SSY281 Model Predictive Control

Assignment 1

Basic Control

$\boxed{\text{Deadline: January 30, 23:59}}$

Systems & Control

Department of Electrical Engineering

Chalmers University of Technology

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Instructions

This assignment is **individual** and must be solved according to the following rules and instructions:

- Written report:
  - For those questions where a text (motivations, explanations, observations from simulations) rather than a numerical value is asked, the answers should be concisely written in the report.
  - Figures included in the report should have legends, and axes should be labelled.
  - The report should be uploaded *before the deadline* to your project document area in PingPong.
  - Name the report as A1_XX.pdf, where XX is your *group* number[1].
- Code:
  - Your code should be written in the Matlab template provided with this assignment following the instructions therein.
  - Name the Matlab script as A1_XX.m, where XX is your *group* number.
- Grading:
  - This assignment if worth **10 points** in total. Each question is worth 1 point.

Objectives

The purpose of this assignment is to refresh your knowledge about basic control concepts that are needed in the course.

1 Discrete state-space model

A pendulum with length l and point mass m, subject to gravity force and controlled by a motor at the pivot point, giving an external torque τ can be described by the following differential equation:

$$ml^2\ddot{\theta} = \tau + mgl\sin\theta, \tag{1}$$

where θ is the angle relative to the vertical direction. By defining the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$, and using the small angle approximation $\sin\theta \approx \theta$, an approximate linear model in the state-space form can be rewritten as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= g/l \cdot x_1(t) + 1/ml^2 \cdot \tau(t) = \alpha x_1(t) + \beta u(t),\\ y(t) &= x_1(t),\end{aligned} \tag{2}$$

where $u(t) = \tau(t)$.

Question 1. *Using the largest possible sampling interval h and $\alpha = 0.5$, $\beta = 1$ in (2), find the matrices A, B, C in the the following discrete-time model.*

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k),\\ y(k) &= Cx(k).\end{aligned} \tag{3}$$

Explain how you chose h in the report.

Hint: the commands `sys=ss(Ac,Bc,Cc,Dc)` and `sysd=c2d(sys,h)` in Matlab can be used.

For the next questions use the $h = 0.1$ and calculate the matrices of the discrete-time system as in slide 12

Question 2. *In case that the continuous system (2) has an input delay of $0.5h$ second, calculate A_a, B_a, C_a in the following model using MATLAB.*

$$\begin{aligned}\zeta(k+1) &= A_a\zeta(k) + B_a u(k)\\ y(k) &= C_a\zeta(k)\end{aligned}, \quad \zeta(k) = \begin{bmatrix} x(k) \\ u(k-1)\end{bmatrix} \tag{4}$$

Hint: The commands `expm` and `int` in MATLAB can be used.

2 Controllability and Observability

Question 3. *Check the controllability and the observability of systems* (2), (3), *and* (4).

Hint: The commands `ctrb`, `obsv`, and `rank` in MATLAB to check can be used.

Question 4. *Find a non-zero matrix C for the system* (2), *i.e. redefine $y(t)$, such that the system is not observable.*

Answer the following questions

1. *Can you conclude that the same output (C matrix) makes the system* (3) *unobservable as well?*
2. *Can you conclude that the same output (C matrix) makes the system* (4) *unobservable as well?*

Briefly motivate your answers in the report.

Question 5. *Calculate the states of the model* (3) *at time t using the output at times t and $t+1$.*

Calculate the states of the model (3) *with the matrix C found in the previous question and explain your findings in the report.*

3 Feedback Design

In this section, you use the *pole placement* technique to design l*inear controllers and observers* for the pendulum.

Hint: The commands `place` and `step` in MATLAB can be used for pole-placement and plotting the step response, respectively.

Question 6. *Consider the discrete-time model* (3) *of the pendulum in closed-loop with the state-feedback control law $u(k) = -Kx(k)$.*

Find the controller gain K such that the poles of the closed-loop system in discrete time matches the poles $\lambda_{1,2} = -4 \pm 6i$, given for the for the continuous-time system (2), *where symbol i is the imaginary unit.*

Plot the step response of the closed loop discrete-time system together with the response of the delayed system (4) *in closed-loop with the same state-feedback gain. In the report, explain the difference you observe.*

For the delayed system (4), *find a new state-feedback gain to recover the same closed-loop response as before. Add the plot of the step response to the previous figure, compare the results and explain your observations in the report.*

4 Set-point tracking and disturbance rejection

In this section, we try to regulate the pendulum angle at a desired position and reject an input disturbance. For the questions in this section, simulate your system using a simple `for` loop and without `step` function.

Question 7. *For the system* (4)*, find the steady-state and input* (x_s, u_s)*, when* $y_s = \frac{\pi}{6}$*. Use these steady-state input and output and the feedback gain that you calculated in* ***Question 6****, to regulate the output to its desired value when the initial condition is zero. Plot the output and explain your observations in the report.*

Question 8. *Consider the following discrete model where* $d(k)$ *is constant disturbance.*

$$\begin{aligned} \zeta(k+1) &= A_a\zeta(k) + B_a u(k) + B_d d(k) \\ y(k) &= C_a\zeta(k) \end{aligned}, \quad \zeta(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{5}$$

Augment the state vector with the disturbance and define the new augmented system as

$$\begin{aligned} \xi(k+1) &= A_e\xi(k) + B_e u(k) \\ y(k) &= C_e\xi(k) \end{aligned}, \quad \xi(k) = \begin{bmatrix} \zeta(k) \\ d(k) \end{bmatrix}. \tag{6}$$

Find the matrices A_e*,* B_e*, and* C_e *and check the stabilizability and the detectability of the augmented system.*

Question 9. *Consider the discrete-time model* (3) *and augment it as in* (5)*. Design a linear controller to place the poles of the closed-loop system as in* ***Question 6*** *and at one for the new pole. Can we place the last pole somewhere else? Why? Answer these two questions in the report.*

For the following linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{7}$$

when the system is detectable, a linear observer can be designed to estimate the system states, based on its measured output and input.

The observer includes a “copy” of the system dynamics and a difference, between the estimated and measured outputs, to adjust the estimated state as follows

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)), \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned}, \tag{8}$$

where L is a constant matrix which is called the observer gain. By defining the estimation error as $e(k) = x(k) - \hat{x}(k)$, one can calculate the error dynamics as follows.

$$e(k+1) = (A - LC)e(k). \tag{9}$$

Consequently, by designing L such that $e(k)$ is exponentially stable, then the error converges to zero and the state estimation converges to the real state. The observer gain L can be calculated with the `place` command in MATLAB, by using $A^\top$ and $C^\top$ instead of A and B as input arguments, while the its output is the gain $L^\top$.

Question 10. *Design a simple linear observer for the augmented system* (6) *with poles placed at* $\{0.1, 0.2, 0.3, 0.4\}$.