SSY281 - Model Predictive Control
Assignment A01 - Basic Control

Lucas Rath - **Group 09**

January 30, 2019

Question 1

The biggest sampling interval h should be chosen in such a way that the discrete system retains all the frequencies from the original continuous-time signal. Further, one can consider that for this simple second-order system, the maximum frequency that is contributing to the spectrum is around the natural frequency w_n. Finally, according to the nyquist sampling theorem, we should choose the biggest sampling interval as follows:

$$h = \frac{1}{2 * \omega_n / 2} = \frac{\pi}{\omega_n} \tag{1}$$

Listing 1: *Largest possible sampling interval h*

```
wn = damp(Ac);
h  = pi/wn(1);     % [s]

h =
    4.4429
```

Listing 2: *Calculation of discrete state-space matrices using the largest possible sampling interval h*

```
    A =
       11.5920   16.3324
        8.1662   11.5920

    B =
       21.1839
       16.3324

    C =
         1     0
```

Question 2

Listing 3: Discrete state-space matrices for the extended system (4)

```
Aa =
    1.0025    0.1001    0.0038
    0.0500    1.0025    0.0501
         0         0         0

Ba =
    0.0013
    0.0500
    1.0000

Ca =
     1     0     0

Da =
     0
```

Question 3

All the three systems (2), (3), and (4) are both controllable and observable since their reachability and observability matrices are full rank.

Question 4

The system is not observable if the observability matrix is rank deficient $\Rightarrow det(W_o) = 0$. In this way we can calculate a matrix $C = [c_1 \; c_2]$ that will lead in a non observable system in the following way:

$$det(W_0) = det\,[C \quad CA_c]^T = det\begin{bmatrix} c_1 & c_2 \\ \frac{c_2}{2} & c_1 \end{bmatrix} = c1^2 - c2^2/2 = 0 \tag{2}$$

So, any C matrix satisfying $c1^2 - c2^2/2 = 0$ will lead in a unobservable system. Choosing c_1 arbitrarily to be equal 1, we get the following C matrix:

$$C = [1 \quad 1.4142] \tag{3}$$

Moreover, if the continuous time is unobservable, then the discrete formulations (3) and (4) will be also unobservable because the discrete representation is simply a ZOH sampling of the continuous time system.

Question 5

First, the dynamics equations can be written for $y(k)$ and $y(k+1)$ in the following way:

$$\begin{aligned} y(k) &= C\, x(k) \\ y(k+1) &= C\, x(k+1) = C\,[A\, x(k) + B\, u(k)] \end{aligned} \tag{4}$$

which can also be written in a matrix form:

$$\begin{bmatrix} y(k) \\ y(k+1) \end{bmatrix} = \begin{bmatrix} C \\ C\,A \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ C\,B \end{bmatrix} u(k) \tag{5}$$

Finally, we recognize the observability matrix W_o in (5), which results in:

$$x(k) = W_o^{-1} \cdot \left(\begin{bmatrix} y(k) \\ y(k+1) \end{bmatrix} - \begin{bmatrix} 0 \\ C\,B \end{bmatrix} u(k) \right) \tag{6}$$

Clearly, if we choose a matrix C that leads in a non-observable matrix W_o, then by definition W_o will be rank deficient and therefore not invertible. As a result, it will not be possible to recover the states $x(k)$ based on the last inputs and outputs.

Question 6

Listing 4: *Controller gains for the systems (2) and (4)*

```
K1 =
    34.7708    7.2399

K2 =
    34.9736    8.9834    0.4055
```

