SSY281 - Model Predictive Control
Assignment A06 - MPT and Persistent Feasibility

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Question 1

The H and V representation of the given polyhedron are:

$$H = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{1}$$

The H representation is defined as the intersection of a finite number of halfspaces and hyperplanes. The H matrix is the concatenation of the matrices $[A\ b]$, where the i-th line represents the half-plane equation ($A_i x < b_i$). The polyhedron is defined as $P = \{x \in \mathbb{R}^n | Ax \leq b, A_e x = b_e\}$.

On the other hand, the V-representation is the convex combination of a finite number of vertices V and rays R, such that $\mathcal{P} = \{x \in \mathbb{R}^n | x = \lambda^T V + \gamma^T R,\ \gamma \geq 0,\ 1^T \lambda = 1\}$.

Question 2

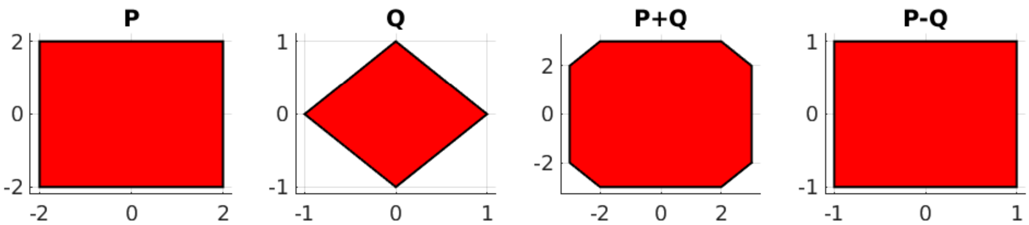

Figure 1: *Minkowiski sum and the Pontryagin difference between the polyhedrons P and Q*

Question 3

The set $\mathcal{S}$ is invariant if $\forall x \in \mathcal{S} \Rightarrow x^+ \in \mathcal{S}$. Since we know that $A_{in}x \leq b_{in}$ and $x^+ = Ax$, we can conclude that $A_{in}A^{-1}x^+ \leq b_{in}$. As shown below in figure 2, $Reach(\mathcal{S}) = \{x \mid A_{in}A^{-1}x \leq b_{in}\} \subseteq \mathcal{S}$ and therefore $\mathcal{S}$ is invariant. Moreover, since the eigenvalues of A are stable, we can then ensure that the system will always converge to the origin and therefore the system can be said to be positively invariant.

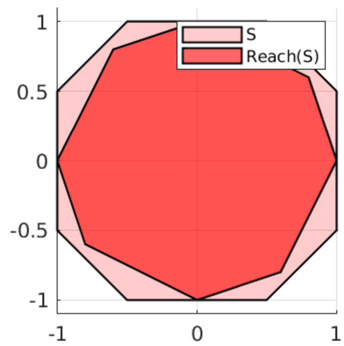

Figure 2: *Set S and the set reachable from S*

Question 4

Given the initial constraint set $\mathcal{S} = \{x \mid A_s\, x \leq b_s\}$ and $\mathbb{U} = \{x \mid A_u\, u \leq b_u\}$ and the system dynamics $x^+ = Ax + Bu$ we can calculate the $Reach(\mathcal{S}) = \{x^+ = Ax + Bu \mid x \in \mathcal{S}, u \in U\}$ by rearranging the equations listed:

$$x^+ = Ax + Bu \quad \Rightarrow \quad x = A^{-1}x^+ - A^{-1}Bu \tag{2}$$

$$\Rightarrow \quad A_s\, x \leq b_s \quad \Rightarrow \quad A_s(A^{-1}x^+ - A^{-1}Bu) \leq b_s \tag{3}$$

Joining equation 3 and the definition of $\mathbb{U}$, we get the new polyhedron which defines the constraints for the next state x^+ given the input u at the actual state x:

$$\begin{bmatrix} A_s A^{-1} & -A_s A^{-1} B \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x^+ \\ u \end{bmatrix} \leq \begin{bmatrix} b_s \\ b_u \end{bmatrix} \tag{4}$$

The $Reach(\mathcal{S})$ is then projection of the polyhedron defined in (4) into x^+. This set represents all the states x^+, such that there is an admissible input $u \in \mathbb{U}$ that will lead $x \in \mathcal{S}$ into x^+. The following results are obtained for the given problem:

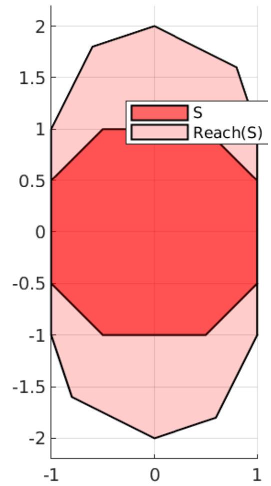

Figure 3: *Set S and the set reachable from S*

Clearly, the procedure shown above only works if the matrix A is invertible (if there are no zero eingenvalues), which is the case for this question. For this reason, the standard way of calculating the reach is by the following operation:

$$Reach(\mathcal{S}) = A \odot \mathcal{S} \oplus B \odot \mathbb{U} \tag{5}$$

where $A \odot \mathcal{S}$ is an affine mapping of the set $\mathcal{S}$ with the matrix A. For the case when $\mathcal{S}$ is convex, the affine mapping is done by simply scaling/multiplying the vertices of $\mathcal{S}$ by A. Further, the $\oplus$ operator between two polyhedrons is simply the Minkowski summation. For the case when $\mathcal{S}$ is convex, we achieve this by adding the vertices between each polyhedron for all the possible combinations. We then further remove redundant vertices.

