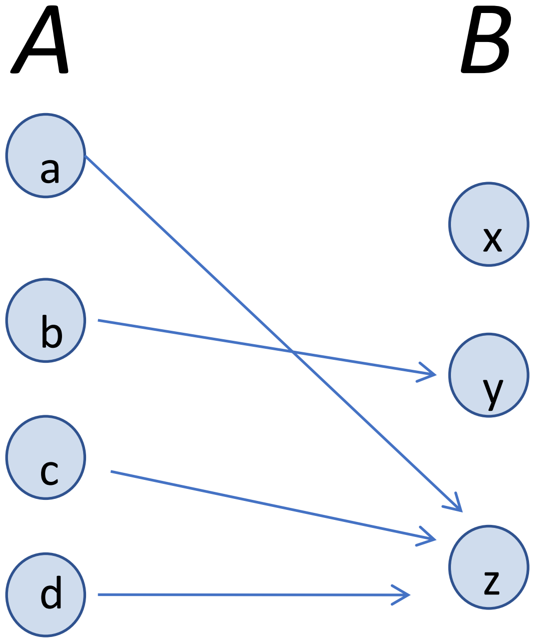
Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

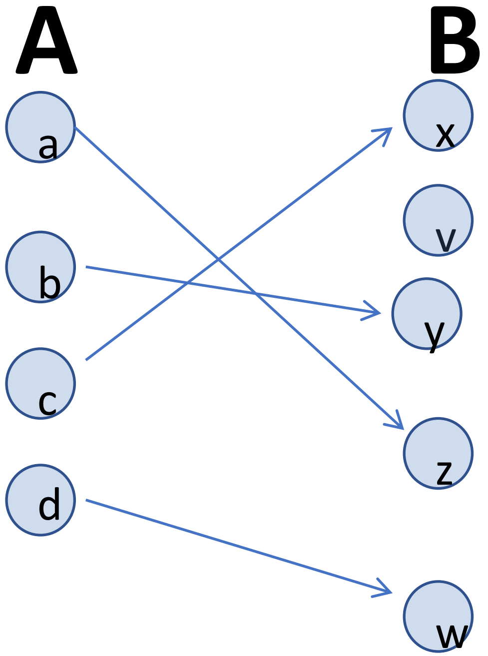
f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}



Injections

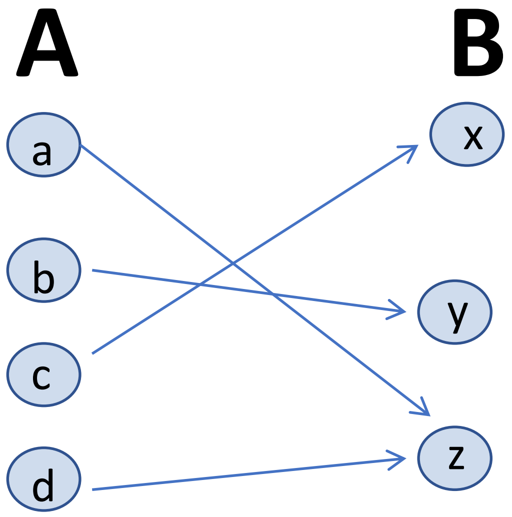
Definition: A function f is said to be *one-to-one* , or *injective*, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.



Surjections

A function $f : A \to B$ is called *onto* or *surjective* iff for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

A function f is called a *surjection* if it is **onto**.



Example

Example 1: for $f : \{a,b,c,d\} \rightarrow \{1,2,3\}$, $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

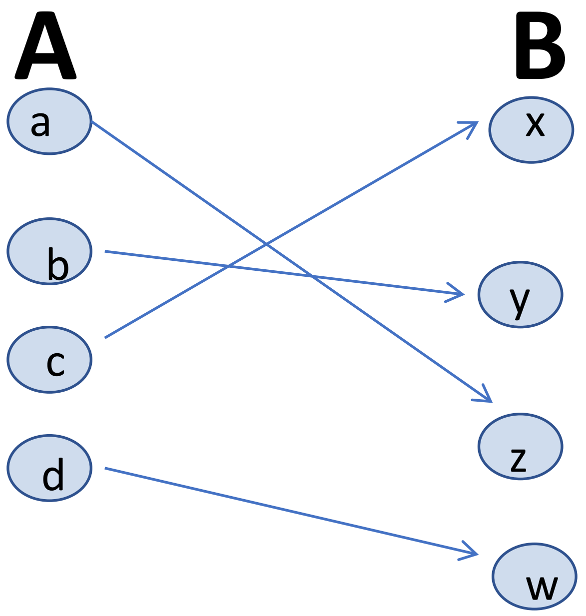
Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain.

