ICALC Note

Math Contest Level 2

Number Theory 2

Lesson 1

www.icalc.ca

How to Listen

Listen to advice and accept instruction, that you may gain wisdom in the future.
(Proverbs 19:20)

- Listen with your ears
- Listen with your eyes
- Listen with your heart
- Listen with all attention

Q1 prime vs. composite.

Lesson 1: Divisor Problems

Review

- Prime Factorization Q2 what's that ?

Key Points Express natural # as product of primes.

5.1 Introduction

theorey is study of +ve int!

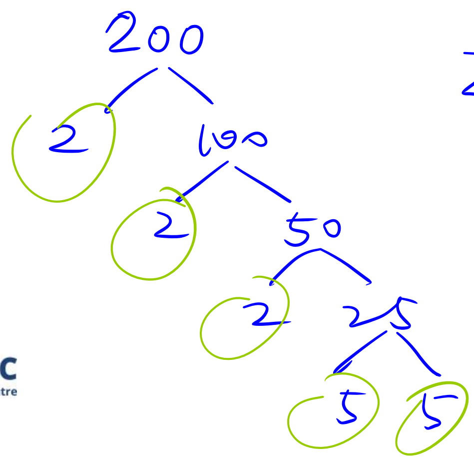
In this chapter we discuss a method (and develop a formula) for **counting the number of positive divisors of an integer**. We then use this method flexibly to count divisors that possess certain properties (multiples of 2, squares, cubes, etc.) and solve other problems.

While working the problems in this chapter, remember that your goal is to find ways to organize divisors of integers so that you can use their properties in creative ways to solve problems.

5.2 Counting Divisors

Problem 5.1

(a)Find the prime factorization of 200.



$200 = 2^3 \cdot 5^2$

(b)Find all of the divisors of 200 using its prime factorization.

$200 = 2 \times 2 \times 2 \times 5 \times 5$

each positive divisor of 200 takes some "2" & "5"

e.g. $2 \times 5 = 10$, $2 \times 2 \times 5 = 20 = 2^2 \cdot 5^1$

$2 \times 2 \times 2 = 8 = 2^3 \cdot 5^0$

(c)How are the exponents in the prime factorization of 200 related to the number of positive divisors of 200? How many positive integers are divisors of 200?

How can we make sure we don't miss any?

$a \setminus b$	0	1	2
0	$2^0 \cdot 5^0 = 1$	$2^0 \cdot 5^1 = 5$	$2^0 \cdot 5^2 = 25$
1	$2^1 \cdot 5^0 = 2$	$2^1 \cdot 5^1 = 10$	$2^1 \cdot 5^2 = 50$
2	$2^2 \cdot 5^0 = 4$	$2^2 \cdot 5^1 = 20$	$2^2 \cdot 5^2 = 100$
3	$2^3 \cdot 5^0 = 8$	$2^3 \cdot 5^1 = 40$	$2^3 \cdot 5^2 = 200$

Each combination of values for a and b represents one divisor of 200 that we can construct. Since there are 4 values for a to choose from and 3 values for b to choose from, there are

a: 0, 1, 2, 3 b: 0, 1, 2

$$4 \cdot 3 = 12$$

$n = p_1^a \cdot p_2^b$

$2 \cdot 5$

total positive divisors of 200.

Concept:

While counting divisors is not all that difficult for small natural numbers, it quickly becomes a chore for larger ones. Understanding the relationships between the prime factorization of an integer and each of its divisors helps us count the divisors more easily.

Important:

Since giving a quantity a name makes it easier to talk about, we denote the **number of positive divisors of a natural number** n **as** $t(n)$.

$t(200) = 12$

Problem 5.2 How many positive integers are divisors of 192?

$192 = 2^6 \cdot 3^1$

For positive divisor: 2 : power $0 \to 6$ 7 choices

3 : power $0 \to 1$ 2 choices

$(6+1)(1+1) = 7 \times 2 = 14$

Problem 5.3 Let n be a natural number with prime factorization,

$$n = {p_1}^{e_1} \cdot {p_2}^{e_2} \cdots {p_m}^{e_m}.$$

$192 = 2^6 \cdot 3^1$

Find a formula for the number of positive divisors of n.

Let's imagine how we can construct each positive divisor of n.
The prime factorization of n includes anywhere from 0 to e_1 powers of p_1, so there are $e_1 + 1$ choices for the exponent of p_1 in the prime factorization of a divisor.

Likewise, there are $e_2 + 1$ choices for the exponent of p_2, there are $e_3 + 1$ choices for the exponent of p_3, . . ., and $e_m + 1$ choices for the exponent of p_m. All these choices correspond to the possible sets of exponents in the prime factorizations of the divisors of n.

