

Diffusion Equation with Finite Difference Method

TF4062

Iwan Prasetyo
Fadjar Fathurrahman

1 Explicit method for 1d diffusion equation

1.1 Initial-boundary value problem for 1d diffusion

1d diffusion equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + f(x, t) \quad (1)$$

Initial condition:

$$u(x, 0) = I(x), \quad x \in [0, L] \quad (2)$$

Boundary condition:

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \quad (3)$$

Spatial grid:

$$x_i = (i - 1)\Delta x, \quad i = 1, \dots, N_x \quad (4)$$

Temporal grid:

$$t_n = (n - 1)\Delta t, \quad n = 1, \dots, N_t \quad (5)$$

1.2 Explicit or forward Euler scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + f_i^n \quad (6)$$

By using the following definition of *mesh Fourier number*:

$$F = \alpha \frac{\Delta t}{\Delta x^2} \quad (7)$$

we can rearrange the equation (6) to:

$$u_i^{n+1} = u_i^n + F (u_{i+1}^n - 2u_i^n + u_{i-1}^n) + f_i^n \Delta t \quad (8)$$

The equation (8) can be used

1.3 Implicit or backward Euler scheme

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + f_i^n \quad (9)$$

$$-F u_{i-1}^n + (1 + 2F) u_i^n - F u_{i+1}^n = u_i^{n-1} + f_i^n \Delta t \quad (10)$$

$$\mathbf{A}\mathbf{u} = \mathbf{b} \quad (11)$$

$$\begin{bmatrix} A_{1,1} & A_{1,2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1,2} & A_{2,2} & A_{2,3} & \cdots & \cdots & \cdots & & & \vdots \\ 0 & A_{3,2} & A_{3,3} & A_{3,4} & \cdots & \cdots & & & \vdots \\ \vdots & \ddots & & \ddots & & 0 & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & 0 & A_{i,j-1} & A_{i,j} & A_{i,j+1} & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & A_{N_x-1,N_x} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & A_{N_x,N_x-1} & A_{N_x,N_x} \end{bmatrix} \quad (12)$$

$$\begin{aligned} A_{i,i-1} &= -F \\ A_{i,i} &= 1 + 2F \\ A_{i,i+1} &= -F \end{aligned}$$

$$\begin{aligned} A_{1,1} &= 1 \\ A_{1,2} &= 0 \\ A_{N_x-1,N_x-1} &= 0 \\ A_{N_x,N_x} &= 1 \end{aligned}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_{N_x} \end{bmatrix} \quad (13)$$

Right-hand side $b_1 = 0$ and $b_{N_x} = 0$

$$b_i = u_i^{n-1} + f_i^{n-1} \Delta t, \quad i = 2, \dots, N_x - 1 \quad (14)$$

1.4 Crank-Nicolson

$$u_i^{n+1} - \frac{1}{2}F(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}) = u_i^n + \frac{1}{2}F(u_{i-1}^n - 2u_i^n + u_{i+1}^n) + \frac{1}{2}f_i^{n+1} + \frac{1}{2}f_i^n \quad (15)$$

$$\mathbf{A}\mathbf{u} = \mathbf{b} \quad (16)$$

$$\begin{aligned} A_{i,i-1} &= -\frac{1}{2}F \\ A_{i,i} &= 1 + F \\ A_{i,i+1} &= -\frac{1}{2}F \end{aligned}$$

Right-hand side $b_1 = 0$ and $b_{N_x} = 0$

$$b_i = u_i^n + \frac{1}{2}F(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \frac{1}{2}(f_i^{n+1} + f_i^n)\Delta t \quad i = 2, \dots, N_x - 1 \quad (17)$$