Example 2: Is the function $f(x) = x^2$ from the set of integers onto?

Solution: No, f is not onto since there is no integer x with $x^2 = -1$, for example.

Bijections

A function f is a ***one-to-one correspondence***, or a *bijection*, if it is both one-to-one and onto (surjective and injective).

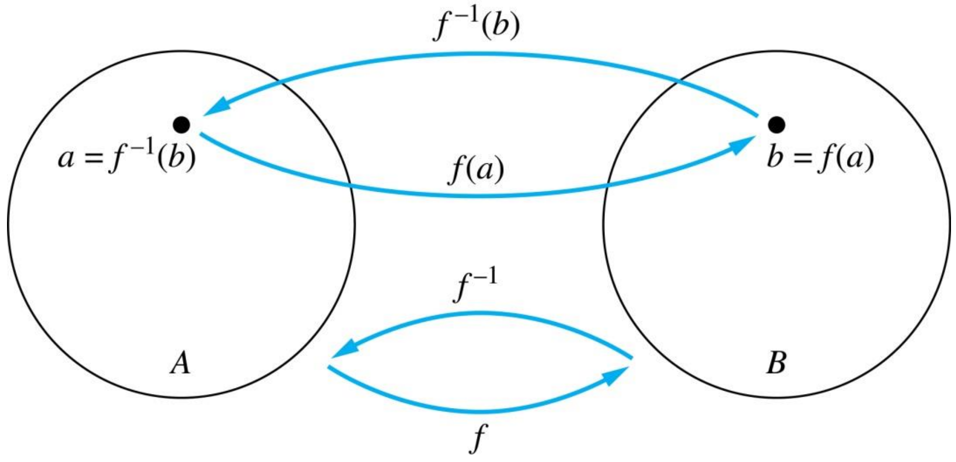


Inverse Functions

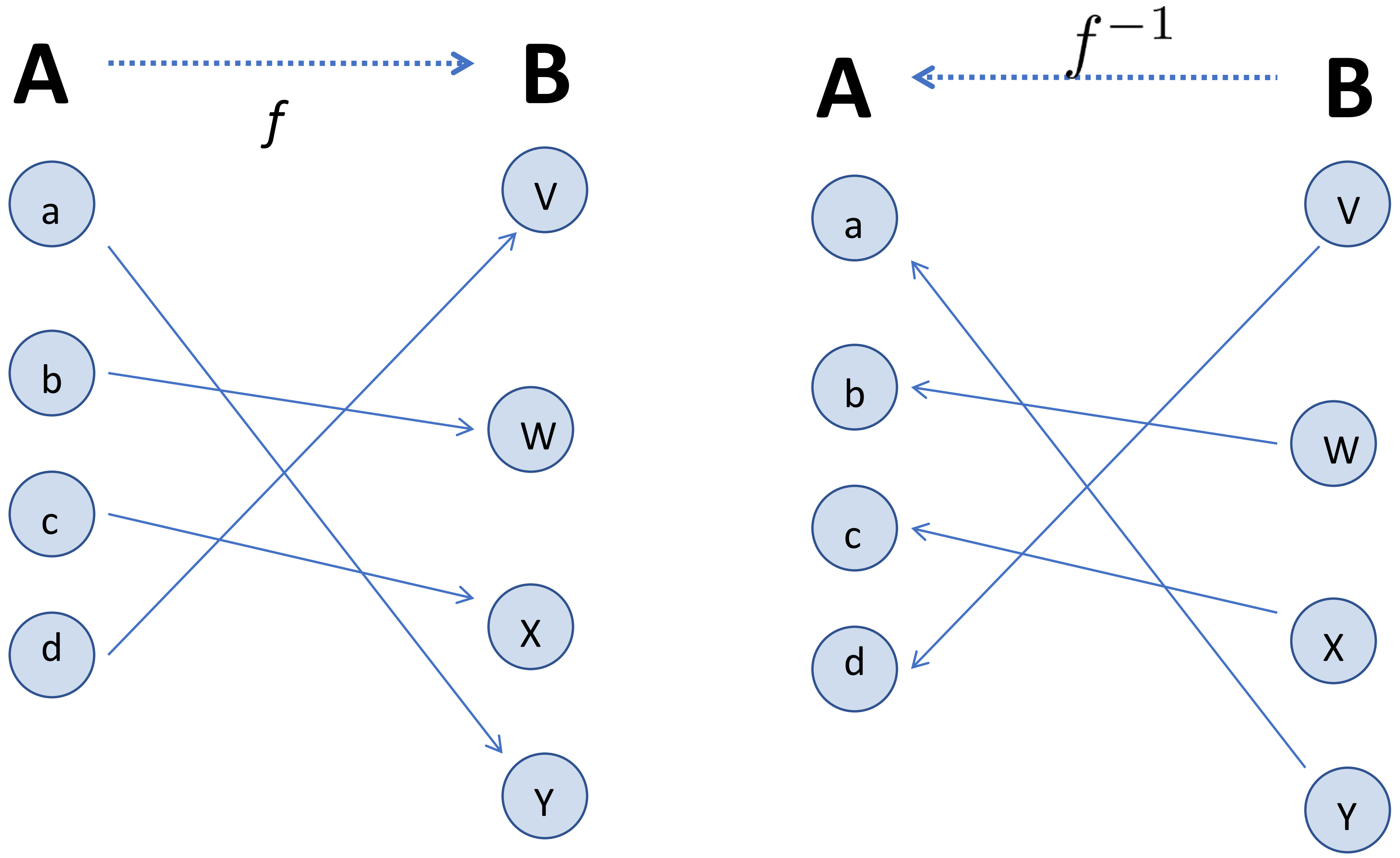
Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted, is the function from *B* to *A* defined as