From these numbers of choices, we compute $t(n)$, the total number of positive divisors of n:

$(6+1)(1+1) = 14$

$$t(n) = (e_1 + 1)(e_2 + 1) \cdots (e_m + 1).$$

In other words, we find the total number of positive divisors of a natural number by taking the product of 1 more than each of the exponents in its prime factorization.

Concept: While it is convenient to know mathematical formulas (such as the one we derived for counting divisors), it is **far more important to understand the method used**.

Over the course of this chapter we examine many divisor counting problems (where the formula we developed does not apply) that require creative solutions involving the method we developed.

5.3 Divisor Counting Problems

There are many kinds of divisor counting problems aside from the task of counting all of the positive divisors of an integer. These problems require not simply knowledge of the formula for counting positive divisors, but a full understanding of the divisor counting process we used to create the formula.

Problem 5.4

(a)Find the prime factorization of 168.

168 → 2, 84 → 2, 42 → 2, 21 → 3, 7

$168 = 2^3 \cdot 3^1 \cdot 7^1$

(b)Find the number of positive divisors of 168.

$0 \to 3 \quad 0 \to 1 \quad 0 \to 1$

positive divisor $= 2^? \cdot 3^? \cdot 7^?$ $= 2^a \cdot 3^b \cdot 7^c$

$a = 0, 1, 2, 3$

$b = 0, 1$

$c = 0, 1$

$4 \times 2 \times 2 = 16$

(c)What do we know about the prime factorization of an even divisor of 168?

which of 2, 3.. 7 can create even #?

Must be 2, so we need at least one "2"

the power of 2 is at least 1,

(d)How many of the positive divisors of 168 are even?

# of choices for the exponent of 2 ? $1 \to 3$

3 ? $0 \to 1$

7 ? $0 \to 1$

$3 \times 2 \times 2 = 12$

\# 2 is the only even prime #

Concept: A second solution to this problem that involves a technique we call **"complementary counting."** Complementary counting is the process of counting what we do not want (instead of what we do want) and subtracting what we don' t want from the whole:

$$\text{Total things} - \text{things we don't want} = \text{things we do want.}$$

To use complementary counting to count the positive even divisors of 168, we first count the **total number of positive divisors (16)** and then subtract from that the **total number of odd divisors**.

A divisor of 168 must have the form $2^a \cdot 3^b \cdot 7^c$,where a is 0, 1, 2, or 3, b is 0 or 1, and c is 0 or 1.

odd $d = 2^a \cdot 3^b \cdot 7^c$ $a: 0$, $b: 0, 1$, $c: 0, 1$ } 4 choices for d.

In order for a divisor to be odd, a must be 0. This leaves 1 choice for a, 2 choices for b, and 2 choices for c, thus there are $1 \cdot 2 \cdot 2 = 4$ odd positive divisors of 168.

Subtracting the number of odd divisors from the total gives us $16 - 4 = \mathbf{12}$ positive even divisors.

operation of exponents

① $(2^2)^3 = 4^3 = 64$ $= 2^{2\times 3} = 2^6 = 64.$

$(a^b)^c = a^{b \cdot c}$

Problem 5.5

(a)What can we say about the exponents in the prime factorization of a perfect square that isn' t true for non-square integers?

② $2^2 \cdot 2^3 = 4 \cdot 8 = 32$ $= 2^{2+3} = 2^5 = 32$

③ $\frac{2^3}{2^2} = \frac{8}{4} = 2$ $= 2^{3-2} = 2^1 = 2$

6: 1 2 / 6 3

12: 1 2 3 / 12 6 4

4: 1 2 / 4

16: 1 2 4 / 16 8

$a^b \cdot a^c = a^{b+c}$

$a^b \div a^c = a^{b-c}$

(b)Show that any positive perfect square has an odd number of positive divisors.

$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_m^{e_m}$

$n^2 = (p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_m^{e_m})(p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_m^{e_m}) = p_1^{2e_1} \cdot p_2^{2e_2} \cdots p_m^{2e_m}$

$t(n^2) = (2e_1+1)(2e_2+1)(2e_3+1)\cdots$ ODD #

Important: A positive integer is a perfect square if and only if it has an odd number of positive divisors.

Recognizing this fact helps us determine useful information about integers.

Problem 5.6

(a)What can we say about the exponents in the prime factorization of a natural number that has at least one divisor that is a perfect square greater than 1?

simple example

$4 = 2^2 \to$

$8 = 2^3 \leftarrow 2 \cdot 2^2$

$72 = 2^3 \cdot 3^2 \to$ has factor

$4 = 2^2$

$9 = 3^2$

$36 = 6^2$

$15 = 3^1 \cdot 5^1$

at least one of the power must be 2 or larger

(b)Find the prime factorization of 5400.

