J. Group Theory **18** (2015), 887–904
DOI 10.1515/jgth-2015-0015

The absolute center of finite groups

Hangyang Meng and Xiuyun Guo

Communicated by Evgenii I. Khukhro

Abstract. In this paper we first investigate the relationship between the absolute center $L(G)$ and the Frattini subgroup $\Phi(G)$ for a finite group G, and then we describe the structure of finite groups G satisfying $L(G) \leq \Phi(G)$. For example, we prove that a finite group G is an $\mathcal{F}$-group if and only if $G/L(G)$ is an $\mathcal{F}$-group, where $\mathcal{F}$ is a saturated formation containing Z_2. Next we determine all finite groups X such that $X/L(X)$ is isomorphic to $Z_{p^\omega} \times Z_p$, where ω is a positive integer and Z_{p^ω} is a cyclic group of order p^ω.

1 Introduction

A group G is said to be capable if there exists a group H such that $H/Z(H)$ is isomorphic to G. The study of capable groups plays a central role in various group-theoretical problems. Many mathematicians, such as Baer, Hall, Senior, Schur, Isaacs and so on, have investigated capable groups [1, 4, 8] and many interesting results have been given. For example, a classical result due to Schur states that if the central quotient $G/Z(G)$ of a group G is finite, then the commutator subgroup G' is also finite, see [7]. In 1994, Hegarty [5] introduced the autocommutator subgroup

$$K(G) = \langle g^{-1}g^{\alpha} : g \in G,\ \alpha \in \mathrm{Aut}(G)\rangle$$

of a group G and its absolute center

$$L(G) = \{g \in G : g^{\alpha} = g \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

Furthermore, Hegarty proved an analogue of Schur's theorem for the absolute center and the autocommutator subgroup of a group, that is, if G is a group such that $G/L(G)$ is finite, then $K(G)$ is also finite.

Following Chaboksavar, Ghouchan and Saeedi [3], we say that a group G is autocapable if there exists a group M such that $M/L(M)$ is isomorphic to G. In 1997, Hegarty [6] proved that for any finite autocapable group G there are finitely

The research of the work was partially supported by the National Natural Science Foundation of China (11371237) and a grant of "The First-Class Discipline of Universities in Shanghai".

many finite groups M such that $M/L(M)$ is isomorphic to G. However, it is not easy to see if a group G is autocapable. In 2014, Chaboksavar, Ghouchan and Saeedi proved that there is no group G such that $G/L(G)$ is isomorphic to Q_8, and there are infinitely many finite and infinite non-autocapable groups. Now the following questions naturally arise.

- Can we describe the structure of a finite group G if $G/L(G)$ is known?
- Which finite groups X satisfy the equation $X/L(X) \cong G$ for a given finite autocapable group G?

In this paper we first investigate the relationship between $L(G)$ and the Frattini subgroup $\Phi(G)$ for a finite group G, and then we describe the structure of finite groups G satisfying $L(G) \leq \Phi(G)$. For example, we prove that a finite group G is an $\mathscr{F}$-group if and only if $G/L(G)$ is an $\mathscr{F}$-group, where $\mathscr{F}$ is a saturated formation containing Z_2 (Corollary 3.8). Next we determine the structure of the absolute center of all finite minimal non-abelian p-groups, and then we determine all finite groups X such that $X/L(X)$ is isomorphic to $Z_{p^\omega} \times Z_p$, where ω is a positive integer and Z_{p^ω} is a cyclic group of order p^ω.

2 Preliminary results

In this section we list some basic results, which are frequently used in this paper.

Lemma 2.1. *Let $H_1, \dots, H_s$ be subgroups of a finite group G such that*

$$G = H_1 \times \cdots \times H_s.$$

Then

$$L(G) \leq L(H_1) \times \cdots \times L(H_s).$$

Moreover, if H_i is characteristic in G for $i = 1, \dots, s$, then

$$L(G) = L(H_1) \times \cdots \times L(H_s).$$

Proof. Let α_i be an automorphism of H_i for $i = 1, \dots, s$. Then we define a map

$$\alpha : G \to G,\ \ x = (h_1, \dots, h_s) \mapsto \alpha(x) = (h_1, \dots, h_{i-1}, \alpha_i(h_i), h_{i+1}, \dots, h_s).$$

It is easy to see that α is an automorphism of G. If $x = (h_1, \dots, h_s) \in L(G)$, then it follows from

$$(h_1, \dots, h_s) = x = \alpha(x) = (h_1, \dots, \alpha_i(h_i), \dots, h_s)$$

that $\alpha_i(h_i) = h_i$ and so $h_i \in L(H_i)$. Thus $L(G) \leq L(H_1) \times \cdots \times L(H_s)$.

If H_i is characteristic in G for each i, then $\mathrm{Aut}(G) = \mathrm{Aut}(H_1) \times \cdots \times \mathrm{Aut}(H_s)$, which implies that $L(H_1) \times \cdots \times L(H_s) \leq L(G)$. Whence the result follows. □

Lemma 2.2 ([2, Theorem 3.2.]). *Let H and K be subgroups of a finite group G such that $G = H \times K$. If H and K have no common direct factor, then*

$$\mathrm{Aut}(G) \cong \Big\{\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) : \alpha \in \mathrm{Aut}(H),\ \beta \in \hom(K, Z(H)),\\ \gamma \in \hom(H, Z(K)),\ \delta \in \mathrm{Aut}(K)\Big\}.$$

Lemma 2.3. *Let G be a finite group. Then $L(G) = G$ if and only if $G \cong 1$ or Z_2.*

Proof. This lemma is a special case of [3, Lemma 2.2]. □

3 The absolute center and the Frattini subgroup

Recall that a group H is said to be a direct factor of a group G if there exist subgroups K and T of G such that $K \cong H$ and $G = K \times T$. In this section, we investigate the relationship between $L(G)$ and the Frattini subgroup $\Phi(G)$ for a finite group G. The following lemma is the key lemma in this paper.

