Arch. Math. 116 (2021), 363–367

0003-889X/21/040363-5
published online November 24, 2020
https://doi.org/10.1007/s00013-020-01552-2

Archiv der Mathematik

Large linear groups of nilpotence class two

Hangyang Meng

Abstract. Let V be a non-trivial finite-dimensional vector space over a finite field F of characteristic p and let G be an irreducible subgroup of $GL(V)$ having nilpotence class at most two. We prove that if $|G| > |V|/2$, then G is cyclic, or $|V| = 3^2$ or 5^2. This is a refinement of Glauberman's result for the tight bound of linear groups of nilpotence class two.

Mathematics Subject Classification. 20C15, 20D10.

Keywords. Linear groups, Modules, Nilpotence class two.

1. Introduction. Let V be a non-trivial finite-dimensional vector space over a finite field F of characteristic p, p a prime. In [1, Proposition 1], Glauberman proved that if G is a p'-subgroup of $GL(V)$ having nilpotence class at most two, the order of G is at most $|V| - 1$. Note that this bound is the best. For example, there indeed exists a cyclic subgroup of $GL(V)$ with order $|V| - 1$, and if $|V| = 3^2$, $GL(V)$ has subgroups isomorphic to Q_8 and D_8. Our first result shows that the linear groups whose orders reach the best bound only belong to the two cases mentioned above.

Theorem 1. *Let V be a non-trivial finite-dimensional vector space over a finite field F of characteristic p, p a prime, and let G be a p'-subgroup of $GL(V)$ having nilpotence class at most two with order $|V| - 1$. Then*

(a) *G is cyclic; or*
(b) *$|V| = 3^2$ and G is isomorphic to Q_8 or D_8.*

Note that if G is a linear subgroup of order less than $|V|$, then every proper subgroup of G has order at most $|V|/2$. In order to find the linear groups with large order, we study the irreducible linear group such that $|G| > |V|/2$ in Theorem 2.

This research is sponsored by the Shanghai Sailing Program (20YF1413400) and the Young Scientists Fund of NSFC (12001359).

Recall that a linear group is called *irreducible* if it is irreducible on its natural module. Given two linear groups $G_i \leq GL(V_i), i = 1, 2$, these two actions (or modules) of G_i on V_i are called *similar* if there exist a group-isomorphism $\alpha : G_1 \to G_2$ and a linear isomorphism $\phi : V_1 \to V_2$ such that

$$(vg)\phi = (v\phi)(g\alpha) \quad \forall g \in G_1, v \in V_1.$$

Theorem 2. *Let V be a non-trivial finite-dimensional vector space over a finite field F of characteristic p, p a prime, and let G be an irreducible subgroup of $GL(V)$ with nilpotence class at most two. Then $|G| \leq |V|/2$ unless one of the following cases holds:*

(a) *G is cyclic and $|G| = |V| - 1$;*
(b) *$|V| = 3^2$ and $G \cong D_8$ or Q_8;*
(c) *$|V| = 5^2$ and the action of G on V is similar to the action of the following subgroup of order 2^4 in $GL(2, 5)$ on $GF(5) \oplus GF(5)$:*

$$\left\langle \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \text{ or } \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

We can extend Theorem 2 to the general case.

Corollary 3. *Let V be a non-trivial finite-dimensional vector space over a finite field F of characteristic p, p a prime, and let G be a non-abelian p'-subgroup of $GL(V)$ having nilpotence class two such that $|G| > |V|/2$. Then $G/Z(G)$ is an elementary abelian 2-group and $p = 3$ or 5.*

2. Proof of Theorem 2. Most of the notation and terminology in this paper follows [2]. The following two lemmas are well-known.

Lemma 4 ([3, Chap.2, Theorem 3.10]). *Let G be an abelian group and let V be a faithful, irreducible G-module. Then G is cyclic of order dividing $|V| - 1$.*

Lemma 5 ([2, Chap.3, Theorem 1.3]). *Let G be a group and let V be a faithful, irreducible G-module over a field of characteristic p. Then $O_p(G) = 1$.*

Now we start to prove Theorem 2.

Proof of Theorem 2. Let (G, V) be a counterexample with $|G||V|$ minimal. Then $|G| > |V|/2$ and (G, V) does not satisfies $(a), (b)$, or (c) in Theorem 2.
(1) G is a non-abelian p'-group.
If G is abelian, Lemma 4 implies that G is cyclic of order dividing $|V| - 1$. As $|G| > |V|/2$, G satisfies (a), a contradiction. As G is nilpotent, by Lemma 5, G is a p'-group.

