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OVERGROUPS OF WEAK SECOND MAXIMAL SUBGROUPS

HANGYANG MENG✉ and XIUYUN GUO

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Abstract

A subgroup H is called a weak second maximal subgroup of G if H is a maximal subgroup of a maximal subgroup of G. Let $m(G, H)$ denote the number of maximal subgroups of G containing H. We prove that $m(G, H) - 1$ divides the index of some maximal subgroup of G when H is a weak second maximal subgroup of G. This partially answers a question of Flavell ['Overgroups of second maximal subgroups', *Arch. Math.* **64**(4) (1995), 277–282] and extends a result of Pálfy and Pudlák ['Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups', *Algebra Universalis* **11**(1) (1980), 22–27].

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1. Introduction

All groups considered in this paper are finite.

Let G be a group and let H be a proper subgroup of G. We denote by $\mathrm{Max}(G, H)$ the set of all maximal subgroups of G containing H and by $m(G, H)$ the cardinality of the set $\mathrm{Max}(G, H)$.

A subgroup H is called a *second maximal subgroup* of G if H is the maximal subgroup of every member of $\mathrm{Max}(G, H)$ and we say that H is a *weak second maximal subgroup* of G if H is a maximal subgroup of some member of $\mathrm{Max}(G, H)$. A second maximal subgroup is a weak second maximal subgroup but the converse is not true in general.

The aim of this paper is to study $m(G, H)$, that is, the number of maximal subgroups of G containing H, when H is a weak second maximal subgroup of G. The following is our main result.

Theorem 1.1. *Let G be a group and H a weak second maximal subgroup of G such that $G/\operatorname{Core}_G(H)$ is solvable. Then $m(G, H) - 1$ divides $|G : M|$ for some $M \in \mathrm{Max}(G, H)$.*

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If $A \leq B$, we write $\mathrm{Core}_B(A) = \bigcap_{b \in B} A^b$, the core of A in B. If $\mathrm{Core}_A(B) = 1$, we call A *core-free* in B.

Our motivation comes from Flavell [1, Theorem A], where it is proved that if H is a second maximal subgroup of a group G, then $m(G, H) - 1$ is at most

$$\max\{|G : X| : X \in \mathrm{Max}(G, H)\}.$$

Flavell asks the natural question: 'Does the inequality proved in the above result still hold if second maximal is replaced by weak second maximal?'. As a corollary of Theorem 1.1, we give an affirmative answer when $G/\,\mathrm{Core}_G(H)$ is solvable.

Corollary 1.2. *Let G be a group and H a weak second maximal subgroup of G such that $G/\,\mathrm{Core}_G(H)$ is solvable. Then $m(G, H)$ is at most*

$$1 + \max\{|G : X| : X \in \mathrm{Max}(G, H)\}.$$

Pálfy and Pudlák [5, Theorem 3] have shown that if H is a second maximal subgroup of G such that $G/\,\mathrm{Core}_G(H)$ is solvable, then $m(G, H) - 1$ is a prime power. It is well known that the index of every maximal subgroup of a solvable group is a prime power. Thus, as another corollary of Theorem 1.1, we can extend the result of Pálfy and Pudlák to weak second maximal subgroups.

Corollary 1.3. *Let G be a group and H a weak second maximal subgroup of G such that $G/\,\mathrm{Core}_G(H)$ is solvable. Then $m(G, H) - 1$ is a prime power.*

2. Preliminaries: modules

We recall some results about modules. The notation and terminology agree with [2, Ch. 3].

Throughout this section, we assume that F is a field and V is a finite-dimensional vector space over F, denoted simply by V/F. Let G be a group and ϕ a representation of G on V/F. Then V is called a G-module over F (with respect to ϕ) by the law $v \cdot g = v(g\phi)$, where $g \in G$ and $v \in V$, and we say simply that V/F is a G-module.

We sometimes use a more general 'module'. A G-module V is *of mixed characteristic* if $V = V_1 \oplus \cdots \oplus V_n$, where for each i there exists a field F_i such that V_i/F_i is a G-module.

