Crane Keller Box Formualtion

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0.1 Keller Box Method

Recalling our governing boundary layer PDE's for Cranes flow (**??**)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{1b}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \mathrm{Pr}^{-1}\frac{\partial^2 T}{\partial y^2}+\mathrm{Ec}\,\mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{1c}$$

and corresponding boundary conditions

$$u-x=v=T-1=0 \quad \text{at } y = 0, \tag{1d}$$

$$u\to T\to 0 \quad \text{as } y \to 0. \tag{1e}$$

To validate our similarity solutions we compared them to numerical solutions of the boundary layer equations using the Keller Box method Keller and Cebeci (1971). The Keller box method is an implicit finite difference scheme which has been used to solve a wide variety of nonlinear boundary layer problems. The method is described in detail in Cebeci and Bradshaw (2012), however we will provide an explanation of how the method was applied to our extension of Crane's model including both viscothermal and dissipative effects. To apply the Keller Box method we define

$$a=v,\quad b=u,\quad c = u',$$

$$d=T,\quad e = T',$$

and write our system (**??**) as

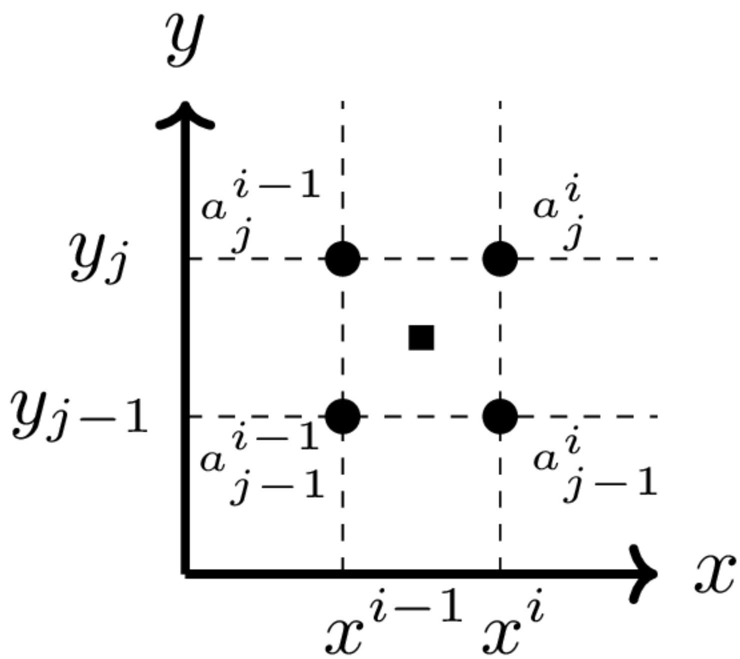

Figure 1: Diagram outlining how the Keller box scheme is discretised.

$$a' + b_x = 0, \tag{2a}$$
$$b' - c = 0, \tag{2b}$$
$$\mu(d)(c' - mec) - bb_x - ac = 0, \tag{2c}$$
$$d' - e = 0, \tag{2d}$$
$$\frac{1}{\mathrm{Pr}}e' - bd_x - ae + \mathrm{Ec}\,\mu(d)(cx)^2 = 0. \tag{2e}$$

To initialise the scheme we set the initial temperature and velocity profiles at $x = 0$, equal to the similarity solutions computed above for a given value of the sensitivity parameter. This is then used as the initial guess for the profile at $x^i = x^{i-1} + \Delta x$. The ODE's are then evaluated at $(x^i, y_{j-1/2})$ while the PDE's are evaluated at $(x^{i-1/2}, y_{j-1/2})$. As such it is useful to define the following quantities

$$
\begin{aligned}
\Delta_y a_j^i &= \frac{1}{2\Delta y_j}(a_j^i + a_j^{i-1} - a_{j-1}^i - a_{j-1}^{i-1}),\\
\Delta_x a_j^i &= \frac{1}{2\Delta x_i}(a_j^i - a_j^{i-1} + a_{j-1}^i - a_{j-1}^{i-1}),\\
a_{j-1/2}^{i-1/2} &= \frac{1}{4}(a_j^i + a_j^{i-1} + a_{j-1}^i + a_{j-1}^{i-1}).
\end{aligned}
$$

which will be used to simplify the notation in the explanation that follows. Similar anonymous functions appear in the Matlab script to improve the readability of the code. The next step is to linearise (2)

