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Coprime actions with supersolvable fixed-point groups

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Abstract. Let A be an elementary abelian r-group acting on a finite r'-group G. Suppose that the fixed-point group $\mathrm{C}_G(a)$ is supersolvable for each $a \in A^\#$. We show that G is supersolvable if $|A| \geqslant r^4$ and that $G' \leqslant \mathbf{F}_3(G)$ if $|A| \geqslant r^3$. Moreover, we prove some other results for cases when the fixed-point group $\mathrm{C}_G(a)$ is abelian, p-nilpotent or satisfies the Sylow tower property.

1 Introduction

All groups considered in this paper are finite.

Let G be a group admitting an action of a group A and let

$$\mathrm{C}_G(A) = \{x \in G \mid x^a = x \text{ for all } a \in A\}$$

be the fixed-point group of A in G. Thompson proved in [8] that if A is cyclic of prime order and $\mathrm{C}_G(A) = 1$, then G must be nilpotent. In general it seems interesting to investigate how the structure of $\mathrm{C}_G(A)$ influences the structure of G.

Throughout this paper, r is a fixed prime, A is an elementary abelian r-group and G is an r'-group. We write $A^\#$ for the set of non-identity elements of A.

From Glauberman's solvable signalizer functor theorem [3] it follows that G must be solvable if $|A| \geqslant r^3$ and $\mathrm{C}_G(a)$ is solvable for each $a \in A^\#$. Moreover, it was proved in [4] using the classification of the finite simple groups that the above result is still true when $|A| \geqslant r^2$. On the other hand, Ward [9] showed that if $|A| \geqslant r^3$ and $\mathrm{C}_G(a)$ is nilpotent for each $a \in A^\#$, then G is nilpotent. Ward also proved in [10] that if $|A| \geqslant r^4$ and $\mathrm{C}_G(a)'$ is nilpotent for each $a \in A^\#$, then the derived group G' of G is nilpotent.

Recall a group G is called *supersolvable* if every chief factor of G is cyclic. In this paper, we mainly focus on the following question.

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Question. What can be said about the structure of G if $\mathrm{C}_G(a)$ is supersolvable for each $a \in A^{\#}$?

We shall prove the following result.

Theorem 1.1. *Let A be an elementary abelian r-group of order at least r^4 acting on an r'-group G. Suppose that $\mathrm{C}_G(a)$ is supersolvable for each $a \in A^{\#}$. Then G is supersolvable.*

The following example essentially due to Ward [10] shows that the conclusion of Theorem 1.1 may fail if $|A| = r^3$. Let r, q, s, p be distinct primes such that

$$r \mid q-1, \quad r, q \mid s-1 \quad \text{and} \quad r, q, s \mid p-1;$$

it is easy to obtain such primes using Dirichlet's theorem on primes in arithmetic progression. Let $\alpha_1, \alpha_2, \alpha_3$ be primitive r-th roots of unity in the fields $\mathbb{F}_q, \mathbb{F}_s, \mathbb{F}_p$, respectively.

Let $U_2 = W_1 \rtimes \langle \phi_1 \rangle$, where $W_1 = \langle x \rangle$, $o(x) = q$, $o(\phi_1) = r$ and $x^{\phi_1} = x^{\alpha_1}$, and consider the regular wreath product $V_2 = C_s \wr_{\mathrm{reg}} U_2$ (see [1, Chapter A, Definition 18.7]), where C_s is the cyclic group of order s. Let W_2 be the normal Sylow s-subgroup of V_2; clearly $V_2 = W_2 U_2$. Define an automorphism ϕ_2 of V_2 by

$$wu \mapsto w^{\alpha_2} u \quad \text{for } w \in W_2,\ u \in U_2.$$

Similarly, let $U_3 = W_2 \rtimes \langle \phi_2 \rangle$, and set $V_3 = C_p \wr_{\mathrm{reg}} U_3$. Let W_3 be the normal Sylow p-subgroup of V_3 and let ϕ_3 be an automorphism of V_3 given by