Let $E = Hom_G(V, V)$, the set of all G-homomorphisms of V into itself, which is a subring of $Hom_F(V, V)$. By Schur's lemma, E is a division ring. The finiteness implies that E is a field. Regard V as an EG-module and define

$$q = |E| \text{ and } d = dim_E V.$$

(2) V is a faithful, irreducible G-module over E.
Note that V is also a faithful G-module over E. If U is a non-trivial proper EG-submodule of V, as F can be viewed as a subfield of $Hom_G(V, V) = E$, U

is also an FG-submodule of V, contrary to the irreducibility of the FG-module V. Hence V is a faithful, irreducible G-module over E.

(3) $d \geq 2$ **and** $Z(G)$ **is cyclic of order dividing** $q-1$.

Note that $d \geq 2$ otherwise G is a subgroup of the multiplicative group $E-\{0\}$, which is cyclic, contrary to Step (1). Note that $Z(G) \subseteq E$. Hence $Z(G)$ is a subgroup of the multiplicative group $E-\{0\}$, which implies that $Z(G)$ is cyclic of order dividing $q-1$.

(4) G **is imprimitive**.

Assume that G is primitive, which implies that every abelian normal subgroup of G is cyclic. Then $O_{2'}(G)$ is cyclic and $O_2(G)$ is cyclic or isomorphic to $D_{2^n}(n \geq 4), SD_{2^n}(n \geq 4)$, or $Q_{2^n}(n \geq 3)$. As G is non-abelian and nilpotent of class two, $O_2(G) \cong Q_8$, which implies that $|G/Z(G)| = 4$ and $|O_2(Z(G))| = 2$. Then we have that $q \geq 3$ is odd and

$$q^2/2 \leq q^d/2 = |V|/2 < |G| = 4|Z(G)| \leq 4(q-1),$$

which implies that $q = 3$ or 5.

If $q = 5$, $Z(G)$ is a 2-group and so $|Z(G)| = |O_2(Z(G))| = 2$. But $|V|/2 \geq q^2/2 = 25/2 > 8 = 4|Z(G)| = |G|$, contrary to the choice of G. Assume that $q = 3$. It follows that $|Z(G)| = 2$, and so $G \cong Q_8$ and $d = 2$, which belongs to the case (b), a contradiction.

By Step (4), there exists a decomposition $V = V_1 \oplus \cdots \oplus V_r (r \geq 2)$ such that the set $\Omega = \{V_1, ..., V_r\}$ is permuted transitively by G. If we choose such r as small as possible, $H = N_G(V_1)$ is maximal in G and $|G : H| = r$. It is not difficult to see that the irreducibility of G on V implies the irreducibility of H on V_1. Write $K = C_H(V_1)$ and we have:

(5) $1 \neq K \leq Z(H), |K| \leq r$, **and** r **divides** $q-1$.

If $K = 1$, then H acts faithfully and irreducibly on V_1. By the minimality of (G, V), $|H| \leq |V_1| - 1$. Then it is easy to see that

$$|G| \leq r|H| \leq r(|V_1| - 1) \leq |V_1|^r/2 = |V|/2,$$

contrary to the choice of G. Hence $K \neq 1$.

As G is nilpotent, $H \trianglelefteq G$ and r is a prime. Hence V_i is H-invariant for each i and G/H acts regularly on Ω. Take an r-element $x \in G - H$. Then $\langle x \rangle$ permutes Ω transitively. Then

$$C_K(x) \leq \bigcap_{i=1}^{r} K^{x^i} = \bigcap_{i=1}^{r} C_H(V_1)^{x^i} = \bigcap_{i=1}^{r} C_H(V_i) = 1.$$

Since $K \trianglelefteq H$ and G is nilpotent of class two, $[K, H] \leq K \cap Z(G) \leq C_K(x) = 1$. Thus $K \leq Z(H)$.

For each $k \in K$, $[k, x] \in Z(G) \cap O_r(G) = O_r(Z(G))$. As $x^r \in H$ and $k \in Z(H)$, we have that $[k, x]^r = [k, x^r] = 1$ and so $[k, x] \in \Omega_1(O_r(Z(G)))$. Now consider the map $\phi : K \to \Omega_1(O_r(Z(G)))$ such that $\phi(k) = [k, x], k \in K$. As $C_K(x) = 1$, ϕ is injective. Note that $K \neq 1$ implies that $\Omega_1(O_r(Z(G))) \neq 1$, by Step (3), $|\Omega_1(O_r(Z(G)))| = r$ divides $q-1$. It follows that

$$|K| \leq |\Omega_1(O_r(Z(G)))| = r.$$

(5) Final contradiction.
Note that H/K acts faithfully and irreducibly on V_1. By the minimality of (G,V), $|H/K| \leq |V_1|-1$. Thus $|G| = |G/H||H/K||K| \leq r^2(|V_1|-1)$. Now it follows that

$$|V_1|^r/2 = |V|/2 < |G| \leq r^2(|V_1|-1) \leq (q-1)^2(|V_1|-1) \leq (|V_1|-1)^3,$$

which forces that $r = 2$.