Lemma 3.1. *Let G be a finite group. If Z_2 is not a direct factor of G, then*

$$L(G) \le \Phi(G).$$

Proof. Suppose that the lemma is false. Let G be a group such that Z_2 is not a direct factor of G and $L(G)$ is not contained in $\Phi(G)$. Then there exists a maximal subgroup M of G such that $L(G) \not\le M$. Since $L(G) \le Z(G)$, the maximality of M implies that $G = ML(G)$ and $M \lhd G$. Hence we assume that $|G/M| = p$, where p is a prime number. Now we consider the following two cases.

Case 1: $p \mid |L(G) \cap M|$. Choose $z \in L(G) \bigcap M$ such that $o(z) = p$ and fix $g \in L(G) \backslash M$. It is clear that

$$G = \bigsqcup_{i=0}^{p-1} Mg^i.$$

where "$\sqcup$" means disjoint union of sets. For any $x \in G$, there exist unique $m \in M$ and $i \in \{0, 1, \ldots, p-1\}$ such that $x = mg^i$. Define a map

$$\alpha : G \to G, \quad x = mg^i \mapsto mg^i z^i.$$

We claim that α is an automorphism of G. In fact, for any two elements $x = mg^i$ and $y = ng^j$ of G, where $i, j \in \{0, 1, \ldots, p-1\}$ and $m, n \in M$, we see

$$xy = mg^i ng^j = mng^{i+j}.$$

Since $0 \le i + j < 2p - 1$ and $g^p \in M$, we consider two possibilities.

If $i + j \geq p$, then

$$\begin{aligned}\alpha(xy) &= \alpha((mng^p)g^{i+j-p}) \\ &= (mng^p)g^{i+j-p}z^{i+j-p} \\ &= (mg^i z^i)(ng^j z^j) \\ &= \alpha(x)\alpha(y).\end{aligned}$$

If $i + j < p$, then

$$\begin{aligned}\alpha(xy) &= \alpha((mn)g^{i+j}) \\ &= (mn)g^{i+j}z^{i+j} \\ &= (mg^i z^i)(ng^j z^j) \\ &= \alpha(x)\alpha(y).\end{aligned}$$

In a word, $\alpha(xy) = \alpha(x)\alpha(y)$. In addition, for any element $x(= mg^i) \in \mathrm{Ker}(\alpha)$, in which $m \in M$ and $i \in \{0, 1, \ldots, p-1\}$,

$$1 = \alpha(x) = \alpha(mg^i) = mg^i z^i,$$

which implies $i = 0$ and $m = 1$. Thus $\mathrm{Ker}(\alpha) = 1$, and so the finiteness of G implies that α is an automorphism of G.

Since $g \in L(G)$, it follows from the definition of $L(G)$ and α that

$$g = \alpha(g) = \alpha(1 \cdot g^1) = gz.$$

Hence $z = 1$, which is a contradiction.

Case 2: $p \nmid |L(G) \bigcap M|$. In this case, since

$$p = |G/M| = |ML(G)/M| = |L(G)/L(G) \bigcap M|,$$

we see that p divides $|L(G)|$ and we may choose $z \in L(G)$ such that $o(z) = p$ and $z \notin M$. Hence $G = \langle M, z\rangle = M \times \langle z\rangle$.

The hypothesis of the lemma implies $p > 2$. It is clear that for any $x \in G$, there exist unique $m \in M$ and $i \in \{0, 1, \ldots, p-1\}$ such that $x = mz^i$. Consider the map

$$\beta : G \to G, \quad x(= mz^i) \mapsto mz^{2i}.$$

We claim that the map β is an automorphism of G; the proof is similar to Case 1.

Since $z \in L(G)$, it is clear that $z = \alpha(z) = \alpha(1 \cdot z^1) = z^2$, a contradiction. The proof of the lemma is complete. □

Remark 3.2. The hypothesis that Z_2 is not a direct factor of G in Lemma 3.1 can not be removed. In fact, let a be an element of order 2, let H be a group of odd order and let

$$G = \langle a \rangle \times H.$$

Then

$$L(G) = L(\langle a \rangle) \times L(H) = \langle a \rangle \times L(H)$$

by Lemmas 2.1 and 2.2, and

$$\Phi(G) = \Phi(\langle a \rangle) \times \Phi(H) = \Phi(H).$$

Hence $L(G) \not\leq \Phi(G)$.

Lemma 3.3. *Let G be a finite group such that $G \cong E_n \times H$, where H is a group such that Z_2 is not a direct factor of H and*

$$E_n \cong \underbrace{Z_2 \times Z_2 \times \cdots \times Z_2}_{n}.$$

If $n > 1$, then $L(G) \leq \Phi(G)$.

Proof. Lemma 3.1 implies that $L(H) \leq \Phi(H)$ and [3, Lemma 2.1] implies that $L(E_n) = 1$. Hence, by Lemma 2.1, we see that

$$L(G) \leq L(E_n) \times L(H) = 1 \times L(H) \leq \Phi(E_n) \times \Phi(H) = \Phi(G).$$

The lemma follows. □

Lemma 3.4. *Let $G = \langle a \rangle \times H$ be a finite group, where a is an element of order two and H has no normal subgroup of order two. Then $L(G) = \langle a \rangle \times L(H)$.*

Proof. By Lemma 2.1, $L(G) \leq L(\langle a \rangle) \times L(H) = \langle a \rangle \times L(H)$. So we only need to prove $\langle a \rangle \times L(H) \leq L(G)$. Since H has no normal subgroup of order two, $\langle a \rangle$ and H have no common direct factor. Using Lemma 2.2, we see that every automorphism θ of G can be expressed as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha \in \mathrm{Aut}(\langle a \rangle) = 1$, $\beta \in \hom(H, \langle a \rangle)$, $\gamma \in \hom(\langle a \rangle, Z(H))$, $\delta \in \mathrm{Aut}(H)$, which is defined by the rule

$$\theta \begin{pmatrix} a^i \\ h \end{pmatrix} \triangleq \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a^i \\ h \end{pmatrix} = \begin{pmatrix} \alpha(a^i)\beta(h) \\ \gamma(a^i)\delta(h) \end{pmatrix}$$

for all $i = 0, 1$ and $h \in H$.

