# Two-Body Problem

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## 1 Introduction

The two body problem is an isolated system of two particles which interact through a central potential. For two particles with mass $m_1$ and $m_2$, the equations of motion is

$$m_1\ddot{r_1} = F_{21};\ m_2\ddot{r_2} = F_{12}. \tag{1}$$

In Newtonian gravity, the potentials are therefore

$$U_{12} = U_{21} = \frac{Gm_1m_2}{|r_1 - r_2|}. \tag{2}$$

Since the 2 particles interact via a central potential, the total momentum is conserved. We consider the center of mass,

$$R = \frac{m_1r_1 + m_2r_2}{M} \tag{3}$$

where $M = m_1 + m_2$.
The time derivative is therefore

$$V_{CM} = \frac{m_1v_1 + m_2v_2}{M}. \tag{4}$$

Since our system is isolated, the net force and total acceleration is zero, thus $V_{CM}$ is a constant. The motion of the center of mass is then

$$R(t) = V_{CM}t. \tag{5}$$

We define the vector of distance between the particles:

$$r = r_1 - r_2. \tag{6}$$

Thus,

$$r_1 = R + \frac{m_2r}{m_1 + m_2};\ r_2 = R - \frac{m_1r}{m_1 + m_2}. \tag{7}$$

Since we already know $R$, our problem reduces to finding $r$.

If we multiply the masses to eq. 1, we get

$$m_1 m_2 \ddot{r_1} = m_2 F_{21};\ m_1 m_2 \ddot{r_2} = m_1 F_{12}. \tag{8}$$

Subtract the second from the first equation and simplify, we get

$$\frac{m_1 m_2}{m_2 + m_1} \ddot{r} = F_{21} \tag{9}$$

The force derived from the central potential only depends on distance between the 2 particles.

$$F_{21} = -\frac{\partial}{\partial r_1} U_{12}(|r_1 - r_2|) = \nabla U_{12} \tag{10}$$

From above, we can summarize them as

$$m_* \ddot{r} = -\frac{\partial}{\partial r} U(|r|) = F(r) \tag{11}$$

where

$$m_* = \frac{m_1 m_2}{m_1 + m_2} \tag{12}$$

is the reduced mass.

# 2 Conservation of Angular Momentum

In this section, we will rewrite the system above in terms of the angular momentum.

$$L = m_* r \times V;\ V = \dot{r}. \tag{13}$$

Since $F$ is in the $r$ direction, the torque has to be 0, and the angular momentum must be constant.

$$\frac{d}{dt} L = 0 \tag{14}$$

Next, we use the *scalar triple product identity*,

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \tag{15}$$

and apply it to the angular momentum, we get

$$r \cdot L = m_* r \cdot (r \times V) = m_* V \cdot (r \times r) = 0. \tag{16}$$

Hence $r$ is always *perpendicular* to angular momentum $L$.

Because the angular momentum is constant, there must be a fixed vector of which $r$ is perpendicular to in space. Since the position vector is always perpendicular to a certain direction in space, we can reduce the probelm into a 2D plane in polar coordinates where

$$r_x = r \cos\theta;\ r_y = r \sin\theta. \tag{17}$$

We choose the $x$ - $y$ plane for the particles to be on in accordance to conventions, and the angular momentum is oriented to the $z$ direction. The velocity in $\theta$ direction is always positive, therefore the particles always rotates around the center of our coordinate system in the same direction.

$$\dot{\theta} = \frac{|L|}{m_* r^2} \tag{18}$$

## 3 Conservation of Energy

To begin with, we know that Energy is the sum of kinetic and potential energies.

$$E = \frac{1}{2} m_* v^2 + U(r) \tag{19}$$

In polar coordinates, the velocity is

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\theta}^2. \tag{20}$$

The energy conservation becomes

$$E = \frac{1}{2} m_* \dot{r}^2 + \frac{1}{2} m_* r^2 \dot{\theta}^2 + U(r). \tag{21}$$

We call part of this equation the **effective potential energy**.

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} + U(r) \tag{22}$$

Since the angular momentum is constant, in spherical coordinates,

$$L = mr^2 \dot{\phi} = const; \ \dot{\phi} = \frac{L}{mr^2} \tag{23}$$

Here the $\phi$ in spherical coordinates is equivalent to $\theta$ in polar coordinates.

