Weekly Homework 9

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Mathematics and Statistics Theory

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Question 1 (20 pts)

An experiment consists of throwing a fair coin four times. Find the frequency function (PMF) and the cumulative distribution function (CDF) of the following random variables:

1. the number of heads before the first tail (if there is no tail in four throws, you can assume $X = 4$) Possibilities for $X : 0, 1, 2, 3, 4$

   PMF:

$$P(X = 0) = P(T) = \frac{1}{2}$$
$$P(X = 1) = P(HT) = \frac{1}{4}$$
$$P(X = 2) = P(HHT) = \frac{1}{8}$$
$$P(X = 3) = P(HHHT) = \frac{1}{16}$$
$$P(X = 4) = P(HHHH) = \frac{1}{16}$$

   CDF:

$$F(0) = P(X \leq 0) = \frac{1}{2}$$
$$F(1) = P(X \leq 1) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$F(2) = P(X \leq 2) = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$
$$F(3) = P(X \leq 3) = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$$
$$F(4) = P(X \leq 4) = \frac{15}{16} + \frac{1}{16} = \frac{16}{16} = 1$$

2. the number of heads following the first tail (if there is no tail in four throws, you can assume the $X = 0$) Possibilities for $X : 0, 1, 2, 3$

PMF:

$$
\begin{aligned}
P(X=0) &= P(TTTT) = \frac{1}{16} \\
P(X=1) &= P(HT) + P(THT) + P(TTHT) = \frac{4}{16} = \frac{1}{4} \\
P(X=2) &= P(HHT) + P(THHT) = \frac{6}{16} = \frac{3}{8} \\
P(X=3) &= P(THHHT) = \frac{4}{16} = \frac{1}{4}
\end{aligned}
$$

CDF:

$$
\begin{aligned}
F(0) &= P(X \leq 0) = \frac{1}{16} \\
F(1) &= P(X \leq 1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16} \\
F(2) &= P(X \leq 2) = \frac{5}{16} + \frac{6}{16} = \frac{11}{16} \\
F(3) &= P(X \leq 3) = \frac{11}{16} + \frac{4}{16} = \frac{15}{16}
\end{aligned}
$$

3. the number of heads minus the number of tails. Possibilities for $X : -4, -2, 0, 2, 4$

   PMF:

$$
\begin{aligned}
P(X=-4) &= P(TTTT) = \frac{1}{16} \\
P(X=-2) &= P(HTTT) + P(THTT) + P(TTHT) + P(TTTH) = \frac{4}{16} = \frac{1}{4} \\
P(X=0) &= P(HHTT) + P(HTHT) + P(HTTH) + P(THHT) + P(THTH) = \frac{6}{16} = \frac{3}{8} \\
P(X=2) &= P(HHHT) + P(HHTH) + P(HTHH) + P(THHH) = \frac{4}{16} = \frac{1}{4} \\
P(X=4) &= P(HHHH) = \frac{1}{16}
\end{aligned}
$$

   CDF:

$$
\begin{aligned}
F(-4) &= P(X \leq -4) = \frac{1}{16} \\
F(-2) &= P(X \leq -2) = \frac{1}{16} + \frac{1}{4} = \frac{5}{16} \\
F(0) &= P(X \leq 0) = \frac{5}{16} + \frac{3}{8} = \frac{11}{16} \\
F(2) &= P(X \leq 2) = \frac{11}{16} + \frac{1}{4} = \frac{15}{16} \\
F(4) &= P(X \leq 4) = \frac{15}{16} + \frac{1}{16} = \frac{16}{16} = 1
\end{aligned}
$$

4. the number of tails times the number of heads. Possibilities for $X : 0, 1, 2, 3, 4, 6$

   PMF:

$$P(X = 0) = P(HHHH) + P(TTTT) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$
$$P(X = 1) = P(HTTT) + P(THTT) + P(TTHT) + P(TTTH) = \frac{4}{16} = \frac{1}{4}$$
$$P(X = 2) = P(HTHT) + P(THTH) = \frac{6}{16} = \frac{3}{8}$$
$$P(X = 3) = P(HHT) + P(THHT) + P(THHT) = \frac{3}{16}$$
$$P(X = 4) = P(HHTT) + P(THHT) + P(THTT) = \frac{2}{16} = \frac{1}{8}$$
$$P(X = 6) = P(THHH) + P(TTHH) + P(TTTH) = \frac{1}{16}$$

