

# Two-Body Problem

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## 1 Introduction

The two body problem is an isolated system of two particles which interact through a central potential. For two particles with mass  $m_1$  and  $m_2$ , the equations of motion is

$$m_1 \ddot{r}_1 = F_{21}; m_2 \ddot{r}_2 = F_{12}. \quad (1)$$

In Newtonian gravity, the potentials are therefore

$$U_{12} = u_{21} = \frac{Gm_1m_2}{|r_1 - r_2|^2}. \quad (2)$$

Since the 2 particles interact via a central potential, the total momentum is conserved. We consider the center of mass,

$$R = \frac{m_1 r_1 + m_2 r_2}{M} \quad (3)$$

where  $M = m_1 + m_2$ .

The time derivative is therefore

$$V_{CM} = \frac{m_1 v_1 + m_2 v_2}{M}. \quad (4)$$

Since our system is isolated, the net force and total acceleration is zero, thus  $V_{CM}$  is a constant. The motion of the center of mass is then

$$R(t) = V_{CM}t. \quad (5)$$

We define the vector of distance between the particles:

$$r = r_1 - r_2. \quad (6)$$

Thus,

$$r_1 = R + \frac{m_2 r}{m_1 + m_2}; r_2 = R - \frac{m_1 r}{m_1 + m_2}. \quad (7)$$

Since we already know  $R$ , our problem reduces to finding  $r$ .

If we multiply the masses to ??, we get

$$m_1 m_2 \ddot{r}_1 = m_2 F_{21}; \quad m_1 m_2 \ddot{r}_2 = m_1 F_{12}. \quad (8)$$

Subtract the second from the first equation and simplify, we get

$$\frac{m_1 m_2}{m_2 + m_1} \ddot{r} = F_{21} \quad (9)$$

The force derived from the central potential only depends on distance between the 2 particles.

$$F_{21} = -\frac{\partial}{\partial r_1} U_{12}(|r_1 - r_2|) = \nabla U_{12} \quad (10)$$

From above, we can summarize them as

$$m_* \ddot{r} = -\frac{\partial}{\partial r} U(|r|) = F(r) \quad (11)$$

where

$$m_* = \frac{m_1 m_2}{m_1 + m_2} \quad (12)$$

is the reduced mass.

## 2 Conservation of Angular Momentum

In this section, we will rewrite the system above in terms of the angular momentum.

$$L = m_* r \times V; \quad V = \dot{r}. \quad (13)$$

Since  $F$  is in the  $r$  direction, the torque has to be 0, and the angular momentum must be constant.

$$\frac{d}{dt} L = 0 \quad (14)$$

Next, we use the *scalar triple product identity*,

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \quad (15)$$

and apply it to the angular momentum, we get

$$r \cdot L = m_* r \cdot (r \times V) = m_* V \cdot (r \times r) = 0. \quad (16)$$

Hence  $r$  is always *perpendicular* to angular momentum  $L$ .

Because the angular momentum is constant, there must be a fixed vector of which  $r$  is perpendicular to in space. Since the position vector is always perpendicular to a certain direction in space, we can reduce the problem into a 2D plane in polar coordinates where

$$r_x = r \cos \theta; \quad r_y = r \sin \theta. \quad (17)$$

We choose the  $x - y$  plane for the particles to be on in accordance to conventions, and the angular momentum is oriented to the  $z$  direction. The velocity in  $\theta$  direction is always positive, therefore the particles always rotates around the center of our coordinate system in the same direction.

$$\dot{\theta} = \frac{|L|}{m_* r^2} \quad (18)$$

### 3 Conservation of Energy

To begin with, we know that Energy is the sum of kinetic and potential energies.

$$E = \frac{1}{2} m_* v^2 + U(r) \quad (19)$$

In polar coordinates, the velocity is

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\theta}^2. \quad (20)$$

The energy conservation becomes

$$E = \frac{1}{2} m_* \dot{r}^2 + \frac{1}{2} m_* r^2 \dot{\theta}^2 + U(r). \quad (21)$$

### 4 Solutions for Two-Body Problem

Since  $\dot{\theta} = \frac{L}{mr^2}$ , we get the acceleration in  $\theta$  direction

$$\ddot{\theta} = -\frac{\dot{r}L}{mr^3} = -\frac{2\dot{r}\dot{\theta}}{r}. \quad (22)$$

Acceleration in  $r$  direction is

$$\ddot{r} = \frac{F}{m} = r\dot{\theta}^2 - \frac{Gm}{r^2} \quad (23)$$

because  $G\ddot{M}/r^2$  and  $r = \frac{x}{\cos \theta}$ .

Our problem then becomes a system of two second-order ordinary differential equations stated above. To numerically solve it using RK-4, we have to use 4 first-order differential equations such as

$$V_r = \dot{r} \quad (24)$$

$$V_\theta = \dot{\theta} \quad (25)$$

$$\dot{V}_r = rV_\theta^2 - \frac{Gm}{r^2} \quad (26)$$

$$\dot{V}_\theta = -\frac{V_r L}{mr^3} \quad (27)$$