## 4 Solutions for Two-Body Problem (Analytical)

We have derived that energy of the system is constant

$$\begin{aligned} E &= \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\phi}^2) + U(r) = const. \\ \dot{r} &= \frac{dr}{dt} = \sqrt{\frac{2}{m}(E - U(r)) \frac{L^2}{m^2 r^2}} \\ \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{L^2}{m^2 r^2}}} &= \int dt = t. \end{aligned}$$

We want to find trajectory $r(\phi)$.

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{L}{mr^2}$$
$$E = \frac{1}{2}m\left(\frac{dr}{d\phi}\right)^2\left(\frac{L}{mr^2}\right)^2 + \frac{L^2}{2mr^2} + U(r)$$
$$E = \frac{L}{2m}\left(\frac{dr}{r^2 d\phi}\right)^2 + \frac{L^2}{2mr^2} + U(r) \tag{24}$$

We introduce $b = \frac{1}{r}$ and $db = \frac{-dr}{r^2}$. The equation of energy thus becomes

$$E = \frac{L^2}{2m}\left[\left(\frac{db}{d\phi}\right)^2 + b^2\right] + U(\frac{1}{b}), \tag{25}$$

where if we find $r(\phi)$, and $r(\phi) = \frac{1}{b(\phi)}$, we can find $\phi(t)$. We can do this by integrating $\int \dot{\phi} dt = \phi(t)$ where $\dot{\phi} = \frac{L}{mr^2}$. We then take the derivative of $E$ with respect to $\phi$.

$$\frac{L^2}{2m}\left[2\frac{db}{d\phi}\frac{d^2b}{d\phi^2} + 2b\frac{db}{d\phi}\right] + \frac{dU}{db}\frac{db}{d\phi} = 0. \tag{26}$$

Divide both sides by $\frac{db}{d\phi}$, assuming $b \neq const.$ and $\frac{db}{d\phi} \neq 0$, we get

$$\frac{L^2}{m}\left(\frac{d^2b}{d\phi^2} + b\right) = -\frac{du}{db}. \tag{27}$$

Here we obtain **Binet's equation for central field**.

$$\frac{d^2b}{d\phi^2} + b = -\frac{m}{L^2}\frac{dU}{db} \tag{28}$$

## 4.1 Kepler Motion

In a gravitationally bounded two-body system, we take potential

$U(r) = -\frac{Gm_* m}{r} = -\frac{\alpha}{r}$ where $\alpha > 0$ and $\alpha = Gm_* m = const.$

Using **Binet's equation** and $U = -\alpha b$ where $\frac{dU}{db} = -\alpha$,

$$\frac{d^2b}{d\phi^2} + b = \frac{m\alpha}{L^2} = \frac{1}{p};\ p = \frac{L^2}{m\alpha};\ b = \frac{1 + e\cos\phi}{p}. \tag{29}$$

by the definition of **orbital eccentricity** $e$. Since $r = \frac{1}{b}$, we get $r = \frac{p}{1+e\cos\phi}$.

The minimum of $r$ is limited by the cosine function, hence,

$$r_{min} = \frac{p}{1+e};\ r_{max} = \frac{p}{1-e}. \tag{30}$$

If we define $a$ as the semi-major axis of the elliptical orbit, we get

$$\frac{p}{1+e} + \frac{p}{1-e} = 2a;\ p = a(1-e^2). \tag{31}$$

## 4.2 Energy

The constant $p$ depends on the orbial eccentricity $e$ and angular momentum $L$. In this section, we show that the energy conservation can depend on parameter semi-major axis $a$.

Using eqn. 24 and results obtained above, we can further write energy as

$$E = \frac{L^2(1+e\cos\phi)^4}{2mp^4}\left[\frac{p^2e^2\sin^2\phi}{(1+e\cos\phi)^4} + \frac{p^2}{(1+e\cos\phi)^2}\right] - \frac{\alpha(1+e\cos\phi)}{p}. \quad (32)$$

simplify, we obtain

$$E = \frac{(e^2-1)m\alpha^2}{2a(1-e^2)m\alpha} = -\frac{\alpha}{2a}. \quad (33)$$

## 4.3 Summary

From the two conserved quantities, namely energy $E$ and angular momentum $L$, we obtain a solution to the Binet's equation of orbital motion for $r$ as a function of $\phi$.

$$r = \frac{p}{1+e\cos\phi} \quad (34)$$

where $p = \frac{L^2}{m\alpha}$ and $E = -\frac{\alpha}{2a}$. Hence different $L$ and $E$ will give rise to different shapes of orbits.

