Gas - Particle Interaction Notes

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1 Particles

Particles (black holes, planets, stars, etc) are modelled as point masses. There are N particles, indexed by i. Each particle is entirely described by its mass M_i, position $\mathbf{x}_i$, and momentum $\mathbf{p}_i$.

Each particle produces a gravitational potential Φ_i and its associated gravitational field $\mathbf{g}_i$:

$$\Phi_i(\mathbf{x}) \equiv -\frac{GM_i}{|\mathbf{x} - \mathbf{x}_i|} \tag{1}$$

$$\mathbf{g}_i(\mathbf{x}) \equiv -\nabla\Phi_i(\mathbf{x}) = -\frac{GM_i}{|\mathbf{x} - \mathbf{x}_i|^3}\left(\mathbf{x} - \mathbf{x}_i\right) \tag{2}$$

1.1 Auxiliary Quantities

Particles also have energies, angular momenta, etc which are expressible in terms of $(M_i, \mathbf{x}_i, \mathbf{p}_i)$. We'll need:

$$\mathbf{v}_i = \frac{1}{M_i}\mathbf{p}_i \qquad \text{Velocity} \tag{3}$$

$$L_i = \mathbf{x}_i \times \mathbf{p}_i \qquad \text{(Orbital) Angular Momentum} \tag{4}$$

$$K_i = \frac{\mathbf{p}_i \cdot \mathbf{p}_i}{2M_i} \qquad \text{Kinetic Energy} \tag{5}$$

Each pair of particles has a gravitational potential energy (binding energy) between them:

$$U_{ij} = M_i\Phi_j(\mathbf{x}_i) = M_j\Phi_i(\mathbf{x}_j) = -\frac{GM_iM_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{6}$$

1.2 Time Evolution In Vacuum

If the only thing in the universe is these N gravitating particles, then they will evolve according to:

$$\dot{M}_i = 0 \tag{7}$$

$$\dot{\mathbf{x}}_i = \frac{1}{M_i}\mathbf{p}_i \equiv \mathbf{v}_i \tag{8}$$

$$\dot{\mathbf{p}}_i = \sum_{j\neq i} M_i\mathbf{g}_j(\mathbf{x}_i) \tag{9}$$

In this case, the total momentum $\mathbf{P} = \sum_i \mathbf{p}_i$, total angular momentum $\mathbf{L} = \sum_i \mathbf{L}_i$, and total energy $E = \sum_i K_i + \sum_{i,j<i} U_{ij}$ are conserved (constant in time).

1.3 Time Evolution With Gas

Gas has mass, so it produces a gravitational field $\mathbf{g}_{\text{gas}}(\mathbf{x})$ as well. It also can directly accrete onto a particle, altering its mass, position, and momentum.

The gas' gravity does not directly change the particle's mass or position, but it does provide a force:

$$\mathbf{f}_{\text{gas}-\text{grav},i} = M_i \mathbf{g}_{\text{gas}}(\mathbf{x}_i) \tag{10}$$

Accretion, on the other hand, adds mass to the particles ($\dot{M}_i \neq 0$) and can directly change their position by shifting their center-of-mass:

$$\dot{M}_{\text{acc},i} = \int dV \, \dot{\rho}_{\text{acc},i}(\mathbf{x}) \qquad \text{Mass Accretion} \tag{11}$$

$$\dot{\mathbf{M}}^1_{\text{acc},i} = \int dV \, \dot{\rho}_{\text{acc},i}(\mathbf{x}) \, (\mathbf{x} - \mathbf{x}_i) \qquad \text{Mass Accretion - First Moment} \tag{12}$$

$$\mathbf{f}_{\text{acc,i}} = \int dV \, \dot{\rho}_{\text{acc},i}(\mathbf{x}) \mathbf{v}_{\text{acc},i}(\mathbf{x}) \qquad \text{Momentum Accretion A.K.A Force} \tag{13}$$