$5400 = 2^3 \cdot 3^3 \cdot 5^2$

(c)What is the total number of divisors of 5400 that are not multiples of any perfect square greater than 1?

$d = 2^a \cdot 3^b \cdot 5^c$

a. b. c must be 0, or 1.

2 choices for each of a. b. c

$2 \times 2 \times 2 = 8$

Concept: Organizing information mathematically helps you more easily work toward a solution to a problem.

Problem 5.7 Let n be a natural number with exactly 7 positive divisors.

(a)What do we know about the prime factorization of n?

$$n = p_1^{e_1}\cdot p_2^{e_2}\cdot p_3^{e_3}\cdots p_m^{e_m}$$

$$t(n) = 7 = (e_1+1)(e_2+1)(e_3+1)\cdots$$

$$7 \times 1 \times 1 \cdots$$

$$e_1 = 6 \quad e_2 = 0 \quad e_3 = 0 \cdots$$

$$n = p_1^6$$

(b)How many positive divisors does n^2 have?

$$n^2 = (p_1^6)^2 = p_1^{12}$$

$$t(n^2) = 12+1 = 13$$

Problem 5.8

Let n be a natural number with exactly 28 positive divisors.

(a)What are the possible groups of exponents in the prime factorization of n?

$$t(n) = 28 = 2\cdot 14 = 4\cdot 7.$$

$28 = 1\times 28 \rightarrow n = p_1^{27} = (p_1^9)^3$

$= 2\times 14 \quad n = p_1^1 p_2^{13} \rightarrow ?$

$= 2\times 2\times 7 \quad n = p_1^1 p_2^1 p_3^6 \rightarrow ?$

$= 7\times 4 \quad n = p_1^6 p_2^3 = (p_1^2 p_2^1)^3$

(b)If n is a perfect cube, what are the possible groups of exponents in the prime factorization of n?

$$n = (*)^3$$

(c)If n is a perfect cube that is divisible by exactly 2 primes, what are the exponents in the prime factorization of n?

$$P_1^{1} P_2^{13} \neq (P_1^{a} P_2^{b})^3$$

$$\begin{array}{ccc} \boxed{2,14} & \rightarrow & 1,13 \\ 4,7 & \rightarrow & 3,6 \end{array}$$

$$P_1^{6} P_2^{3} = (P_1^{2} P_2^{1})^3$$

(d)If n is a perfect cube that is divisible by exactly 2 primes, how many positive divisors does $\sqrt[3]{n}$ have?

$$n = p_1^6 \cdot p_2^3 = (p_1^2 \cdot p_2^1)^3.$$

$$\sqrt[3]{n} = p_1^2 \cdot p_2^1,$$

$$t(\sqrt[3]{n}) = (2+1)(3+1) = 6$$

Problem 5.9

(a)What are the possible groups of exponents in the prime factorization of an integer with exactly 6 positive divisors?

$6 = 6 \times 1 \longrightarrow P_1^{5} \cancel{P_2^{0}}$

$2 \times 3 \longrightarrow P_1^{1} P_2^{2}$

(b)How many of the positive divisors of 960 themselves have 6 positive divisors?

$$960 = 2^6 \cdot 3^1 \cdot 5^1.$$

A divisor d of 960 has the form:

$$d = 2^a \cdot 3^b \cdot 5^c.$$

① $P_1^{5} : 2^{5}$

② $P_1^{1} \cdot P_2^{2}$

$3^{1} \cdot 2^{2}$

$5^{1} \cdot 2^{2}$

$$6 = 1 \cdot 1 \cdot 6 = 1 \cdot 2 \cdot 3.$$

These products imply that (a, b, c) must include either two 0's and one 5 or else one 0, one 1, and one 2. This leaves us with only three possibilities: (5, 0, 0), (2, 0, 1), and (2, 1, 0).

We can check these solutions to be sure that we found the divisors we wish to count:

$$\begin{aligned} t(32) &= t(2^5) &= 6 \\ t(12) &= t(2^2 \cdot 3^1) &= 6 \\ t(20) &= t(2^2 \cdot 5^1) &= 6 \end{aligned}$$

So, the divisors of 960 which themselves have 6 positive divisors are: 12, 20, and 32.

Concept:

Check your solutions to be sure that they solve the problem. This is even more important with unusual or difficult problems. So far, no math student was born who didn't make mistakes, but good problem solvers catch many of their own errors. How to check?

Problem 5.10 Let m and n be two relatively prime natural numbers such that $t(m) = 10$ and $t(n) = 6$.

(a)How many primes do the prime factorizations of m and n have in common? $\boxed{NO}$

example: relative prime $25 = 5^2$, $36 = 2^2 \cdot 3^2$

$6 = 2 \times 3$
$21 = 3 \times 7$
Not relative prime.

