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Coprime actions with p-nilpotent centralizers [☆]

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Let A be an elementary abelian group of order r^2 acting coprimely on a finite p-solvable group G. We prove that if $C_G(a)$ is p-nilpotent for each non-trivial element a in A, then G is meta-p-nilpotent, i.e. p-nilpotent-by-p-nilpotent. In fact, our results are more general in this paper.

1. Introduction

All groups considered here are finite.

Let a group A act on a group G coprimely, i.e. $(|A|, |G|) = 1$. Throughout this paper, r is a fixed prime, A is an elementary abelian r-group and $A^\#$ denotes the set of non-identity elements of A.

From Glauberman's solvable signalizer functor theorem [2], it follows that G must be solvable if $|A| \geqslant r^3$ and $C_G(a)$ is solvable for each $a \in A^\#$. Moreover, it was proved in

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[4] by using the classification of the finite simple groups that the above result is still true if $|A| \geqslant r^3$ is replaced by $|A| \geqslant r^2$. On the other hand, J. N. Ward proved that if $\mathrm{C}_G(a)$ is nilpotent for each $a \in A^{\#}$, then G is nilpotent when $|A| \geqslant r^3$ ([14]); and a further result [7, IX, Theorem 6.13] also due to J. N. Ward shows that G is meta-nilpotent, i.e. nilpotent-by-nilpotent, if $|A| = r^2$ and G is solvable. As an analog of Ward's result, authors proved in [11, Theorem 1.3] that if $|A| \geqslant r^3$ and $\mathrm{C}_G(a)$ is p-nilpotent for each $a \in A^{\#}$, then G is p-nilpotent. It is natural to ask the following question:

- What will happen for the p-nilpotent analog of Ward's results when $|A| = r^2$?

In this paper, we will extend Ward's results to a more general case. In order to state it, we have to recall some definitions of classes of groups. Let $\mathfrak{X}$ be a formation, which is a class of groups closed under taking epimorphic images and subdirect products. Then every group G has a smallest normal subgroup with quotient in $\mathfrak{X}$. This subgroup is called the $\mathfrak{X}$-residual of G, denoted by $G^{\mathfrak{X}}$. Clearly $G \in \mathfrak{X}$ if and only if $G^{\mathfrak{X}} = 1$, in this case, G is also called an $\mathfrak{X}$-group. More details about formations can be seen in [1].

Theorem A. *Let p, r be two primes and let $\mathfrak{X}$ be a formation consisting of p-nilpotent groups. Suppose that an elementary abelian r-group A of order r^2 acts coprimely on a p-solvable group G in such a way that $\mathrm{C}_G(a)$ is an $\mathfrak{X}$-group for each $a \in A^{\#}$. Then $G^{\mathfrak{X}} \leq \mathcal{O}_{p',p}(G)$.*

Note that the hypothesis "the p-solvability of G" in Theorem A can be removed by using the argument in [4, Corollary 2.8], which depends on the classification of the finite simple groups. As a direct corollary of Theorem A, we have:

Corollary 1. *Suppose that an elementary abelian r-group A of order r^2 acts coprimely on a p-solvable group G. If $\mathrm{C}_G(a)$ is p-nilpotent for each $a \in A^{\#}$, then G is p-nilpotent-by-p-nilpotent.*

P. Shumyatsky [12, Theorem 1.1] proved that if $|A| \geqslant r^3$ and $\mathrm{C}_G(a)$ is nilpotent of class at most c for each $a \in A^{\#}$, then G is nilpotent of $\{c, r\}$-bounded class. Authors [11, Corollary 2.5] showed that G is abelian if $|A| \geqslant r^3$ and $\mathrm{C}_G(a)$ is abelian for each $a \in A^{\#}$. Note that, in this special case, the nilpotency class of G is not depending on r. The following corollary is to study the case $|A| = r^2$.