$$wu \mapsto w^{\alpha_3} u \quad \text{for } w \in W_3,\ u \in U_3.$$

Let

$$H = V_3 \rtimes \langle \phi_3 \rangle = W_3 W_2 W_1 \langle \phi_1 \rangle \langle \phi_2 \rangle \langle \phi_3 \rangle$$

and take $G = W_3 W_2 W_1$ and $A = \langle \phi_1 \rangle \langle \phi_2 \rangle \langle \phi_3 \rangle$. Observe that G is an r'-group and A is elementary abelian of order r^3. The action of A on G is defined by conjugation in H. For each $a = \phi_1^i \phi_2^j \phi_3^k \in A^{\#}$, where $i, j, k \in \{0, 1, \ldots, r-1\}$, Lemma 2.1 (d) below gives

$$\mathrm{C}_G(a) = \begin{cases} \mathrm{C}_{W_3}(a)\,\mathrm{C}_{W_2}(a) & \text{if } i \neq 0, \\ \mathrm{C}_{W_3}(a) W_1 & \text{if } i = 0 \text{ and } j \neq 0, \\ W_2 W_1 & \text{if } i = j = 0 \text{ and } k \neq 0. \end{cases}$$

It is easy to see from Lemma 3.1 below that $\mathrm{C}_G(a)$ is supersolvable and it is also easy to show that G' is not nilpotent. Hence G is not supersolvable.

Denote by $\mathbf{F}(G)$ the Fitting subgroup of a group G and denote by $\mathbf{F}_i(G)$ the inverse image of $\mathbf{F}(G/\mathbf{F}_{i-1}(G))$ in G for $i \geqslant 2$.

Theorem 1.2. *Let A be an elementary abelian r-group of order at least r^3 acting on an r'-group G. Suppose that $\mathrm{C}_G(a)$ is supersolvable for each $a \in A^{\#}$. Then $G' \leqslant \mathbf{F}_3(G)$. Furthermore, $G' \leqslant \mathbf{F}_2(G)$ if one of the following statement holds:*

(a) *$\mathrm{C}_G(a)'$ is of odd order for each $a \in A^{\#}$, or*

(b) *r is not a half-Fermat prime.*

Recall that a prime r is called *half-Fermat* if there exists a prime p such that $r = \frac{1}{2}(p^m + 1)$ for some $m \in \mathbb{N}$. In the proof of Theorems 1.1 and 1.2, the following theorem plays an important role.

Theorem 1.3. *Let A be an elementary abelian r-group of order at least r^3 acting on an r'-group G. If p is a prime and $\mathrm{C}_G(a)$ is p-nilpotent for each $a \in A^{\#}$, then G is p-nilpotent.*

2 Proof of Theorem 1.3

First we recall some basic results on coprime action.

Lemma 2.1. *Let G be a group admitting an action of a group A with $(|G|,|A|) = 1$. Then the following assertions hold.*

(a) *Suppose that A is abelian and G is a non-trivial elementary abelian group. If $\mathrm{C}_G(a) = 1$ for every $a \in A^{\#}$, then A is cyclic.*

(b) *If G has the Sylow π-property for a set π of primes, then G has an A-invariant Hall π-subgroup.*

(c) *If $N \trianglelefteq G$ and N is A-invariant, then $\mathrm{C}_{G/N}(A) = \mathrm{C}_G(A)N/N$.*

(d) *If $G = XY$, where X and Y are A-invariant subgroups, then*
$$\mathrm{C}_G(A) = \mathrm{C}_X(A)\,\mathrm{C}_Y(A).$$

(e) *Suppose that A is non-cyclic. Then*
$$G = \langle \mathrm{C}_G(B) \mid B \leqslant A \text{ and } A/B \text{ is cyclic} \rangle.$$
In particular, $G = \langle \mathrm{C}_G(a) \mid a \in A^{\#} \rangle$.