Since a, b, c, d, e are known at the previous step in x, x_{i-1}, we use this as our initial guess at x_i by setting

$$
a_j^i = a_j^{i(n)} + \delta a_j^i
$$

where $a_j^{i(0)} = a_j^{i-1}$ and δ denotes a correction to our current guess, which is assumed to be small. To simplify notation the superscripts on the perturbed quantities will be droped since their x location is known. The viscosity term is written

$$
\begin{aligned}
\mu(d_{j-1/2}^{i-1/2}) &= e^{-m\left(d_{j-1/2}^{i-1/2} + \frac{\delta d_j + \delta d_{j-1}}{4}\right)}.\\
&= e^{-m d_{j-1/2}^{i-1/2}} e^{-\frac{m}{4}(\delta d_j + \delta d_{j-1})},\\
&= e^{-m d_{j-1/2}^{i-1/2}}(1 - \frac{m}{4}[\delta d_j + \delta d_{j-1}]) + \mathcal{O}(\delta^2),\\
\mu(d_{j-1/2}^{i-1/2}) &= \left[1 + m d_{j-1/2}^{i-1/2} + \frac{m}{4}\left(\delta d_j + \delta d_{j-1}\right)\right]^{-1},\\
&= \left[1 + m d_{j-1/2}^{i-1/2}\right]^{-1} \left[1 + \frac{m}{4}\frac{\delta d_j + \delta d_{j-1}}{1 + m d_{j-1/2}^{i-1/2}}\right]^{-1},\\
&= \left[1 + m d_{j-1/2}^{i-1/2}\right]^{-1} \left(1 - \frac{m}{4}\frac{\delta d_j + \delta d_{j-1}}{1 + m d_{j-1/2}^{i-1/2}}\right) + \mathcal{O}(\delta^2),
\end{aligned}
$$

for the exponential and inverse distributions respectively. Substituting this into our first order system yields

$$
\begin{aligned}
\alpha_1\delta a_j + \alpha_2\delta a_{j-1} + \alpha_3\delta b_j + \alpha_4\delta b_{j-1} &= r_1, && \text{(3a)}\\
\frac{1}{\Delta y}(\delta b_j - \delta b_{j-1}) - \frac{1}{2}(\delta c_j + \delta c_{j-1}) &= r_2 = 0, && \text{(3b)}\\
\beta_1\delta a_j + \beta_2\delta a_{j-1} + \beta_3\delta b_j + \beta_4 b_{j-1} + \beta_5\delta c_j + \beta_6\delta c_{j-1} & \\
+\beta_7\delta d_j + \beta_8\delta d_{j-1} + \beta_9\delta e_j + \beta_{10}\delta e_{j-1} &= r_3, && \text{(3c)}\\
\frac{1}{\Delta y}(\delta d_j - \delta d_{j-1}) - \frac{1}{2}(\delta e_j + \delta e_{j-1}) &= r_4 = 0, && \text{(3d)}\\
\gamma_1\delta a_j + \gamma_2\delta a_{j-1} + \gamma_3\delta b_j + \gamma_4\delta b_{j-1} + \gamma_5\delta c_j + \gamma_6\delta c_{j-1} & \\
+\gamma_7\delta d_j + \gamma_8\delta d_{j-1}\gamma_9\delta e_j + \gamma_{10}\delta e_{j-1} &= r_5, && \text{(3e)}
\end{aligned}
$$

where the coefficients of the corrections in the continuity, momentum, and energy equations are outlined below. Note that we have only imposed that the perturbations satisfy the ordinary differential equations in (3b) and (3d). While this is a nonstandard way of applying the Keller Box method, it prevented oscillations from developing in the corrections which in turn prevented the scheme from converging at a sufficiently large distance downstream. Applying the Keller Box method as suggested in Cebeci and Bradshaw (2012) would instead result in

$$
\begin{aligned}
r_2 &= -\frac{1}{\Delta y}(b^i_j - b^i_{j-1}) + \frac{1}{2}(c^i_j - c^i_{j-1}),\\
r_4 &= -\frac{1}{\Delta y}(d^i_j - d^i_{j-1}) + \frac{1}{2}(e^i_j - e^i_{j-1}),
\end{aligned}
$$

however we will show that our modified version of the method strongly agrees with our similarity solutions for $\mathrm{Ec} = 0$ and a finite difference scheme which can only be used when the temperature and fluid flow are uncoupled as it is unable to handle the nonlinear terms which arise in the coupled case. The errors for the remaining equations are defined