Corollary 2. *Suppose that an elementary abelian r-group A of order r^2 acts coprimely on a solvable group G. If $\mathrm{C}_G(a)$ is nilpotent of class at most c for each $a \in A^{\#}$, then $G/\mathbf{F}(G)$ is nilpotent of class at most c.*

Proof. Let $\mathfrak{X}$ be a class of groups consisting of all nilpotent groups of class at most c. Applying Theorem A for every prime p, we have that $G^{\mathfrak{X}} \leq \mathcal{O}_{p',p}(G)$. Hence

$$G^{\mathfrak{X}} \leq \bigcap_{p} \mathcal{O}_{p',p}(G) = \mathbf{F}(G),$$

as desired. $\square$

It is worth noting that the nilpotency class of $\mathbf{F}(G)$ can not be bounded by c and r. In fact, for any odd prime p, E.I. Khukhro constructed a p-group G of class $p-1$ acted on by an elementary abelian group A of order 4 such that $\mathrm{C}_G(a)$ is abelian for each $a \in A^{\#}$(see [9, Example 5.10, 5.11]).

The next corollary is motivated by the results of J.G. Thompson [13] and G. Higman [5, Theorem 3], which is proved that a group G admitting a fixed-point-free automorphism of prime order r is nilpotent and its nilpotency class of G is bounded by $k(r)$, which is a function of r.

Corollary 3. *Suppose that an elementary abelian r-group A of order r^2 acts on a group G with $\mathrm{C}_G(A) = 1$. Then $G/\mathbf{F}(G)$ is nilpotent of class bounded by a function of r.*

Proof. It is easy to see that $(|A|, |G|) = 1$. For each $a \in A^{\#}$, there exists $b \in A^{\#}$ such that $A = \langle a, b\rangle$. Then

$$\mathrm{C}_{\mathrm{C}_G(a)}(b) = \mathrm{C}_G(\langle a, b\rangle) = \mathrm{C}_G(A) = 1.$$

By Higman's Theorem [5, Theorem 3], $\mathrm{C}_G(a)$ is nilpotent of class at most $k(r)$. By Theorem 6 in the next section, G is solvable. Then it follows from Corollary 2 that $G/\mathbf{F}(G)$ is nilpotent of class bounded by $k(r)$, as desired. $\square$

2. Preliminaries

Recall that $\mathcal{O}_\pi(G)$ is the maximal normal π-subgroup of G, where π is a set of primes. More general, if $\pi_1, \pi_2, \ldots$ are sets of primes, we can define a characteristic subgroup $\mathcal{O}_{\pi_1,\ldots,\pi_i}(G)$ of G by induction: for $i > 1$,

$$\mathcal{O}_{\pi_1,\ldots,\pi_i}(G)/\mathcal{O}_{\pi_1,\ldots,\pi_{i-1}}(G) = \mathcal{O}_{\pi_i}(G/\mathcal{O}_{\pi_1,\ldots,\pi_{i-1}}(G)),$$

which follows from [7, IX, Notation 1.1].

Lemma 4. *Let G be a group and p a prime. Then*

(a) $\mathcal{O}_{p',p}(G/\mathcal{O}_{p'}(G)) = \mathcal{O}_{p',p}(G)/\mathcal{O}_{p'}(G)$;
(b) $\mathcal{O}_{p',p}(G/\Phi(G)) = \mathcal{O}_{p',p}(G)/\Phi(G)$;
(c) If G is π-solvable and $\mathcal{O}_{\pi'}(G) = 1$, then $\mathrm{C}_G(\mathcal{O}_\pi(G)) \leq \mathcal{O}_\pi(G)$.

Proof. Part (a) is clear; Part (b) easily follows from [6, VI, Lemma 6.3] and Part (c) is a consequence of [7, IX, Lemma 1.3]. $\square$

Recall some results about coprime actions.

Lemma 5. *Let a group A act coprimely on a group G.*

(a) If N is an A-invariant normal subgroup of G then

$$\mathrm{C}_{G/N}(A) = \mathrm{C}_G(A)N/N.$$

In particular, $G = \mathrm{C}_G(A)[G, A]$.

(b) If A is abelian and noncyclic then $G = \langle \mathrm{C}_G(a) : a \in A^{\#} \rangle$.

(c) If the semidirect product AG acts on a non-empty set Ω and G is transitive on Ω, then A has a fixed point on Ω.