(f) *If H is an A-invariant Hall π-subgroup of G, then $\mathrm{C}_H(A)$ is a Hall π-subgroup of $\mathrm{C}_G(A)$.*

Proof. Assertions (a)–(f) follow from [6, 8.3.2, 8.2.6 (a), 8.2.2, 8.2.11, 8.3.4 (a,b), 8.2.6 (e)], respectively. □

Lemma 2.2 ([2, Theorem 2.3]). *Let $\mathbb{F}$ be a field, G a group and V an $\mathbb{F}[G]$-module. Suppose that $H \leqslant G$,* $\operatorname{char}(\mathbb{F})$ *does not divide $|G\colon H|$ and V is a completely reducible $\mathbb{F}[H]$-module. Then V is a completely reducible $\mathbb{F}[G]$-module.*

Proof of Theorem 1.3. Suppose the theorem false and let G be a counter-example of minimal order. By the properties of coprime action and the minimality of G, all proper A-invariant subgroups of G and all quotient groups G/N modulo non-trivial A-invariant normal subgroups N of G are p-nilpotent. We complete the proof in three steps.

Step 1: We claim that G is p-solvable and $\mathrm{O}_{p'}(G) = 1$.

If $p = 2$, then $\mathrm{C}_G(a)$ is solvable and by [6, Theorem 11.3.3], G is solvable, as desired. Now assume that p is odd. By Lemma 2.1 (b), there exists an A-invariant Sylow p-subgroup P of G. If $P = 1$, then clearly G is p-solvable so we may assume that $P \neq 1$; thus $\mathrm{Z}(J(P)) \neq 1$. Observe that $\mathrm{Z}(J(P))$ and $\mathrm{N}_G(\mathrm{Z}(J(P)))$ are both A-invariant. If $\mathrm{N}_G(\mathrm{Z}(J(P))) < G$, then $\mathrm{N}_G(\mathrm{Z}(J(P)))$ is p-nilpotent by the minimality of G, and then G is p-nilpotent by Glauberman's ZJ-theorem [5, Chapter X, Theorem 9.10], which is a contradiction. Hence $\mathrm{N}_G(\mathrm{Z}(J(P))) = G$ and it follows that $G/\mathrm{Z}(J(P))$ is p-nilpotent, so that G is p-solvable. If $\mathrm{O}_{p'}(G) \neq 1$, then by the minimality of G, $G/\mathrm{O}_{p'}(G)$ is p-nilpotent, which implies that G is p-nilpotent, a contradiction.

Step 2: We claim that G is p-closed.

Observe that $\mathrm{O}_p(G) \neq 1$ by Step 1. It follows from the minimality of G that $G/\mathrm{O}_p(G)$ is p-nilpotent. Let $L/\mathrm{O}_p(G)$ be the Hall p'-subgroup of $G/\mathrm{O}_p(G)$; thus G/L is a p-group and L is A-invariant. If $L < G$, then L is p-nilpotent. For the normal p-complement $L_{p'}$ of L we have $L_{p'} \leqslant O_{p'}(G) = 1$. Thus L is a p-subgroup and so is G, which is a contradiction. Hence $G = L$ is p-closed.

Step 3: Final contradiction.

Write $P = \mathrm{O}_p(G)$. As $\mathrm{C}_G(P) \trianglelefteq G$, we have $\mathrm{O}_{p'}(\mathrm{C}_G(P)) \leqslant \mathrm{O}_{p'}(G) = 1$. Hence $\mathrm{C}_G(P) = \mathrm{Z}(P)$. Observe that

$$G = \langle \mathrm{C}_G(B) \mid B \leqslant A \text{ and } A/B \text{ is cyclic} \rangle$$

by Lemma 2.1 (e). Since G is not p-nilpotent, there exists a subgroup B of A such that $\mathrm{C}_G(B) \nleqslant P$. By hypothesis $\mathrm{C}_G(B)$ has a normal p-complement $K \neq 1$. Now $K \leqslant \mathrm{C}_G(B) \leqslant \mathrm{C}_G(b)$ for all $b \in B^{\#}$. Since G is p-closed, each $\mathrm{C}_G(b)$ is p-closed and has a normal p-complement. As $K \leqslant \mathrm{C}_G(b)$, K centralizes $\mathrm{C}_P(b)$. This means that K centralizes $\langle \mathrm{C}_P(b) \mid b \in B^{\#} \rangle$. As $|B| = r^2$, we have that K centralizes $P = \langle \mathrm{C}_P(b) \mid b \in B^{\#} \rangle$ and this contradicts $\mathrm{C}_G(P) = \mathrm{Z}(P)$. This is the final contradiction. □