$$
\begin{aligned}
r_1 &= -\Delta_y a^i_j - \Delta_x b^i_j, && \text{(4a)}\\
r_3 &= -\mu^{i-1/2}_{j-1/2}\left(\Delta_y c^i_j - m\mu^{i-1/2}_{j-1/2} c^{i-1/2}_{j-1/2} e^{i-1/2}_{j-1/2}\right) + b^{i-1/2}_{j-1/2}\Delta_x b^i_j + a^{i-1/2}_{j-1/2} c^{i-1/2}_{j-1/2}, && \text{(4b)}\\
r_5 &= -\mathrm{Pr}^{-1}\,\Delta_y e^i_j + b^{i-1/2}_{j-1/2}\Delta_x d^i_j + a^{i-1/2}_{j-1/2} e^{i-1/2}_{j-1/2} - \mathrm{Ec}\,\mu^{i-1/2}_{j-1/2}\left(c^{i-1/2}_{j-1/2}\right)^2. && \text{(4c)}
\end{aligned}
$$

The coefficients of the continuity equation are

$$
\begin{aligned}
\alpha_1 = -\alpha_2 &= \frac{1}{2\Delta y}, && \text{(5a)}\\
\alpha_3 = \alpha_4 &= \frac{1}{2\Delta x}, && \text{(5b)}
\end{aligned}
$$

for the momentum equation we have

$$
\begin{aligned}
\beta_1 = \beta_2 &= -\frac{1}{4} c_{j-1/2}^{i-1/2}, && (6a)\\
\beta_3 = \beta_4 &= -\frac{1}{4}\Delta_x b_j^i - \frac{1}{2\Delta x} b_{j-1/2}^{i-1/2}, && (6b)\\
\beta_5 &= \mu_{j-1/2}^{i-1/2}\left[\frac{1}{2\Delta y} - \frac{m}{4}\mu_{j-1/2}^{i-1/2} e_{j-1/2}^{i-1/2}\right] + \frac{1}{4} a_{j-1/2}^{i-1/2}, && (6c)\\
\beta_6 &= \mu_{j-1/2}^{i-1/2}\left[-\frac{1}{2\Delta y} - \frac{m}{4}\mu_{j-1/2}^{i-1/2} e_{j-1/2}^{i-1/2}\right] + \frac{1}{4} a_{j-1/2}^{i-1/2}, && (6d)\\
\beta_7 = \beta_8 &= -\frac{m}{4}\left(\mu_{j-1/2}^{i-1/2}\right)^2 \left(\Delta_y c_j^i - 2m\mu_{j-1/2}^{i-1/2} c_{j-1/2}^{i-1/2} e_{j-1/2}^{i-1/2}\right), && (6e)\\
\beta_9 = \beta_{10} &= -\frac{m}{4}\left(\mu_{j-1/2}^{i-1/2}\right)^2 c_{j-1/2}^{i-1/2}. && (6f)
\end{aligned}
$$

Finally for the energy equation we have

$$
\begin{aligned}
\gamma_1 = \gamma_2 &= -\frac{1}{4} e_{j-1/2}^{i-1/2}, && (7a)\\
\gamma_3 = \gamma_4 &= -\frac{1}{4}\Delta_x d_j^i, && (7b)\\
\gamma_5 = \gamma_6 &= \frac{\mathrm{Ec}}{2}\mu_{j-1/2}^{i-1/2} c_{j-1/2}^{i-1/2}, && (7c)\\
\gamma_7 = \gamma_8 &= -\frac{1}{2\Delta x} b_{j-1/2}^{i-1/2} - \frac{\mathrm{Ec}}{4} m\left(\mu_{j-1/2}^{i-1/2} c_{j-1/2}^{i-1/2}\right)^2, && (7d)\\
\gamma_9 &= \frac{1}{2\,\mathrm{Pr}\,\Delta y} - \frac{1}{4} a_{j-1/2}^{i-1/2}, && (7e)\\
\gamma_{10} &= -\frac{1}{2\,\mathrm{Pr}\,\Delta y} - \frac{1}{4} a_{j-1/2}^{i-1/2}, && (7f)
\end{aligned}
$$

Where we have used the inverse rather than the exponential viscosity distribution in our derivation of the numerical scheme here. However this could be easily interchanged. We now must write our system in block tridiagonal form such that

$$
\bar{A}\bar{\Delta} = \bar{R} \tag{8}
$$

$$
= \begin{bmatrix} B_1 & C_1 & 0 & \cdots & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A_J & B_J \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_J \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_J \end{bmatrix}. \tag{9}
$$

