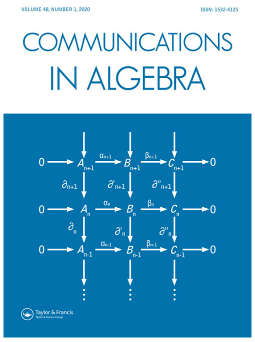

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Normal complements for finite groups under coprime action

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ABSTRACT

Let a finite group A act on a finite group G via automorphism with $(|A|, |G|) = 1$ and let H be a Hall subgroup of G. We prove that if H is a subgroup of $C_G(A)$ having a normal complement in $C_G(A)$, then H has a normal complement in G.

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1. Introduction

All groups considered in this paper are finite.

Let a group A act coprimely on a group G (via automorphism), i.e. $(|G|, |A|) = 1$. M. Y. Kizmaz [4] presents that if $C_G(A)$ is a Hall subgroup of G, then $C_G(A)$ has a normal complement in G. We give a further result:

Theorem A. *Let a group A act coprimely on a group G and let H be a Hall subgroup of G. Suppose that $H \leq C_G(A)$ and H has a normal complement in $C_G(A)$. Then H has a normal complement in G.*

The proof of Theorem A depends on the classification of the simple groups. As remark in [4, Remark 2.2], if H is solvable, the classification of finite simple groups also can be avoided in the proof by using focal subgroup theorem. This part also will be shown in Section 3(see Theorem 5).

2. Proof of Theorem A

Before proving Theorem A, we need the following lemma about coprime automorphism groups of non-abelian simple groups, which depends on the classification of finite simple group.

Lemma 1. *Let G be a non-abelian simple group and let $1 \neq A \leq \mathrm{Aut}(G)$ with $(|A|, |G|) = 1$. Then*

a. *G is a simple group of Lie type and A is a cyclic group of field automorphisms.*
b. *$C_G(A)$ is not a non-trivial Hall subgroup of G.*

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Table 1. Exceptional centralizers.

G	$C_G(A)$	N	$\lvert N\rvert$	$\lvert C_G(A) : N\rvert$
$L_2(2^r)$	$L_2(2) \cong S_3$	C_3	3	2
$L_2(3^r)$	$L_2(3) \cong A_4$	$C_2 \times C_2$	2^2	3
$Sz(2^r)$	$Sz(2) \cong 5 : 4$	C_5	5	2^2
$U_3(2^r)$	$U_3(2) \cong 3^2 : Q_8$	$C_3 \times C_3$	3^2	2^3
$Sp_4(2^r)$	$Sp_4(2)$	$Sp_4(2)' \cong A_6$	$2^3 \cdot 3^2 \cdot 5$	2
${}^2G_2(3^r)$	${}^2G_2(3)$	${}^2G_2(3)' \cong L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
$G_2(2^r)$	$G_2(2)$	$G_2(2)' \cong U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2
${}^2F_4(2^r)$	${}^2F_4(2)$	${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2

c. $C_G(A)$ *possesses a unique minimal normal subgroup N. Except for the cases listed in Table 1,* $C_G(A) = N$ is a non-abelian simple group.

Proof. Note that $(a), (b)$ can be observed from [1, Table 5], also [4, Lemma 2.1]. Part (c) follows from [2, Theorem 2.2.7]. □

The following results about coprime actions are well-known.

Lemma 2. *Let a group A act coprimely on a group G and let N be an A-invariant normal subgroup of G. Then*

a. $C_{G/N}(A) = C_G(A)N/N$;
b. $[G, A, A] = [G, A]$;
c. *if* $C_G(N) \leq N$ *and A acts trivially on N, then A acts trivially on G.*

Proof. Part $(a), (b)$ follows from [5, 8.2.2, 8.2.7, pp. 140–142]. Now we prove Part (c). As $[A, N] = 1$, it implies that $[A, N, G] = [N, G, A] = 1$. Three-Subgroup Lemma implies that $[G, A] \leq C_G(N) \leq N$. Hence $[G, A, A] = 1$ and so Part (b) implies that $[G, A] = 1$, as desired. □

Proof of Theorem A. Let (A, G) be a counterexample with $|AG|$ minimal. Obviously G is not a π-group or a π'-group, $H \neq 1, C_G(A) \neq G$. The minimality implies that A acts faithfully on G and clearly $A \neq 1$. We will reach a contradiction through the following steps:

Step (1). If N is a non-trivial A-invariant normal subgroup of G, then HN/N has a complement in G/N. In particular, $O_{\pi'}(G) = 1$.

Let $\bar{G} = G/N$ and consider the action of A on $\bar{G}$. The coprime action implies that $C_{\bar{G}}(A) = \overline{C_G(A)}$ by Lemma 2 (a). Then $\bar{H} \leq \overline{C_G(A)} = C_{\bar{G}}(A)$ and $\bar{H}$ has a normal complement in $C_{\bar{G}}(A)$, by the minimality, $\bar{H}$ has a normal complement in $\bar{G}$.

In particular if $O_{\pi'}(G) \neq 1$, then $G/O_{\pi'}(G)$ has a normal Hall π'-subgroup. Hence G has a normal Hall π'-subgroup, a contradiction.

Step (2). If N is a proper A-invariant normal subgroup of G, then N is a π-group. In particular, $N \leq H \leq C_G(A)$.