No inverse exists unless *f* is a bijection. Why?

$$f^{-1}(y) = x \text{ iff } f(x) = y$$



Inverse Functions



Questions

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

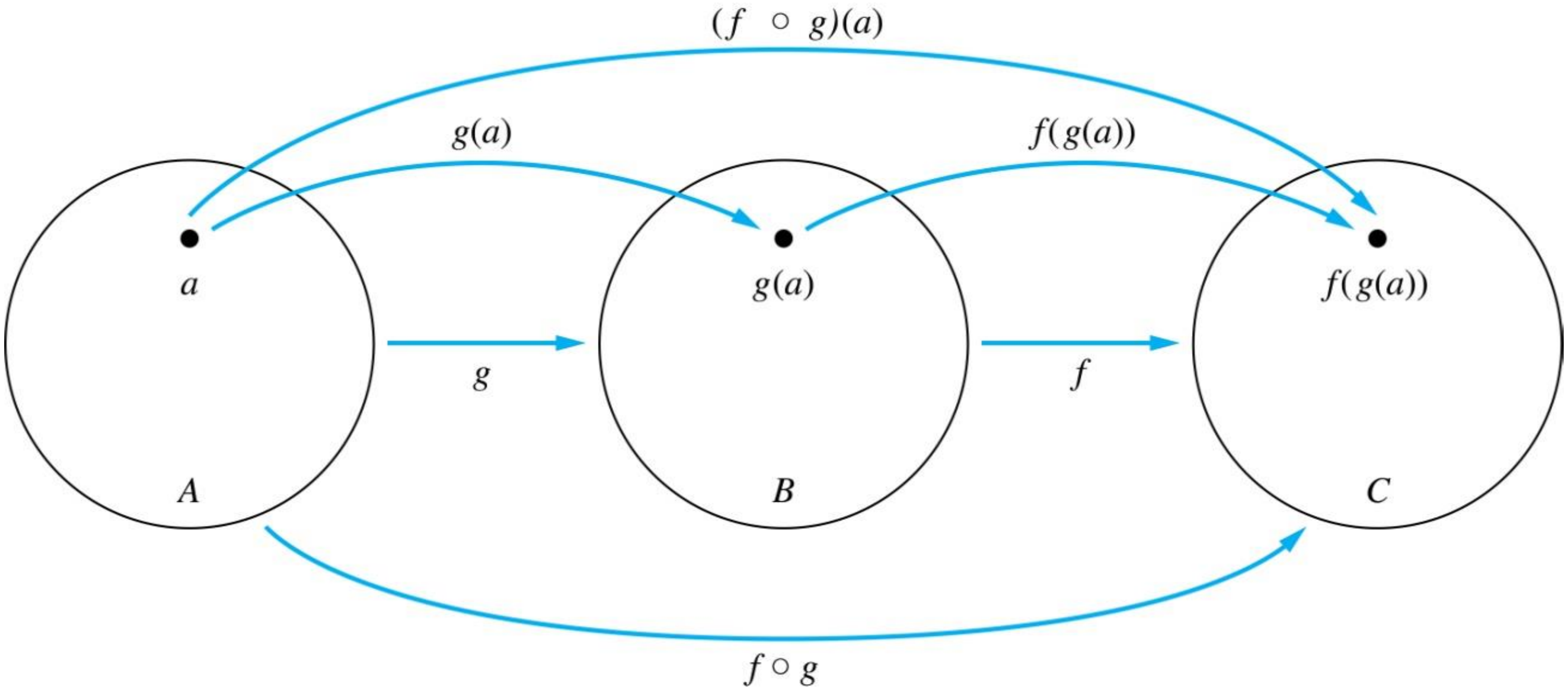
Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y – 1$.

