Disco Primer

Geoffrey Ryan

May 29, 2023

1 Basic Equations

Most simple fluids are are defined by a mass density ρ, a velocity $\mathbf{v}$, and pressure P. These are each functions of space and time: $\rho = \rho(t, x, y, z)$, etc. In a numerical code the set of variables $(\rho, \mathbf{v}, P)$ are often called the *primitive* variables. The gas also has a local temperature T, internal energy density e, sound speed c_s, and many other quantities, which are computed via an *equation of state* (EOS). For example, $e = e(\rho, P)$.

1.1 Equations of State

A common EOS is the Γ-law: $P = (\Gamma - 1)e$ with $\Gamma \in [1, 2]$, for such a gas the sound speed is $c_s^2 = \Gamma P/\rho$. A monatomic non-relativistic gas (e.g. atomic hydrogen H) obeys $\Gamma = 5/3$, a diatomic non-relativistic gas (e.g molecular nitrogen N_2) obeys $\Gamma = 7/5$, a relativistic gas (one whose sound speed is near the speed of light) obeys $\Gamma = 4/3$.

An *ideal* gas obeys the ideal gas law: $P = \rho T/m$, where m is the mass of a gas particle ($n = \rho/m$ is the number density). An ideal, Γ-law gas has $c_s^2 = \Gamma T/m$, the sound speed is proportional to the temperature, so we often refer to gases with large sound speed as "hot" and low sound speed as "cold".

An *isothermal* gas is a gas connected (somehow) to a large reserve of thermal energy so the temperature maintains some fixed value. Such a gas corresponds to $\Gamma = 1$ and has $c_s^2 = P/\rho$.

1.2 Evolution and Conservation

How does a fluid evolve in time? We have 5 real variables: ρ, P, and 3 components of $\mathbf{v}$, we need 5 equations (two scalar, one vectorial). We also have a number of *conservation laws*: conservation of particle number (not creating new fluid out of nothing), conservation of momentum (a vector), and conservation of energy. The momentum density of a fluid is $\rho\mathbf{v}$ and its kinetic energy density is $\rho v^2/2$. These conservation laws are just enough to give us precisely the equations we need: the Continuity Equation, Euler Equations, and the Energy Equation:

$$\frac{d\rho}{dt} \leftarrow \quad \partial_t \rho + \nabla(\rho\mathbf{v}) = 0 \tag{1}$$

Continuity (mass conser.)

$$\frac{d\rho v}{dt} \leftarrow \quad \partial_t(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v} + P) = \rho\mathbf{g} \tag{2}$$

Euler (Momentum density $m \cdot v$)

$$\partial_t\left(\frac{1}{2}\rho v^2 + e\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho v^2 + e + P\right)\mathbf{v}\right] = \rho\mathbf{v}\cdot\mathbf{g} - \dot{q} \leftarrow \text{cooling term } (\sigma T^4) \tag{3}$$

internal energy (thermal)

Change of energy.

$\nabla \cdot P = \vec{F}$

g = grav. acceler.

We have added a *gravitational field* $\mathbf{g} = -\nabla\Phi_g$, where $\Phi_g(t, x, y, z)$ is the *gravitational potential*, and a *radiative cooling* term $\dot{q}$ which can allow the fluid to lose energy. The mass density, momentum density, and total fluid energy density are called the *conservative* variables in numerical codes. The quantities inside the divergence brackets are the *fluxes*. The right hand side of each are the *sources* or *source terms*, they break the conservation in controlled ways (increasing the momentum by providing a force, for instance).

Five variables, five equations! Time derivatives of each in terms of the divergence of some flux which is easy to compute. This gives a system which is easy to time evolve, so long as you're careful of some numerical pitfalls. The practice of correctly and stably solving these equations numerically is an entire subfield of applied mathematics which is a lot of fun!

1.3 Viscosity and Dissipation

The equations above, specifically the Euler equations, are *inviscid*: they assume no viscosity is present. *Viscosity* is a phenomenon exhibited by some fluids where they exhibit a form of friction: their motion slows and their temperature rises. This does not break any conservation laws: the lost kinetic energy is turned into thermal energy, and while the total momentum is unchanged the momentum passed from one fluid element to another does change. We will specifically worry about *shear* viscosity: viscosity against shearing motions.