By hypothesis, $Z(H)$ has no subgroup of order two. Thus $\gamma(\langle a\rangle) = 1$ and $\gamma = 0$ (where 0 denotes the zero homomorphism). Hence, any $\theta \in \mathrm{Aut}(G)$ can be expressed as

$$\begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix},$$

where $\beta \in \hom(H, \langle a\rangle), \delta \in \mathrm{Aut}(H)$. Therefore, for any $\binom{a^i}{h} \in \langle a\rangle \times L(H)$,

$$\theta \begin{pmatrix} a^i \\ h \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} a^i \\ h \end{pmatrix} = \begin{pmatrix} a^i \beta(h) \\ \delta(h) \end{pmatrix} = \begin{pmatrix} a^i \beta(h) \\ h \end{pmatrix}.$$

Noticing that $H = \mathrm{Ker}(\beta)$ or $\mathrm{Ker}(\beta)$ is a maximal subgroup of H for any element $\beta \in \hom(H, \langle a\rangle)$, we see $\Phi(H) \leq \mathrm{Ker}(\beta)$. By Lemma 3.1, we have

$$L(H) \leq \Phi(H) \leq \mathrm{Ker}(\beta).$$

Thus $\beta(h) = 1$ for all $\beta \in \hom(H, \langle a\rangle)$ and $h \in L(H)$ and therefore

$$\langle a\rangle \times L(H) \leq L(G).$$

The lemma is proved. □

From the above lemmas, we obtain the key theorem in this paper.

Theorem 3.5. *Let G be a finite group. The following conclusions are equivalent:*

(1) $G \cong Z_2 \times H$, *where H is a group such that $|Z(H)|$ is odd.*

(2) $G \cong Z_2 \times H$, *where H has no normal subgroup of order two.*

(3) $L(G) \nleq \Phi(G)$.

Proof. It is clear that (1) implies (2) and that (3) follows from (2) by Lemma 3.4. Next we prove that (3) implies (1). If $L(G) \nleq \Phi(G)$, we may assume $G = \langle a\rangle \times H$ by Lemma 3.1 and Lemma 3.3, where $o(a) = 2$ and Z_2 is not a direct factor of H. If $2 \mid |Z(H)|$, then we may choose $b \in Z(H)$ such that $o(b) = 2$. Define a map $\gamma_0 \in \hom(\langle a\rangle, Z(H))$,

$$\gamma_0 : \langle a\rangle \to Z(H), \quad a^i \mapsto b^i, \quad i = 0, 1.$$

According to Lemma 2.2, there is an automorphism θ_0 of G such that it can be expressed as

$$\begin{pmatrix} 1 & 0 \\ \gamma_0 & 1 \end{pmatrix}.$$

We claim that $L(G) \leq H$. In fact, for any $x = \binom{a^i}{h} \in L(G)$ with $i = 0, 1$ and $h \in H$, we have

$$\begin{pmatrix} a^i \\ h \end{pmatrix} = \theta_0 \begin{pmatrix} a^i \\ h \end{pmatrix} = \begin{pmatrix} a^i \\ \gamma_0(a^i)h \end{pmatrix} = \begin{pmatrix} a^i \\ b^i h \end{pmatrix}.$$

Thus $b^i = 1$ and $i = 0$. It follows that $x \in H$ and therefore $L(G) \leq H$. On the other hand, by Lemma 2.1, we have $L(G) \leq \langle a \rangle \times L(H)$. Thus

$$L(G) \leq H \cap (\langle a \rangle L(H)) = L(H)\,(H \cap \langle a \rangle) = L(H).$$

Since H has no direct factor of order two, $L(H) \leq \Phi(H)$ by Lemma 3.1. Hence $L(G) \leq L(H) \leq \Phi(H) = \Phi(G)$, in contradiction to $L(G) \not\leq \Phi(G)$. □

Now we give some applications of Theorem 3.5.

Corollary 3.6. *If G is a group of odd order, then $L(G) \leq \Phi(G)$.*

Proof. It is clear that the corollary is true by Lemma 3.1. □

Corollary 3.7. *Let p be a prime number and let G be a finite p-group except Z_2. Then $L(G) \leq \Phi(G)$.*

Proof. If p is an odd prime, then it is clear that $L(G) \leq \Phi(G)$ by Lemma 3.1. Now we assume that G is a 2-group. If $L(G) \not\leq \Phi(G)$, then, by Lemma 3.5, we may assume that $G = \langle a \rangle \times H$ such that $o(a) = 2$ and $|Z(H)|$ is odd. It follows that $Z(H) = 1$ and therefore $H = 1$. Hence $G = Z_2$, a contradiction. □

Recall that a formation is a class of groups closed under taking homomorphic images and such that if G/M and G/N are in the formation, then so is $G/M \cap N$. Let $\mathcal{F}$ be a formation; a group G is called an $\mathcal{F}$-group if $G \in \mathcal{F}$. Then $\mathcal{F}$ is said to be saturated provided the following condition is satisfied: If G is a group but not an $\mathcal{F}$-group and M is a minimal normal subgroup of G such that G/M is an $\mathcal{F}$-group, then M has a complement and all such complements are conjugate in G. A famous result states that if $\mathcal{F}$ is a saturated formation, then G is an $\mathcal{F}$-group if and only if $G/\Phi(G)$ is an $\mathcal{F}$-group. The following corollary is an analogue of that result.

Corollary 3.8. *Let $\mathcal{F}$ be a saturated formation containing Z_2. Then a finite group G is an $\mathcal{F}$-group if and only if $G/L(G)$ is an $\mathcal{F}$-group.*

Proof. It is clear that we only need to prove the sufficiency part. Suppose that the sufficiency is false and let G be a counterexample of minimal order.

