Discrete Mathematics

# Predicate Logic

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# Motivation



- Does this argument make sense?

Suppose that

All hummingbirds are richly colored,

No large brids live on honey, and

Birds that do not live on honey are dull in color

Thus, hummingbirds are small.

# Predicate

- A **predicate** is a propositional function over variables
  - once values are assigned to the predicate variables, a predicate becomes a proposition which evaluates to either true or false
  - the domain of a predicate variable is a set of values on which the property is asserted
  - *P*(*x*) is said to be the value of the propositional function *P* at *x*.
- Example:
  - Let *P(x)* denote "$x > 0$" and the domain of *x* be the integers.
    P(-3) is false. P(0) is false. P(3) is true.
  - Let $Q(x, y)$ denote $x = y + 3$, and the domain be the all pairs of integers
    $Q(4, 1)$ is true and $Q(2, 3)$ is false.

# Examples of Propositional Functions



- Let "$x + y = z$" be denoted by $R(x, y, z)$ and $U$ (for all three variables) be the integers. Find these truth values:

  $R(2,-1,5)$ **Solution: F**

  $R(3,4,7)$ **Solution: T**

  $R(x, 3, z)$ **Solution: Not a proposition**

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

  $Q(2,-1,3)$ **Solution: T**

  $Q(3,4,7)$ **Solution: F**

  $Q(x, 3, z)$ **Solution: Not a proposition**

# Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If $P(x)$ denotes "$x > 0$," find these truth values:

  $P(3) \lor P(-1)$ **Solution**: T
  $P(3) \land P(-1)$ **Solution**: F
  $P(3) \rightarrow P(-1)$ **Solution**: F

- Expressions with variables are not propositions and therefore do not have truth values. For example,
  $P(3) \land P(y)$
  $P(x) \rightarrow P(y)$

# Quantification (1/2)



- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are mortal.", "Some cats do not have fur."

- A **quantification** expresses the extent to which a predicate is true over a range of elements represented with a variable.
  - A variable is *bound* if a quantifier is used on the variable. A variable is a free variable otherwise
  - Structure:

    *<Quantifier> <Variables w/ domain condition> ( <Predicate> )*

$$\exists x \in \mathbb{N} \, ( x^2 = x^3 )$$

# Universal Quantification



- The universal quantification of $P(x)$, denoted as $\forall x\, P(x)$, is the statement that $P(x)$ holds for all values of $x$ in the domain.
  - $\forall$ is called the universal quantifier
  - E.g., $\forall x \in \mathbb{R}\ (\ x^2 \geq 0\ )$

- $\forall x\, P(x)$ is read as "For all $x$, $P(x)$" or "For every $x$, $P(x)$"
  - If $P(x)$ denotes "$x > 0$" and $U$ is the integers, $\forall x\, P(x)$ is false.
  - If $P(x)$ denotes "$x > 0$" and $U$ is the positive integers, $\forall x\, P(x)$ is true
  - If $P(x)$ denotes "$x$ is even" and $U$ is the integers, $\forall\, x\, P(x)$ is false.

# Existential Quantification



- The existential quantification of $P(x)$, denoted as $\exists x. P(x)$, is the statement that $P(x)$ holds for a value of $x$ in the domain.
  - $\exists$ is called the existential quantifier
  - E.g., $\exists x \in \mathbb{R} \ ( \ x^2 = 1 \ )$

- $\exists x$ *P*(*x*) is read as "For some *x*, P(*x*)", or as "There is an *x* such that P(*x*)," or "For at least one *x*, P(*x*)."
  - If *P*(*x*) denotes "$x > 0$" and $U$ is the integers, $\exists x$ *P*(*x*) is true
    It is also true if U is the positive integers
  - If *P*(*x*) denotes "$x < 0$" and $U$ is the positive integers, $\exists x$ *P*(*x*) is false
  - If *P*(*x*) denotes "*x* is even" and $U$ is the integers, $\exists x$ *P*(*x*) is true

# Uniqueness Quantifier



- $\exists! x\, P(x)$ means that $P(x)$ is true for one and only one $x$ in the universe of discourse.
  - "There is a unique $x$ such that $P(x)$."
  - "There is one and only one $x$ such that $P(x)$"
  - Example.
    - $\exists!\, x \in \mathbb{Q}\ (x + 1 = 0)$ is true.
    - $\exists!\, x \in \mathbb{Q}\ (x > 0)$ is false

- The uniqueness quantifier is not really needed as the restriction that there is a unique $x$ such that $P(x)$ can be expressed as:

$$\exists!\, x\, P(x) \equiv \exists x\, (P(x) \land \forall y\, (P(y) \to y = x))$$

# Properties of Quantifiers



- The truth value of $\exists x\, P(x)$ and $\forall\, x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain $U$.
- Examples

1. If *U* is the positive integers and *P(x)* is the statement "*x* < 2", $\exists x\, P(x)$ is true, and $\forall\, x\, P(x)$ is false.