(d) If G has an A-invariant normal subgroup series:

$$1 = G_0 \leq G_1 ... \leq G_n = G$$

such that $[G_i, A] \leq G_{i-1}$ for $i = 1, ..., n$. Then A acts trivially on G.

(e) If A acts trivially on a subgroup K of G then A acts trivially on $\mathrm{N}_G(K)/\mathrm{C}_G(K)$.

The proof of the following theorem due to Martineau does not depend on the classification of finite simple groups.

Theorem 6. *Suppose that an elementary abelian group A acts on a group G with $\mathrm{C}_G(A) = 1$. Then G is solvable.*

Proof. Without loss of generality, we can assume that A acts faithfully on G as an automorphism group. Thus the result follows from [8, X, Theorem 11.18]. □

The next result is well-known.

Lemma 7. *Suppose that a group $A \neq 1$ acts on a group $G \neq 1$ such that $\mathrm{C}_G(a) = 1$ for all $a \in A^{\#}$. Then*

(a) G is nilpotent;

(b) The Sylow subgroups of A are cyclic or generalized quaternion groups, in particular if A is abelian, then A is cyclic.

Proof. It is a consequence of [6, V, Theorem 8.7]. □

In the following lemma, even though only the case $n = 2$ will be used in the main proof, we prove the general case.

Lemma 8. *Let an elementary abelian r-group A of order r^n act faithfully on a group G, where $n \geq 2$. Then there exist n elements $a_1, ..., a_n \in A^\#$ such that $\mathrm{C}_G(a_i) \neq 1$ for each i and $A = \langle a_1, ..., a_n \rangle$.*

Proof. If $\mathrm{C}_G(a) \neq 1$ for all $a \in A^\#$, then the result is trivial. Hence we may assume that $\mathrm{C}_G(t) = 1$ for some $t \in A^\#$. By Lemma 7 (a), G is nilpotent and clearly $(r, |G|) = 1$.

Work now by induction on n. First we prove that the result is true for $n = 2$. Note that $G \neq 1$, by Lemma 7 (b), there exists $a \in A^\#$ such that $\mathrm{C}_G(a) \neq 1$. As A acts faithfully on G, $\mathrm{C}_G(a) < G$. Take a minimal A-invariant subgroup M such that $\mathrm{C}_G(a) < M \leq G$. Observe that $\mathrm{C}_G(a)\Phi(M) < M$. By the minimality of M, $\Phi(M) \leq \mathrm{C}_G(a)$. The nilpotency of M implies that $\mathrm{C}_G(a) \lhd M$ and $M/\mathrm{C}_G(a)$ is abelian. Write $X = M/\mathrm{C}_G(a)$, and note that $\mathrm{C}_X(a) = 1$ by Lemma 5 (a). Considering the action of A on X, it follows from Lemma 7 (b) that $\mathrm{C}_X(b) \neq 1$ for some $b \in A^\#$. Hence $\mathrm{C}_G(b) \neq 1$. Since $\mathrm{C}_X(a) = 1$, it implies that $b \notin \langle a \rangle$. As $|A| = r^2$, $A = \langle a, b \rangle$, as desired.

Now assume $n \geq 3$ and take B a maximal subgroup of A. By induction, there exist $n-1$ elements $a_1, ..., a_{n-1} \in B^\#$ such that $\mathrm{C}_G(a_i) \neq 1$ for each i and $B = \langle a_1, ..., a_{n-1} \rangle$. Let $C = \mathrm{C}_G(B)$. The faithful action implies that $C < G$. Take a minimal A-invariant subgroup N such that $C < N \leq G$. By the minimality and the nilpotency of N, $C \lhd N$ and N/C is abelian. Write $V = N/C \neq 1$ and $\mathrm{C}_V(B) = 1$.