Corollary 2.3 ([9]). *Let A be an elementary abelian r-group of order at least r^3 acting on an r'-group G. If $\mathrm{C}_G(a)$ is nilpotent for each $a \in A^\#$, then G is nilpotent.*

Let $\preceq$ be a total order relation on the set $\mathbb{P}$ of all primes. Recall that a group G is said to satisfy the Sylow tower property with respect to $\preceq$ if G has a normal Hall π_p-subgroup for each prime p, where $\pi_p = \{q \mid p \preceq q\}$. If $\preceq$ is the natural order relation on $\mathbb{P}$, then G is said to satisfy the Sylow tower property of supersolvable type.

Corollary 2.4. *Let A be an elementary abelian r-group of order at least r^3 acting on an r'-group G. Let $\preceq$ be a total order relation on $\mathbb{P}$. Suppose that $\mathrm{C}_G(a)$ satisfies the Sylow tower property with respect to $\preceq$ for each $a \in A^\#$. Then G satisfies the Sylow tower property with respect to $\preceq$.*

Proof. We work by induction on $|G|$. Let p be the smallest prime with respect to $\preceq$ dividing the order of G. Observe that $\mathrm{C}_G(a)$ is p-nilpotent for each $a \in A^\#$. It follows from Theorem 1.3 that G is p-nilpotent. The normal Hall p'-subgroup H of G is clearly A-invariant. For each $a \in A^\#$, $\mathrm{C}_H(a)$ is a subgroup of $\mathrm{C}_G(a)$ satisfying the Sylow tower property with respect to $\preceq$. Observe that $H < G$ since p divides the order of G. By induction, H satisfies the Sylow tower property with respect to $\preceq$. Thus G satisfies the Sylow tower property with respect to $\preceq$. □

Corollary 2.5. *Let A be an elementary abelian r-group of order at least r^3 acting on an r'-group G. Suppose that n is a positive integer. If $\mathrm{C}_G(a)$ is abelian of exponent dividing n for each $a \in A^\#$, then G is abelian of exponent dividing n. In particular, if $\mathrm{C}_G(a)$ is abelian for each $a \in A^\#$, then G is abelian.*

Proof. We have $G = \langle \mathrm{C}_G(B) \mid B \leqslant A \text{ and } A/B \text{ is cyclic}\rangle$ by Lemma 2.1 (e). By hypothesis $\mathrm{C}_G(B)$ is abelian of exponent dividing n for all subgroups B of A with A/B cyclic. Suppose that B_1 and B_2 are maximal subgroups of A. Then, as $|A| = r^3$, there exists $b \in A^\#$ such that $b \in B_1 \cap B_2$. Hence $\mathrm{C}_G(B_1)\,\mathrm{C}_G(B_2) \leqslant \mathrm{C}_G(b)$, which is abelian by hypothesis. Therefore $\mathrm{C}_G(B_1)$ and $\mathrm{C}_G(B_2)$ commute and it follows that G is abelian of exponent dividing n. □

3 Proof of Theorem 1.1

The following is a reformulation of a supersolvability criterion due to Reinhold Baer.

Lemma 3.1. *The group G is supersolvable if and only if*

(a) *G satisfies the Sylow tower property of supersolvable type and*

(b) $\mathrm{N}_G(G_p)/\mathrm{C}_G(G_p)G_p$ *is abelian of exponent dividing* $p-1$ *for every prime* p *and every Sylow* p*-subgroup* G_p *of* G.

Proof of Theorem 1.1. Suppose Theorem 1.1 is false. Let G be a counter-example of minimal order. By the minimality of G, all proper A-invariant subgroups of G and all quotient groups G/N modulo non-trivial A-invariant normal subgroups N of G are supersolvable. Let p be the largest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. It follows from Corollary 2.4 that G satisfies the Sylow tower property of supersolvable type, and so $P \trianglelefteq G$ and $P \leqslant \mathbf{F}(G)$. The rest of the proof is divided into three steps:

Step 1: We claim that $\Phi(G) = 1$ and $P = \mathbf{F}(G)$ is the unique minimal A-invariant normal subgroup of G.