   CDF:

$$F(0) = P(X \leq 0) = \frac{1}{8}$$
$$F(1) = P(X \leq 1) = \frac{2}{16} + \frac{4}{16} = \frac{6}{16} = \frac{3}{8}$$
$$F(2) = P(X \leq 2) = \frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$$
$$F(3) = P(X \leq 3) = \frac{5}{8} + \frac{3}{16} = \frac{13}{16}$$
$$F(4) = P(X \leq 4) = \frac{5}{8} + \frac{1}{16} = \frac{13}{16}$$
$$F(6) = P(X \leq 6) = \frac{13}{16} + \frac{1}{16} = \frac{14}{16} = \frac{7}{8}$$

Question 2 (20 pts)

Two teams, A and B, play a series of games. If team A has probability 0.4 of winning each game, is it to its advantage to play **the best three out of five games** or **the best four out of seven?** Assume the outcomes of successive games are independent. You need to answer the question and provide the probability of winning the series for each team. Please also state the distribution used in this question.

Scenario 1: Best Three Out of Five Games

In a best three out of five series, Team A must win at least 3 out of 5 games to win the series.

PMF

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Where n is the number of trials (games), k is the number of wins, and p is the probability of wins/success in each trial. Calculate the probability of Team A winning 3-5 out of 5 games:

$$P(X=3) = \binom{5}{3}(0.4)^3(0.6)^2 = \frac{5!}{3!2!}(0.4)^3(0.6)^2 = 10 \cdot 0.064 \cdot 0.36 = 0.2304$$
$$P(X=4) = \binom{5}{4}(0.4)^4(0.6)^1 = \frac{5!}{4!1!}(0.4)^4(0.6)^1 = 5 \cdot 0.0256 \cdot 0.6 = 0.0768$$
$$P(X=5) = \binom{5}{5}(0.4)^5(0.6)^0 = \frac{5!}{5!0!}(0.4)^5(0.6)^0 = 1 \cdot 0.01024 \cdot 1 = 0.01024$$

The sum of these probabilities provide the probability of Team A winning the series:

$$P(\text{Team A wins}) = P(X=3)+P(X=4)+P(X=5) = 0.2304+0.0768+0.01024 = 0.31744$$

Scenario 2: Best Four Out of Seven Games

In a best four out of seven series play, Team A must win at least 4 out of 7 games to win the series. Calculate the probability of Team A winning 4-7 out of 7 games:

$$P(X=4) = \binom{7}{4}(0.4)^4(0.6)^3 = \frac{7!}{4!3!}(0.4)^4(0.6)^3 = 35 \cdot 0.0256 \cdot 0.216 = 0.193536$$
$$P(X=5) = \binom{7}{5}(0.4)^5(0.6)^2 = \frac{7!}{5!2!}(0.4)^5(0.6)^2 = 21 \cdot 0.01024 \cdot 0.36 = 0.0774144$$
$$P(X=6) = \binom{7}{6}(0.4)^6(0.6)^1 = \frac{7!}{6!1!}(0.4)^6(0.6)^1 = 7 \cdot 0.004096 \cdot 0.6 = 0.0172896$$
$$P(X=7) = \binom{7}{7}(0.4)^7(0.6)^0 = \frac{7!}{7!0!}(0.4)^7(0.6)^0 = 1 \cdot 0.0016384 \cdot 1 = 0.0016384$$

The sum of these probabilities provide the probability of Team A winning the series:

P(Team A wins) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) = 0.193536 + 0.0774144 + 0.0172896 + 0.0016384 = 0.28987844

Summarizing the probabilities:

- **Best Three Out of Five**: $P(\text{Team A wins}) = 0.31744$
- **Best Four Out of Seven**: $P(\text{Team A wins}) = 0.2898784$

Team A has a higher probability of winning the series in the best three out of five format compared to the best four out of seven format. Therefore, it is to Team A's advantage to play the best three out of five games.

The distribution used in this question is the **binomial distribution**.

Question 3 (15 pts

Suppose that in a sequence of independent Bernoulli trials, each with probability of success p, the number of failures up to the first success is counted. What is the frequency function for this random variable?

$$P(X = k) = (1 - p)^k p$$

where:

a. p is the probability of success in each trial,

b. $1 - p$ is the probability of failure in each trial,

c. k is the number of failures before the first success.