Lemma 2.1. *Let V be a completely reducible G-module, possibly of mixed characteristic. Then V is the direct sum of G-modules V_i, $1 \leq i \leq r$, satisfying the following conditions:*

(a) *$V_i = X_{i1} \oplus \cdots \oplus X_{it_i}$, where X_{ij} is an irreducible G-submodule for $1 \leq i \leq r$ and $1 \leq j \leq t_i$. Moreover, $X_{ij}, X_{i'j'}$ are isomorphic G-submodules if and only if $i = i'$;*
(b) *any irreducible G-submodule of V lies in V_i for some i;*
(c) *the number of all irreducible G-submodules of V is the sum of the number of all irreducible G-submodules of the V_i for $1 \leq i \leq r$.*

Proof. Part (a) follows from the canonical decomposition of completely reducible modules and part (b) implies part (c). Hence, it suffices to prove part (b). Let U be an irreducible G-submodule of V. Observe that

$$V = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{t_i} X_{ij}.$$

Thus, U must be G-isomorphic to X_{k1} for some k and we will prove that $U \subseteq V_k$. Let p_{ij} be the projection of V onto X_{ij}, where $1 \leq i \leq r$ and $1 \leq j \leq t_i$. It is not difficult to see that Up_{ij} is a G-submodule of X_{ij}, which implies that $Up_{ij} = 0$ or $Up_{ij} = X_{ij}$ by the irreducibility of X_{ij}. Since U is not G-isomorphic to X_{i1} if $i \neq k$, we must have $Up_{ij} = 0$ if $i \neq k$. Now

$$U \subseteq \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{t_i} Up_{ij} = \bigoplus_{j=1}^{t_k} Up_{kj} \subseteq V_k,$$

as desired. □

Suppose that V/F and W/F are G-modules. A homomorphism ψ of V/F into W/F (that is, a linear transformation of V/F into W/F) is called a G-homomorphism if $(v \cdot g)\psi = (v)\psi \cdot g$. We denote the set of all G-homomorphisms of V into W by $\mathrm{Hom}_G(V, W)$.

Recall that a division ring D is a ring such that all its nonzero elements form a group under multiplication. We state the following well-known lemmas.

Lemma 2.2. *Let V/F be an irreducible G-module and F a finite field. Then* $\mathrm{Hom}_G(V, V)$ *is a finite field.*

Proof. From [2, Ch. 3, Theorem 5.2], $\mathrm{Hom}_G(V, V)$ is a division ring. Since $\mathrm{Hom}_G(V, V)$ is finite, $\mathrm{Hom}_G(V, V)$ is a finite field by Wedderburn's theorem. □

Lemma 2.3. *Let F be a finite field and V/F an irreducible G-module. Then* $|\mathrm{Hom}_G(V, V)|$ *divides* $|V|$.

Proof. Write $E = \mathrm{Hom}_G(V, V)$. By Lemma 2.2, E is a finite field. Observe that V is a faithful E-module under the natural action. It follows that V is a vector space of finite dimension over the field E. Hence, $|E|$ divides $|V|$, as desired. □

The following lemma due to Green will play an important role in proving Theorem 1.1.

Lemma 2.4 [2, Theorem 5.6]. *Let V be the direct sum of the isomorphic irreducible G-modules V_i/F, $1 \leq i \leq t$, where F is a finite field. Then the number of distinct irreducible G-submodules of V is exactly $(q^t - 1)/(q - 1)$, where* $q = |\mathrm{Hom}_G(V_1, V_1)|$.

3. Proof of Theorem 1.1

Before proving Theorem 1.1, we start with an easy but useful lemma.

LEMMA 3.1. *Let $G = MN$ be a group such that $M \cap N = 1$ and N is a solvable normal subgroup of G. Then N is a minimal normal subgroup of G if and only if M is a maximal subgroup of G.*

PROOF. Firstly we assume that N is a minimal normal subgroup of G. By hypothesis, N is abelian. Let $M \leq X < G$. It suffices to show that $X = M$. By Dedekind's lemma, $X = M(X \cap N)$. Since both M and N normalise $X \cap N$, it follows that $X \cap N \trianglelefteq G$. Observe that $X \cap N < N$ since $X = M(X \cap N) < G$. By the minimality of N, we have $X \cap N = 1$ and clearly $X = M$.

Conversely, assume that M is a maximal subgroup of G. Let X be a minimal normal subgroup of G contained in N. Since $X \nleq M$, it follows that $G = XM$ by the maximality of M. As $N \cap M = 1$, we have $N = X(N \cap M) = X$, as desired. □

Lemma 3.4 is the key lemma to deal with weak second maximal subgroups. It follows easily from two results in [4].