If $\Phi(G) \neq 1$, then $G/\Phi(G)$ is supersolvable by the minimality of G. Hence G is supersolvable, which is a contradiction. Let

$$\mathfrak{X} = \{1 < N \trianglelefteq G \mid N^A = N\}.$$

Suppose that there exist minimal elements $N_1, N_2 \in \mathfrak{X}$ such that $N_1 \neq N_2$. Then $N_1 \cap N_2 = 1$ and since both G/N_1 and G/N_2 are supersolvable, so is $G = G/(N_1 \cap N_2)$, contrary to the choice of G. Therefore $\mathfrak{X}$ has a unique minimal member, and $\mathbf{F}(G) = P$ is a p-group since $P \leqslant \mathbf{F}(G)$.

Since $\Phi(P) \leqslant \Phi(G) = 1$, P is an elementary abelian p-group, and it can be viewed as a $\mathbb{F}_p[AG]$-module. By [7, Theorem 1.12], $P = \mathbf{F}(G)$ is a completely reducible $\mathbb{F}_p[G]$-module; since $(|A|, |G|) = 1$, it follows from Lemma 2.2 that P is a completely reducible $\mathbb{F}_p[AG]$-module. However $P \in \mathfrak{X}$ and since $\mathfrak{X}$ has a unique minimal member, P must be an irreducible $\mathbb{F}_p[AG]$-module, that is, P is the unique minimal A-invariant normal subgroup of G, as claimed.

Step 2: Let H be the A-invariant Hall p'-subgroup of G. We claim that every A-invariant proper subgroup of H is abelian of exponent dividing $p-1$.

Let X be an A-invariant proper subgroup of H. Then $Y = PX$ is an A-invariant proper subgroup of G. By the minimality of G, Y is supersolvable. It follows from Lemma 3.1 that $Y/P\,\mathrm{C}_Y(P) = \mathrm{N}_Y(P)/P\,\mathrm{C}_Y(P)$ is abelian of exponent dividing $p-1$. Observe that $\mathrm{C}_G(P) \leqslant P$ since $P = \mathbf{F}(G)$. Hence $\mathrm{C}_Y(P) \leqslant P$ and thus $\mathrm{C}_Y(P) = P$ since P is abelian. Then we have that $X \cong Y/P$ is abelian of exponent dividing $p-1$, as claimed.

Step 3: Final contradiction.

Let $B = \mathrm{C}_A(H)$. First assume that $|B| \geqslant r^2$. For each $b \in B^{\#}$, we have the identity $\mathrm{C}_G(b) = \mathrm{C}_P(b)H$ since $H \leqslant \mathrm{C}_G(b)$. This implies that $\mathrm{C}_P(b)$ is normalized by H since $\mathrm{C}_G(b)$ is supersolvable. Observe that P is abelian. Immediately we have that $\mathrm{C}_P(b) \trianglelefteq G$. It is easy to see that $\mathrm{C}_P(b)$ is an A-invariant normal subgroup of G. By Step 1, P is the minimal A-invariant normal subgroup of G. Thus $\mathrm{C}_P(b) = 1$ or $\mathrm{C}_P(b) = P$ for each $b \in B^{\#}$. It follows from Lemma 2.1 (a) that $\mathrm{C}_P(b_1) = P$ for some $b_1 \in B^{\#}$. This implies that $G = PH \leqslant \mathrm{C}_G(b_1)$ is supersolvable, which is a contradiction. Hence $|B| \leqslant r$.

For each $b \in A \setminus B$, we have that $\mathrm{C}_H(b)$ is an A-invariant proper subgroup of H. By Step 2, $\mathrm{C}_H(b)$ is abelian of exponent dividing $p-1$. Considering the faithful action of A/B on H, we see that $\mathrm{C}_H(bB) = \mathrm{C}_H(b)$ is abelian of exponent dividing $p-1$ for each $bB \in (A/B)^{\#}$. Since $|A/B| \geqslant r^3$, it follows from Corollary 2.5 that H is abelian of exponent dividing $p-1$. It now follows easily from Lemma 3.1 that G is supersolvable. This contradiction completes the proof. □

4 Proof of Theorem 1.2

The following lemma is a special case of [2, Theorem A].