The evolution of the particles is then given by:

$$\dot{M}_i = \dot{M}_{\text{acc},i} \tag{14}$$

$$\dot{\mathbf{x}}_i = \frac{1}{M_i}\mathbf{p}_i + \frac{1}{M_i}\dot{\mathbf{M}}^1_{\text{acc},i} \equiv \mathbf{v}_i + \dot{\mathbf{x}}_{\text{acc},i} \tag{15}$$

$$\dot{p}_i = \sum_{j\neq i} M_i \mathbf{g}_j(\mathbf{x}_i) + M_i \mathbf{g}_{\text{gas}}(\mathbf{x}_i) + \mathbf{f}_{\text{acc},i} \tag{16}$$

1.4 Evolution of Auxiliary Quantities

The energies, angular momenta, etc inherit time evolution from the mass, position, and momentum:

The velocity:

$$\dot{\mathbf{v}}_i = \frac{1}{M_i}\dot{\mathbf{p}}_i - \frac{1}{M_i^2}\mathbf{p}_i \dot{M}_i \tag{17}$$

$$= \sum_{j\neq i} \mathbf{g}_j(\mathbf{x}_i) + \mathbf{g}_{\text{gas}}(\mathbf{x}_i) + \frac{1}{M_i}\mathbf{f}_{\text{acc},i} - \frac{1}{M_i}\mathbf{v}_i \dot{M}_{\text{acc},i} \tag{18}$$

Orbital angular momentum:

$$\dot{L}_i = \dot{\mathbf{x}}_i \times \mathbf{p}_i + \mathbf{x}_i \times \dot{\mathbf{p}}_i \tag{19}$$

$$= \frac{1}{M_i}\mathbf{p}_i \times \mathbf{p}_i + \frac{1}{M_i}\dot{\mathbf{M}}^1_{\text{acc},i} \times \mathbf{p}_i + \mathbf{x}_i \times \sum_{j\neq i} M_i \mathbf{g}_j(\mathbf{x}_i) + \mathbf{x}_i \times M_i \mathbf{g}_{\text{gas}}(\mathbf{x}_i) + \mathbf{x}_i \times \mathbf{f}_{\text{acc},i} \tag{20}$$

$$= \sum_{j\neq i} M_i \mathbf{x}_i \times \mathbf{g}_j(\mathbf{x}_i) + M_i \mathbf{x}_i \times \mathbf{g}_{\text{gas}}(\mathbf{x}_i) + \dot{\mathbf{x}}_{\text{acc},i} \times \mathbf{p}_i + \mathbf{x}_i \times \mathbf{f}_{\text{acc},i} \tag{21}$$

Kinetic Energy:

$$\dot{K}_i = \frac{1}{M_i}\mathbf{p}_i \cdot \dot{\mathbf{p}}_i - \frac{1}{2M_i^2} p_i^2 \dot{M}_i \tag{22}$$

$$= \mathbf{v}_i \cdot \sum_{j \neq i} M_i \mathbf{g}_j(\mathbf{x}_i) + \mathbf{v}_i \cdot M_i \mathbf{g}_{\text{gas}}(\mathbf{x}_i) + \mathbf{v}_i \cdot \mathbf{f}_{\text{acc},i} - \frac{1}{2} v_i^2 \dot{M}_{\text{acc},i} \tag{23}$$

$$= \sum_{j \neq i} M_i \mathbf{v}_i \cdot \mathbf{g}_j(\mathbf{x}_i) + M_i \mathbf{v}_i \cdot \mathbf{g}_{\text{gas}}(\mathbf{x}_i) + \mathbf{v}_i \cdot \mathbf{f}_{\text{acc},i} - \frac{1}{2} v_i^2 \dot{M}_{\text{acc},i} \tag{24}$$