Question 5

Given the constraint set $\mathcal{S} = \{x \mid A_s\, x \leq b_s\}$ and $\mathbb{U} = \{x \mid A_u\, u \leq b_u\}$ and the system dynamics $x^+ = Ax + Bu$, the backward reachability of $\mathcal{S}$ is defined as $Pre(\mathcal{S}) = \{x \mid x^+ = Ax + Bu \in \mathcal{S}, u \in \mathbb{U}\}$ and can be derived as follows:

$$
\begin{aligned}
x+ \in \mathcal{S} \quad &\Rightarrow \quad A_s(Ax + Bu) \leq b_s && (6)\\
u \in \mathbb{U} \quad &\Rightarrow \quad A_u u \leq b_u && (7)
\end{aligned}
$$

And therefore it can be rewritten as a new polyhedron:

$$
\begin{bmatrix} A_s A & A_s B \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} b_s \\ b_u \end{bmatrix} \tag{8}
$$

$Pre(\mathcal{S})$ is then the projection of the polyhedron defined in (8) into x. This set represents all the possible states that are able to reach any state in $\mathcal{S}$ with an admissible input $u \in \mathbb{U}$.

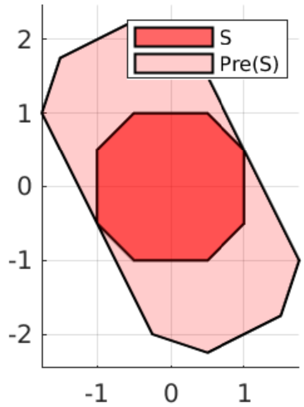

Figure 4: *Set S and the set backwards reachable from S*

Question 6

1. The shortest prediction horizon such that the RH controller is feasible until convergence to the origin is $N = 26$.

2. Using the function `invariantSet` to get the maximal invariant set $\mathcal{X}_f$, we find out that the given initial state $x_0 = [2\ 0]^\mathsf{T}$ belongs to the $\mathcal{X}_f$ and therefore also belongs to the feasibility set $\mathcal{X}_N$, see figure 5.

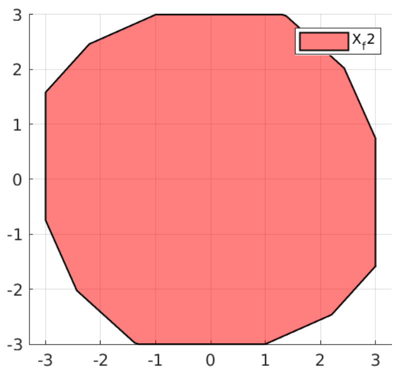

Figure 5: *Maximal invariant set $\mathcal{X}_f$*

Moreover, since $\mathcal{X}_f$ is control invariant the system will never leave $\mathcal{X}_f$ and therefore, the MPC will be equal to the unconstrained finite RH controller case. Further, we can calculate the controller gain and check the stability of the close-loop system.

```
N=2
[K0,~]=DP_09(A,B,N,Q,R,Pf)

K0 =
   -0.2000    0.4500

abs(eig(A-B*K0)) =
    0.6964
    0.6964
```

As seen, the close-loop is stable and therefore this RH controller will be feasible and convergence to the origin.

3. The feasibility set for the two designed controllers can be seen in the figure below:

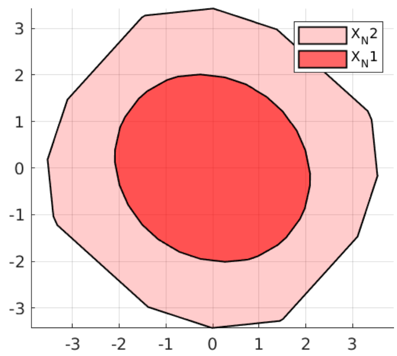

Figure 6: *Feasibility sets for the two RH controllers*

As expected, the size of the feasibility set for the first controller is much smaller than for the second one. This happens because the smaller the terminal state constraint set the smaller will be the feasibility set and in this case $\mathcal{X}_f1$ contains only the origin while $\mathcal{X}_f2$ is a relatively large set.

The number of variables to optimize in both cases are the same, which is equal to two variables. However, the set $\mathcal{X}_N1$ has 52 irredundant inequalities while $\mathcal{X}_N2$ has only 16:

```
>> XN1
        Polyhedron in R^2 with representations:
            H-rep (irredundant) : Inequalities  52 | Equalities   0
            V-rep (irredundant) : Vertices   52 | Rays   0
        Functions : none
>> XN2
        Polyhedron in R^2 with representations:
            H-rep (irredundant) : Inequalities  16 | Equalities   0
            V-rep (irredundant) : Vertices   16 | Rays   0
        Functions : none
```

We can then conclude that the first controller has a smaller feasibility region due to the small terminal state constraint and also will be much more computationally expensive to calculate the input for each time-step since it has many more inequality constraints.