If $L(G) \leq \Phi(G)$, then it is clear that $G/L(G) \in \mathcal{F}$ implies that $G/\Phi(G) \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, it follows that $G \in \mathcal{F}$. Now we assume that $L(G) \nleq \Phi(G)$. By Theorem 3.5, we may assume that $G = \langle a \rangle \times H$ such that $o(a) = 2$ and H has no normal subgroup of order two. By Lemma 3.4, we have $L(G) = \langle a \rangle \times L(H)$. Thus we have

$$G/L(G) = \langle a \rangle L(H)H/\langle a \rangle L(H) \cong H/(L(H)\langle a \rangle \cap H) = H/L(H),$$

and so $H/L(H) \in \mathcal{F}$. By induction, $H \in \mathcal{F}$. It follows from $G/H \cong Z_2 \in \mathcal{F}$ and $G/\langle a \rangle \cong H \in \mathcal{F}$ that $G \cong G/(H \cap \langle a \rangle) \in \mathcal{F}$, a contradiction. □

Remark 3.9. The condition "$Z_2 \in \mathcal{F}$" in Corollary 3.8 cannot be removed. In fact, let $\mathcal{F}$ be the class of all groups of odd order. It is clear that $\mathcal{F}$ is a saturated formation and $Z_2 \notin \mathcal{F}$. We take a group $G = \langle a \rangle \times H$ such that $o(a) = 2$ and H is a group of odd order. It is clear that $G \notin \mathcal{F}$ but $G/L(G)$ is a group of odd order.

Recall that a group G is said to be p-solvable group if its upper p-series reaches G:

$$1 = P_0(G) \trianglelefteq M_0(G) \trianglelefteq P_1(G) \trianglelefteq M_1(G) \trianglelefteq \cdots \trianglelefteq P_l(G) \trianglelefteq M_l(G) = G,$$

where

$$M_i(G)/P_i(G) = O_{p'}(G/P_i(G)), \quad P_i(G)/M_{i-1}(G) = O_p(G/M_{i-1}(G)).$$

The minimum number l is called the p-length of G and denoted by $l_p(G)$. The p-length of p-solvable groups is an interesting topic in finite group theory. The following corollary shows that for $p \neq 2$ a p-solvable group G and its quotient group $G/L(G)$ have the same p-length.

Corollary 3.10. *Let G be a p-solvable group, where p is an odd prime. Then $l_p(G) = l_p(G/L(G))$.*

Proof. It is clear that $l_p(G) \geq l_p(G/L(G))$. If $L(G) \leq \Phi(G)$, then $G/\Phi(G)$ is a factor group of $G/L(G)$. Thus $l_p(G/L(G)) \geq l_p(G/\Phi(G)) = l_p(G)$ and so we get $l_p(G) = l_p(G/L(G))$. Now we assume that $L(G) \nleq \Phi(G)$. By Theorem 3.5, we may assume that $G = \langle a \rangle \times H$ such that $o(a) = 2$ and Z_2 is not a direct factor of the group H. By Lemma 3.4, we have $L(G) = \langle a \rangle \times L(H)$, which implies that $G/L(G) \cong H/L(H)$ and hence $l_p(G/L(G)) = l_p(H/L(H))$. By Lemma 3.1, $L(H) \leq \Phi(H)$ and thus $l_p(H) = l_p(H/L(H))$. Furthermore, it follows from $l_p(G) = \max\{l_p(\langle a \rangle), l_p(H)\} = l_p(H)$ (as $l_p(\langle a \rangle) = 0$) that

$$l_p(G/L(G)) = l_p(H/L(H)) = l_p(H) = l_p(G).$$

The proof is now complete. □

4 The absolute center of finite minimal non-abelian p-groups

Recall that a p-group is called a minimal non-abelian p-group if it is a non-abelian group and all its maximal subgroups are abelian. It is well-known that finite minimal non-abelian p-groups play an important role in finite group theory. In this section, we determine the absolute center of finite minimal non-abelian p-groups. First we list the following result due to Redei [9].

Lemma 4.1 ([9]). *Let G be a finite minimal non-abelian p-group. Then G is one of the following groups:*

(1) Q_8,

(2) $M_p(n,m) = \langle a,b : a^{p^n} = b^{p^m} = 1,\ [a,b] = a^{p^{n-1}}\rangle$, *where* $n \geq 2$, $m \geq 1$ *(metacyclic),*

(3) $M_p(n,m,1) = \langle a,b,c : a^{p^n} = b^{p^m} = c^p = 1,\ [a,b] = c,\ [a,c] = 1,\ [b,c] = 1\rangle$, *where* $n \geq m \geq 1$, $m + n > 2$ *(non-metacyclic).*

The following equivalent conditions about finite minimal non-abelian p-groups are always used.

Lemma 4.2 ([9]). *Let G be a finite p-group. Then the following conditions are equivalent:*

(1) *G is a minimal non-abelian p-group.*

(2) $d(G) = 2$ *and* $|G'| = p$.

(3) $d(G) = 2$ *and* $Z(G) = \Phi(G)$.

The following lemma plays an important role in the theory of automorphism groups of finite groups.

Lemma 4.3. *Let $G = \langle a_1, \ldots, a_n : f_i(a_1, \ldots, a_n) = 1,\ i \in I\rangle$ be a finite group, where $\{f_i : i \in I\}$ is a set of generated relations of G. Let* $\mathrm{Gen}(G) = \{a_1, \ldots, a_n\}$ *be a set of generators. If φ is a map from* $\mathrm{Gen}(G)$ *to G such that*

$$G = \langle \varphi(a_1), \ldots, \varphi(a_n) : f_i(\varphi(a_1), \ldots, \varphi(a_n)) = 1,\ i \in I\rangle, \tag{4.1}$$

then there exists a unique automorphism $\widetilde{\varphi} \in Aut(G)$ such that $\widetilde{\varphi}|_{\mathrm{Gen}(G)} = \varphi$.

For convenience, we use

$$\mathrm{Gen}(G) \times \varphi(\mathrm{Gen}(G)) = \{(a_1, \varphi(a_1)), \ldots, (a_n, \varphi(a_n))\}$$

to denote the automorphism $\widetilde{\varphi}$ of G determined by the set $\mathrm{Gen}(G)$ and the map φ from $\mathrm{Gen}(G)$ to G with (4.1).