As r divides $q-1$, we have that $q \geq 3$ is odd and so is $|V_1|$. Since

$$|V_1|^2/2 = |V|/2 < |G| \leq r^2(|V_1|-1) = 4(|V_1|-1),$$

it implies that $|V_1| = 3$ or 5. If $|V_1| = 3$, then $|V| = 3^2$ and $|G| = 2^3$, which is the case (b).

Now we assume that $|V_1| = 5$. In this case, $|V| = 5^2$ and H/K is cyclic of order dividing $|V_1|-1 = 4$. Hence

$$H \lesssim H/C_H(V_1) \times H/C_H(V_2) \lesssim C_4 \times C_4,$$

as $|K| = 2$, H is abelian of order dividing 8. Since

$$5^2/2 = |V|/2 < |G| = 2|H| \text{ divides } 16,$$

it implies that $|G| = 16$ and $|H| = 8$. Then the action of G on V is similar to the action of some 2-subgroup of $GL(2,5)$ on $GF(5) \oplus GF(5)$. Hence G could be viewed as a maximal subgroup of

$$P = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong C_4 \wr C_2,$$

where P is a Sylow 2-subgroup of $GL(2,5)$. Note that

$$\Phi(P) = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Then G is one of the following groups:

$$P_1 = \left\langle \Phi(P), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, P_2 = \left\langle \Phi(P), \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \right\rangle, P_3 = \left\langle \Phi(P), \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Note that P_3 is reducible. Hence $G = P_1$ or P_2, which implies that (G,V) satisfies (c). This is the final contradiction. $\square$

3. Proof of Theorem 1.

Proof of Theorem 1. It is easy to see that Theorem 1 is true when $|V| \leq 3$ and we assume that $|V| \geq 4$. As $(|G|,|V|) = 1$, by Maschke's theorem, V is completely reducible for G. If V is not irreducible for G, we may assume that $V = V_1 \oplus V_2$, where each V_i is a non-trivial G-module. Since $G/C_G(V_i)$ acts faithfully on V_i, by [1, Proposition 1], $|G/C_G(V_i)| \leq |V_i|-1$. Then, as $|V_i| \geq 2$,

$$|G| \leq |G/C_G(V_1)||G/C_G(V_2)| \leq (|V_1|-1)(|V_2|-1) \leq |V|-3,$$

contrary to the hypothesis. Hence V is irreducible for G. Note that $|G| = |V|-1 > |V|/2$ as $|V| \geq 4$. Hence Theorem 1 follows from Theorem 2 immediately. $\square$

Proof of Corollary 3. As $(|G|, |V|) = 1$, by Maschke's Theorem, V is complete reducible for G. Assume that $V = V_1 \oplus \cdots \oplus V_n$, where each V_i is an irreducible G-module. Write $C_i = C_G(V_i)$ and $G_i = G/C_i$. It follows from [1, Proposition 1] that $|G_i| < |V_i|$. Since

$$|V|/2 < |G| \text{ divides } \prod_{i=1}^{n} |G_i| < \prod_{i=1}^{n} |V_i| = |V|,$$

we can deduce that $G \cong G_1 \times \cdots \times G_n$ and for each i,

$$|G_i| > |V_i|/2.$$

It follows from Theorem 2 that for each i,

(a) G_i is cyclic; or

(b) $|V_i| = 3^2$ or 5^2, G_i is a 2-group such that $G_i/Z(G_i)$ is elementary.

As G is non-abelian, there is some j such that (G_j, V_j) satisfies (b). Hence $p = 3$ or 5. Since each $G_i/Z(G_i)$ is an elementary 2-group (or trivial), $G/Z(G)$ is an elementary 2-group, as desired. □

Acknowledgements. The author is grateful to the referee for his/her valuable suggestions.

References

[1] Glauberman, G.: On Burnside's other $p^a q^b$-theorem. Pac. J. Math. **56**(2), 469–476 (1975)

[2] Gorenstein, D.: Finite Groups. Chelsea, New York (1980)

[3] Huppert. B.: Endliche Gruppen I. Die Grundlehren der Mathematischen Wissenschaften, vol. 134. Springer, Berlin, New-York (1967)

Hangyang Meng
Department of Mathematics
Shanghai University
Shanghai 200444
People's Republic of China
e-mail: hymeng2009@shu.edu.cn

Received: 3 August 2020

Revised: 7 October 2020

Accepted: 27 October 2020.