SSY281 - Model Predictive Control
Assignment A05 - MPC Stability

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Question 1

Given the discrete linear system, the Lyapunov function $V(x(k)) = x^\intercal(k)Sx(k)$ can be determined by solving the following equation:

$$V(x^+) - V(x) = x^\intercal(A^\intercal SA - S)x = -x^\intercal Qx \tag{1}$$

where Q must be a positive definite matrix. By defining Q arbitrarily as the identity matrix, we can calculate S:

$$S = \begin{bmatrix} 2.8526 & 0.0711 \\ 0.0711 & 1.0684 \end{bmatrix} \tag{2}$$

Question 2

First, $V(x)$ can be defined as a function of x_0 exclusively:

$$V(x_0) = \sum_{i=0}^{\infty}(x(i)^\intercal Qx(i) + u(i)^\intercal Ru(i)) = \sum_{i=0}^{\infty}(x(i)^\intercal Qx(i) + (-Kx(i))^\intercal R(-Kx(i))) \tag{3}$$

where K is the LQ controller obtained solving the dare with the given parameters.

Moreover, if $V(x)$ is a Lyapunov function, then V must decay along solutions of the system:

$$V(x^+) - V(x) = -x_0^\intercal Qx_0 - (Kx_0)^\intercal R(Kx_0) \tag{4}$$

Hence, since its given that Q and R are positive definite matrices, then the energy of the system V will decay along solutions and therefore V can be said to be a Lyapunov function.

Question 3

A) The shortest N such the RH controller stabilizes the system is $N_3 = 5$.

B) When N=1, Q does not affect the controller and therefore does not affect also the system poles.

C) If the terminal cost $V_f = x^\intercal Px$ is chosen such that P is the solution of the stationary `dare`, the system will be stable for any $N \geq 1$. Under specific conditions stated in Theorem 10.10 in the lecture notes, then the finite time solution will inherit some of the nice properties of the infinite time solution, e.g. stability.

D) Another R, for which the system is stable is $R = 0.1$

Question 4

When $N = 1$ the LQ unconstrained LQ optimal controller is calculate using the following equation (equation 19 of the lecture notes):

$$K = (R + B^\top P_f B)^{-1} B^\top P_f A \tag{5}$$

For the given configuration, it can be easily verified that $B^\top P_f B = p_n$ and $B^\top P_f = [0, \cdots, 0, p_n]$. Therefore $B^\top P_f A = 0_{1\times n}$ and K reduces to:

$$K = \frac{0_{1\times n}}{R + p_n} = 0_{1\times n} \tag{6}$$

The system is stable when all the eigenvalues of the close loop system are inside the unit circle. However, given that $K = 0_{1\times n}$, the close loop poles are simply the poles of A.

$$\lambda(A - BK) = \lambda(A) \tag{7}$$

However, as given in the exercise, A is unstable and so will be the close loop system. Therefore there are no Q and R that will make the close loop system stable for the given system parameters.

Question 5

A) For $N = 1$, the LQ controller is simply given by:

$$K(0) = (R + B^\top P_f B)^{-1} B^\top P_f A \tag{8}$$

If the system is stable, then the close loop has all eigenvalues inside the unit circle, i.e. $|\lambda_i(A-BK)| < 1$, $\forall i$. Further, substituting the numerical values of matrices A, B, P_f we get:

$$|\lambda_{1,2}(A - BK)| = \left| \frac{6\,\mathrm{R} + 1 \pm \sqrt{12\,\mathrm{R}^2 - 12\,\mathrm{R} + 1}}{2\,(\mathrm{R} + 1)} \right| < 1 \tag{9}$$

Lets then split the analysis of the equation above in two cases. First, when $12\,\mathrm{R}^2 - 12\,\mathrm{R} + 1 >= 0$, both poles are real and by analyzing (9), one can check that they are also positive since $R \geq 0$ and therefore it is not possible that the poles assume negative values. Thus:

$$\frac{6\,\mathrm{Rs} + 1 \pm \sqrt{12\,\mathrm{R}^2 - 12\,\mathrm{R} + 1}}{2\,(\mathrm{R} + 1)} < 1 \tag{10}$$

$$\Rightarrow \quad -4R + 1 > \pm\sqrt{12\,\mathrm{R}^2 - 12\,\mathrm{R} + 1} \tag{11}$$

Again, since the square root is always positive, one can ignore the "minus" case of equation 11, since it will always be less than the "plus" case.

$$-4R + 1 > \sqrt{12\,\mathrm{R}^2 - 12\,\mathrm{R} + 1} \tag{12}$$

$$\Rightarrow \quad 16R^2 - 8R + 1 > 12R^2 - 12R + 1 \tag{13}$$

$$\Rightarrow \quad R^2 + R > 0 \tag{14}$$

$$\Rightarrow \quad \begin{cases} R > 0 \\ R < -1 \qquad \text{(not possible, since } R > 0) \end{cases} \tag{15}$$

Indeed, when $12\,\mathrm{R}^2 - 12\,\mathrm{R} + 1 >= 0$ implies that $(R \le \frac{3-\sqrt{6}}{6}) \vee (R \ge \frac{3+\sqrt{6}}{6})$. Additionally, the left side of equation 12 can never be negative, and therefore $R < 1/4$. Joining all the results, we get that for real eigenvalues:

$$0 < R \le \frac{3-\sqrt{6}}{6} \qquad \text{(real eigenvalues case: } R \le \frac{3-\sqrt{6}}{6} \vee R \ge \frac{3+\sqrt{6}}{6}\text{)} \tag{16}$$