Example 3: Let f: **R** → **R** be such that $f(x) = x^2$.
Is f invertible, and if so, what is its inverse?

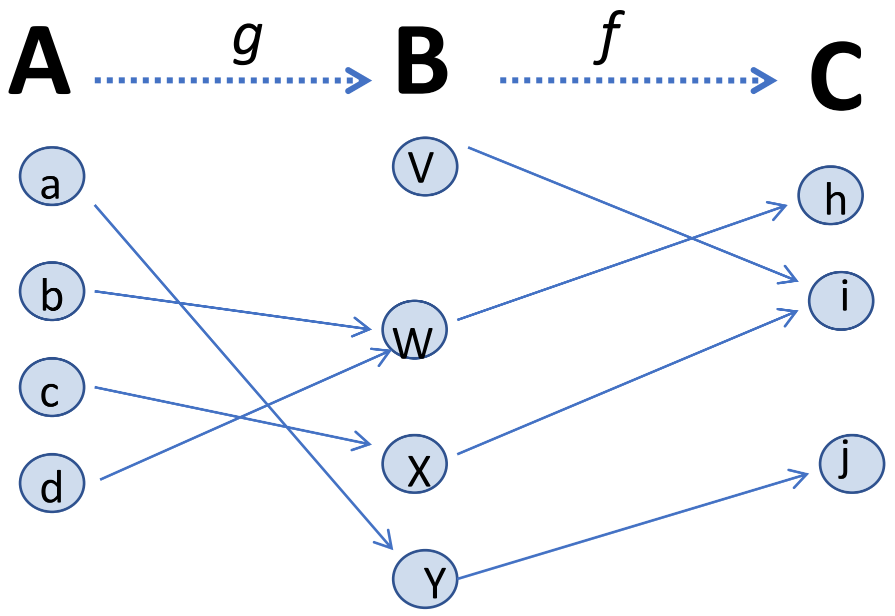
Solution: The function f is not invertible because it is not one-to-one.

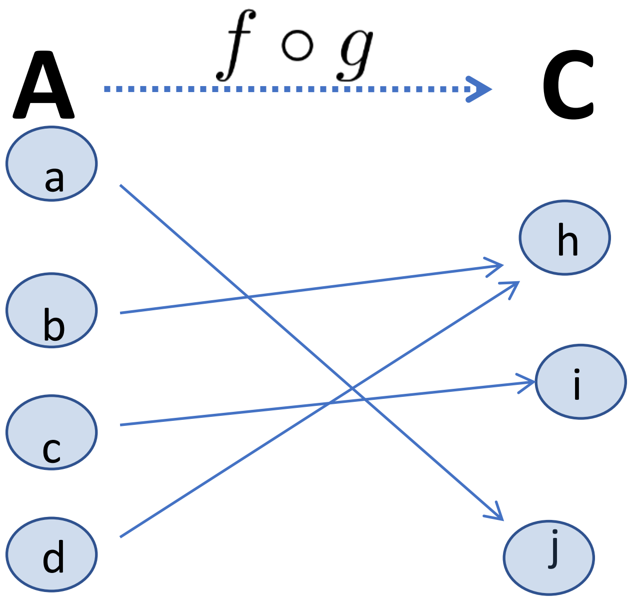
Composition

- **Definition**: Let *f*: *B* → *C*, *g*: *A* → *B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by $f \circ g(x) = f(g(x))$



Composition





Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

and $f(g(x)) = (2x+1)^2$

$g(f(x)) = 2x^2 + 1$

Composition Questions

Example 2: Let g be a function from $\{a,b,c\}$ to itself s.t.
$g(a) = b$, $g(b) = c$, and $g(c) = a$.