Lemma 4.1. *Let R be a group of order r that acts on the solvable r'-group H. Let V be an RH-module and a faithful, completely reducible H-module. Suppose that K is a nilpotent normal subgroup of $\mathrm{C}_H(R)$ such that*

$$K \leqslant \mathrm{Ker}(\mathrm{C}_H(R) \text{ on } \mathrm{C}_V(R)).$$

Then $K \leqslant \mathbf{F}_2(H)$. Furthermore, if r is not a half-Fermat prime or K is of odd order, then $K \leqslant \mathbf{F}(H)$.

Proof. Let L be the subnormal closure of K in H. By [2, Theorem A] we have $L = K[L,R]$ and $[L,R]$ is nilpotent. Since $L/[L,R]$ is isomorphic to a quotient of K and K is nilpotent, $L/[L,R]$ is nilpotent. Observe that $L = \mathbf{F}_2(L) \leqslant \mathbf{F}_2(H)$ since L is subnormal in H. Hence $K \leqslant L \leqslant \mathbf{F}_2(H)$.

Write $[L,R] = S \times P$ with S a 2-group and P a $2'$-group. Using [2, Theorem A] again, we conclude from the nilpotency of K that $S = 1$ and we also find that $P = 1$ if r is not a half-Fermat prime or K is of odd order. Hence in these cases, $[L,R] = 1$ and $K = L$ is subnormal in H and so $K \leqslant \mathbf{F}(H)$. □

For the proof of Theorem 1.2, we need the following two results.

Theorem 4.2. *Let A be an elementary abelian r-group of order at least r^3 acting on an r'-group G. Suppose that $\mathrm{C}_G(a)$ is supersolvable for each $a \in A^{\#}$. Then $G' \leqslant \mathbf{F}_3(G)$.*

Proof. Suppose that the theorem is false and let G be a counter-example of minimal order. Thus all proper A-invariant subgroups M of G satisfy $M' \leqslant \mathbf{F}_3(M)$, and all quotient groups G/N with N a non-trivial A-invariant normal subgroup N of G satisfy $(G/N)' \leqslant \mathbf{F}_3(G/N)$. By Corollary 2.4, G satisfies the Sylow tower property of supersolvable type. Let p be the largest prime dividing $|G|$ and let V be the Sylow p-subgroup of G. Clearly V is an A-invariant normal subgroup of G and $V \leqslant \mathbf{F}(G)$.

Step 1: We claim that $\Phi(G) = 1$ and $V = \mathbf{F}(G)$ is an irreducible $\mathbb{F}_p[AG]$-module.

Suppose that $\Phi(G) \neq 1$. Considering the quotient group $G/\Phi(G)$, we conclude from the minimality of G that $(G/\Phi(G))' \leqslant \mathbf{F}_3(G/\Phi(G)) = \mathbf{F}_3(G)/\Phi(G)$. This implies that $G' \leqslant \mathbf{F}_3(G)$, contrary to the choice of G. Hence $\mathbf{F}(G)$ can be viewed as an AG-module, possibly of mixed characteristic. By [7, Theorem 1.12] and Lemma 2.2, $\mathbf{F}(G)$ is a completely reducible AG-module. If the AG-module $\mathbf{F}(G)$ is reducible, then we may assume $\mathbf{F}(G) = V_1 \times V_2$, where $1 \neq V_i$ is an AG-module for each i. Consider the group G/V_i and let $T_i/V_i = \mathbf{F}_3(G/V_i)$. By the minimality of G we have $G' \leqslant G'V_i \leqslant T_i$. Observe that

