Diffusion Equation with Finite Difference Method
TF4062

Iwan Prasetyo
Fadjar Fathurrahman

1 Initial-boundary value problem for 1d diffusion

1d diffusion equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial t^2} + f(x,t) \tag{1}$$

Initial condition:

$$u(x,0) = I(x), \qquad x \in [0, L] \tag{2}$$

Boundary condition:

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t > 0 \tag{3}$$

Spatial grid:

$$x_i = (i-1)\Delta x, \qquad i = 1, \ldots, N_x \tag{4}$$

Temporal grid:

$$t_n = (n-1)\Delta t, \qquad n = 1, \ldots, N_t \tag{5}$$

2 Forward Euler scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + f_i^n \tag{6}$$

By using the following definition of *mesh Fourier number*:

$$F = \alpha \frac{\Delta t}{\Delta x^2} \tag{7}$$

we can rearrange the equation (6) to:

$$u_i^{n+1} = u_i + F\left(u_{i+1}^n - 2u_i^n + u_{i+1}^n\right) + f_i^n \Delta t \tag{8}$$

The equation (8) can be used

3 Backward Euler scheme

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + f_i^n \tag{9}$$

$$-Fu_{i-1}^n + (1+2F)u_i^n - Fu_{i+1}^n = u_{i-1}^{n-1} + f_i^n \tag{10}$$

$$\mathbf{Au} = \mathbf{b} \tag{11}$$

$$
\begin{bmatrix}
A_{1,1} & A_{1,2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A_{1,2} & A_{2,2} & A_{2,3} & \cdots & \cdots & \cdots & & & \vdots \\
0 & A_{3,2} & A_{3,3} & A_{3,4} & \cdots & \cdots & & & \vdots \\
\vdots & \ddots & & \ddots & & 0 & & & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & 0 & A_{i,j-1} & A_{i,j} & A_{i,j+1} & \ddots & \vdots \\
\vdots & & & & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & & & & \ddots & \ddots & \ddots & A_{N_x-1,N_x} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & A_{N_x,N_x-1} & A_{N_x,N_x}
\end{bmatrix} \tag{12}
$$

$$
\begin{aligned}
A_{i,i-1} &= -F \\
A_{i,i} &= 1 + 2F \\
A_{i,i+1} &= -F
\end{aligned}
$$

$$
\begin{aligned}
A_{1,1} &= 1 \\
A_{1,2} &= 0 \\
A_{N_x-1,N_x-1} &= 0 \\
A_{N_x,N_x} & 1
\end{aligned}
$$

$$
\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_{N_x} \end{bmatrix} \tag{13}
$$

Right-hand side $b_1 = 0$ and $b_{N_x} = 0$

$$
b_i = u_i^{n-1} + f_i^{n-1}\Delta t, \qquad i = 2, \ldots, N_x - 1 \tag{14}
$$

4 Crank-Nicolson (CN) method

In the Crank-Nicolson method we require the PDE to be satisfied at the spatial mesh point x_i but midway between the points in the time mesh ($t_{n+\frac{1}{2}}$):

$$
\frac{\partial}{\partial t} u_i^{n+\frac{1}{2}} = \alpha \frac{\partial^2}{\partial x^2} u_i^{n+\frac{1}{2}} + f_i^{n+\frac{1}{2}} \tag{15}
$$

Using centered difference in space and time:

$$
\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{\Delta x^2}\left(u_{i+1}^{n+\frac{1}{2}} - 2u_i^{n+\frac{1}{2}} + u_{i-1}^{n+\frac{1}{2}}\right) + f_i^{n+\frac{1}{2}} \tag{16}
$$

Because $u_i^{n+\frac{1}{2}}$ is not the quantity that we want to calculate, we must approximate it. We can approximate it by an average between the value at t_n and t_{n+1}:

$$
u_i^{n+\frac{1}{2}} \approx \frac{1}{2}(u_i^n + u_i^{n+1}) \tag{17}
$$

We also can use the same approximation for $f_i^{n+\frac{1}{2}}$:

$$f_i^{n+\frac{1}{2}} \approx \frac{1}{2}(f_i^n + f_i^{n+1}) \tag{18}$$

Substituting these approximations we obtain:

$$u_i^{n+1} - \frac{1}{2}F\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right) = u_i^n + \frac{1}{2}F\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) + \frac{1}{2}f_i^{n+1} + \frac{1}{2}f_i^n \tag{19}$$

We notice that the equation (19) has similar structure as the one we obtained for backward Euler method:

$$\mathbf{A}\mathbf{u} = \mathbf{b} \tag{20}$$

The element of the matrix $\mathbf{A}$ are:

$$\begin{aligned} A_{i,i-1} &= -\frac{1}{2}F \\ A_{i,i} &= 1 + F \\ A_{i,i+1} &= -\frac{1}{2}F \end{aligned}$$

for internal points $i = 2, \ldots, N_x - 1$. For boundary points we have:

$$\begin{aligned} A_{1,1} &= 1 \\ A_{1,2} &= 0 \\ A_{N_x,N_x-1} &= 0 \\ A_{N_x,N_x} &= 1 \end{aligned}$$

For the right-hand side vector $\mathbf{b}$ we have $b_1 = 0$ and $b_{N_x} = 0$ and

$$b_i = u_i^n + \frac{1}{2}F\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \frac{1}{2}\left(f_i^{n+1} + f_i^n\right)\Delta t \tag{21}$$

for internal points $i = 2, \ldots, N_x - 1$.