Let $\mathcal{A} = \{B, \langle x \rangle : x \in A - B\}$. It is easy to see that $\mathcal{A}$ is a partition of A and

$$|\mathcal{A}| = 1 + \frac{|A| - |B|}{r - 1} = 1 + r^{n-1}.$$

Hence $(|\mathcal{A}| - 1, |V|) = 1$. As $\mathrm{C}_V(B) = 1$, by [10, 8.3.1], there exists $a_n \in A - B$ such that $\mathrm{C}_V(a_n) \neq 1$. Hence $A = \langle B, a_n \rangle = \langle a_1, ..., a_{n-1}, a_n \rangle$ and $\mathrm{C}_G(a_i) \neq 1$ for all i, as desired. $\square$

3. Modules

Let F be a field and V is a finite-dimensional vector space over F, denoted simply by V/F. Let G be a group and ϕ a representation of G on V/F. Then V is said to be a (right) G-module over F (with respect to ϕ) by the law $vg = v(g\phi)$, where $g \in G$ and $v \in V$, and we say simply that V/F is a G-module. In this paper, all modules considered here are finite-dimension vector spaces over a field.

If K is an extension field of F, then define

$$V_{\mathrm{K}} = V \otimes_{\mathrm{F}} \mathrm{K},$$

which becomes a G-module over K by law

$$(\sum_i v_i \otimes k_i)g = \sum_i v_i g \otimes k_i.$$

Lemma 9. *Let* $\mathrm{F} \subseteq \mathrm{K}$ *be a field extension and* G *be a group and* V/F *be an* G*-module. Then*

(a) $\mathrm{C}_{V_{\mathrm{K}}}(g) = \mathrm{C}_V(g) \otimes_{\mathrm{F}} \mathrm{K}$*, where* $g \in G$*;*
(b) $\mathrm{Ker}(G \textit{ on } V) = \mathrm{Ker}(G \textit{ on } V_{\mathrm{K}})$*.*

Proof. (a) Obviously $\mathrm{C}_V(g) \otimes_{\mathrm{F}} \mathrm{K} \subseteq \mathrm{C}_{V_{\mathrm{K}}}(g)$. Let $\{e_1, ..., e_n\}$ be an F-basis of V. Then $\{e_1 \otimes 1, ..., e_n \otimes 1\}$ is a K-basis of V_K. We take $v \in \mathrm{C}_{V_{\mathrm{K}}}(g)$ and we may assume that $v = \sum_i \lambda_i(e_i \otimes 1)$ for some $\lambda_i \in \mathrm{K}$. Let T be a F-subspace of K spanned by $\{\lambda_i : i = 1, ..., n\}$ and assume that $\{k_1, ..., k_m\}$ is an F-basis of T. Observe that

$$v \in V \otimes_{\mathrm{F}} T = V \otimes_{\mathrm{F}} (\bigoplus_{i=1}^{m} \mathrm{F}k_i) = \bigoplus_{i=1}^{m} V \otimes_{\mathrm{F}} \mathrm{F}k_i = \bigoplus_{i=1}^{m} V \otimes k_i.$$

Then

$$v \in \mathrm{C}_{V\otimes_{\mathrm{F}}T}(g) = \bigoplus_{i=1}^{m} \mathrm{C}_{V\otimes k_i}(g) = \bigoplus_{i=1}^{m} \mathrm{C}_V(g) \otimes k_i \subseteq \mathrm{C}_V(g) \otimes_{\mathrm{F}} \mathrm{K},$$

as desired.

(b) It is easy to check it by definition. □

Recall that a G-module V is called *homogeneous* if V is the direct sum of some isomorphic irreducible G-modules.

Lemma 10. *Suppose that a group* A *acts coprimely on a group* G *and let* V/F *be an* AG*-module on which* G *is faithful and homogeneous, where* F *is algebraically closed. Suppose that* L *is a non-trivial* A*-invariant abelian normal subgroup of* G*. Then* $\mathrm{C}_L(A) \neq 1$*.*

Proof. Since G is homogeneous on V and $L \trianglelefteq G$, V is a completely reducible L-module. Then,

$$V = V_1 \oplus ... \oplus V_s,$$

where each V_i is the direct sum of isomorphic irreducible L-submodule of V. Write $\Omega = \{V_1, ..., V_s\}$. Since L is a normal subgroup of the semidirect product AG, for each $x \in AG$, $V_i x$ is also the direct sum of some isomorphic irreducible L-modules. Hence $V_i x \subseteq V_j$ for some j. Note that $V_j x^{-1}$ is also the direct sum of some isomorphic irreducible L-submodules and $V_j x^{-1}, V_i$ have isomorphic irreducible L-submodule, which implies that $V_j = V_i x$. Hence AG permutes the set Ω.

