Weekly Homework 12

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Mathematics and Statistics Theory

November 24, 2024

Question 1 (10 pts)

Find the density function of the random variable $Y = e^X$, where $X \sim N(\mu, \sigma^2)$. This is called the lognormal density, since $\log Y$ is normally distributed.

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln y - \mu)2}{2\sigma 2}\right), \quad y > 0$$

Question 2 (15 pts)

The Weibull cumulative distribution function is $F(x) = 1 - e^{-(x/\alpha)^\beta}$, $x \geq 0, \alpha > 0, \beta > 0$

1. Find the density function: The density function $f(x)$ is obtained by differentiating $F(x)$:

$$f(x) = \frac{d}{dx}\left[1 - e^{-(x/\alpha)^\beta}\right] = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^\beta}, \quad x \geq 0$$

2. Show that if X follows a Weibull distribution, then $Y = \left(\frac{X}{\alpha}\right)^\beta$ follows an exponential distribution: If X is Weibull-distributed with parameters α and β, then:

$$Y = \left(\frac{X}{\alpha}\right)^\beta$$

Then the cumulative distribution function of Y is:

$$F_Y(y) = P(Y \leq y) = P\left(\left(\frac{X}{\alpha}\right)^\beta \leq y\right) = P\left(X \leq \alpha y^{1/\beta}\right)$$

Since X follows a Weibull distribution, its CDF is:

$$F_X(x) = 1 - e^{-(x/\alpha)^\beta}$$

Therefore, the CDF of Y is:

$$F_Y(y) = 1 - e^{-y}, \quad y \geq 0$$

This represents the CDF of an exponential distribution with parameter 1.

3. Generating Weibull random variables from a uniform random number generator: If U is a uniform random variable on $(0, 1)$, then $-\ln(U)$ is an exponential random variable. Using this property, we can generate Weibull random variables:

$$X = \alpha \left(-\ln(U)\right)^{1/\beta}$$

Here is the Python code:

```
import numpy as np

def generate_weibull(alpha, beta, size=1):
   U = np.random.uniform(low=0.0, high=1.0, size=size)
   X = alpha * (-np.log(U))**(1/beta)
   return X

# Example
   alpha = 2
   beta = 3
   size = 10
   weibull_samples = generate_weibull(alpha, beta, size)
   print(weibull_samples)
```

Question 3 (20 pts)

Three players, A, B, and C play 10 independent rounds of a game, and each player has a probability $\frac{1}{3}$ of winning each round. Let X be the number of games won by A, Y the number of games won by B, and Z the number of games won by C.

1. Find the joint distribution of the numbers of games won by each of the three players: Since each game is won independently and each player has a $\frac{1}{3}$ chance of winning, the joint distribution of X, Y, and Z aligns with a multinomial distribution:

$$P(X = x, Y = y, Z = z) = \frac{10!}{x!y!z!} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{3}\right)^z,$$

where $x + y + z = 10$ and $x, y, z \geq 0$.

2. Find the marginal distribution of the number of games won by A: The marginal distribution of X follows a binomial distribution with parameters $n = 10$ and $p = \frac{1}{3}$:

$$P(X = x) = \binom{10}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{10-x},$$

for $x = 0, 1, 2, \ldots, 10$.

3. Find the conditional distribution of the number of games won by A given that B won 3 games: Given $Y = 3$, the number of remaining games is $10 - 3 = 7$. The conditional distribution of X given $Y = 3$ is a binomial distribution with parameters $n = 7$ and $p = \frac{1}{3}$:

$$P(X = x \mid Y = 3) = \binom{7}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{7-x},$$

for $x = 0, 1, 2, \ldots, 7$.

4. Are the numbers of games won by A and B independent? No, they are not independent. Since $X + Y + Z = 10$, knowing the number of games won by one player gives us knowledge about the number of games won by the others, therefore, they are dependent.

Question 4 (25 pts)

An urn contains 5 black balls, 6 white balls, and 7 red balls; and 6 balls are chosen without replacement.

1. **Find the joint distribution of the numbers of black balls (X), white balls (Y), and red balls (Z) in the sample:** The joint distribution of X, Y, and Z follows a multivariate hypergeometric distribution:

$$P(X = x, Y = y, Z = z) = \frac{\binom{5}{x}\binom{6}{y}\binom{7}{z}}{\binom{18}{6}},$$

where $x + y + z = 6$ and $0 \le x \le 5$, $0 \le y \le 6$, $0 \le z \le 7$.