Question 4 (15 pts)

Find an expression for the cumulative distribution function of a geometric random variable.

$$F(k) = P(X \leq k)$$

X represents the number of failures before the first success, and $X \leq k$ means having up to k failures before the first success. The probability of having exactly d failures before the first success is given by:

$$P(X = d) = (1 - p)^d p$$

To find the CDF, we sum the probabilities for all possible values from 0 to k:

$$F(k) = \sum_{d=0}^{k} P(X = d)$$

Substituting $P(X = d)$:

$$F(k) = \sum_{d=0}^{k} (1 - p)^d p$$

This is a geometric series with the first term $a = p$ and the common ratio $r = 1 - p$. The sum of the first $k + 1$ terms of a geometric series is given by:

$$S_{k+1} = a\frac{1 - r^{k+1}}{1 - r}$$

Substituting $a = p$ and $r = 1 - p$:

$$F(k) = p\frac{1 - (1 - p)^{k+1}}{1 - (1 - p)}$$

Since $1-(1-p)=p$, this simplifies to:

$$F(k)=1-(1-p)^{k+1}$$

Question 5 (15 pts)

The probability of being dealt a royal straight flush (ace, king, queen, jack, and ten of the same suit) in poker is about $1.3X10^-8$. Suppose that an avid poker player sees 100 hands a week, 52 weeks a year, for 20 years.

1. What is the probability that player is never dealt a royal straight flush?
2. What is the probability that player is dealt exactly two royal straight flushes?

Calculate the total number of hands the player sees:

$$\text{Total hands} = 100 \text{ hands/week} \times 52 \text{ weeks/year} \times 20 \text{ years} = 104,000 \text{ hands}$$

1. Probability that player is never dealt a royal straight flush

The probability of not being dealt a royal straight flush in one hand is:

$$P(\text{no RSF}) = 1 - 1.3 \times 10^{-8}$$

The probability of never being dealt a royal straight flush in 104,000 hands is:

$$P(\text{never RSF}) = (1 - 1.3 \times 10^{-8})^{104{,}000}$$

Using the approximation $(1-x)^n \approx e^{-nx}$ for x, we have:

$$P(\text{never RSF}) \approx e^{-104{,}000\times 1.3\times 10^{-8}} = e^{-0.001352}$$

So, the probability is:

$$P(\text{never RSF}) \approx e^{-0.001352} \approx 0.99865$$

2. Probability of player being dealt exactly two royal straight flushes

The number of royal straight flushes follows a binomial distribution with parameters $n = 104,000$ and $p = 1.3 \times 10^{-8}$.

Using the Poisson approximation for the binomial distribution, where $\lambda = np$:

$$\lambda = 104,000 \times 1.3 \times 10^{-8} = 0.001352$$

The probability of being dealt exactly k royal straight flushes is given by the Poisson distribution:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

For $k = 2$:

$$P(X = 2) = \frac{(0.001352)^2 e^{-0.001352}}{2!} = \frac{0.000001827904 e^{-0.001352}}{2} = 0.000000913952 e^{-0.001352}$$

Therefore, the probability is:

$$P(X = 2) \approx 0.000000913952 \times 0.99865 \approx 0.00000091276$$

Question 6 (15 pts)

The university administration assures a mathematician that he has only 1 chance in 10,000 of being trapped in a much-maligned elevator in the mathematics building. If he goes to work 5 days a week, 52 weeks a year, for 10 years, and always rides the elevator up to his office when he first arrives, what is the probability that

1. he will never be trapped?
2. he will be trapped once?
3. he will be trapped twice?

Assume that the outcomes on all the days are mutually independent (a dubious assumption in practice. But you can just assume it is.)

1. Probability that he will never be trapped

The probability of not being trapped on a single day is $1 - p = 0.9999$.

The probability of never being trapped over the span of 2,600 days is:

$$P(X = 0) = (1 - p)^n = 0.9999^{2600}$$

Using the approximation $(1 - x)^n \approx e^{-nx}$ for x:

$$P(X = 0) \approx e^{-2600 \times 0.0001} = e^{-0.26} \approx 0.7711$$

2. Probability that he will be trapped once

The probability of being trapped exactly once over the span of 2,600 days is:

$$P(X = 1) = \binom{n}{1} p(1 - p)^{n-1} = \binom{2600}{1} (0.0001)(0.9999)^{2599}$$

Using the approximation $(1 - x)^{n-1} \approx e^{-nx}$ for x:

$$P(X = 1) = 2600 \times 0.0001 \times e^{-0.26} \approx 2600 \times 0.0001 \times 0.7711 = 0.2005$$

3. Probability that he will be trapped twice

The probability of being trapped exactly twice over the span of 2,600 days is:

$$P(X = 2) = \binom{n}{2} p^2 (1-p)^{n-2} = \binom{2600}{2} (0.0001)^2 (0.9999)^{2598}$$

Using the approximation $(1-x)^{n-2} \approx e^{-nx}$ for x:

$$P(X = 2) = \frac{2600 \times 2599}{2} \times (0.0001)^2 \times e^{-0.26} \approx \frac{2600 \times 2599}{2} \times 0.00000001 \times 0.7711 \approx 0.0260$$