We next claim that G is transitive on Ω. Suppose the contrary; let Ω_0 be a G-orbit on Ω. Let $W = \Sigma_{U\in\Omega_0} U$. As $\Omega_0 \neq \Omega$, $W \neq V$. Since V is completely reducible for G, there exists a G-submodule $W' \neq 0$ such that $V = W \oplus W'$. As G is homogeneous on V, W

and W' have isomorphic irreducible G-submodule. Hence W and W' have isomorphic irreducible L-submodule, which is contrary to the choice of W, as claimed.

Considering the action of AG on Ω, it follows from Lemma 5 (c) that there exist some $V_i \in \Omega$ such that $V_i a = V_i$ for all $a \in A$, which implies V_i is AL-invariant.

We may assume that V_i is the direct sum of L-submodules which are all isomorphic to the irreducible L-module X. Since L is abelian and F is algebraically closed, it follows that $\dim_{\mathrm{F}} X = 1$ and for any $x \in X, l \in L$, $xl = \lambda(l)x$, where λ is a homomorphism L from $\mathrm{F}^{\times}$. It implies that $vl = \lambda(l)v$ for any $v \in V_i, l \in L$. Hence we have that

$$v(al) = (va)l = \lambda(l)(va) = (\lambda(l)v)a = (vl)a = v(la),$$

for each $v \in V_i, l \in L$ and $a \in A$. It follows that $[L, A] \leq \mathrm{Ker}(AL \text{ } on \text{ } V_i)$. If $[L, A] = L$, then L acts trivially on V_i. It implies that $L = L^g$ acts trivially on $V_i g$ for any $g \in G$ as $L \trianglelefteq G$. Recall that G is transitive on Ω. It implies that $L \neq 1$ acts trivially on $V = \oplus_i V_i$, contrary to the faithful action of G on V. Hence $[L, A] < L$. It follows from Lemma 5 (a) that $\mathrm{C}_L(A) \neq 1$, as desired. □

4. Proof of Theorem A

Lemma 11. *Let A be a non-cyclic abelian group and V be an A-module over a field* F *such that* $\mathrm{Char}\,\mathrm{F} \nmid |A|$. *Then* $V = \sum_{a \in A^{\#}} \mathrm{C}_V(a)$.

Proof. Work by induction on $\dim_{\mathrm{F}} V$. As $\mathrm{Char}\,\mathrm{F} \nmid |A|$, V is completely reducible. If V is not irreducible, we may assume $V = V_1 \oplus V_2$, where V_i is a proper A-submodule of V. By induction, $V_i = \sum_{a \in A^{\#}} \mathrm{C}_{V_i}(a)$ for $i = 1, 2$ and thus $V = \sum_{a \in A^{\#}} \mathrm{C}_V(a)$.

Now assume that V is an irreducible A-module. It follows from [3, III, Theorem 2.3] that $A/\mathrm{Ker}(A \text{ } on \text{ } V)$ is cyclic. As A is non-cyclic, $\mathrm{Ker}(A \text{ } on \text{ } V) \neq 1$. Thus $V = \mathrm{C}_V(a)$ for some $1 \neq a \in \mathrm{Ker}(A \text{ } on \text{ } V)$ and the lemma is proved. □

Theorem 12. *Let an elementary abelian r-group A of order r^2 acts coprimely on a p-solvable group G and let $\mathfrak{X}$ be a formation consisting of p-nilpotent groups. Let V be a faithful AG-module over a field* F *of characteristic p, where $p \neq r$, and $\mathcal{O}_p(G) = 1$. Suppose that*

(i) $\mathrm{C}_G(a)$ *is a $\mathfrak{X}$-group for each $a \in A^{\#}$;*
(ii) $\mathrm{C}_V(a) \leq \mathrm{C}_V(g)$ *for each $a \in A^{\#}$ and each p'-element $g \in \mathrm{C}_G(a)$.*

Then G is a $\mathfrak{X}$-group.