2. If *U* is the negative integers and *P(x)* is the statement "*x* < 2", both $\exists x\, P(x)$ and $\forall\, x\, P(x)$ are true.

3. If *U* is {3, 4, 5} and *P(x)* is the statement "*x* > 2", both $\exists x\, P(x)$ and $\forall\, x\, P(x)$ are true.
   But if *P(x)* is the statement "*x* < 2", both $\exists x\, P(x)$ and $\forall\, x\, P(x)$ are false.

# Thinking about Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.

- To evaluate $\forall x\, P(x)$ loop through all $x$ in the domain.
  - If at every step P($x$) is true, then $\forall x\, P(x)$ is true.
  - If at a step P($x$) is false, then $\forall x\, P(x)$ is false and the loop terminates.

- To evaluate $\exists x\, P(x)$ loop through all $x$ in the domain.
  - If at some step, P($x$) is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an $x$ for which P($x$) is true, then $\exists x\, P(x)$ is false.

- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

# Precedence of Quantifiers



- Quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators
  - E.g., $\forall x\, P(x) \lor Q(x)$ means $(\forall x\, P(x)) \lor Q(x)$ or
    $\forall x\, (P(x) \lor Q(x))$ ?

# Translating from English to Predicate Logic



- Example: Every student in this class has taken a Java class
- Solution
  - If *U* is all students in this class, define a propositional function $J(x)$ denoting "x has taken Java" and translate as $\forall x\, J(x)$
  - If *U* is all people, define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$

# Equivalences in Predicate Logic



- Predicate statements are *logically equivalent* if and only if they have the same truth value
  - $\forall x(P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x Q(x)$
  - $\exists\, x\big(P(x) \lor Q(x)\big) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$
  - $\neg\forall x P(x) \equiv \exists x \neg P(x)$
  - $\neg\exists x P(x) \equiv \forall x \neg P(x)$

# Nested Quantifiers



- A nested quantifier has one quantifier within the scope of another quantifier
  - E.g., $\forall x(\exists y(x + y = 0))$
  - E.g., *every man has exactly one wife*
  - E.g., every real number except zero has a multiplication inverse

- Depending on quantification orders, the statements containing the same predicate may have different truth values
  - E.g., for that $x$ and $y$ are integers, $\forall x\big(\exists y(x + y = 0)\big)$ and $\exists y\big(\forall x(x + y = 0)\big)$

# Questions on Order of Quantifiers



**Example 1**: Let *U* be the real numbers,

Define $P(x, y)$ *as* $x \times y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$ **Answer:** False
2. $\forall x \exists y P(x, y)$ **Answer:** True
3. $\exists x \forall y \, P(x, y)$ **Answer:** True
4. $\exists x \, \exists \, y \, P(x, y)$ **Answer:** True

# Negating Nested Quantifiers



- E.g., $\forall w \neg \forall a \exists f (P(w, f) \land Q(f, a))$

$$\begin{aligned} &\equiv \forall w \exists a \neg \exists f (P(w, f) \land Q(f, a)) \\ &\equiv \forall w \exists a \forall f \neg (P(w, f) \land Q(f, a)) \\ &\equiv \forall w \exists a \forall f (\neg P(w, f) \lor \neg Q(f, a)). \end{aligned}$$

# Exercises 1.5.33



- Rewrite each of these statements so that negations appear only within predicates

  (b) $\neg\forall y\, \exists x\, P(x, y)$

  (d) $\neg(\exists x \exists y\, \neg P(x, y) \wedge \forall x \forall y Q(x, y))$

# Revisting Motivating Example



All hummingbirds are richly colored.

$$\forall\, x \in Bird\ (Hummingbird(x) \rightarrow RichlyColored(x))$$

No large brids live on honey.

$$\nexists x \in Bird\ \big(Large(x) \wedge Honey(x)\big)$$
$$\forall x \in Bird\ \big(\neg Large(x) \vee \neg Honey(x)\big)$$

Birds that do not live on honey are dull in color.

$$\forall\, x \in Bird\ (\neg Honey(x) \rightarrow \neg RichlyColored(x))$$

Then, hummingbirds are small.

$$\forall\, x \in Bird\ ($$
$$\big(Hummingbird(x) \rightarrow RichlyColored(x)\big) \wedge$$
$$\big(\neg Large(x) \vee \neg Honey(x)\big) \wedge$$
$$\big(\neg Honey \rightarrow \neg RichlyColored(x)\big)\ )$$

$$(Honey(x) \rightarrow \neg Large(x))$$

$$(RichlyColored(x) \rightarrow Honey(x))$$