Let f be a function from $\{a,b,c\}$ to $\{1,2,3\}$ s.t.
$f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

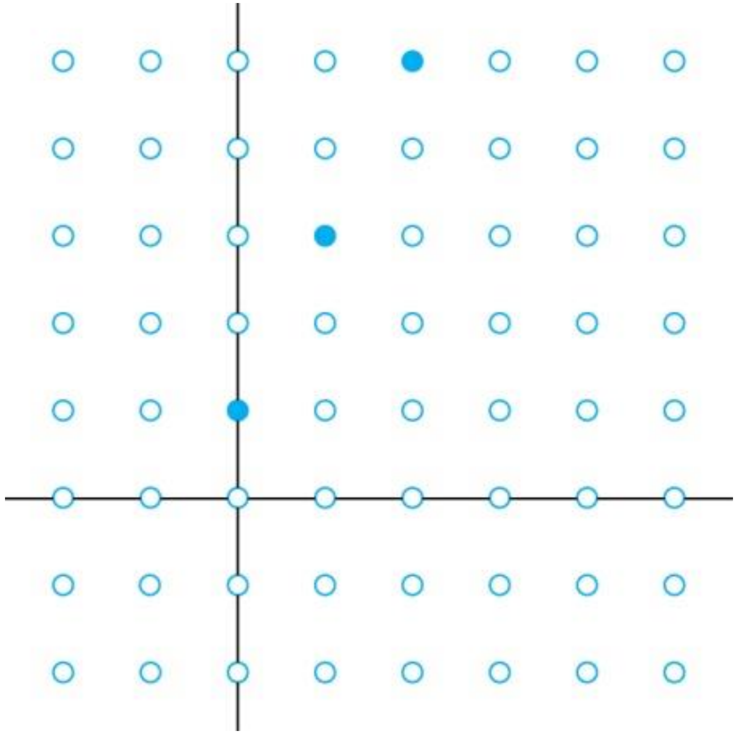
Solution: The composition $f \circ g$ is defined by

$$f \circ g\ (a) = f(g(a)) = f(b) = 2.$$
$$f \circ g\ (b) = f(g(b)) = f(c) = 1.$$
$$f \circ g\ (c) = f(g(c)) = f(a) = 3.$$

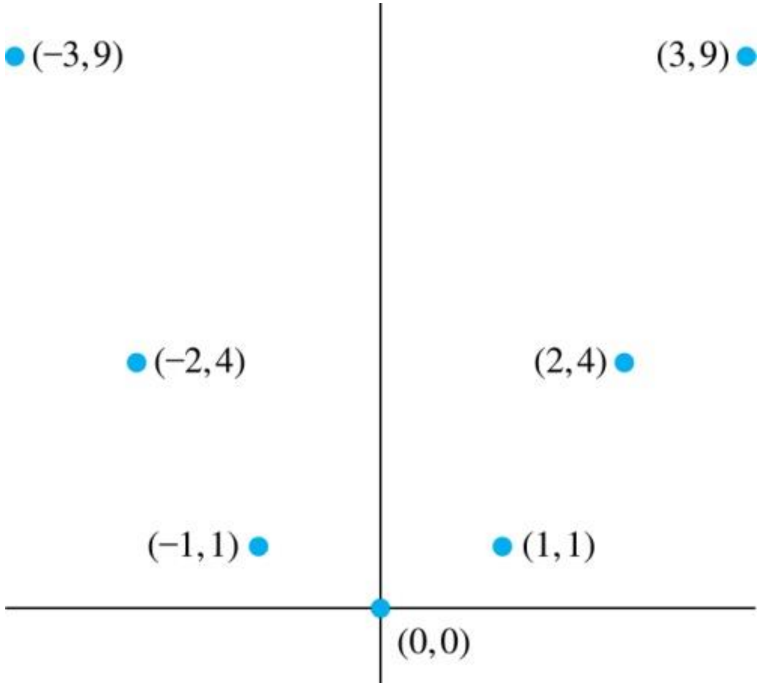
Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Partial Functions

A *partial function f* from a set *A* to a set *B*, *denoted* $f : A \multimap\!\!\longrightarrow B$ is an assignment to each element *a* in a subset of *A* on a unique element *b* in *B*.

- *The subset of A is called the domain of definition of f*
- *f* is *undefined* for elements in *A* that are not in the domain of definition of *f*.
- When the domain of definition of *f* equals *A*, we say that *f* is a *total function*.

Example: *f*: **N** → **R** where $f(n) = \sqrt{n}$ is a partial function from **Z** to **R** where the domain of definition is the set of nonnegative integers. Note that *f* is undefined for negative integers.