LEMMA 3.2 [4, Lemma 1]. *Let G be a group and H a subgroup of G. If there exist $M, X \in \mathrm{Max}(G, H)$ such that H is maximal in M but not maximal in X, then* $\mathrm{Core}_G(H) = \mathrm{Core}_G(M)$.

LEMMA 3.3 [4, Theorem B]. *Let G be a solvable group and H a weak second maximal subgroup of G. Then there exists at most one member X of* $\mathrm{Max}(G, H)$ *such that H is not maximal in X.*

LEMMA 3.4. *Let G be a solvable group and H a weak second maximal subgroup of G with* $\mathrm{Core}_G(H) = 1$. *Then:*

(a) *there exists some member of* $\mathrm{Max}(G, H)$ *which is not core-free in G;*
(b) *if H is not a second maximal subgroup of G, then there exists a unique member of* $\mathrm{Max}(G, H)$ *which is not core-free in G.*

PROOF. We first prove part (a). Let N be a minimal normal subgroup of G. If $G = NH$, then $N \cap H \leq \mathrm{Core}_G(H) = 1$ and it follows from Lemma 3.1 that H is a maximal subgroup of G, contrary to the hypothesis. Thus, there is a maximal subgroup M of G containing NH. Clearly $N \leq \mathrm{Core}_G(M)$. Hence, $M \in \mathrm{Max}(G, H)$ and $\mathrm{Core}_G(M) \neq 1$.

Now we prove part (b). Since H is not a second maximal subgroup of G, by [4, Theorem B], there exists a unique member X of $\mathrm{Max}(G, H)$ such that H is not maximal in X. Thus, H is maximal in M for every $M \in \mathrm{Max}(G, H) - \{X\}$. It follows from Lemma 3.2 that $\mathrm{Core}_G(M) = 1$. By part (a), X is the unique member of $\mathrm{Max}(G, H)$ which is not core-free in G, as desired. □

PROOF OF THEOREM 1.1. We may assume that $\mathrm{Core}_G(H) = 1$ and G is solvable. Suppose that $\mathrm{Max}(G, H) = \{M_1, \ldots, M_r\}$, where $r = m(G, H)$. If $r = 2$, the result is trivial. Thus, we assume that $r \geq 3$. By Lemma 3.4(a), there exists at least one member of $\mathrm{Max}(G, H)$ which is not core-free in G. Hence, we can prove the theorem by considering the following two cases.

Case I. There exists a unique member of $\mathrm{Max}(G, H)$ which is not core-free in G.

Without loss of generality, we may assume that $\mathrm{Core}_G(M_1) \neq 1$ and $\mathrm{Core}_G(M_i) = 1$ for $2 \le i \le r$. Take a minimal normal subgroup N of G contained in $\mathrm{Core}_G(M_1)$. Then $G = NM_i$ and $N \cap M_i = 1$ for $2 \le i \le r$. In this case, G is a solvable primitive group. By Galois' theorem [3, Ch. II, Theorem 3.2], any two complements of N in G are conjugate by an element of N.

Set $\mathfrak{X} = \mathrm{Max}(G, H) - \{M_1\}$ and consider the action of the group $\mathrm{C}_N(H)$ on the set $\mathfrak{X}$ via conjugation. (This action is well defined since M^x is a core-free maximal subgroup containing H for every $M \in \mathfrak{X}$ and $x \in \mathrm{C}_N(H)$.) We claim that $\mathrm{C}_N(H)$ acts transitively on the set $\mathfrak{X}$. In fact, take $M_i, M_j \in \mathfrak{X}$ for $2 \le i, j \le r$. Then $M_i = M_j^n$ for some $n \in N$ since all complements of N in G are conjugate in N. Since $H \le M_i, M_j$, it follows that $\langle H, H^n \rangle \le M_i$. For any $h \in H$, we have $[h, n] = h^{-1}h^n \in M_i \cap N = 1$ since $N \trianglelefteq G$. Thus, $n \in \mathrm{C}_N(H)$, as claimed. Hence, $\mathrm{C}_N(H)$ acts transitively on $\mathfrak{X}$ and it follows that $|\mathfrak{X}|$ divides $|\,\mathrm{C}_N(H)|$. So, $m(G, H) - 1 = |\mathfrak{X}|$ divides $|N| = |G : M_2|$, as desired.