2. **Find the joint distribution of the numbers of black (X) and white balls (Y) in the sample:** The joint distribution of X and Y marginalizing over Z:

$$P(X = x, Y = y) = \sum_{z=0}^{6-x-y} \frac{\binom{5}{x}\binom{6}{y}\binom{7}{z}}{\binom{18}{6}},$$

where $x + y + z = 6$ and $0 \le x \le 5$, $0 \le y \le 6$, and $0 \le z \le 7 - x - y$.

3. **Find the marginal distribution of the number of white balls in the sample:** The marginal distribution of Y:

$$P(Y = y) = \sum_{x=0}^{5} \sum_{z=0}^{6-x-y} \frac{\binom{5}{x}\binom{6}{y}\binom{7}{z}}{\binom{18}{6}},$$

where $0 \le y \le 6$ and $x + y + z = 6$.

4. Find the conditional distribution of the number of white balls in the sample given that there are 2 black balls in the sample:

Since $X = 2$, the conditional distribution of Y:

$$P(Y = y \mid X = 2) = \frac{P(X = 2, Y = y)}{P(X = 2)} = \frac{\sum_{z=0}^{6-2-y} \frac{\binom{5}{2}\binom{6}{y}\binom{7}{z}}{\binom{18}{6}}}{\sum_{y=0}^{6-2} \sum_{z=0}^{6-2-y} \frac{\binom{5}{2}\binom{6}{y}\binom{7}{z}}{\binom{18}{6}}},$$

where $0 \le y \le 4$.

5. Are the numbers of black and white balls in the sample independent? No, they are not independent. Since the sampling is done without replacement, the occurrence of one event affects the probabilities of the other events, meaning that they are dependent.

Question 5 (15 pts)

Consider the bivariate density function:

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

1. Find the marginal density functions of X and Y:

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 \frac{3}{2}(x^2+y^2)\,dy = \frac{3}{2}\left[x^2y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left(x^2+\frac{1}{3}\right) = \frac{3}{2}x^2 + \frac{1}{2}$$

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 \frac{3}{2}(x^2+y^2)\,dx = \frac{3}{2}\left[\frac{x^3}{3} + y^2x\right]_0^1 = \frac{3}{2}\left(\frac{1}{3}+y^2\right) = \frac{1}{2} + \frac{3}{2}y^2$$

2. Find the conditional density function of X given $Y=y$:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{1}{2}+\frac{3}{2}y^2} = \frac{3(x^2+y^2)}{1+3y^2}, \quad 0<x<1$$

3. Are X and Y independent? No, X and Y are not independent because the joint density function $f(x,y)$ is not a product of the marginal density functions $f_X(x)$ and $f_Y(y)$.

Question 6 (15 pts)

Let X and Y have the joint density function

$$f(x,y) = \begin{cases} \frac{6}{7}(x+y)^2, & 0\le x\le 1,\ 0\le y\le 1 \\ 0, & \text{otherwise} \end{cases}$$

1. Find the marginal density functions of X and Y:

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 \frac{6}{7}(x+y)^2\,dy = \frac{6}{7}\int_0^1 (x+y)^2\,dy$$

$$f_X(x) = \frac{6}{7}\left[\frac{(x+y)^3}{3}\right]_0^1 = \frac{6}{7}\left(\frac{(x+1)^3}{3} - \frac{x^3}{3}\right) = \frac{6}{7}\left(\frac{x^3+3x^2+3x+1}{3} - \frac{x^3}{3}\right) = \frac{6}{7}\left(x^2+x+\frac{1}{3}\right)$$

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 \frac{6}{7}(x+y)^2\,dx = \frac{6}{7}\int_0^1 (x+y)^2\,dx$$

$$f_Y(y) = \frac{6}{7}\left[\frac{(x+y)^3}{3}\right]_0^1 = \frac{6}{7}\left(\frac{(1+y)^3}{3} - \frac{y^3}{3}\right) = \frac{6}{7}\left(\frac{y^3+3y^2+3y+1}{3} - \frac{y^3}{3}\right) = \frac{6}{7}\left(y^2+y+\frac{1}{3}\right)$$

2. Find the conditional density function of X given $Y = y$:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{6}{7}(x+y)^2}{\frac{6}{7}\left(y^2+y+\frac{1}{3}\right)} = \frac{(x+y)^2}{y^2+y+\frac{1}{3}}, \quad 0 \leq x \leq 1$$

3. Are X and Y independent? No, X and Y are not independent because the joint density function $f(x,y)$ is not a product of the marginal density functions $f_X(x)$ and $f_Y(y)$. Thus, they are dependent.