Lemma 4.4. *Let $G = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle$ be a finite abelian p-group, in which*

$$o(a_i) = p^{n_i} \quad \text{and} \quad n_1 \geq \cdots \geq n_k.$$

Then

$$L(G) = \begin{cases} 1 & \text{if } p \text{ is odd, or } p = 2 \text{ and } n_1 = n_2, \\ \langle a_1^{p^{n_1-1}} \rangle & \text{if } p = 2, \text{ and } k = 1 \text{ or } n_1 > n_2. \end{cases}$$

Proof. If p is odd or $p = 2$ with $n_1 = n_2$, then $L(G) = 1$ by [3, Lemma 2.1]. If $p = 2$ and $k = 1$, then $L(G) \cong Z_2$ by [3, Lemma 2.1]. Now we assume that $p = 2$ and $n_1 > n_2$. In this case we assume that

$$\alpha = \{(a_1, a_1^{s_{11}} a_2^{s_{12}} \cdots a_k^{s_{1k}}), \ldots, (a_k, a_1^{s_{k1}} a_2^{s_{k2}} \cdots a_k^{s_{kk}})\}$$

is an automorphism of G. Lemma 4.3 implies $o(\alpha(a_1)) = 2^{n_1}$ so that

$$(a_1^{s_{11}} a_2^{s_{12}} \cdots a_k^{s_{1k}})^{2^{n_1-1}} = a_1^{2^{n_1-1} s_{11}} \neq 1.$$

Thus $2 \nmid s_{11}$ and so

$$\alpha(a_1^{2^{n_1-1}}) = (\alpha(a_1))^{2^{n_1-1}} = (a_1^{s_{11}} a_2^{s_{12}} \cdots a_k^{s_{1k}})^{2^{n_1-1}} = a_1^{2^{n_1-1} s_{11}} = a_1^{2^{n_1-1}}.$$

Hence $\langle a_1^{2^{n_1-1}} \rangle \leq L(G)$. By [3, Lemma 2.1], the lemma follows. □

Lemma 4.5. *Let $G = \langle a, b : a^{p^n} = b^{p^m} = 1,\ [a,b] = a^{p^{n-1}} \rangle$ be a finite group, where $m > n \geq 2$ or $p = 2, n = m = 2$. If*

$$\beta = \beta(i, j, s, t) = \{(a, a^i b^j), (b, a^s b^t)\}$$

is an automorphism of G with $1 \leq i, s \leq p^n$ and $1 \leq j, t \leq p^m$, then

$$(i, p) = 1, \quad t \equiv 1 \pmod{p}, \quad p^{m-n+1} \mid j. \tag{4.2}$$

Proof. By Lemma 4.3, β satisfies relation (4.1).

If $m > n \geq 2$, then $o(\beta(a)) = p^n$ implies that

$$\begin{cases} 1 = (a^i b^j)^{p^n} = a^{ip^n} b^{jp^n} [b^j, a^i]^{\frac{1}{2} p^n (p^n - 1)} = b^{jp^n} a^{-\frac{1}{2} ijp^{2n-1}(p^n-1)}, \\ 1 \neq (a^i b^j)^{p^{n-1}} = a^{ip^{n-1}} b^{jp^{n-1}} [b^j, a^i]^{\frac{1}{2} p^{n-1}(p^{n-1}-1)}. \end{cases}$$

We see $p \mid j$ from $m > n$ and thus $[b^j, a^i] = [b,a]^{ij} = 1$. It follows that

$$\begin{cases} p^{m-n} \mid j, \\ (i, p) = 1 \text{ or } p^{m-n+1} \nmid j. \end{cases}$$

In addition, $[\beta(a), \beta(b)] = \beta(a)^{p^{n-1}}$ implies that

$$[a^i b^j, a^s b^t] = (a^i b^j)^{p^{n-1}}.$$

Considering $p \mid j$, we see

$$(a^i b^j)^{p^{n-1}} = (a)^{ip^{n-1}} (b)^{jp^{n-1}}$$

and

$$[a^i b^j, a^s b^t] = [a^i, a^s b^t]^{b^j} [b^j, a^s b^t] = [a, b]^{it} = a^{itp^{n-1}},$$

which implies that $a^{ip^{n-1}(t-1)} b^{jp^{n-1}} = 1$. Clearly, p^{m-n+1} divides j and thus $(i, p) = 1$, which implies that $t \equiv 1 \pmod{p}$. Hence β satisfies relation (4.2).

If $p = 2$ and $m = n = 2$, then $o(\beta(a)) = 4$ implies that

$$1 \neq (a^i b^j)^2 = a^{2i} b^{2j} [b^j, a^i]^{\frac{2(2-1)}{2}} = a^{2i(1-j)} b^{2j},$$

and therefore $2 \nmid i$ or $2 \nmid j$. Also $o(\beta(b)) = 4$ implies that $2 \nmid s$ or $2 \nmid t$. It follows from $[\beta(a), \beta(b)] = \beta(a)^2$ that $[a^i b^j, a^s b^t] = (a^i b^j)^2$. Notice that

$$(a^i b^j)^2 = (a)^{2i} (b)^{2j} [b^j, a^i] = a^{2i(1-j)} b^{2j}$$

and

$$[a^i b^j, a^s b^t] = [a^i, a^s b^t]^{b^j} [b^j, a^s b^t] = [a, b]^{it} = a^{2it},$$

we see $a^{2i(t+j-1)} b^{2j} = 1$. Clearly, $2 \mid j$ and thus $2 \nmid i$. Furthermore, $2 \mid t - 1$. Hence β satisfies relation (4.2). □

Lemma 4.6. *Let*

$$G = \langle a, b, c : a^{2^n} = b^{2^m} = c^2 = 1,\ [a, b] = c,\ [a, c] = 1,\ [b, c] = 1 \rangle$$

be a finite group, where $n > m \geq 1, n \geq 3$. *If*

$$\gamma = \gamma(i, j, k, r, s, t) = \{(a, a^i b^j c^k), (b, a^r b^s c^t), (c, c)\}$$

is an automorphism of G, where $1 \leq i, r \leq 2^n$, $1 \leq j, s \leq 2^m$, $1 \leq k, t \leq 2$, *then*

$$2 \nmid i, \quad 2 \nmid s, \quad 2^{n-m} \mid r. \tag{4.3}$$

Proof. By Lemma 4.3, γ satisfies relation (4.1). Since $n > m \geq 1$, $o(\gamma(b)) = 2^m$ implies that

$$(a^r b^s c^t)^{2^m} = a^{2^m r} b^{2^m s} c^{2^m t} [b^s, a^r]^{2^{m-1}(2^m-1)} = a^{2^m r} c^{2^{m-1}(2^m-1)rs} = 1.$$