Case II. There exist at least two members of $\mathrm{Max}(G, H)$ which are not core-free in G.

In this case, we may assume that $\mathrm{Core}_G(M_i) \neq 1$, where $i = 1, 2$. By Lemma 3.4(b), H is a second maximal subgroup of G, that is, H is maximal in every member of $\mathrm{Max}(G, H)$. Since $\mathrm{Core}_G(M_1) \cap \mathrm{Core}_G(M_2) \le M_1 \cap M_2 = H$, it follows that $\mathrm{Core}_G(M_1) \cap \mathrm{Core}_G(M_2) \le \mathrm{Core}_G(H) = 1$. Let N_i be a minimal normal subgroup of G contained in $\mathrm{Core}_G(M_i)$ for $i = 1, 2$. Then $N_1 \nleq M_2, N_2 \nleq M_1$ and $N_1 \cap N_2 = 1$. Moreover, $G = N_1M_2 = N_2M_1$. Since H is maximal in M_i with $\mathrm{Core}_G(H) = 1$, we have $N_i \nleq H$ and $M_i = HN_i$ and we conclude that $G = HN_1N_2$.

Write $N = N_1N_2$, so that $N = N_1 \times N_2$ is abelian. It is easy to see that $N \cap H \le \mathrm{C}_H(N) \le \mathrm{Core}_G(H) = 1$. Hence, N can be viewed as a faithful H-module, possibly of mixed characteristic. Observe that N_1, N_2 are both irreducible H-modules. Thus, N is a completely reducible H-module.

Let $\mathfrak{N}$ denote the set of all irreducible H-submodules of N. We will prove that $|\mathfrak{N}| = r$. Let φ be the map from $\mathrm{Max}(G, H)$ to $\mathfrak{N}$ given by $\varphi(M) = M \cap N$ for $M \in \mathrm{Max}(G, H)$. Since $M \cap N \trianglelefteq M$ and $H \le M$, it follows that $M \cap N$ is an H-submodule of N and $M = (M \cap N)H$ since H is maximal in M. By Lemma 3.1, $M \cap N$ is a minimal normal subgroup of M. Hence, $M \cap N$ is an irreducible H-submodule of N and φ is well defined.

To complete the proof, we show that φ is bijective. If $\varphi(M) = \varphi(K)$ for some $M, K \in \mathrm{Max}(G, H)$, then $M \cap N = K \cap N$. Since H is maximal in both M and K, $M = (M \cap N)H = (K \cap N)H = K$, which implies that φ is injective. For any $U \in \mathfrak{N}$, by the complete reducibility of N, we have $N = U \times U_1$, where U_1 is an H-module. Since N is the direct product of two irreducible H-modules, U_1 is an irreducible H-module by the Krull–Remak–Schmidt theorem, which implies that U_1 is a minimal normal subgroup of G. Write $X = UH$. Then $G = U_1X$ and $U_1 \cap X = 1$. It follows from Lemma 3.1 that X is maximal in G. Hence, $X \in \mathrm{Max}(G, H)$ and $\varphi(X) = U$. Thus, φ is surjective.

If N_1, N_2, as H-modules, are not isomorphic, then it follows from Lemma 2.1 that $\mathfrak{N} = \{N_1, N_2\}$ and $r = |\mathfrak{N}| = 2$, contrary to $r \geq 3$. Thus, we may assume that N_1, N_2 are isomorphic H-modules. Then we can assume that N_1, N_2 and N are elementary p-groups for some prime p and so N is an H-module over $\mathrm{GF}(p)$. It follows from Lemma 2.4 that $r = |\mathfrak{N}| = (q^2 - 1)/(q - 1) = 1 + q$, where $q = |\mathrm{Hom}_H(N_1, N_1)|$. Since N_1 is an irreducible H-module over a finite field $\mathrm{GF}(p)$, by applying Lemma 2.3, we see that q divides $|N_1| = |G : M_2|$. Thus, $r - 1$ divides $|G : M_2|$ and the theorem is proved.

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HANGYANG MENG, Department of Mathematics,
Shanghai University, Shanghai 200444, PR China
e-mail: hangyangmenges@gmail.com

XIUYUN GUO, Department of Mathematics,
Shanghai University, Shanghai 200444, PR China
e-mail: xyguo@staff.shu.edu.cn