Proof. Let K be an extension field of F. Let $V_{\mathrm{K}} = V \otimes_{\mathrm{F}} \mathrm{K}$. By Lemma 9, V_{K} is a faithful AG-module over K and $\mathrm{C}_{V_{\mathrm{K}}}(g) = \mathrm{C}_V(g) \otimes_{\mathrm{F}} \mathrm{K}$ for each $g \in G$, which implies that AG-module V_{K} also satisfies hypothesis (i) and (ii). Hence, without loss of generality, we may assume that F is algebraically closed.

Suppose now the theorem is false, and let G be a counterexample of minimal order. We can take a faithful AG-module V of minimal dimension satisfying hypothesis (i) and (ii). Write $N = \mathcal{O}_{p'}(G)$. Since $\mathcal{O}_p(G) = 1$ and $G \neq 1$ is p-solvable. Then $N \neq 1$ and $\mathrm{C}_G(N) \leq N$ by Lemma 4. Now we will derive the contradiction in the following steps:

(1) There exist $a_1, a_2 \in A^\#$ such that $\mathrm{C}_N(a_i) \neq 1$ for $i = 1, 2$ and $A = \langle a_1, a_2 \rangle$.

By Lemma 8, it suffices to show that A acts faithful on N. Suppose the contrary; we can take $a \in A^\#$ such that a acts trivially on N. By Lemma 5 (e), a acts trivially on $G/\mathrm{C}_G(N)$. As $\mathrm{C}_G(N) \leq N$, we have that a acts trivially on G/N. It follows from Lemma 5 (d) that a acts trivially on G. It implies that $G = \mathrm{C}_G(a) \in \mathfrak{X}$, contrary to the choice of G.

(2) V is an irreducible AG-module.

Suppose the contrary; Take an AG-submodule series:

$$1 = V_0 \leq V_1 .... \leq V_t = V$$

such that V_i/V_{i-1} is an irreducible AG-module for $i = 1, ..., t$, where $t \geq 2$. Write $K_i = \mathrm{Ker}(AG \textit{ on } V_i/V_{i-1})$ for each i and $K = \bigcap_i K_i$.

We will prove that $G/K_i \in \mathfrak{X}$ for each i. In fact, if $K_i \cap A \neq 1$, there exists $a \in (K_i \cap A)^\#$ such that $[G, a] \leq K_i$. Then we have

$$G/K_i = \mathrm{C}_{G/K_i}(a) = \mathrm{C}_G(a)K_i/K_i \in \mathfrak{X}.$$

Now we may assume that $K_i \cap A = 1$, that is, $|AK_i/K_i| = r^2$. We will check that AG/K_i-module V_i/V_{i-1} satisfies the hypothesis of the theorem. Observe that V_i/V_{i-1} is a faithful and irreducible AG/K_i-module, and it implies that $\mathcal{O}_p(G/K_i) \leq \mathcal{O}_p(AG/K_i) = 1$. Moreover, for each $aK_i \in (AK_i/K_i)^\#$, where $a \in A^\#$,

$$\mathrm{C}_{G/K_i}(aK_i) = \mathrm{C}_G(a)K_i/K_i \in \mathfrak{X},$$

and

$$\mathrm{C}_{V_i/V_{i-1}}(aK_i) = (\mathrm{C}_{V_i}(a) + V_{i-1})/V_{i-1}.$$

Since by hypothesis (ii), every p'-element of $\mathrm{C}_G(a)$ centralizes $\mathrm{C}_{V_i}(a)$, it implies that every p'-element of $\mathrm{C}_{G/K_i}(aK_i)$ centralizes $\mathrm{C}_{V_i/V_{i-1}}(aK_i)$. Hence AG/K_i-module V_i/V_{i-1} satisfies the hypothesis of the theorem, by minimality of G, $G/K_i \in \mathfrak{X}$, as claimed.

Hence $G/K = G/\bigcap_i K_i \in \mathfrak{X}$. As K acts trivially on V_i/V_{i-1} for each i and is faithful on V, it follows from Lemma 5 (d) that K is a p-group, which implies that $K \leq \mathcal{O}_p(G) = 1$. Hence $G \in \mathfrak{X}$, contrary to the choice of G.