Using Newtonian gravity where the kinetic energy and gravitational potential are equal to each other,

$$\frac{1}{2}mv_e^2 = \frac{GmM}{r}, \quad (35)$$

we obtain the equation for escape velocity of a system.

$$v_e = \sqrt{\frac{2Gm}{r}}. \quad (36)$$

In case of gravitationally bounded objects, the total energy is equal to kinetic energy of the object orbiting the total mass minus the potential energy.

$$E_{tot} = \frac{1}{2}mv^2 - \frac{GMm}{r}. \quad (37)$$

1. If $E < 0$, the orbit is a bounded elliptical orbit.

2. If $E = 0$, the orbit is parabolic where it is marginally unbounded. It comes in from infinitely far away and orbit through and heads back to infinitely away.

3. If $E > 0$, the orbit is hyperbolic where the object does not orbit.

### 4.3.1 Energy & Momentum as a function of $a$ and $e$

From above results, we can write the energy and momentum in terms of semi-major axis and eccentricity.

$$E = -\frac{\alpha}{2a} \tag{38}$$

$$L = \sqrt{(1-e^2)am\alpha} \tag{39}$$

In terms of the two masses $m_1$ and $m_2$,

$$E = -\frac{Gm_1m_2}{a} \tag{40}$$

$$L = m_1m_2\sqrt{(1-e^2)\frac{Ga}{m_1+m_2}} \tag{41}$$

### 4.3.2 $a$ and $e$ as a function of energy & momentum

We can hence do the inverse of above equations to show that eccentricity and semi-major axis can be written as a function of energy and angular momentum.

$$a = -\frac{\alpha}{2E} \tag{42}$$

$$e = \sqrt{1+\frac{2EL^2}{m\alpha^2}} \tag{43}$$

In terms of $m_1$ and $m_2$,

$$a = -\frac{Gm_1m_2}{2E} \tag{44}$$

$$e = \sqrt{1+\frac{2EL^2(m_1+m_2)}{m_1^3m_2^3G^2}} \tag{45}$$

The above results limit the conserved energy to be negative to have a bounded elliptical orbit.

### 4.3.3 Time derivatives of $a$ and $e$

In the more realistic situations, the eccentricity and semi-major axis of the orbit can change over time. We can hence write the time derivatives of $a$ and $e$ as a function of individual masses, angular momentum, and energy.

$$\dot{a} = -\frac{G}{2}\left[\frac{\dot{m}_1m_2+m_1\dot{m}_2}{E} - \frac{m_1m_2\dot{E}}{E^2}\right] = a\left(\frac{\dot{m}_1}{m_1}+\frac{\dot{m}_2}{m_2}-\frac{\dot{E}}{E}\right) \tag{46}$$

where $\dot{E} = \dot{E}_1 + \dot{E}_2$. The time derivative of energy is the sum of time derivatives of individual energies.

For eccentricity,

$$\dot{e} = \frac{1}{G^2 m_1^6 m_2^6 e} \cdot \Big( m_1^3 m_2^3 E L^2 (\dot{m}_1 + \dot{m}_2) + (m_1 + m_2)(\dot{E} L^2 - 2 E L \dot{L}) - E L^2 (m_1 + m_2)(3 m_1^2 \dot{m}_1 m_2^3 + 3 m_2^2 \dot{m}_2 m_1^3) \Big) \tag{47}$$

# 5 Solutions for Two-Body Problem (Numerical)

Since $\dot{\theta} = \frac{L}{mr^2}$, we get the acceleration in $\theta$ direction

$$\ddot{\theta} = -\frac{\dot{r} L}{m r^3} = -\frac{2 \dot{r} \dot{\theta}}{r}. \tag{48}$$

Acceleration in $r$ direction is

$$\ddot{r} = \frac{F}{m} = r \dot{\theta}^2 - \frac{Gm}{r^2} \tag{49}$$

because $G\ddot{M}/r^2$ and $r = \frac{x}{\cos\theta}$.

Our problem then becomes a system of two second-order ordinary differential equations stated above. To numerically solve it using RK-4,

we have to use 4 first-order differential equations such as

$$V_r = \dot{r} \tag{50}$$

$$V_\theta = \dot{\theta} \tag{51}$$

$$\dot{V}_r = r V_\theta^2 - \frac{Gm}{r^2} \tag{52}$$

$$\dot{V}_\theta = -\frac{2 V_r L}{m r^3} \tag{53}$$