Fiction uniforms speed.

Shear is a particular type of non-uniformity in the velocity field. Fluid elements in a shearing fluid will slide past and overtake each other. Viscosity acts as friction to reduce this shear: the slower fluid gets pulled faster and the fast fluid gets pulled slower, so in the end each has the same velocity. Shear is measured by the *shear tensor*:

$$\sigma_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right) - \frac{1}{3}\gamma_{ij}\nabla\cdot\mathbf{v} \tag{4}$$

Here we used *index notation*, the ∇_i is a covariant derivative, and γ_{ij} is the metric tensor for the coordinate system. In Cartesian coordinates $\gamma_{ij} = \mathrm{diag}(1, 1, 1)$.

The magnitude of viscosity is parameterized by the *kinetic viscosity* ν. The fluid equations with viscosity included are the Navier-Stokes equations:

covariant

Generalized to different coords.

$$\partial_t \rho + \nabla_j \left(\rho v^j\right) = 0 \tag{5}$$

$$\partial_t \left(\rho v_i\right) + \nabla_j \left(\rho v_i v^j + P\delta_i^j - 2\rho\nu\, \sigma_i^j\right) = \rho g_i \tag{6}$$

$$\partial_t \left(\frac{1}{2}\rho v^2 + e\right) + \nabla_j \left[\left(\frac{1}{2}\rho v^2 + e + P\right) v^j - 2\rho\nu\, \sigma_{jk} v^k\right] = \rho v^j g_j - \dot{q} \tag{7}$$

Here given in index notation to make the tensors explicit, and used the summation convention: $a_i b^i \equiv \sum_{i=1}^{3} a_i b^i = a_1 b^1 + a_2 b^2 + a_3 b^3$.

This set of equations, exactly as written here, describe just about every fluid known to humankind. The air in weather & climate modelling, and in aerospace design. Water from microfluidics to the ocean, simulations of the heart & arteries. The earth and distant planets, the sun and distant stars, the interstellar and intergalactic media, and, of course, astrophysical disks!

1.4 Coordinates and Dimensions

Here we just give the equations explicitly in a couple coordinate systems, with the covariant derivatives ∇_i written in terms of $\partial_i \equiv \partial/\partial x^i$ and whatever extra source terms might be required.

In Disco we work in 2D most of the time, so we're going to set v_z and all ∂_z to 0. The resulting equations are the *vertically-integrated* equations: technically we have integrated in z from the bottom to top of the flow, assuming $v_z \approx 0$. As a result we will change variables slightly: the density is now the *surface density* $\Sigma = \int \rho dz$ and the pressure is now the vertical pressure $\Pi = \int P dz$. The internal energy density is also integrated, we will write it as $\Sigma\epsilon$, where $\epsilon = e/\rho$ is the *specific internal energy.*

1.4.1 Cartesian

Cartesian coordinates x, y, z are very simple to work in for many problems! In Cartesian coordinates $v_x = v^x$, $\nabla_i = \partial_i$ and similarly for the other components. Here are all 4 equations written out:

Vertical integrated
$\Rightarrow$ surface density

$$\partial_t \Sigma + \partial_x (\Sigma v_x) + \partial_y (\Sigma v_y) = 0 \tag{8}$$

$$\partial_t (\Sigma v_x) + \partial_x \left(\Sigma v_x^2 + \Pi - 2\Sigma\nu\, \sigma_{xx}\right) + \partial_y (\Sigma v_x v_y - 2\Sigma\nu\, \sigma_{xy}) = \Sigma g_x \tag{9}$$

$$\partial_t (\Sigma v_y) + \partial_x (\Sigma v_x v_y - 2\Sigma\nu\, \sigma_{xy}) + \partial_y \left(\Sigma v_y^2 + \Pi - 2\Sigma\nu\, \sigma_{yy}\right) = \Sigma g_y \tag{10}$$

$$\begin{aligned} \partial_t \left(\frac{1}{2}\Sigma v^2 + \Sigma\epsilon\right) + \partial_x \left[\left(\frac{1}{2}\Sigma v^2 + \Sigma\epsilon + \Pi\right) v_x - 2\Sigma\nu(\sigma\cdot\mathbf{v})_x\right] & \\ + \partial_y \left[\left(\frac{1}{2}\Sigma v^2 + \Sigma\epsilon + \Pi\right) v_y - 2\Sigma\nu(\sigma\cdot\mathbf{v})_y\right] &= \Sigma\mathbf{v}\cdot\mathbf{g} - \dot{q} \end{aligned} \tag{11}$$