(3) G is homogeneous on V.

By Step (2), V is an irreducible AG-module. As $G \lhd AG$, by Clifford's Theorem [6, V, Theorem 17.3],

$$V = W_1 \oplus ... \oplus W_s,$$

where each W_i is the direct sum of isomorphic irreducible G-submodule, and A permutates transitively on the set $\{W_1, ..., W_s\}$. Let $B = \mathrm{Stab}_A(W_1)$. As A is abelian, actually $B = \mathrm{Stab}_A(W_i)$ for each i and $s = |A : B|$. We want to prove that $A = B$ and Step (3) follows. By Step (1), there exists $a_1, a_2 \in A^{\#}$ such that $A = \langle a_1, a_2 \rangle$ and $\mathrm{C}_N(a_i) \neq 1$ for $i = 1, 2$. It suffices to show that $a_i \in B$ for $i = 1, 2$.

Assume that $a_i \notin B$ for some i. Let $x \in \mathrm{C}_N(a_i)^{\#}$. As G is faithful on V, there exists W_j and some element $w \in W_j$ such that $wx \neq w$. As $a_i \notin B = \mathrm{Stab}_G(W_i)$, $W_j, W_ja_i, ...W_ja_i^{p-1}$ are pairwise distinct. Let $u = w + wa_i + ...wa_i^{p-1} \in W_j \oplus W_ja_i ... \oplus W_ja_i^{p-1}$; then clearly $u \in \mathrm{C}_V(a_i)$. By hypothesis (ii), $\mathrm{C}_V(a_i)$ is centralized by $\mathrm{C}_N(a_i)$, which implies that $ux = u$. As $x \in \mathrm{C}_N(a_i)$,

$$u = ux = wx + wxa_i + ... + wxa_i^{p-1}.$$

It implies that

$$0 = ux - u = (wx - w) + (wx - w)a_i + ... + (wx - w)a_i^{p-1} \in W_j \oplus W_ja_i ... \oplus W_ja_i^{p-1}.$$

Observe that $(wx - w)a_i^k \in W_ja_i^k$ for $k = 1, ..., p - 1$ and $wx - w \neq 0$, which is a contradiction to the direct sum. Hence $a_i \in B$ and Step (3) is proved.

(4) $\mathrm{C}_{\mathcal{O}_{p'}(G)}(A) \neq 1$.

Suppose that $\mathrm{C}_{\mathcal{O}_{p'}(G)}(A) = 1$. It follows from Theorem 6 that $\mathcal{O}_{p'}(G)$ is solvable. Since $G \neq 1$ is p-solvable with $\mathcal{O}_p(G) = 1$, $\mathcal{O}_{p'}(G) \neq 1$. The solvability of $\mathcal{O}_{p'}(G)$ implies that the center of its Fitting subgroup $\mathbf{Z}(\mathbf{F}(\mathcal{O}_{p'}(G))) \neq 1$. Write $L = \mathbf{Z}(\mathbf{F}(\mathcal{O}_{p'}(G)))$, and observe that L is an non-trivial A-invariant abelian normal subgroup of G. It follows from Lemma 10 that $\mathrm{C}_L(A) \neq 1$. Hence $\mathrm{C}_{\mathcal{O}_{p'}(G)}(A) \neq 1$, which is a contradiction.

(5) Final contradiction.

As $\mathrm{Char}\,\mathrm{F} = p \nmid |A|$, it follows from Lemma 11 that $V = \sum_{a \in A^{\#}} \mathrm{C}_V(a)$. For each $a \in A^{\#}$, by hypothesis (ii), $\mathrm{C}_{\mathcal{O}_{p'}(G)}(A) \leq \mathrm{C}_{\mathcal{O}_{p'}(G)}(a)$ centralizes $\mathrm{C}_V(a)$. Hence $1 \neq \mathrm{C}_{\mathcal{O}_{p'}(G)}(A)$ centralizes V, which is a contradiction as G is faithful on V. $\square$