Potential Energy:

$$\dot{U}_{ij} = \Phi_j(\mathbf{x}_i)\dot{M}_i + \Phi_i(\mathbf{x}_j)\dot{M}_j - M_i \mathbf{g}_j(\mathbf{x}_i) \cdot \dot{\mathbf{x}}_i - M_j \mathbf{g}_i(\mathbf{x}_j) \cdot \dot{\mathbf{x}}_j \tag{25}$$

$$= \Phi_j(\mathbf{x}_i)\dot{M}_{\text{acc},i} + \Phi_i(\mathbf{x}_j)\dot{M}_{\text{acc},j} - \mathbf{g}_j(\mathbf{x}_i) \cdot \mathbf{p}_i - \mathbf{g}_j(\mathbf{x}_i) \cdot \dot{\mathbf{M}}^1_{\text{acc},i} - \mathbf{g}_i(\mathbf{x}_j) \cdot \mathbf{p}_j - \mathbf{g}_i(\mathbf{x}_j) \cdot \dot{\mathbf{M}}^1_{\text{acc},j} \tag{26}$$

$$= -\mathbf{g}_j(\mathbf{x}_i) \cdot \mathbf{p}_i - \mathbf{g}_i(\mathbf{x}_j) \cdot \mathbf{p}_j + \Phi_j(\mathbf{x}_i)\dot{M}_{\text{acc},i} + \Phi_i(\mathbf{x}_j)\dot{M}_{\text{acc},j} - \mathbf{g}_j(\mathbf{x}_i) \cdot \dot{\mathbf{M}}^1_{\text{acc},i} - \mathbf{g}_i(\mathbf{x}_j) \cdot \dot{\mathbf{M}}^1_{\text{acc},j} \tag{27}$$

2 DISCO Dictionary

Each of the values in the `report.dat` is one part of the time derivative integrated over the timestep Δt.

For the angular momentum: the particles *actually* have a total angular momentum J and spin angular momentum $J = L + S$. For gravitational interactions: $\dot{S}_{\text{grav}} = 0$ so $\dot{J}_{\text{grav}} = \dot{L}_{\text{grav}}$. For accretion (sinks) in principle $\dot{J}_{\text{acc}} = \dot{L}_{\text{acc}} + \dot{S}_{\text{acc}}$, but we build our sinks so that $\dot{S}_{\text{acc}} = 0$ as well.

Each potential energy U_{ij} depends on variation of both the i'th and j'th particles. We'll write $\dot{U}_{ij}^{(i)}$ for the part of $\dot{U}_{ij}^{(i)}$ that depends on variation of particle i and vice-versa for particle j. The total $\dot{U}_{ij} = \dot{U}_{ij}^{(i)} + \dot{U}_{ij}^{(j)}$.

There is also a potential energy $U_{\text{fluid},i}$ between each particle and the gravity of the gas itself. `Disco` stores the variation of *this* potential energy, as well as its sum with the particle-particle potential energy:

$$\dot{U}_{\text{tot},i} \equiv \dot{U}_{\text{fluid},i} + \sum_{j \neq i} U_{ij}^{(i)} \tag{28}$$

To get the change in the particle potential (ie. *what we care about*) you must subtract the two:

$$\dot{U}_{\text{particle}} = \sum_{i,j<i} \dot{U}_{ij} = \sum_i \left(\dot{U}_{\text{tot},i} - U_{\text{fluid},i}\right) \tag{29}$$

And finally, here is the dictionary between the possible entries that can go in the `report.dat` and the notation here:

$\mathbf{x}$	r_i	`planet[i].r`	(30)
	ϕ_i	`planet[i].phi`	(31)
$\mathbf{v}$	$v_{r,i}$	`planet[i].vr`	(32)
	Ω_i	`planet[i].omega`	(33)
$\dot{M}$	$\dot{M}_{\mathrm{acc},i}\Delta t$	`PL_SNK_M`	(34)
$\dot{\mathbf{x}}$	$\dot{M}^1_{x,\mathrm{acc},i}\Delta t$	`PL_SNK_MX`	(35)
	$\dot{M}^1_{y,\mathrm{acc},i}\Delta t$	`PL_SNK_MY`	(36)
$\dot{\mathbf{p}}$	$M_i g_{x,\mathrm{gas}}(\mathbf{x}_i)\Delta t$	`PL_GRV_PX`	(37)
	$M_i g_{y,\mathrm{gas}}(\mathbf{x}_i)\Delta t$	`PL_GRV_PY`	(38)
	$f_{x,\mathrm{acc},i}\Delta t$	`PL_SNK_PX`	(39)
	$f_{y,\mathrm{acc},i}\Delta t$	`PL_SNK_PY`	(40)
$\dot{L}_z$	$\left[\mathbf{x}_i \times (M_i \mathbf{g}_{\mathrm{gas}}(\mathbf{x}_i))\right]_z \Delta t$	`PL_GRV_JZ`	(41)
	$\left(\left[\dot{\mathbf{x}}_{\mathrm{acc},i} \times \mathbf{p}_i\right]_z + \left[\mathbf{x}_i \times \mathbf{f}_{\mathrm{acc},i}\right]_z\right)\Delta t$	`PL_SNK_LZ`	(42)
$\dot{K}$	$M_i \mathbf{v}_i \cdot \mathbf{g}_{\mathrm{gas}}(\vec{x}_i)\Delta t$	`PL_GRV_K`	(43)
	$\left(\mathbf{v}_i \cdot \mathbf{f}_{\mathrm{acc},i} - \frac{1}{2} v_i^2 \dot{M}_{\mathrm{acc},i}\right)\Delta t$	`PL_SNK_K`	(44)
$\dot{U}_{\mathrm{fluid}}$	$\dot{U}_{\mathrm{fluid},i}\Delta t$	`PL_SNK_UGAS`	(45)
$\dot{U}_{\mathrm{tot}}$	$\dot{U}_{\mathrm{tot},i}\Delta t$	`PL_SNK_U`	(46)

3 Eccentricity Vector

You can write the eccentricity vector as:

$$\mathbf{e} = \frac{1}{GMm^2}\mathbf{p} \times \mathbf{L} - \hat{\mathbf{r}} \tag{47}$$

Here $M = M_1 + M_2$ and $m = M_1 M_2 / M$. The momentum p and angular momentum L are those for the reduced particle in the central potential but are equal to the total momentum and angular momentum of the system so long as the barycenter is fixed at the origin.

For an aligned system, $\mathbf{L} = L_z \hat{\mathbf{z}}$, hence:

$$e_x = \frac{1}{GMm^2} p_y L_z - \cos\phi \tag{48}$$

$$e_y = -\frac{1}{GMm^2} p_x L_z - \sin\phi \tag{49}$$

The time derivatives are:

$$\dot{e}_x = -\frac{p_y L_z}{GMm^2}\left(\frac{\dot{M}}{M} + 2\frac{\dot{m}}{m}\right) + \frac{1}{GMm^2}\left(L_z \dot{p}_y + p_y \dot{L}_z\right) + (\sin\phi)\,\dot{\phi} \tag{50}$$

$$\dot{e}_y = \frac{p_x L_z}{GMm^2}\left(\frac{\dot{M}}{M} + 2\frac{\dot{m}}{m}\right) - \frac{1}{GMm^2}\left(L_z \dot{p}_x + p_x \dot{L}_z\right) + (\cos\phi)\,\dot{\phi} \tag{51}$$

We already know $\dot{\mathbf{p}}$ and $\dot{L}_z$ from the above dictionary. Also:

$$\dot{M} = \dot{M}_1 + \dot{M}_2 \tag{52}$$

$$\dot{m} = m\left(\frac{\dot{M}_1}{M_1} + \frac{\dot{M}_2}{M_2} - \frac{\dot{M}}{M}\right) \tag{53}$$

$$\dot{\phi} = -\frac{y}{r^2}\dot{x} + \frac{x}{r^2}\dot{y} \tag{54}$$

These $\dot{M}_1$, $\dot{M}_2$, $\dot{x}$, and $\dot{y}$ need contributions from the accretion/sink terms according to the above dictionary.