Now, analyzing the second case, when the eigenvalues are a complex and conjugate pair:

$$\left| \frac{6\,\mathrm{R} + 1 \pm i\sqrt{-(12\,\mathrm{R}^2 - 12\,\mathrm{R} + 1)}}{2\,(\mathrm{R}+1)} \right| < 1 \tag{17}$$

$$\Rightarrow \quad \frac{(6R+1)^2 - (12R^2 - 12R + 1)}{(2R+2)^2} < 1 \tag{18}$$

$$\Rightarrow \quad 6R - R - 1 < 0 \tag{19}$$

$$\Rightarrow \quad \begin{cases} R > -1 \\ R < 0.2 \end{cases} \tag{20}$$

Indeed, the result above is valid for the interval when the eigenvalues are complex. We therefore end up with:

$$\frac{3-\sqrt{6}}{6} < R < 0.2 \qquad \text{(complex eigenvalues case: } \frac{3-\sqrt{6}}{6} \le R < \frac{3+\sqrt{6}}{6}\text{)} \tag{21}$$

Finally, making the union of both solutions in (16) and (21) we get:

$$\boxed{0 < R < 0.2} \tag{22}$$

B) The procedure for N=2 is quite similar compared to item A. However, the receding horizon will now be 2. Therefore we need to calculate the optimal cost to go from N=2 to 1 to be able to calculate the controller gain at the actual state K(0):

$$P(1) = Q + A^\mathsf{T} P_f A - A^\mathsf{T} P_f B (R + B^\mathsf{T} P_f B)^{-1}\, B^\mathsf{T} P_f A \tag{23}$$

$$K = K(0) = (R + B^\mathsf{T} P(1) B)^{-1} B^\mathsf{T}\, P(1)\, A \tag{24}$$

Again, we want the close loop poles to be stable. Therefore:

$$|\lambda_{1,2}(A - BK)| = \left| \frac{6\,\mathrm{R}^2 + 37\,\mathrm{R} + 1 \pm \sqrt{12\,\mathrm{R}^4 - 252\,\mathrm{R}^3 + 661\,\mathrm{R}^2 + 26\,\mathrm{R} + 1}}{2\,(\mathrm{R}^2 + 28\,\mathrm{R} + 2)} \right| < 1 \tag{25}$$

Lets analyze the two cases separately, when the polynomial inside the root in the equation above is positive and negative. First, solving for the positive case with Wolframalpha, we get the following solution intervals:

$$\Rightarrow \begin{cases} 2.618 < R < 3.12 \\ -0.07 < R < 0.382 \end{cases} \qquad \text{(real eigenvalues case: } R < 3.12 \vee R > 17.92\text{)} \tag{26}$$

Considering now the case when the root leads to a complex number, we can rewrite 25 as:

$$|\lambda_{1,2}(A - BK)| = \left| \frac{6\,\mathrm{R}^2 + 37\,\mathrm{R} + 1 \pm i\sqrt{-(12\,\mathrm{R}^4 - 252\,\mathrm{R}^3 + 661\,\mathrm{R}^2 + 26\,\mathrm{R} + 1)}}{2\,(\mathrm{R}^2 + 28\,\mathrm{R} + 2)} \right| < 1 \tag{27}$$

$$\Rightarrow \quad \frac{(6\,\mathrm{R}^2 + 37\,\mathrm{R} + 1)^2 - (12\,\mathrm{R}^4 - 252\,\mathrm{R}^3 + 661\,\mathrm{R}^2 + 26\,\mathrm{R} + 1)}{(2R^2 + 56R + 4)^2} < 1 \tag{28}$$

Solving equation above with Wolframalpha leads to:

$$\Rightarrow \left\{ \; 3.12 < R < 4.49 \qquad \text{(complex eigenvalues case: } 3.12 < R < 17.92\text{)} \right. \tag{29}$$

We can now make the union between the two cases in (26) and (29), but not forgetting to respect the interval for each case and assume that R is positive. We finally get:

$$\boxed{0 < R < 0.382 \;\cup\; 2.618 < R < 4.49} \tag{30}$$

Question 6

An infinite-time LQR controller is given by:

$$K = (R + B^\intercal PB)^{-1} B^\intercal PA \tag{31}$$

where P is the solution of the `dare`:

$$P = Q + A^\intercal PA - A^\intercal PBK \tag{32}$$

The close loop system matrix A_t will be then:

$$A_t = A - BK = A - B(A^\intercal PB)^{-1}(Q + A^\intercal PA - P) \tag{33}$$

To show that the close-loop system is stable for any Q, R, A and B we first propose $V(x) = x^\intercal Px$, with P definite-positive given by the stationary dare, as the Lyapunov function. If the system is stable, V must decay along the solutions of the system, i.e. $V(x^+) - V(x) < 0$:

$$\begin{aligned} V(x^+) - V(x) \; &= (A_t x)^\intercal P(A_t x) - x^\intercal Px & (34)\\ &= x^\intercal (A - BK)^\intercal P(A - BK)x - x^\intercal Px & (35)\\ &= x^\intercal (A^\intercal PA + K^\intercal B^\intercal PBK - A^\intercal PBK - K^\intercal BPA - P)x & (36) \end{aligned}$$

Now, from equation 32, we now that $-A^\intercal PBK = P - Q - A^\intercal PA$. Replacing this term in the equation above leads to:

$$V(x^+) - V(x) \;=\; x^\intercal(-Q - K^\intercal B^\intercal P(A - BK))x \tag{37}$$

Moreover, we can rewrite equation 31 as $B^\intercal P(A - BK) = RK$ and replace into the equation above:

$$V(x^+) - V(x) \;=\; x^\intercal \underbrace{(-Q - K^\intercal RK)}_{\leq 0} x \tag{38}$$

Hence, since Q and R are positive-definite matrices, the difference $V(x^+) - V(x)$ will be always negative and therefore V will decay along solutions of the system. The system is therefore stable when controlled by and infinite-time LQ controller.