SSY281 - Model Predictive Control
Assignment A04 - Optimization basics and QP problems

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Question 1

A) If the optimization problem is convex, then for all $x_1, x_2 \in S \Rightarrow z = \theta x_1 + (1-\theta)x_2 \in S,\ 0 \leq \theta \leq 1$, where S is the feasible set $S = \{\ x \mid g_i(x) \leq 0, h_i(x) = 0, \forall i\}$. Therefore, for $\theta = 0.5 \Rightarrow z = (x_1 + x_2)/2$ and z must belong to the feasible set S.

B) If the optimization problem is convex and $a + b = 1,\ \ 0 \leq a \leq 1$ then $z = ax_1 + bx_2$ will be never worse than both x_1 and x_2 at the same time. Moreover, since $a + b = 1$ we can rewrite $a = \theta$ and $b = (1 - \theta)$ and according to the convexity theorem, z will be feasible and also:

$$\underbrace{f(\theta x_1 + (1-\theta)x_2)}_{f(z)} \leq \underbrace{\theta f(x_1) + (1-\theta)f(x_2)}_{\Omega}, \qquad \forall\, 0 \leq \theta \leq 1 \tag{1}$$

Moreover, one can easily verify that:

$$f(x_1) \leq \Omega \leq f(x_2), \quad \text{if} \quad f(x_2) \geq f(x_1) \tag{2}$$

$$f(x_2) \leq \Omega \leq f(x_1), \quad \text{if} \quad f(x_1) \geq f(x_2) \tag{3}$$

Therefore, if $f(z) \leq \Omega$, it follows that necessarily $f(z)$ is less than either $f(x_1)$ or $f(x_2)$. More formally: $f(z) \leq \max(f(x_1), f(x_2))$.

Indeed, it can happen that z is better than both x_1 and x_2. For instance, if x_1 and x_2 are chosen such that $f(x_1) = f(x_2)$, then from (1), it follows that $f(z) \leq f(x_1) = f(x_2)$. In this particular case, $f(z)$ will be necessarily equal or better than both x_1 and x_2.

C) Clearly, this condition is just a particular case from last item, when $\theta = 0.5$. Therefore all the assumptions made in the last item still hold.

Question 2

A certain set is said to be convex if all the line segments between two points of the set are also part of the set:

$$x_1, x_2 \in S \quad \Rightarrow \quad \theta x_1 + (1-\theta)x_2 \ \in\ S, \quad 0 \leq \theta \leq 1 \tag{4}$$

A) First, lets reformulate the definition of $S = \{x \in \Re^n \mid \alpha \leq a^T x \leq \beta\}$ as an intersection of two sets S_1 and S_2:

$$\begin{aligned} &S = S_1 \cap S_2 \\ &S_1 = \{x \in \Re^n \mid f_1(x) \leq 0\} \qquad f_1(x) = a^T x - \beta \\ &S_2 = \{x \in \Re^n \mid f_2(x) \leq 0\} \qquad f_2(x) = -a^T x + \alpha \end{aligned} \tag{5}$$

Then, it can be proven that S_1 is a convex set, i.e. if $x_1, x_2 \in S_1$, then also $\theta x_1 + (1-\theta)x_2 \ \in \ S_1$:

$$x_1, x_2 \in S_1 \quad \Rightarrow \quad f_1(x_1), f_1(x_2) \leq 0$$

$$f_1\big(\theta x_1 + (1-\theta)x_2\big) \ = \ a^T\,(\theta x_1 + (1-\theta)x_2) - b \ = \ \theta \underbrace{f_1(x_1)}_{\leq 0} + (1-\theta) \underbrace{f_1(x_2)}_{\leq 0} \ \leq \ 0$$

$$\Rightarrow \quad \theta x_1 + (1-\theta)x_2 \ \in \ S_1 \tag{6}$$

It has then be shown that S_1 is convex. Following the same logic, it can be easily proven that S_2 is also a convex set. Finally, the intersection between convex sets is also convex. Therefore, S is convex.

B) The set M is convex if for all $x_1, x_2 \in M$, then also $\theta x_1 + (1-\theta)x_2 \ \in \ M$:

$$M = \{x \mid g(x) \leq 0\}\ , \qquad g(x) = \|x - y\| - f(y) \quad \forall y \in S \subseteq \mathbb{R}^n$$

$$x_1, x_2 \in M \quad \Rightarrow \quad g(x_1), g(x_2) \leq 0$$

$$g\big(\theta x_1 + (1-\theta)x_2\big) \ = \ \|\theta x_1 + (1-\theta)x_2 - y\| - f(y) \tag{7}$$

Now, by using the triangle inequality $|a+b| \leq |a| + |b|$:

$$g\big(\theta x_1 + (1-\theta)x_2\big) \ \leq \ \theta\big(\|x_1 - y\| - f(y)\big) + (1-\theta)\big(\|x_2 - y\| - f(y)\big) \ = \ \theta \underbrace{g(x_1)}_{\leq 0} + (1-\theta) \underbrace{g(x_2)}_{\leq 0} \leq 0$$

$$\Rightarrow \quad g\big(\theta x_1 + (1-\theta)x_2\big) \ \leq \ 0 \tag{8}$$

Equation 8 has then proven that the set M is convex for all $y \in S$.

C) First we can rewrite the definition of the convex set in the following way:

$$M = \{x \mid g(x) \leq 0\} \tag{9}$$

where:

$$\begin{aligned} g(x) &= \|x - x_0\| - \|x - y\| \\ &= (x - x_0)^T(x - x_0) - (x-y)^T(x-y) \\ &= 2(y - x_0)x - (y^T y - x_0^T x_0) \end{aligned} \tag{10}$$

As can be notice, for a fixed y, the set M is a halfspace, which as known to be convex. Therefore, since $y \in S$, the set M can be reformulated as an intersection of halfspaces:

$$\bigcap_{y \in S} \{x \mid \|x - x_0\| - \|x - y\| \leq 0\} \tag{11}$$

and as explained before, the intersection of convex sets is also an convex set. Therefore M is convex.