Then we have $2^{n-m} \mid r$. Considering $[a^i b^j c^k, a^r b^s c^t] = c$, we see

$$\begin{aligned}[a^i b^j c^k, a^r b^s c^t] &= [a^i b^j, a^r b^s] = [a^i b^j, b^s][a^i b^j, a^r]^{b^s} \\ &= [a^i b^j, b^s] = [a^i, b^s] = c^{is},\end{aligned}$$

which implies that $2 \mid is - 1$. It is clear that $2 \nmid i$ and $2 \nmid s$. Thus γ satisfies relation (4.3). □

If G is a finite group and A is a subset of $\mathrm{Aut}(G)$, then we use $C_G(A)$ to denote the centralizer of A in G, which consists of all elements in G fixed by A. The following theorem shows the structure of the absolute center of finite minimal non-abelian p-groups.

Theorem 4.7. *Let G be a finite minimal non-abelian p-group.*

(1) *If $p > 2$, then the following hold.*

(a) *If $G = M_p(n,m)$, $n \geq 2$, $m \geq 1$, then*

$$L(G) = \begin{cases} \langle b^{p^{m-1}} \rangle & \text{if } n < m, \\ 1 & \text{if } n \geq m. \end{cases}$$

(b) *If $G = M_p(n,m,1)$, $n \geq m \geq 1$, $m + n > 2$, then*

$$L(G) = 1.$$

(2) *If $p = 2$, then the following hold.*

(a) *If $G \cong Q_8$, then*

$$L(G) = Z(Q_8) = (Q_8)' = \Phi(Q_8) \cong Z_2.$$

(b) *If $G = M_2(n,m)$, $n \geq 2$, $m \geq 1$, then*

$$L(G) = \begin{cases} \langle a^{2^{n-1}} \rangle \times \langle b^{2^{m-1}} \rangle & \text{if } n < m \text{ or } n = m = 2, \\ \langle a^{2^{n-1}} \rangle & \text{if } n > m \text{ or } n = m \geq 3. \end{cases}$$

(c) *If $G = M_2(n,m,1)$, $n \geq m \geq 1$, $m + n > 2$, then*

$$L(G) = \begin{cases} \langle c \rangle & \text{if } n = m \text{ or } n = 2,\ m = 1, \\ \langle a^{2^{n-1}} \rangle \times \langle c \rangle & \text{otherwise}. \end{cases}$$

Proof. We prove the results by considering the following cases.

Case 1: $p > 2$. Since G is a finite minimal non-abelian p-group and $p > 2$, the map

$$\alpha : G \to G, \quad x \mapsto x^{1+p}$$

is an automorphism of G. For any element x of $L(G)$, $x = \alpha(x) = x^{1+p}$ and so $x^p = 1$. Hence $\exp(L(G)) = p$.

Case (a): $G = M_p(n,m), n \geq 2, m \geq 1$. In this case $\alpha_1 = \{(a, a^2), (b, b)\}$ is an automorphism of G and therefore $L(G) \leq C_G(\alpha_1) = \langle b \rangle$. It follows from $\exp(G) = p$ that $L(G) \leq \langle b^{p^{m-1}} \rangle$. If $n \geq m$, then we consider the automorphism $\alpha_2 = \{(a, a), (b, a^{p^{n-m}} b)\}$ and so

$$L(G) \leq C_G(\alpha_2) \cap \langle b^{p^{m-1}} \rangle = \langle a \rangle \cap \langle b^{p^{m-1}} \rangle = 1.$$

If $n < m$, for any automorphism $\beta = \{(a, a^i b^j), (b, a^s b^t)\}$ of G, β satisfies relation (4.2) by Lemma 4.5. We see

$$\begin{aligned} \beta(b^{p^{m-1}}) &= (\beta(b))^{p^{m-1}} = (a^s b^t)^{p^{m-1}} \\ &= a^{sp^{m-1}} b^{tp^{m-1}} [b, a]^{\frac{1}{2} stp^{m-1}(p^{m-1}-1)}. \end{aligned}$$

It follows from $m - 1 \geq n \geq 2$ and $t \equiv 1 \pmod p$ that

$$\beta(b^{p^{m-1}}) = b^{p^{m-1}}.$$

Therefore $L(G) = \langle b^{p^{m-1}} \rangle$ and (a) follows.

Case (b): $G = M_p(n,m,1), n \geq m \geq 1, m + n > 2$. In this case, considering the following two automorphisms and their centralizers in G, we have

$$\begin{aligned} \alpha_3 &= \{(a, a^2), (b, b), (c, c^2)\}, \quad C_G(\alpha_3) = \langle b \rangle, \\ \alpha_4 &= \{(a, a), (b, b^2), (c, c^2)\}, \quad C_G(\alpha_4) = \langle a \rangle. \end{aligned}$$

Then $L(G) \leq C_G(\alpha_3) \cap C_G(\alpha_4) = 1$ and (b) follows.

Case 2: $p = 2$. Noticing that $|G'| = 2$ and G' char G, we see $G' \leq L(G)$.

Case (a): $G = Q_8$. It is easy to verify (a).

Case (b): $G = M_2(n,m), n \geq 2, m \geq 1$. In this case $\alpha_5 = \{(a, a^3), (b, b^3)\}$ is an automorphism of G and therefore

$$L(G) \leq C_G(\alpha_5) = \langle a^{2^{n-1}} \rangle \times \langle b^{2^{m-1}} \rangle.$$

Thus

$$G' = \langle a^{2^{n-1}} \rangle \leq L(G) \leq \langle a^{2^{n-1}} \rangle \times \langle b^{2^{m-1}} \rangle.$$

If $n > m$ or $n = m \geq 3$, then $\alpha_6 = \{(a, a), (b, a^{2^{n-m}} b)\}$ is an automorphism of G and $C_G(\alpha_6) = \langle a \rangle$. We see

$$L(G) \leq C_G(\alpha_6) \cap (\langle a^{2^{n-1}} \rangle \times \langle b^{2^{m-1}} \rangle) = \langle a^{2^{n-1}} \rangle.$$

Therefore $L(G) = \langle a^{2^{n-1}} \rangle$.