The relevant shear tensor components are:

$$\sigma_{xx} = \frac{2}{3}\partial_x v_x - \frac{1}{3}\partial_y v_y \qquad \sigma_{yy} = -\frac{1}{3}\partial_x v_x + \frac{2}{3}\partial_y v_y \qquad \sigma_{xy} = \frac{1}{2}\left(\partial_x v_y + \partial_y v_x\right) \tag{12}$$

1.4.2 Cylindrical Polar

This is where `Disco` really spins. Cylindrical polar coordinates r, ϕ, z where $x = r\cos\phi$ and $y = r\sin\phi$ are perfect for disks. In these coordinates the metric is $\gamma_{ij} = \mathrm{diag}(1, r^2, 1)$ and $v_r = v^r$ but $v_\phi = r^2 v^\phi$. To avoid ambiguity we will define $\Omega \equiv v^\phi$: the *angular velocity* with dimensions 1/time (measured, for instance, in radians per second). Then $v_\phi = r^2\Omega$, this is called the *specific angular momentum* and it has dimensions length^2/time (measured for instance in cm^2 per second).

The 4 equations are:

Solve in DISCO

Continuity

$$\partial_t \Sigma + \frac{1}{r}\partial_r \left(r \Sigma v_r \right) + \partial_\phi \left(\Sigma \Omega \right) = 0 \tag{13}$$

Radial Momentum

$$\partial_t \left(\Sigma v_r \right) + \frac{1}{r}\partial_r \left(r\Sigma v_r^2 + r\Pi - 2r\Sigma\nu\, \sigma_{rr} \right) + \partial_\phi \left(\Sigma v_r \Omega - 2\Sigma\nu\, \sigma^\phi_r \right) = r\Sigma\Omega^2 + \frac{1}{r}\Pi + \Sigma g_r \tag{14}$$

Angular Momentum

$$\partial_t \left(\Sigma r^2 \Omega \right) + \frac{1}{r}\partial_r \left(r^3 \Sigma v_r \Omega - 2r\Sigma\nu\, \sigma_{r\phi} \right) + \partial_\phi \left(\Sigma r^2 \Omega^2 + \Pi - 2\Sigma\nu\, \sigma^\phi_\phi \right) = \Sigma g_\phi \tag{15}$$

Energy

$$\begin{aligned} \partial_t \left(\frac{1}{2}\Sigma v^2 + \Sigma\epsilon \right) &+ \frac{1}{r}\partial_r \left(r \left(\frac{1}{2}\Sigma v^2 + \Sigma\epsilon + \Pi \right) v_r - 2r\Sigma\nu(\sigma\cdot\mathbf{v})_r \right) \\ &+ \partial_\phi \left(\left(\frac{1}{2}\Sigma v^2 + \Sigma\epsilon + \Pi \right) \Omega - 2\Sigma\nu(\sigma\cdot\mathbf{v})^\phi \right) = \Sigma\mathbf{v}\cdot\mathbf{g} - \dot{q} \end{aligned} \tag{16}$$

Things are a little more complicated! We have to be careful about $_\phi$ vs $^\phi$, and the radial momentum equation got some new source terms. These are an artifact of the coordinates: radial momentum is not a thing that is really conserved. The $r\Sigma\Omega^2$ term is the centripetal acceleration: fluid being spun in an arc will experience a force radially outwards (v_r will grow).

The gravitational term Σg_ϕ is the *gravitational torque*. In terms of the potential: $g_\phi = -\partial_\phi \Phi_g$. It is zero if the potential is coming from a single object at $r = 0$.