Question 3

1-Norm)

The 1-norm objective is given by:

$$\begin{aligned} \min_{x} \quad & |Ax|_1 \\ s.t. \quad & Fx \le g \end{aligned} \tag{12}$$

whose cost function can be rewritten as:

$$|Ax|_1 = \sum_i |a^i x| \tag{13}$$

where a^i is the i-th row of the matrix A. Each term $|a^i x|$ can then be read as equal to the smaller number σ_i that satisfies the following equation:

$$|a^i x| = \begin{cases} \min\limits_{\sigma_i} & \sigma_i \\ s.t. & \sigma_i \ge |a^i x| \end{cases} \tag{14}$$

Coupling this statement above in (12) and (13), we have:

$$\begin{aligned} \min_{x} \quad & \sum_i \min_{\sigma_i} \sigma_i \\ s.t. \quad & \begin{matrix} Fx \le g & \\ \sigma_i \ge +a^i x & \quad 1 \le i \le N \\ \sigma_i \ge -a^i x & \end{matrix} \end{aligned} \tag{15}$$

which can be finally rewritten in the standard form as:

$$\begin{aligned} \min_{x,\sigma} \quad & \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}^T \begin{bmatrix} x \\ \sigma \end{bmatrix} \\ s.t. \quad & \begin{bmatrix} F & 0 \\ A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} \le \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{16}$$

where $\mathbf{0}$ and $\mathbf{1}$ are column vectors of zeros and ones respectively, with the same dimension as $x, \sigma \in \mathbb{R}^n$. Note that now the optimization variables are x and σ and the problem is clearly a linear optimization problem.

∞-Norm)

The infinity norm optimization problem is given bellow:

$$\begin{aligned} \min_{x} \quad & |Ax|_\infty \\ s.t. \quad & Fx \le g \end{aligned} \tag{17}$$

which cost function can be also defined as:

$$|Ax|_\infty = \max_i |a^i x| = \begin{cases} \min\limits_{\gamma} & \gamma \\ s.t. & \mathbf{1}\gamma \ge |Ax| \end{cases} \tag{18}$$

where again a^i is the i-th row of the matrix A and $\gamma \in \mathbb{R}^1$ is an auxiliary variable. Coupling (18) in (17), and using matrix representation, we can rewrite the optimization function:

$$\begin{aligned} \min_{x,\gamma} \quad & \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ \gamma \end{bmatrix} \\ s.t. \quad & \begin{bmatrix} F & 0 \\ A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} \leq \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{19}$$

Question 4

A) To adapt to this new cost function, trivial changes must be made in equation (16):

$$\begin{aligned} \min_{x,\sigma} \quad & \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}^T \begin{bmatrix} x \\ \sigma \end{bmatrix} \\ s.t. \quad & \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{aligned} \tag{20}$$

B) The same can be made with equation (19):

$$\begin{aligned} \min_{x,\gamma} \quad & \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ \gamma \end{bmatrix} \\ s.t. \quad & \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{aligned} \tag{21}$$

Question 5

A) Solving the quadratic optimization problem leads to the following results:

```
x =
    2.5000
         0
         0
         0
fval =
    3.1250
```

B) The lagrange multipliers calculated with `quadprog` are:

```
lambda.ineqlin =
         0
    0.0000
    0.0000
    0.0000
    2.5000
         0
         0
         0
```

which correspond to the following constraints (in standard form):

$$\begin{aligned} x_1 &\leq 5 \\ x_2 &\leq 1 \\ u_0 &\leq 2 \\ u_0 &\leq 2 \\ -x_1 &\leq -2.5 \\ -x_2 &\leq 1 \\ -u_0 &\leq 2 \\ -u_0 &\leq 2 \end{aligned} \tag{22}$$

To check if the solution satisfies the KKT conditions we need to verify 4 conditions. Very straightforward, the following conditions can be verified to hold:

$$\begin{aligned} \mu^* &\geq 0 \\ g(x^*) &\leq 0 \\ \mu_i \, g_i(x^*) &= 0 \end{aligned} \tag{23}$$

We can easily see that all μ^* are greater than zero, all the constraints $g_i(x^*)$ are satisfied, and that either μ^* or $g_i(x^*)$ are zero for each constraint.

The last KTT condition follows:

$$\begin{aligned} &\nabla f(x^*) + \nabla g(x^*)\mu^* = 0 \\ = &\begin{bmatrix} x1^* \\ x_2^* \\ u_0^* \\ u_1^* \end{bmatrix} + [I \ \ -I]\, \mu^* = \begin{bmatrix} 2.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \end{aligned} \tag{24}$$

Therefore, all the KKT conditions are satisfied.

As can be easily confirmed, the only active constraint is $-x_1 \leq -2.5$, since $\mu \neq 0$ for this constraint.

C)

If the lower bound of x_1 was removed, the problem would be unfeasible since the optimal value p^0 would be equal to minus infinity:

$$p^0 = \inf\{f(x)|g_i(x) \leq 0, \quad i = 1, ..., m\} = -\infty \tag{25}$$

The reason is because x_1 could then be decreased unboundedly always contributing to the reduction of the loss function $f(x, u)$.

On the other hand, removing the upper bound constraint of x_1 would in this case not change the result and the optimal value $\infty < p^0 < \infty$.