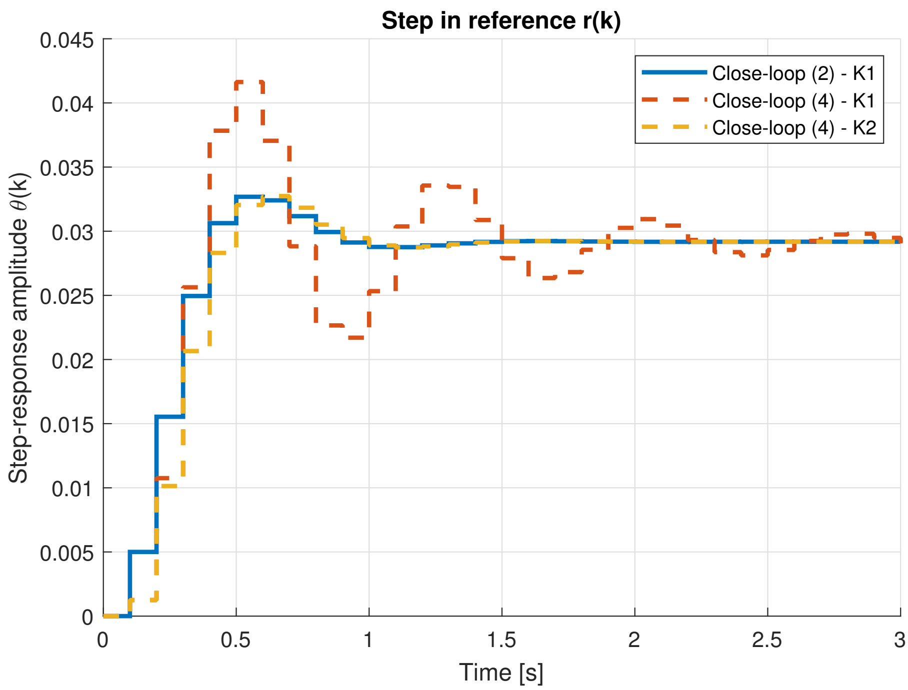

Figure 1: *Step-response $\theta(k)$ for different systems and controllers*

Clearly, if we zero-pad the controller K1 and use for the system (4) we modify the location of the desired poles. This can be checked by comparing the blue and the red curves in figure 1, since the system (4) presents much worse close-loop behavior. Alternatively, one can verify that the eigenvalues $eig(Aa - Ba * [K_1\ 0])$ are different than the desired poles p1, p2 and 0.

In order to maintain the same close-loop characteristics, one needs to design the controller again. Instead of zero-padding the controller gain, an extra desired pole infinitely fast (which corresponds to $z = 0$) should be added using the Ackermann's formula. Clearly, the yellow and the blue curves in figure 1 now present similar performance and also the same close-loop eigenvalues. As expected, the system (4) present some small delay in comparison to (2), due to the formulated input delay of 0.5h.

```
eig(Aa-Ba*[K1 0]) =      % K1 = place(A, B, [p1,p2]);
   0.5930 + 0.6472i
   0.5930 - 0.6472i
   0.4135 + 0.0000i

>> eig(Aa-Ba*K2) =       % K2 = place(Aa, Ba, [p1 p2 0]);
   0.5532 + 0.3785i
   0.5532 - 0.3785i
  -0.0000 + 0.0000i
```

Question 7

Listing 5: *Steady-state and input (xs us) for ys*

```
xs =
    0.5236
   -0.0000
   -0.2618

us =
   -0.2618
```

Since it is known that the system with the deviation variables δx and δu presents the same dynamics as the original system, we can use this property to define the new control input as:

$$\begin{aligned} \delta u(k) &= -K_2\, \delta x(k) \\ \delta x(k+1) &= A_a\, \delta x(k) + B_a\, \delta u \\ y &= Ca\ \ (\delta x(k) - x_s) \end{aligned} \tag{7}$$

As can be seen in figure 2, the system presented the same close-loop performance as in the last question, but now with steady-state response equal to the desired value $y_s = \pi/6 = 0.5236$.

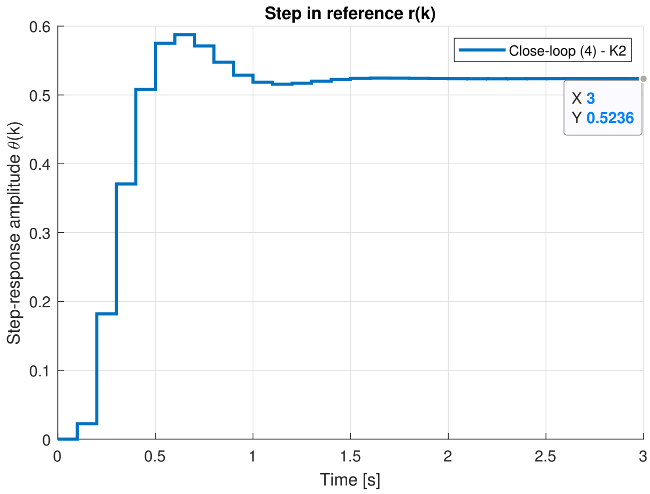

Figure 2

Question 8

Listing 6: *Discrete state-space matrices for the augmented system (6)*

```
Ae =
    1.0025    0.1001    0.0038         0
    0.0500    1.0025    0.0501    1.0000
         0         0         0         0
         0         0         0    1.0000


Be =
    0.0013
    0.0500
    1.0000
         0


Ce =
     1     0     0     0
De =
     0
```

This new system is not controllable since the determinant of the reachability matrix is rank-deficient. However, it can still be stabilizable if the uncontrollable modes are stable. This can be checked with help of the PBH test:

Listing 7: *PBH test*

```
eig(Ae) =
    1.0733      % Unstable mode
    0.9317
         0
    1.0000      % Marginally stable mode

rank([1.0733*eye(4)-Ae Be])  % => unstable pole is controllable
rank([1*eye(4)-Ae Be])       % => marginally stable pole is NOT controllable
```

- System is NOT stabilizable: there is one marginally stable mode is not controllable
- System is detectable: observability matrix is full-rank.

Question 9

Listing 8: *Designed controller for the system (5)*

```
K3 =
   0.5532 + 0.3785i
   0.5532 - 0.3785i
  -0.0000 + 0.0000i
   1.0000 + 0.0000i
```

We can not place the desired close-loop poles anywhere. We can only design the controller if the pole from the last mode is exactly equal to 1.

This happens because, as stated in the last question, we have one marginally stable and uncontrollable mode in our system at z=1. In this way, there is no way create a controller that will change the location of this pole. For this reason, we need to leave the last desired pole as 1.

Question 10

Listing 9: *Designed observer for system (6)*

```
    L =
        2.0050
       13.6608
       -1.9198
        3.0215
```