DISCO Summary

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1 Solutions to the Disco Primer

Fluids are defined by spacetime dependent quantites: mass density ρ, velocity v, and pressure P. In case of the accretion disks of a black hole, we have 5 real variables: ρ, P, and 3 components of v. We also have 5 equations for the 5 variables along with sonservation laws:

1. Conservation of particle nonumber
2. Conservation of momentum
3. Conservation of energy.

These conservation laws give us the equations needed to solve: the Continuity Equation, Euler Equations, and the Energy Equation.

$$
\begin{aligned}
&\partial_t \rho + \nabla(\cdot \rho v) = 0 \\
&\partial_t(\rho v) + \nabla \cdot (\rho v v + P) = \rho g \\
&\partial_t(\frac{1}{2}\rho v^2 + e) + \nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + e + P\right) v\right] = \rho v \cdot g - \dot{q}
\end{aligned} \tag{1}
$$

The goal for DISCO code is to solve these equations for a hydrodynamic system. Below I have documented the process of which I preliminarily solve these equations using analytical methods to give rise to a simple disk model.

1.1 Steady and axissymmetric

We assume the disk model to be long-lived and axisymmetric. That is, the disk will look the same from any angle at any moment of time. Thus, we set derivatives of time and angle ϕ in cylindrical coordinates to 0. Here we will have four ordinary differential equations:

$$
\begin{aligned}
&\textit{Continuity: } \frac{1}{r}\partial_r(r\Sigma V_r) = 0 \\
&\textit{Radial Momentum: } \frac{1}{r}\partial_r\left(r\Sigma V_r^2 + r\Pi - 2r\Sigma\nu\left(\frac{2}{3}\partial_r V_r - \frac{1}{3}\partial_\phi\Omega\right)\right) = 0 \\
&\textit{Angular Momentum: } \frac{1}{r}\partial_r\left(r^3\Sigma V_r\Omega - 2r\Sigma\nu\left(\frac{1}{2}r^2\partial_r\Omega\right)\right) = 0 \\
&\textit{Energy: } \frac{1}{r}\partial_r\left(r\left(\frac{1}{2}\Sigma v^2 + \Sigma\epsilon + \Pi\right)V_r - 2r\Sigma\nu(\sigma\cdot v)r\right) = \Sigma v\cdot g - \dot{q} \qquad (2)
\end{aligned}
$$

where ν is viscosity and $\dot{q}$ is a cooling term, Ω is angular velocity, and

$$
\begin{aligned}
v\cdot g &= -\frac{GM}{r^2}\cdot v \\
\Sigma &= \int \rho dz. \qquad (3)
\end{aligned}
$$

1.2 The razor thin limit

We are assuming that the disk is razor thin and inviscid. Here we take the approximations of $H \simeq 0$ and $\nu = 0$. We also set the sounds speed $c_s = 0$, as well as Π and ϵ. Here we find the solutions of $V_r(r)$ and $\Omega(r)$.

We can do so by taking the insides of the first and third equation as constants, and solve for the other two equations using results obtained from it. Eventually, we get one of the solutions similar to Saturn's rings: $V_r = 0$ and

$$\Omega = \sqrt{\frac{GM}{r^3}}. \qquad (4)$$

As we take the raidal velocity as 0, we are assuming that the gas stays in a steady orbit without moving inward or outward. This makes sense in our razor thin and axisymmetric approximation but it is unrealistic since accretion does not occur with $V_r = 0$.

1.3 The Accretion Rate and Torque

We define the accretion rate $\dot{M}$ as the rate at which matter passes a praticular radius mobing toward the central object: $\dot{M} = -2\pi r\Sigma v_r$. From the continuity eqaution from step 1, we can derive

$$\partial_r(r\Sigma V_r) = 0 \qquad (5)$$

and $\dot{M} = -2\pi r\Sigma V_r = \textit{constant}$. This means that for a disk that is steady and axisymmetric, the accretion rate is constant across all radii. Later, we will use the expression of V_r to solve for other quantities.

$$V_r = \frac{\dot{M}}{-2\pi r\Sigma} \qquad (6)$$

Similarly, we define the torque through the disk $\dot{J}$ as the rate at which the central object accretes angular momentum, which is equal to 2π times the angular momentum flux. From the angular momentum equation from step 1, we have

$$\frac{1}{r}\partial_r(r^3\Sigma V_r\Omega - r^3\Sigma\nu\partial_r\Omega) = \partial_r\dot{J} = 0. \tag{7}$$

Using V_r from above, we get

$$\dot{J} = \dot{M}r^2\Omega + 2\pi r^3\Sigma\nu\partial_r\Omega. \tag{8}$$

Rearrange, we have an expression for Σ

$$\Sigma = \frac{\dot{J} - \dot{M}r^2\Omega}{2\pi r^3\nu\partial_r\Omega}. \tag{9}$$

1.4 An accretion disk

So far, we have two algebraic equations involving $\dot{M}$ and $\dot{J}$ and the radial momentum and energy equations. It is time to solve the final radial momentum equation assuming the disk is thin $H \ll r$ abd the radial velocity subsonic $|v_r| < c_s$. The centripital acceleration term should balance the term with the largest other term, in this case, $r\Sigma\frac{GM}{r^2}$. This confirms the expression for angular velocity $\Omega = \sqrt{\frac{GM}{r^3}}$.

After simplifying, we obtain the final solution for Σ

$$\Sigma = \frac{-1}{3\pi\nu}\left(\frac{\dot{J}}{\sqrt{GMr}} - \dot{M}\right), \tag{10}$$

and for V_r

$$V_r = -\frac{3\nu}{2r}. \tag{11}$$

From here, we have obtained a series of constants and expressions that will help the DISCO code solve the hydrodynamic equations.

We define function f:

$$f = 1 - \frac{\dot{J}}{\dot{M}\sqrt{GMr}}. \tag{12}$$

Finally, some expressions in terms of f:

$$\begin{aligned} \Sigma &= \frac{\dot{M}f}{3\pi\nu} \\ V_r &= \frac{3\nu}{-2rf} \\ \Omega &= \sqrt{\frac{GM}{r^3}}. \end{aligned} \tag{13}$$

2 Instructions on using the DISCO code

Now that we have analytically solved solutions of a simple disk model, it is time that we put these varaibles into the disco code.

In order to run the code, we need to provide initial conditions and the instructions for the computer to run the simulations. I have listed below some of the useful tips for this project running on disco.

The $Makefile_opt.infilecontainstheinitialconditionfilewewilluse.Wecanspecifythefilenameafter$

```
INITIAL = {FILENAME}
```