Weekly Homework 11

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Mathematics and Statistics Theory

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Question 1 (15 pts)

Suppose that the life time of an electronic device has an exponential distribution with $\lambda = 0.1$.

1. **What is the probability that the device will last more than 10 years?**

   The exponential distribution's cumulative distribution function (CDF) is given by:

   $$F(t) = 1 - e^{-\lambda t}$$

   The probability that the device lasts more than t years is:

   $$P(T > t) = 1 - F(t) = e^{-\lambda t}$$

   For $t = 10$ years:

   $$P(T > 10) = e^{-0.1\times 10} = e^{-1} \approx 0.3679$$

2. **What is the probability that the device will last between 5 and 10 years?**

   The probability that the device lasts between a and b years is:

   $$P(a < T \leq b) = F(b) - F(a)$$

   Using CDF:

   $$F(t) = 1 - e^{-\lambda t}$$

   For $a = 5$ and $b = 10$ years:

   $$F(10) = 1 - e^{-0.1\times 10} = 1 - e^{-1}$$

   $$F(5) = 1 - e^{-0.1\times 5} = 1 - e^{-0.5}$$

   Therefore, the probability is:

$$P(5 < T \leq 10) = F(10) - F(5) = (1 - e^{-1}) - (1 - e^{-0.5}) = e^{-0.5} - e^{-1} \approx 0.6065 - 0.3679 = 0.2386$$

3. **Find t such that the probability that the device will last more than t years is 0.01.**

Solve for t in the equation:

$$P(T > t) = 0.01$$

Using survival function:

$$e^{-\lambda t} = 0.01$$

Solving for t:

$$-\lambda t = \ln(0.01)$$

$$t = \frac{\ln(0.01)}{-\lambda} = \frac{\ln(0.01)}{-0.1}$$

$$t \approx \frac{-4.6052}{-0.1} = 46.052 \text{ years}$$

Question 2 (15 pts)

Let X be a normal random variable with mean $\mu = 5$ and standard deviation σ. Given $X \sim N(\mu = 5, \sigma = 10)$.

(a) Find $P(X > 10)$:

The standard normal variable Z is given by:

$$Z = \frac{X - \mu}{\sigma}$$

Substitute $\mu = 5$ and $\sigma = 10$:

$$Z = \frac{X - 5}{10}$$

For $P(X > 10)$:

$$P(X > 10) = P\left(Z > \frac{10 - 5}{10}\right) = P(Z > 0.5)$$

Use the standard normal distribution table:

$$P(Z > 0.5) = 1 - P(Z \leq 0.5) \approx 1 - 0.6915 = 0.3085$$

(b) Find $P(-20 < X < 15)$:

For $P(-20 < X < 15)$:

$$P(-20 < X < 15) = P\left(\frac{-20 - 5}{10} < Z < \frac{15 - 5}{10}\right) = P(-2.5 < Z < 1)$$

Now, using the standard normal distribution table:

$$P(-2.5 < Z < 1) = P(Z < 1) - P(Z \leq -2.5)$$

$$P(Z < 1) \approx 0.8413$$
$$P(Z \leq -2.5) \approx 0.0062$$

Therefore,

$$P(-2.5 < Z < 1) = 0.8413 - 0.0062 = 0.8351$$

(c) Find the value of x so that $P(X > x) = 0.5$:

$$P(Z > \frac{x-5}{10}) = 0.5$$

Since $P(Z > 0) = 0.5$:

$$\frac{x-5}{10} = 0$$

Hence,

$$x - 5 = 0$$
$$x = 5$$

Question 3 (20 pts)

Suppose that in a sequence of independent Bernoulli trials, each with probability of success p, the number of failures up to the first success is counted. What is the frequency function for this random variable?

(a) Expected time until a call is answered: The expected value (mean) of a gamma distribution is given by:

$$\mathrm{E}[X] = k \cdot \theta$$

Substitute $k = 2$ and $\theta = 3$:

$$\mathrm{E}[X] = 2 \cdot 3 = 6$$

So, the expected time until a call is answered is 6 minutes.

(b) Variance of the time until a call is answered: The variance of a gamma distribution is given by:

$$\mathrm{Var}(X) = k \cdot \theta^2$$

Substitute $k = 2$ and $\theta = 3$:

$$\mathrm{Var}(X) = 2 \cdot 3^2 = 2 \cdot 9 = 18$$

So, the variance of the time until a call is answered is 18 minutes.

