Two-Body Problem

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1 Introduction

The two body problem is an isolated system of two particles which interact through a central potential. For two particles with mass m_1 and m_2, the equations of motion is

$$m_1\ddot{r_1} = F_{21};\ m_2\ddot{r_2} = F_{12}. \tag{1}$$

In Newtonian gravity, the potentials are therefore

$$U_{12} = U_{21} = \frac{Gm_1m_2}{|r_1 - r_2|}. \tag{2}$$

Since the 2 particles interact via a central potential, the total momentum is conserved. We consider the center of mass,

$$R = \frac{m_1r_1 + m_2r_2}{M} \tag{3}$$

where $M = m_1 + m_2$.
The time derivative is therefore

$$V_{CM} = \frac{m_1v_1 + m_2v_2}{M}. \tag{4}$$

Since our system is isolated, the net force and total acceleration is zero, thus V_{CM} is a constant. The motion of the center of mass is then

$$R(t) = V_{CM}t. \tag{5}$$

We define the vector of distance between the particles:

$$r = r_1 - r_2. \tag{6}$$

Thus,

$$r_1 = R + \frac{m_2r}{m_1 + m_2};\ r_2 = R - \frac{m_1r}{m_1 + m_2}. \tag{7}$$

Since we already know R, our problem reduces to finding r.

If we multiply the masses to eq. 1, we get

$$m_1 m_2 \ddot{r_1} = m_2 F_{21};\ m_1 m_2 \ddot{r_2} = m_1 F_{12}. \tag{8}$$

Subtract the second from the first equation and simplify, we get

$$\frac{m_1 m_2}{m_2 + m_1}\ddot{r} = F_{21} \tag{9}$$

The force derived from the central potential only depends on distance between the 2 particles.

$$F_{21} = -\frac{\partial}{\partial r_1} U_{12}(|r_1 - r_2|) = \nabla U_{12} \tag{10}$$

From above, we can summarize them as

$$m_* \ddot{r} = -\frac{\partial}{\partial r} U(|r|) = F(r) \tag{11}$$

where

$$m_* = \frac{m_1 m_2}{m_1 + m_2} \tag{12}$$

is the reduced mass.

2 Conservation of Angular Momentum

In this section, we will rewrite the system above in terms of the angular momentum.

$$L = m_* r \times V;\ V = \dot{r}. \tag{13}$$

Since F is in the r direction, the torque has to be 0, and the angular momentum must be constant.

$$\frac{d}{dt} L = 0 \tag{14}$$

Next, we use the *scalar triple product identity*,

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \tag{15}$$

and apply it to the angular momentum, we get

$$r \cdot L = m_* r \cdot (r \times V) = m_* V \cdot (r \times r) = 0. \tag{16}$$

Hence r is always *perpendicular* to angular momentum L.

Because the angular momentum is constant, there must be a fixed vector of which r is perpendicular to in space. Since the position vector is always perpendicular to a certain direction in space, we can reduce the probelm into a 2D plane in polar coordinates where

$$r_x = r \cos\theta;\ r_y = r \sin\theta. \tag{17}$$

We choose the x - y plane for the particles to be on in accordance to conventions, and the angular momentum is oriented to the z direction. The velocity in θ direction is always positive, therefore the particles always rotates around the center of our coordinate system in the same direction.

$$\dot{\theta} = \frac{|L|}{m_* r^2} \tag{18}$$

3 Conservation of Energy

To begin with, we know that Energy is the sum of kinetic and potential energies.

$$E = \frac{1}{2} m_* v^2 + U(r) \tag{19}$$

In polar coordinates, the velocity is

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\theta}^2. \tag{20}$$

The energy conservation becomes

$$E = \frac{1}{2} m_* \dot{r}^2 + \frac{1}{2} m_* r^2 \dot{\theta}^2 + U(r). \tag{21}$$

We call part of this equation the **effective potential energy**.

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} + U(r) \tag{22}$$

Since the angular momentum is constant, in spherical coordinates,

$$L = mr^2 \dot{\phi} = const; \ \dot{\phi} = \frac{L}{mr^2} \tag{23}$$

Here the ϕ in spherical coordinates is equivalent to θ in polar coordinates.

4 Solutions for Two-Body Problem (Analytical)

We have derived that energy of the system is constant

$$\begin{aligned}
E &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + U(r) = const. \\
\dot{r} &= \frac{dr}{dt} = \sqrt{\frac{2}{m}(E - U(r)) \frac{L^2}{m^2 r^2}} \\
\int & \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{L^2}{m^2 r^2}}} = \int dt = t.
\end{aligned}$$

We want to find trajectory $r(\phi)$.

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{L}{mr^2}$$
$$E = \frac{1}{2}m\left(\frac{dr}{d\phi}\right)^2\left(\frac{L}{mr^2}\right)^2 + \frac{L^2}{2mr^2} + U(r)$$
$$E = \frac{L}{2m}\left(\frac{dr}{r^2 d\phi}\right)^2 + \frac{L^2}{2mr^2} + U(r) \tag{24}$$

We introduce $b = \frac{1}{r}$ and $db = \frac{-dr}{r^2}$. The equation of energy thus becomes

$$E = \frac{L^2}{2m}\left[\left(\frac{db}{d\phi}\right)^2 + b^2\right] + U(\frac{1}{b}), \tag{25}$$

where if we find $r(\phi)$, and $r(\phi) = \frac{1}{b(\phi)}$, we can find $\phi(t)$.

5 Solutions for Two-Body Problem (Numerical)

Since $\dot{theta} = \frac{L}{mr^2}$, we get the acceleration in θ direction

$$\ddot{\theta} = -\frac{\dot{r}L}{mr^3} = -\frac{2\dot{r}\dot{\theta}}{r}. \tag{26}$$

Acceleration in r direction is

$$\ddot{r} = \frac{F}{m} = r\dot{\theta}^2 - \frac{Gm}{r^2} \tag{27}$$

because $G\ddot{M}/r^2$ and $r = \frac{x}{\cos\theta}$.

Our problem then becomes a system of two second-order ordinary differential equations stated above. To numerically solve it using RK-4,

we have to use 4 first-order differential equations such as

$$V_r = \dot{r} \tag{28}$$

$$V_\theta = \dot{\theta} \tag{29}$$

$$\dot{V}_r = rV_\theta^2 - \frac{Gm}{r^2} \tag{30}$$

$$\dot{V}_\theta = -\frac{V_r L}{mr^3} \tag{31}$$