$$T_1 \cap T_2 \lesssim (T_1 \cap T_2)V_1/V_1 \times (T_1 \cap T_2)V_2/V_2$$

is a subgroup of $T_1/V_1 \times T_2/V_2$ whose Fitting height is at most 3. We have $T_1 \cap T_2 = \mathbf{F}_3(T_1 \cap T_2) \leqslant \mathbf{F}_3(G)$ since $T_1 \cap T_2 \trianglelefteq G$. Observe that $\mathbf{F}_3(G) \leqslant T_1 \cap T_2$. Thus $T_1 \cap T_2 = \mathbf{F}_3(G)$ and so $G' \leqslant T_1 \cap T_2 = \mathbf{F}_3(G)$, which is a contradiction. Hence $\mathbf{F}(G)$ is an irreducible $\mathbb{F}_p[AG]$-module. Since $1 \neq V \leqslant \mathbf{F}(G)$ and V is an A-invariant normal subgroup of G, we have $V = \mathbf{F}(G)$, as claimed.

Step 2: We claim that $\mathbf{F}(\mathrm{C}_G(a)) \leqslant \mathbf{F}_3(G)$ for each $a \in A^{\#}$.

Let H be the A-invariant Hall p'-subgroup of G. For each $a \in A^{\#}$, we see from Lemma 2.1 (f) that $\mathrm{C}_V(a) \in \mathrm{Syl}_p(\mathrm{C}_G(a))$ and $\mathrm{C}_H(a) \in \mathrm{Hall}_{p'}(\mathrm{C}_G(a))$. Since $V \trianglelefteq G$, we have $\mathrm{C}_V(a) \trianglelefteq \mathrm{C}_G(a)$. Clearly $\mathrm{C}_V(a) \leqslant \mathbf{F}(\mathrm{C}_G(a))$ and $\mathrm{C}_V(a) \in \mathrm{Syl}_p(\mathbf{F}(\mathrm{C}_G(a)))$. Let K be the Hall p'-subgroup of $\mathbf{F}(\mathrm{C}_G(a))$. Then K is a nilpotent normal subgroup of $\mathrm{C}_H(a)$ such that

$$K \leqslant \mathrm{Ker}(\mathrm{C}_H(a) \text{ on } \mathrm{C}_V(a)).$$

By Step 1, V is an irreducible AG-module. It is easy to see that V is an irreducible AH-module. Thus V is a completely reducible H-module and H acts faithfully on V since $\mathrm{C}_G(V) \leqslant V$. Applying Lemma 4.1, we have $K \leqslant \mathbf{F}_2(H)$. Thus $\mathbf{F}(\mathrm{C}_G(a)) = \mathrm{C}_V(a)K \leqslant V\mathbf{F}_2(H) \leqslant \mathbf{F}_3(G)$, as claimed.

Step 3: Final contradiction.

Considering the action of A on $G/\mathbf{F}_3(G)$, by Lemma 2.1 (c) we see that

$$\mathrm{C}_{G/\mathbf{F}_3(G)}(a) = \mathrm{C}_G(a)\mathbf{F}_3(G)/\mathbf{F}_3(G) \cong \mathrm{C}_G(a)/\,\mathrm{C}_{\mathbf{F}_3(G)}(a)$$

is a homomorphic image of $C_G(a)/\mathbf{F}(C_G(a))$ since $\mathbf{F}(C_G(a)) \leqslant \mathbf{F}_3(G)$ by Step 2. Observe that $C_G(a)/\mathbf{F}(C_G(a))$ is abelian since $C_G(a)$ is supersolvable. Hence $C_{G/\mathbf{F}_3(G)}(a)$ is abelian for each $a \in A^{\#}$. Since $|A| \geqslant r^3$, by Corollary 2.5 the group $G/\mathbf{F}_3(G)$ is abelian and so $G' \leqslant \mathbf{F}_3(G)$. This is a contradiction and the proof is complete. □

Theorem 4.3. *Let A be an elementary abelian r-group of order at least r^3 acting on an r'-group G. Suppose that $C_G(a)$ is supersolvable for each $a \in A^{\#}$. If either*

(a) *$C_G(a)'$ has odd order for each $a \in A^{\#}$, or*

(b) *r is not a half-Fermat prime,*

then $G' \leqslant \mathbf{F}_2(G)$.