(b)How does the group of exponents in the prime factorization of the product mn relate to the groups of exponents in m and n?

$m = p_1^{a_1} \cdot p_2^{a_2} \cdots p_m^{a_m}$

$n = q_1^{b_1} q_2^{b_2} \cdots q_n^{b_n}$

$m \cdot n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} \cdot q_1^{b_1} q_2^{b_2} \cdots q_m^{b_n}$

p. q are different!

(c)Find $t(mn)$ in terms of $t(m)$ and $t(n)$.

$$t(mn) = \underbrace{(a_1+1)(a_2+1)\cdots(a_m+1)}_{t(m)}\underbrace{(b_1+1)(b_2+1)\cdots(b_n+1)}_{t(n)}$$

$$t(m)\cdot t(n) = 10\times 6 = 60!$$

If m and n are any two natural numbers such that $\gcd(m,n) = 1, t(mn) = t(m)\cdot t(n)$.

Co prime

5.4 Divisor Products

Problem 5.11

(a)List the positive divisors of 60 from least to greatest.

$$1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$$

(b)Beneath each divisor in your list, write the quotient of 60 divided by the divisor.

1	2	3	4	5	6	10	12	15	20	30	60
60	30	20	15	12	10	6	5	4	3	2	1

(c)Find the product of all the positive divisors of 60.

$$\begin{array}{rcccccccccccc} P &=& 1 &\cdot& 2 &\cdot& 3 &\cdots& 20 &\cdot& 30 &\cdot& 60 \\ P &=& 60 &\cdot& 30 &\cdot& 20 &\cdots& 3 &\cdot& 2 &\cdot& 1 \\ P^2 &=& (1\cdot 60) &\cdot& (2\cdot 30) &\cdot& (3\cdot 20) &\cdots& (20\cdot 3) &\cdot& (30\cdot 2) &\cdot& (60\cdot 1) \end{array}$$

Since $t(60) = 12$, we see that $P^2 = 60^{12}$. Taking the square root, we find that the product of the positive divisors of 60 is 60^6.

(d)Find a formula for the product of the positive divisors of a natural number.

simple example: 6 : $d = 1, 2, 3, 6$

$\prod(d_i) = 1 \times 2 \times 3 \times 6 = 6^{\frac{4}{2}}$ → # of the divisor

$$P_n = n^{\frac{t(n)}{2}}$$

Problem 5.12

(a)Describe the prime factorization of a positive divisor of 450.

$450 = 2 \cdot 3^2 \cdot 5^2$

2 more steps: ① $t(450) = (1+1)(2+1)(2+1) = 18$

② $P_{450} = 450^{\frac{18}{2}} = 450^9$

(b)Find the product of all of the positive divisors of 450 that are multiples of 3.

$P_{450} = 450^{\frac{t(450)}{2}} = 450^{\frac{(1+1)(2+1)(2+1)}{2}} = 450^9$

But it contains some divisor without factor 3

let $n = 2 \cdot 5^2 = 50$

$P_{50} = 50^{\frac{(1+1)(2+1)}{2}} = 50^3$

$\frac{P_{450}}{P_{50}} = \frac{450^9}{50^3} = \frac{50^9 \cdot 9^9}{50^3} = 50^6 \cdot 9^9 = 2^6 \cdot 3^{18} \cdot 5^{12}$

Homework (Show your work and upload your homework to google classroom)

5.2.1: Find the number of positive divisors of each integer using its prime factorization.

(a)4

(b)6

(c)12

(d)15

(e)25

(f)30

(g)60

(h)124

(i)180

(j)280

(k)441

(l)504

5.2.2:

Find the number of positive divisors of 2002.

5.2.3: Karlanna places 600 marbles into m total boxes such that each box contains an equal number of marbles. There is more than one box, and each box contains more than one marble. For how many values of m can this be done?

5.3.1: How many of the positive divisors of 252 are even?

5.3.2: How many of the positive divisors of 2160 are multiples of 3?

5.3.3: How many ordered pairs, (x, y), of positive integers satisfy the equation $xy = 144$?

5.3.4: How many positive divisors do 48 and 156 have in common?

5.3.5: What proportion of the positive divisors of 840 are prime?

5.3.6: Let n be an odd integer with exactly 11 positive divisors. Find the number of positive divisors of $8n^3$.

5.3.9: Let n be a natural number with exactly 2 positive prime divisors. If n^2 has 27 divisors, how many does n have?

5.3.10: How many of the positive divisors of 45000 themselves have exactly 12 positive divisors?

5.4.1: Find the product of the positive divisors of 78.

5.4.2: Find the product of the positive divisors of 120.

5.4.5: 48 chickens are kept in n cages such that each cage contains the same number of chickens. What is the product of the possible values of n?