To determine the matrices and vectors in this block system we need only to consider a system with two points in the y direction, from which the full system can be easily generalised.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\Delta y} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{\Delta y} & -\frac{1}{2} \\
0 & -\frac{1}{\Delta y} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{\Delta y} & -\frac{1}{2} & 0 & 0 \\
\beta_2 & \beta_4 & \beta_6 & \beta_8 & \beta_{10} & \beta_1 & \beta_3 & \beta_5 & \beta_7 & \beta_9 \\
\alpha_2 & \alpha_4 & 0 & 0 & 0 & \alpha_1 & \alpha_3 & 0 & 0 & 0 \\
\gamma_2 & \gamma_4 & \gamma_6 & \gamma_8 & \gamma_{10} & \gamma_1 & \gamma_3 & \gamma_5 & \gamma_7 & \gamma_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta a_1 \\ \delta b_1 \\ \delta c_1 \\ \delta d_1 \\ \delta e_1 \\ \delta a_2 \\ \delta b_2 \\ \delta c_2 \\ \delta d_2 \\ \delta e_2
\end{bmatrix}
=
\begin{bmatrix}
-a_1^i \\ x^i - b_1^i \\ 1 - d_1^i \\ 0 \\ 0 \\ r_3 \\ r_1 \\ r_5 \\ -b_2^i \\ -d_2^i
\end{bmatrix}, \quad (10)
$$

where we have exchanged rows in order to ensure that the main diagonal sub matrix is non singular. This required in order to be able to apply of block version of the TDMA algorithm. The ordering is not unique, but also worked for the deforming surface problem and hence its use here. At a general, internal point y_j we have

$$
A_j = \begin{bmatrix}
\beta_1 & \beta_3 & \beta_5 & \beta_7 & \beta_9 \\
\alpha_1 & \alpha_3 & 0 & 0 & 0 \\
\gamma_1 & \gamma_3 & \gamma_5 & \gamma_7 & \gamma_9 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad B_j = \begin{bmatrix}
\beta_2 & \beta_4 & \beta_6 & \beta_8 & \beta_{10} \\
\alpha_2 & \alpha_4 & 0 & 0 & 0 \\
\gamma_2 & \gamma_4 & \gamma_6 & \gamma_8 & \gamma_{10} \\
0 & 0 & 0 & -\frac{1}{\Delta y} & -\frac{1}{2} \\
0 & -\frac{1}{\Delta y} & -\frac{1}{2} & 0 & 0
\end{bmatrix} \quad (11)
$$

$$
C_j = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\Delta y} & -\frac{1}{2} \\
0 & -\frac{1}{\Delta y} & -\frac{1}{2} & 0 & 0
\end{bmatrix}, \quad R_j = \begin{bmatrix}
r_3 \\ r_1 \\ r_5 \\ r_4 \\ r_2
\end{bmatrix}, \quad (12)
$$

with the only changes occurring in B and R at either boundary On the surface of the sheet we have

$$
B_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{\Delta y} & -\frac{1}{2} \\
0 & -\frac{1}{\Delta y} & -\frac{1}{2} & 0 & 0
\end{bmatrix}, \quad R_1 = \begin{bmatrix}
-a_1^i \\ x^i - b_1^i \\ 1 - d_1^i \\ r_4 \\ r_2
\end{bmatrix}, \quad (13)
$$

while at the free stream we get

$$
B_J = \begin{bmatrix}
\beta_2 & \beta_4 & \beta_6 & \beta_8 & \beta_{10} \\
\alpha_2 & \alpha_4 & 0 & 0 & 0 \\
\gamma_2 & \gamma_4 & \gamma_6 & \gamma_8 & \gamma_{10} \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & & 1 & 0
\end{bmatrix}, \quad R_J = \begin{bmatrix}
r_2 \\ r_1 \\ r_5 \\ -b_J^i \\ -d_J^i
\end{bmatrix}, \quad (14)
$$

To problem was solved on for $[x, y] \in [0, 0] \times [10, 20]$ with a step size of 0.01 in the x direction. To reduce the computational cost the y co-ordinate, which was originally linearly spaced, with one quarter the density of the x mesh was transformed using

$$y = y \exp\left(\frac{y - Y_{max}}{Y_{max}}\right)$$

to concentrate the density of the mesh near the surface of the sheet.

References

T. Cebeci and P. Bradshaw. *Physical and computational aspects of convective heat transfer*. Springer Science & Business Media, 2012.

H. B. Keller and T. Cebeci. Accurate numerical methods for boundary layer flows i. two dimensional laminar flows. In *Proceedings of the second international conference on numerical methods in fluid dynamics*, pages 92–100. Springer, 1971.