Proof. Suppose the theorem false and let G be a counter-example of minimal order. Let p be the largest prime dividing $|G|$. Let V be the Sylow p-subgroup of G. Arguing as in the proof of Theorem 4.2, we see that $\Phi(G) = 1$ and $V = \mathbf{F}(G)$ is an irreducible $\mathbb{F}_p[AG]$-module.

Next we claim that $C_G(a)' \leqslant \mathbf{F}_2(G)$ for each $a \in A^{\#}$. Let H be the A-invariant Hall p'-subgroup of G. For each $a \in A^{\#}$, by Lemma 2.1, $C_V(a) \in \mathrm{Syl}_p(C_G(a))$ and $C_H(a) \in \mathrm{Hall}_{p'}(C_G(a))$. Since $V \trianglelefteq G$, we have $C_V(a) \trianglelefteq C_G(a)$ and clearly $C_V(a) \leqslant \mathbf{F}(C_G(a))$.

Let K be a Hall p'-subgroup of $C_G(a)'$. Since $C_G(a)'$ is nilpotent, K is a characteristic subgroup of $C_G(a)'$. Hence K is a nilpotent normal p'-subgroup of $C_G(a)$, moreover, $K \leqslant C_H(a)$. Observe that $C_V(a) \leqslant \mathbf{F}(C_G(a))$ and $K \leqslant C_G(a)' \leqslant \mathbf{F}(C_G(a))$. Hence $[C_V(a), K] = 1$, that is,

$$K \leqslant \mathrm{Ker}(C_H(a) \text{ on } C_V(a)).$$

Since V is an irreducible AG-module, we have $V \leqslant \mathrm{Ker}(AG \text{ on } V)$. Let U be a non-zero AH-module of V. Since U is V-invariant, U is AG-invariant, that is, U is an AG-submodule of V. Thus V is also an irreducible AH-module, and therefore V is a completely reducible H-module and H acts faithfully on V since $C_G(V) \leqslant V$. By hypothesis, either $|K|$ is odd or r is not a half-Fermat prime. Applying Lemma 4.1, we conclude that $K \leqslant \mathbf{F}(H)$. Thus

$$C_G(a)' \leqslant C_V(a)K \leqslant V\mathbf{F}(H) \leqslant \mathbf{F}_2(G),$$

as claimed.

Finally, considering the action of A on $G/\mathbf{F}_2(G)$, for each $a \in A^{\#}$, we see from Lemma 2.1 (c) that $C_{G/\mathbf{F}_2(G)}(a)$ is abelian. Since $|A| \geqslant r^3$, by Corollary 2.5, $G/\mathbf{F}_2(G)$ is abelian, that is, $G' \leqslant \mathbf{F}_2(G)$. This is a contradiction and the proof is complete. □

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Bibliography

[1] K. Doerk and T. O. Hawkes, *Finite Soluble Groups. Volume 4*, Walter de Gruyter, Berlin, 1992.

[2] P. Flavell, Automorphisms of soluble groups, *Proc. Lond. Math. Soc. (3)* **112** (2016), 623–650.

[3] G. Glauberman, On solvable signalizer functors in finite groups, *Proc. Lond. Math. Soc. (3)* **33** (1976), 1–27.

[4] R. Guralnick and P. Shumyatsky, Derived subgroups of fixed points, *Israel J. Math.* **126** (2001), 345–362.

[5] B. Huppert and N. Blackburn, *Finite Groups. Volume 3*, Springer, Berlin, 1982.

[6] H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups: An Introduction*, Springer, New York, 2006.

[7] O. Manz and T. R. Wolf, *Representations of Solvable Groups*, London Math. Soc. Lecture Note Ser. 185, Cambridge University Press, Cambridge, 1993.

[8] J. G. Thompson, Finite groups with fixed-point-free automorphisms of prime order, *Proc. Natl. Acad. Sci. USA* **45** (1959), 578–581.

[9] J. N. Ward, On finite groups admitting automorphisms with nilpotent fixed-point group, *Bull. Aust. Math. Soc.* **5** (1971), 281–282.

[10] J. N. Ward, On finite soluble groups and the fixed-point groups of automorphisms, *Bull. Aust. Math. Soc.* **5** (1971), 375–378.

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