Now we assume $n < m$ or $m = n = 2$. As in Case (1.a), we may have

$$\langle b^{2^{m-1}} \rangle \leq L(G)$$

by Lemma 4.5. Therefore, $L(G) = \langle a^{2^{n-1}} \rangle \times \langle b^{2^{m-1}} \rangle$ and (b) follows.

Case (c): $G = M_2(n, m, 1)$, $n \geq m \geq 1$, $m + n > 2$. In this case it is clear that $\langle c \rangle = G' \leq L(G)$. Consider the following automorphisms and their centralizers in G:

$$\alpha_7 = \{(a, a), (b, a^{2^{n-m}} b), (c, c)\}, \quad C_G(\alpha_7) = \langle a \rangle \times \langle c \rangle,$$
$$\alpha_8 = \{(a, a^3), (b, b), (c, c)\}, \quad C_G(\alpha_8) = \langle a^{2^{n-1}} \rangle \times \langle b \rangle \times \langle c \rangle.$$

Thus $\langle c \rangle \leq L(G) \leq C_G(\alpha_7) \cap C_G(\alpha_8) = \langle a^{2^{n-1}} \rangle \times \langle c \rangle$.

If $m = n$, then $\alpha_9 = \{(a, b), (b, a), (c, c)\}$ is an automorphism of G. It follows that

$$\begin{aligned}\langle c \rangle \leq L(G) &\leq (\langle a^{2^{n-1}} \rangle \times \langle c \rangle) \cap C_G(\alpha_9) \\ &= (\langle a^{2^{n-1}} \rangle \times \langle c \rangle) \cap (\langle a^2 b^2 \rangle \times \langle c \rangle) = \langle c \rangle,\end{aligned}$$

so $L(G) = \langle c \rangle$.

If $n > m \geq 1$ and $n = 2$, then $n = 2, m = 1$ and $\alpha_{10} = \{(a, ab), (b, b), (c, c)\}$ is an automorphism of G. Thus

$$L(G) \leq (\langle a^{2^{n-1}} \rangle \times \langle c \rangle) \cap C_G(\alpha_{10}) = (\langle a^{2^{n-1}} \rangle \times \langle c \rangle) \cap (\langle b \rangle \times \langle c \rangle) = \langle c \rangle$$

and therefore $L(G) = \langle c \rangle$.

If $n > m \geq 1$ and $n \geq 3$, then

$$\gamma = \gamma(i, j, k, r, s, t) = \{(a, a^i b^j c^k), (b, a^r b^s c^t), (c, c)\}$$

as an automorphism of G. By Lemma 4.5, γ satisfies relation (4.3). Notice that

$$\begin{aligned}\gamma(a^{2^{n-1}}) &= (\gamma(a))^{2^{n-1}} = (a^i b^j c^k)^{2^{n-1}} \\ &= a^{2^{n-1} i} b^{2^{n-1} j} [b, a]^{\frac{2^{n-1}(2^{n-1}-1)ij}{2}} c^{2^{n-1} k} = a^{2^{n-1} i}.\end{aligned}$$

It follows from $i \equiv 1 \pmod 2)$ that $\gamma(a^{2^{n-1}}) = a^{2^{n-1}}$. Since γ is arbitrary, we see

$$\langle a^{2^{n-1}} \rangle \leq L(G).$$

Thus $L(G) = \langle a^{2^{n-1}} \rangle \times \langle c \rangle$ and (c) follows. □

Remark 4.8. We can find many interesting examples using Theorem 4.7. For example,

- there exist finite groups of odd order whose absolute centers are non-trivial, such as $M_p(m,n)$ with $p > 2$ and $m > n \geq 2$,
- there exists a finite group G such that $\Phi(G) = L(G) \neq 1$, such as $M_2(2,2)$ or Q_8,
- there exists a finite group G such that $1 < L(G) < \Phi(G)$, such as $M_2(n,m,1)$ with $n \geq m \geq 1$ and $m+n > 2$.

5 Applications

In this section we hope to extend the result of Chaboksavar, Farrokhi Derakhshandeh Ghouchan and Saeedi [3]. In fact, we determine all solutions of the equation $X/L(X) \cong G$ whenever $G \cong Z_{p^\omega} \times Z_p$, where ω is a positive integer and Z_{p^ω} is a cyclic group of order p^ω.

Theorem 5.1. *Let p be a prime number and let ω be a positive integer. Then G is a finite group such that $G/L(G) \cong Z_{p^\omega} \times Z_p$ if and only if G is isomorphic to one of the following groups:*

(I) *When p is odd,*

(1) $Z_{p^\omega} \times Z_p$,

(2) $Z_{p^\omega} \times Z_p \times Z_2$.

(II) *When $p = 2$,*

(1) Q_8 $(\omega = 1)$,

(2) $Z_2 \times Z_2$ $(\omega = 1)$,

(3) $Z_{2^{\omega+1}} \times Z_2$,

(4) $M_2(\omega+1, 1)$,

(5) $M_2(2, \omega+1)$,

(6) $M_2(\omega+1, 1, 1)$ $(\omega \geq 2)$,

(7) $M_2(2, 1, 1)$ $(\omega = 2)$.

Proof. (I) Let p be odd. Since $L(G) \leq Z(G)$ and $G/L(G)$ is abelian, G is nilpotent and $c(G) \leq 2$. Suppose that $G = P \times Q_1 \times \cdots \times Q_s$, where P is a Sylow p-subgroup of G and Q_i is a Sylow q_i-subgroup of G, for $i = 1, \ldots, s$. By Lemma 2.1,

$$L(G) = L(P) \times L(Q_1) \times \cdots \times L(Q_s).$$

Therefore,

$$G/L(G) \cong P/L(P) \times Q_1/L(Q_1) \times \cdots \times Q_s/L(Q_s)$$

is a p-group. It follows that $L(Q_i) = Q_i$ for $i = 1, \ldots, s$. By Lemma 2.3, we see $Q_i = 1$ or $Q_i \cong Z_2$. Hence we may assume that $G = P$ or $G \cong P \times Z_2$, where P is a p-group such that $P/L(P) \cong Z_{p^\omega} \times Z_p$.