The relevant shear tensor components are:

$$\sigma_{rr} = \frac{2}{3}\partial_r v_r - \frac{1}{3}\partial_\phi \Omega - \frac{1}{3}\frac{v_r}{r} \tag{17}$$

$$\sigma^\phi_\phi = -\frac{1}{3}\partial_r v_r + \frac{2}{3}\partial_\phi \Omega + \frac{2}{3}\frac{v_r}{r} \tag{18}$$

$$\sigma_{r\phi} = \frac{1}{2} r^2 \partial_r \Omega + \frac{1}{2}\partial_\phi v_r \tag{19}$$

$$\sigma^\phi_r = \frac{1}{r^2}\sigma_{r\phi} \tag{20}$$

These are the exact equations Disco solves!

2 Disks

Here we will coax the accretion disk solutions out of the Navier-Stokes equations. We will make many approximations to get easy-to-handle analytic formulae. `Disco` will be able to do the full problem with fewer assumptions, but we need to feed `Disco` good initial and boundary conditions so it is very necessary to understand these solutions at the pen & paper level.

An *accretion disk* is a disk-shaped flow of gas around a central gravitating body (or bodies). To be a *disk* it must be generally shorter in height than in radius (otherwise it would be called a column!). This arrangement of material is very common in the universe, just look at galaxies! It tends to happen when a cloud of gas is bound by gravity and has a lot of angular momentum. The angular momentum allows the gas to settle into orbits (rather than falling directly onto the central body). If the gas cloud started out with a random arrangement of internal velocities, the orbits of the gas parcels will intersect and the gas will start colliding with itself. Importantly, these collisions conserve angular momentum! So the total angular momentum of the cloud will be conserved, and the gas will re-distribute and re-arrange itself until the collisions stop: when all the gas is on concentric non-intersecting orbits. Finally, the gas needs some way to lose energy by cooling: the collisions generate a lot of heat! As the gas cools, the cloud orbits will settle into a single plane, and a disk forms.

The *accretion* in "accretion disk" refers to gas slowly accumulating on the central object: the gas does not live on perfect orbits forever. Internal processes (turbulence, magnetic fields, often approximated as "viscosity") causes gas to slowly change its orbit and spiral inwards.

So: we are looking for solutions to the Navier-stokes equations that are: long-lived, short in vertical height, following largely orbital motion around a central object.

2.1 The Scale Height

First thing, let's consider the vertical equation we dropped. Disks are long-lived phenomena: they tend to achieve *hydrostatic equilibrium* in the vertical direction. In vertical hydrostatic equilibrium, the vertical velocity $v_z \equiv 0$. We can also assume the vertical components of the shear tensor $\sigma_{rz} = \sigma_{\phi z} = \sigma_{zz} = 0$. The vertical Euler equation is:

$$\partial_t \rho v_z + \frac{1}{r}\partial_r \left(r\rho v_r v_z\right) + \partial_\phi \left(\rho v_z \Omega\right) + \partial_z \left(\rho v_z^2 + P\right) = \rho g_z \tag{21}$$

Vertical hydrostatic equilibrium ($v_z = 0$) kills most of these terms and simplifies this to:

$$\partial_z P = \rho g_z \tag{22}$$

We want to use this to get a simple measure of the disk height H. Gravity pulls the gas downwards (towards $z = 0$) and pressure holds the disk up. The pressure is zero at the disk surface ($z = H$), and maximal at the center ($z = 0$).

For a single central object of mass M:

$$\Phi_g = -\frac{GM}{\sqrt{r^2 + z^2}} \implies \quad g_r = -\frac{GM}{r^2} + \mathcal{O}\left(\frac{z^2}{r^2}\right), \quad g_\phi = 0, \quad g_z = -\frac{GM}{r^3} z + \mathcal{O}\left(\frac{z^3}{r^3}\right) \tag{23}$$

We've already started approximating: keeping only the leading order behaviour as $z \ll r$. This is valid so long as the disk is *thin*. Disk theory covers a whole range of approximations for how "thin" a disk is: razor-thin, thin, slim, and thick. We will be operating mostly in the "thin" regime.