(c) Probability that a call is answered within 3 minutes: For a gamma distribution, the cumulative distribution function (CDF) is used to find probabilities. The CDF of a gamma distribution is typically denoted by $F(x; k, \theta)$.

Using the shape parameter $k = 2$ and scale parameter $\theta = 3$, the CDF at $x = 3$ can be computed using statistical software or tables. The CDF value gives the probability that X is less than or equal to x. The probability that a call is answered within 3 minutes is:

$$P(X \leq 3) = F(3; 2, 3)$$

Using a calculator or software:

$$P(X \leq 3) \approx 0.199$$

Therefore, the probability of a call being answered within 3 minutes is approx 0.199.

(d) Probability that a call is answered between 2 and 6 minutes: To find the probability that a call is answered between 2 and 6 minutes, we calculate the CDF values at $x = 2$ and $x = 6$, and then find the difference.

$$P(2 < X \leq 6) = F(6; 2, 3) - F(2; 2, 3)$$

Calculate:

$$P(X \leq 6) \approx 0.738$$

$$P(X \leq 2) \approx 0.104$$

Therefore,

$$P(2 < X \leq 6) = 0.738 - 0.104 = 0.634$$

The probability of a call being answered between 2 and 6 minutes is approx. 0.634.

Question 4 (15 pts)

A factory produces light bulbs with a mean life of 1200 hours and a standard deviation of 200 hours. Assume the life of the light bulbs is normally distributed. If a random sample of 36 light bulbs is selected, what is the probability that the sample mean life is:

(a) More than 1250 hours:

1. Calculate the standard error:

$$\sigma_{\bar{X}} = \frac{200}{\sqrt{36}} = \frac{200}{6} = 33.33$$

2. Convert 1250 hours to a Z-score:

$$Z = \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} = \frac{1250 - 1200}{33.33} = \frac{50}{33.33} \approx 1.5$$

3. Find the probability using the Z-score:

$$P(\bar{X} > 1250) = P(Z > 1.5)$$

Use the standard normal distribution table:

$$P(Z > 1.5) \approx 1 - P(Z \leq 1.5) = 1 - 0.9332 = 0.0668$$

Therefore, the probability that the sample mean life is more than 1250 hours is approximately 0.0668.

(b) Between 1150 and 1250 hours:

1. Convert 1150 and 1250 hours to Z-scores:

$$Z_1 = \frac{1150 - 1200}{33.33} = \frac{-50}{33.33} \approx -1.5$$

$$Z_2 = \frac{1250 - 1200}{33.33} = \frac{50}{33.33} \approx 1.5$$

2. Find the probability:

$$P(1150 < \bar{X} < 1250) = P(-1.5 < Z < 1.5)$$

Using the standard normal distribution table:

$$P(-1.5 < Z < 1.5) = P(Z < 1.5) - P(Z < -1.5)$$

$$P(Z < 1.5) \approx 0.9332$$

$$P(Z < -1.5) \approx 0.0668$$

$$P(-1.5 < Z < 1.5) = 0.9332 - 0.0668 = 0.8664$$

Hence, the probability that the sample mean life is between 1150 and 1250 hours is approximately 0.8664.

(c) Less than 1100 hours:

1. Convert 1100 hours to a Z-score:

$$Z = \frac{1100 - 1200}{33.33} = \frac{-100}{33.33} \approx -3$$

2. Find the probability:

$$P(\bar{X} < 1100) = P(Z < -3)$$

Using the standard normal distribution table:

$$P(Z < -3) \approx 0.0013$$

Therefore, the probability that the sample mean life is less than 1100 hours is approximately 0.0013.

Question 5 (15 pts)

The weights of packages delivered by a courier company are uniformly distributed between 2 and 10 pounds. If a random sample of 50 packages is selected, use the Central Limit Theorem to find the probability that the sample mean weight is:

(a) More than 6 pounds: 1. Calculate the mean (μ) and standard deviation (σ) of the uniform distribution $U(2, 10)$:

$$\mu = \frac{2 + 10}{2} = 6$$

$$\sigma = \frac{10 - 2}{\sqrt{12}} = \frac{8}{\sqrt{12}} \approx 2.309$$

2. Calculate the standard error of the mean (SE) for a sample size $n = 50$:

$$\text{SE} = \frac{\sigma}{\sqrt{n}} = \frac{2.309}{\sqrt{50}} \approx 0.326$$

3. Convert 6 pounds to a Z-score:

$$Z = \frac{\bar{X} - \mu}{\text{SE}} = \frac{6 - 6}{0.326} = 0$$

4. Find the probability using the Z-score:

$$P(\bar{X} > 6) = P(Z > 0)$$

Using the Z-table, $P(Z > 0) = 0.5$.
So, the probability that the sample mean weight is more than 6 pounds is 0.5 or 50%.
(b) Between 5 and 7 pounds: 1. Calculate the mean (μ) and standard deviation (σ) of the uniform distribution $U(2, 10)$:

$$\mu = \frac{2 + 10}{2} = 6$$

$$\sigma = \frac{10 - 2}{\sqrt{12}} = \frac{8}{\sqrt{12}} \approx 2.309$$

2. Calculate the standard error of the mean (SE) for a sample size $n = 50$:

$$\text{SE} = \frac{\sigma}{\sqrt{n}} = \frac{2.309}{\sqrt{50}} \approx 0.326$$

3. Convert 5 and 7 pounds to Z-scores:

$$Z_1 = \frac{5 - 6}{0.326} \approx -3.07$$

$$Z_2 = \frac{7 - 6}{0.326} \approx 3.07$$

4. Find the probability using the Z-scores:

$$P(5 < \bar{X} < 7) = P(-3.07 < Z < 3.07)$$

Using the Z-table:

$$P(-3.07 < Z < 3.07) = P(Z < 3.07) - P(Z < -3.07)$$

$$P(Z < 3.07) \approx 0.9989$$

$$P(Z < -3.07) \approx 0.0011$$

Thus,

$$P(-3.07 < Z < 3.07) = 0.9989 - 0.0011 = 0.9978$$

So, the probability that the sample mean weight is between 5 and 7 pounds is approximately 0.9978 or 99.78%.

Question 6 (10 pts)

A line segment of length 1 is cut once at random. What is the probability that the longer piece is more than twice the length of the shorter piece?

Let x be the length of the shorter piece, and $1 - x$ be the length of the longer piece. We are interested in finding the probability that the longer piece is more than twice the length of the shorter piece. This can be expressed as:

$$1 - x > 2x$$

Rearranging the inequality:

$$1 > 3x$$
$$x < \frac{1}{3}$$

Since the cut can occur at any point along the segment with equal probability, x is uniformly distributed between 0 and 1. Therefore, the probability that x is less than $\frac{1}{3}$ is:

$$P\left(x < \frac{1}{3}\right) = \frac{1}{3}$$

So, the probability that the longer piece is more than twice the length of the shorter piece is $\frac{1}{3}$ or approximately 0.3333.

Question 7 (10 pts)

A function is a probability density function if it satisfies the following two properties:

- $f(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Prove that if f and g are densities, then $\alpha f + (1-\alpha)g$ is also a density, where $0 \leq \alpha \leq 1$. Note that you can use the following properties of integrals:

- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Proof

Given:

- f and g are densities
- $0 \leq \alpha \leq 1$

We need to prove that $\alpha f + (1 - \alpha)g$ is also a density.

Step 1: Non-Negativity

We need to show that $\alpha f(x) + (1-\alpha)g(x) \geq 0$ for all x.

Since $f(x) \geq 0$ and $g(x) \geq 0$ for all x, and $0 \leq \alpha \leq 1$, it follows that:

$$\alpha f(x) \geq 0 \quad \text{and} \quad (1-\alpha)g(x) \geq 0$$

Therefore,

$$\alpha f(x) + (1-\alpha)g(x) \geq 0 \quad \text{for all } x$$

Step 2: Integrates to 1

We need to show that $\int_{-\infty}^{\infty} [\alpha f(x) + (1-\alpha)g(x)]\, dx = 1$.

Using the properties of integrals, we have:

$$\int_{-\infty}^{\infty} [\alpha f(x) + (1-\alpha)g(x)]\, dx = \alpha \int_{-\infty}^{\infty} f(x)\, dx + (1-\alpha) \int_{-\infty}^{\infty} g(x)\, dx$$

Since $f(x)$ and $g(x)$ are densities, we know that:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} g(x)\, dx = 1$$

Therefore,

$$\alpha \int_{-\infty}^{\infty} f(x)\, dx + (1-\alpha) \int_{-\infty}^{\infty} g(x)\, dx = \alpha \cdot 1 + (1-\alpha) \cdot 1 = \alpha + (1-\alpha) = 1$$