Proof of Theorem A. Suppose that the theorem is false and let G be a counterexample of minimal order. Then $\mathrm{C}_G(a) \in \mathfrak{X}$ for each $a \in A^{\#}$ but $G^{\mathfrak{X}} \nleq \mathcal{O}_{p',p}(G)$, moreover, we may assume $p \neq r$ otherwise the result is trivial. Let N be a non-trivial A-invariant normal subgroup of G and consider the action of A on G/N. By Lemma 5 (a),

$$\mathrm{C}_{G/N}(a) = \mathrm{C}_G(a)N/N \in \mathfrak{X},$$

as $\mathfrak{X}$ is closed under taking epimorphic images. The minimality of G implies that $G^{\mathfrak{X}}N/N \leq \mathcal{O}_{p',p}(G/N)$. It follows from Lemma 4 (a) and (b) that $\mathcal{O}_{p',p}(G/N) = \mathcal{O}_{p',p}(G)/N$ if $N = \mathcal{O}_{p'}(G)$ or $\Phi(G)$. Hence the minimality of G forces that $\mathcal{O}_{p'}(G) = \Phi(G) = 1$ and $\mathcal{O}_p(G)$ is an elementary abelian p-group.

Write now H the semidirect product AG. It is easy to see that $\mathcal{O}_p(H) \leq G$, which implies that $\mathcal{O}_p(G) = \mathcal{O}_p(H)$, denote by V. Observe that $\mathcal{O}_{p'}(H) \cap G \leq \mathcal{O}_{p'}(G) = 1$, which implies that $\mathrm{C}_G(\mathcal{O}_{p'}(H)) = G$. As $(|A|, |G|) = 1$, we have that $\mathcal{O}_{p'}(H) \leq A$. If $\mathcal{O}_{p'}(H) \neq 1$, then $G = \mathrm{C}_G(\mathcal{O}_{p'}(H)) \leq \mathrm{C}_G(a) \in \mathfrak{X}$ for some $a \in \mathcal{O}_{p'}(H)^{\#} \subseteq A^{\#}$, which is a contradiction. Hence $\mathcal{O}_{p'}(H) = 1$. Notice that H is p-solvable. By Lemma 4 (c), $\mathrm{C}_H(V) \subseteq V$, moreover, $V = \mathrm{C}_H(V)$ as V is abelian. Write $\overline{H} = H/V$ and V can be viewed as a faithful $\overline{H}$-module over the p-element field $\mathrm{GF}(p)$.

Now we want to apply Theorem 12 on $\overline{AG}$-module V. Observe that $\mathcal{O}_p(\overline{G}) = 1$. As, for each $a \in A^{\#}$, $\mathrm{C}_G(a)$ is an $\mathfrak{X}$-group, by Lemma 5 (a), $\mathrm{C}_{\overline{G}}(\overline{a}) = \mathrm{C}_G(a)V/V$ is an $\mathfrak{X}$-group for each non-trivial element $\overline{a} = aV$ of $\overline{A}$. Furthermore, since $\mathrm{C}_G(a)$ is p-nilpotent, obviously $\mathcal{O}_{p'}(\mathrm{C}_{\overline{G}}(\overline{a})) = \mathcal{O}_{p'}(\mathrm{C}_G(a))V/V$. As $\mathrm{C}_V(a) \leq \mathcal{O}_p(\mathrm{C}_G(a))$, we have that

$$[\mathrm{C}_V(a), \mathcal{O}_{p'}(\mathrm{C}_G(a))] \leq [\mathcal{O}_p(\mathrm{C}_G(a)), \mathcal{O}_{p'}(\mathrm{C}_G(a))] = 1,$$

which implies that $\mathrm{C}_V(\overline{a})$ is centralized by $\mathcal{O}_{p'}(\mathrm{C}_{\overline{G}}(\overline{a}))$, i.e. $\mathrm{C}_V(\overline{a})$ is centralized by every p'-element of $\mathrm{C}_{\overline{G}}(\overline{a})$. Now all the hypotheses of Theorem 12 hold for $\overline{AG}$-module V, therefore, $\overline{G}$ is an $\mathfrak{X}$-group, i.e. $G^{\mathfrak{X}} \leq V = \mathcal{O}_p(G)$, contrary to choice of G. Thus the proof is complete. □

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