We want to get a simple measure to relate the height of a disk H to the local density, pressure, and gravitational field. So we start approximating wildly! We can replace the pressure gradient with its average value $\partial_z P \sim -P(z = 0)/H$. The vertical gravitational field is approximately $g_z \sim -GMH/r^3$, and we can approximate the density with its central value $\rho \sim \rho(z = 0)$. Putting it all together:

$$
\begin{aligned}
& -\frac{P(z=0)}{H} \sim -\rho(z=0)\frac{GM}{r^3}H \\
\Longrightarrow \quad & \frac{P(z=0)}{\rho(z=0)} \sim \frac{GM}{r^3}H^2 \\
\Longrightarrow \quad & H \sim \sqrt{\frac{r^3}{GM}}\sqrt{\frac{P(z=0)}{\rho(z=0)}} \\
\Longrightarrow \quad & H \sim \sqrt{\frac{r^3}{GM}}c_s \\
\Longrightarrow \quad & H \sim \frac{c_s}{\Omega_k} \qquad (24)
\end{aligned}
$$

Here we have defined the *Keplerian* angular velocity $\Omega_k \equiv \sqrt{GM/r^3}$: the angular velocity of a circular orbit. We have also replaced P/ρ with the sound speed c_s^2, which drops the Γ factor but that is far from the only evil commited here.

This is a very useful relation! It tells us the local scale height of the disk, and shows that it increases with increasing sound speed (which is related to the temperature). So, indeed, hot disks are thicker than cold disks!

HW

2.2 Instructions For Building A Simple Disk Model

We're going to build a simple thin accretion disk model from the ground up!

1. **Steady and axisymmetric.** Disks are long-lived and tend to be roughly axisymmetric, so let's make that a requirement! Take the cylindrical Navier-stokes equations (Continuity, Radial Momentum, Angular Momentum, and Energy) and re-write them with all derivatives of t and ϕ set to 0. Remember there are some derivatives hidden inside the shear tensor σ. Include a central gravitating object of mass M. You should now have a set of four ordinary differential equations: everything is only a function of r.

2. **The razor thin limit.** Now we're going to go further! Solve the Disk Equations from Step 1, assuming the disk is razor-thin: $H \approx 0$, and inviscid ($\nu = 0$). The disk must have some mass in it, so $\Sigma > 0$ everywhere. Since $H \sim c_s/\Omega_k$, you can also set $c_s \approx 0$, as well as Π and ϵ (since $\Pi \sim \Sigma c_s$ and $\epsilon \sim \Pi/\Sigma \sim c_s^2$). Find v_r and Ω as functions of r, and determine if there are constraints on Σ. What must $\dot{q}$ be? There might be multiple solutions, one looks like Saturn's Rings.

3. **The Accretion Rate**. The *accretion rate* $\dot{M}$ is the rate at which matter passes a particular radius moving towards the central object: $\dot{M} = -2\pi r \Sigma v_r$. It is -2π (for integrating around ϕ) multiplied by the radial mass flux (the bit inside the radial derivative in the Continuity equation). What does the continuity equation (from Step 1) tell you about $\dot{M}$?

4. **The Torque**. The torque through the disk, $\dot{J}$, is another constant of the system. It is the rate at which the central object accretes angular momentum, and equal to -2π times the radial angular momentum flux. Use the angular momentum equation from Step 1 to write down a formula for $\dot{J}$. Is it constant? Do not assume ν is 0. Find an equation for Σ in terms of r, $\dot{M}$, $\dot{J}$, ν, and Ω.

5. **An Accretion Disk**. You now have two algebraic equations (involving $\dot{M}$ and $\dot{J}$) plus the radial momentum and energy equations. We're going to solve the radial momentum equation. Assume the disk is thin: $H \ll r$ and that the radial velocity is subsonic: $|v_r| < c_s$. Take the radial momentum equation from Step 1, and identify the order (in terms of H/r) of each of the terms. Assume $\Sigma \approx \mathcal{O}(1)$ and $GM/r^2 \approx \mathcal{O}(1)$. The centripetal acceleration term must balance the largest (ie. lowest power of H/r) other term. Use this to find the angular velocity Ω. With that, find the surface density Σ and radial velocity v_r in terms of M, r, ν, $\dot{M}$, and $\dot{J}$. This is a decent accretion disk model!

6. **The Luminosity** Give the energy equation from step 1 the same treatment you just gave the radial momentum equation. Find the necessary radiative cooling $\dot{q}$ to balance the largest term(s).