Now we only discuss the structure of the p-group P. By Corollary 3.6, we see $L(P) \leq \Phi(P)$. It is clear that

$$P/\Phi(P) \cong P/L(P)/\Phi(P)/L(P) = P/L(P)/\Phi(P/L(P)) \cong Z_p \times Z_p.$$

Therefore $d(P)$, the rank of P, is two. Now suppose that $P/L(P) = \langle \overline{a} \rangle \times \langle \overline{b} \rangle$, where $\overline{a} = aL(P), \overline{b} = bL(P)$ and $o(\overline{a}) = p^\omega, o(\overline{b}) = p$. Clearly,

$$P = \langle a, b, L(P) \rangle = \langle a, b \rangle$$

and thus $P' = \langle [a,b], P_3 \rangle$, where $P_3 = [[P,P],P]$. It follows from $c(P) \leq 2$ that $P' = \langle [a,b] \rangle$. Since $b^p \in L(P) \leq Z(P)$, we have $[a,b]^p = [a,b^p] = 1$, which implies that $|P'| = 1$ or p.

- If $|P'| = p$, then P is a finite minimal non-abelian p-group by Lemma 4.2. Since $P/L(P)$ is abelian, we have $P' \leq L(P)$, which is contradictory to Theorem 4.7 (1).
- If $|P'| = 1$, then P is an abelian p-group. It follows from Lemma 4.7 that $L(P) = 1$. Hence $P \cong Z_{p^\omega} \times Z_p$.

In a word, G is isomorphic to the group (1) or (2).

(II) Let $p = 2$. As in (I), we have that G is a 2-group. The hypotheses of the theorem imply $G \not\cong Z_2$. It follows from Corollary 3.7 that $L(G) \leq \Phi(G)$. Thus we have $d(G) = 2$ and $|G'| = 1$ or 2 as in (I).

If $|G'| = 1$, it is obvious that G is of type (4) or (5) ($\omega = 1$) by Lemma 4.4.

If $|G'| = 2$, then G is a finite minimal non-abelian 2-group. Using the results of Theorem 4.7 (2), we consider the following cases.

(a) If $G \cong Q_8$, then $G/L(G) \cong Z_2 \times Z_2$, which satisfies the condition when $\omega = 1$. In this case, G is isomorphic to the group (3).

(b) If $G = M_2(n,m), n \geq 2, m \geq 1$, then we discuss the following cases.

- If $m > n$ or $m = n = 2$, then $L(G) = \langle a^{2^{n-1}} \rangle \times \langle b^{2^{m-1}} \rangle$ and hence

$$G/L(G) \cong Z_{2^{n-1}} \times Z_{2^{m-1}}.$$

  We see $n = 2, m = 1 + \omega$ ($\omega \geq 1$) and G is isomorphic to the group (7).

- If $n > m$ or $n = m \geq 3$, then $L(G) = \langle a^{2^{n-1}} \rangle$ and hence

$$G/L(G) \cong Z_{2^{n-1}} \times Z_{2^m}.$$

It is clear that $n = 1 + \omega$, $m = 1$ ($\omega \geq 1$). In this case, G is isomorphic to the group (6).

(c) If $G = M_2(n, m, 1)$, $n \geq m \geq 1$, $n + m > 2$, then we discuss the following two cases.

- If $m = n$ or $n = 2$, $m = 1$, at this time, $L(G) = \langle c \rangle$. Therefore

$$G/L(G) \cong Z_{2^n} \times Z_{2^m}.$$

This implies that $n = 1$, $m = 1$ ($\omega = 1$) or $n = 2$, $m = 1$ ($\omega = 2$), which is impossible because of $n + m > 2$.

- If $n > m \geq 1$ and $n \geq 3$, then $L(G) = \langle a^{2^{n-1}} \rangle \times \langle c \rangle$. Therefore

$$G/L(G) \cong Z_{2^{n-1}} \times Z_{2^m}.$$

Clearly, $n = 1 + \omega$, $m = 1$ ($\omega \geq 2$). Thus G is of type (8) or (9).

Conversely, it is easy to verify that a group isomorphic to one of the groups (1)–(9) satisfies $G/L(G) \cong Z_{p^\omega} \times Z_p$. □

Bibliography

[1] R. Baer, Groups with abelian central quotient group, *Trans. Amer. Math. Soc.* **44** (1938), 357–386.

[2] J. N. S. Bidwell, M. J. Curran and D. J. McCaughan, Automorphisms of direct products of finite group, *Arch. Math.* **86** (2006), 481–489.

[3] M. Chaboksavar, M. Farrokhi Derakhshandeh Ghouchan and F. Saeedi, Finite groups with a given absolute central factor group, *Arch. Math.* **102** (2014), 401–409.

[4] M. Hall and J. K. Senior, *The Groups of Order* 2^n *(*$n \leq 6$*)*, The Macmillan Company, New York, 1964.

[5] P. V. Hegarty, The absolute center of a group, *J. Algebra* **169** (1994), 929–935.

[6] P. V. Hegarty, Autocommutator subgroups of finite groups, *J. Algebra* **190** (1997), 556–562.

[7] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1976.

[8] I. M. Isaacs, Derived subgroups and centers of capable groups, *Proc. Amer. Math. Soc.* **129** (2001), 2853–2859.

[9] L. Redei, *Endliche* p*-Gruppen*, Akademiai Kiado, Budapest, 1989.

Received November 20, 2014; revised February 18, 2015.

Author information

Hangyang Meng, Department of Mathematics, Shanghai University,
Shanghai 200444, P. R. China.

Xiuyun Guo, Department of Mathematics, Shanghai University,
Shanghai 200444, P. R. China.
E-mail: xyguo@staff.shu.edu.cn