As $N \lhd -G, H \cap N$ is a Hall π-subgroup of N and clearly $H \cap N \leq C_N(A)$. Since H has a normal complement in $C_G(A)$, $H \cap N$ has a normal complement in $C_N(A)$. Hence, by minimality, $H \cap N$ has a normal complement in N, that is $N = (N \cap H)O_{\pi'}(N)$. As $O_{\pi'}(N) \leq O_{\pi'}(G) = 1$ by **Step (1)**, $N = H \cap N$ is a π-group.

Step (3). If N is a proper A-invariant normal subgroup of G, then $N = 1$. In particular, G is characteristic simple.

Assume that $N \neq 1$ and, by **Step (2)**, $N \leq H$. Considering the action of A on G/N, it follows from **Step (1)** that H/N has a normal complement in G/N. Write $T/N = O_{\pi'}(G/N)$ and clearly T is an A-invariant normal subgroup of G. If $T \neq G$, it follows from **Step (2)** that T is a π-group. Hence $G = HT$ is a π-group, which is a contradiction.

Now we assume that $T=G$. In this case, G is π-separable and $N=\mathrm{O}_{\pi}(G)$. As $\mathrm{O}_{\pi'}(G)=1$ by **Step (1)**, it follows from that $\mathrm{C}_G(N)\le N$. Since A acts trivially on N by **Step (2)**, it follows from Lemma 2 (c) that A acts trivially on G, which contradicts $\mathrm{C}_G(A)\ne G$.

Step (4). *G* is a non-abelian simple group.

By **Step (3)**, $G=G_1\times\cdots\times G_r$, where all G_i isomorphic to the same simple group. If all G_i are abelian, then G is a p-group for some prime p. Hence, we can deduce that G is a π-group or a π'-group, a contradiction. So all G_i isomorphic to the same non-abelian simple group.

Let $\Omega=\{G_1,\dots,G_r\}$ the set of all minimal normal subgroups of G. Clearly G_i^a is also a minimal normal subgroup of G for each i and each $a\in A$. Therefore, the action of A on G induces an natural action of A on Ω. Note that every A-orbit of Ω produces a non-trivial A-invariant normal subgroup of G. Consequently, by **Step (3)**, the action of A on G is transitive. So we may assume that $G_i=G_1^{a_i}$ for each i, where $a_i\in A$.

If $r\ge 2$, for each $x\in G_1\cap \mathrm{C}_G(A)$, we have that

$$x=x^{a_2}\in G_1\cap G_2=1.$$

Hence $G_1\cap \mathrm{C}_G(A)=1$ and so $|G_1|$ divides $|G:\mathrm{C}_G(A)|$ is a π'-number. Thus G is a π'-group, which is a contradiction. Hence $r=1$ and $G=G_1$ is non-abelian simple.

Step (5). We reach the final contradiction.

By **Step (4)**, G is a non-abelian simple group. Recall that $A\ne 1$ acts faithfully on G with $(|G|,|A|)=1$. By Lemma 1 (b), $H<\mathrm{C}_G(A)$. By hypothesis, H has a normal complement X in $\mathrm{C}_G(A)$ and so, as $1<H<\mathrm{C}_G(A), 1<X<\mathrm{C}_G(A)$ and so $\mathrm{C}_G(A)$ is not simple. It follows from Lemma 1 (c) that $\mathrm{C}_G(A)$ possesses a unique minimal normal subgroup N and $(G,\mathrm{C}_G(A),N)$ belongs to one of the cases listed in Table 1. Thus $N\le X$. Since X is a Hall subgroup of $\mathrm{C}_G(A)$, inspection of these cases yields that $G\cong \mathrm{L}_2(2^r)(r\ge 2), \mathrm{L}_2(3^r)(r\ge 2), \mathrm{Sz}(2^r)(r\ge 3 \textit{ is odd})$ or $\mathrm{U}_3(2^r)(r\ge 4 \textit{ is even})$, and $N=X$. Hence $|H|=|\mathrm{C}_G(A):N|$, and, in such cases, H is a Sylow 2-subgroup or Sylow 3-subgroup of $\mathrm{C}_G(A)$. But, comparing the order of G,

- $G\cong \mathrm{L}_2(2^r), r\ge 2, |H|=2$ and $|G|_2=2^r$;
- $G\cong \mathrm{L}_2(3^r), r\ge 2, |H|=3$ and $|G|_3=3^r$;
- $G\cong \mathrm{Sz}_2(2^r), r\ge 3 \textit{ is odd}, |H|=2^2$ and $|G|_2=2^{2r}$;
- $G\cong \mathrm{U}_3(2^r), r\ge 4 \textit{ is even}, |H|=2^3$ and $|G|_2=2^{\frac{3r}{2}}$.

we can deduce that H is not a Hall subgroup of G, contrary to the hypothesis. This is the final contradiction. □

3. Remarks

In this section, we will show the proof of Theorem A under the hypothesis that the Hall subgroup H is solvable without using the classification of finite simple groups. We have to recall some definitions about fusions and focal subgroups, and we refer reader to the book [3,5].

Let $H\le K\le G$ be groups. We say K controls G-fusion in H if every pair of G-conjugate element of H are K-conjugate. Note that if K controls G-fusion in H, then K controls G-fusion in every subgroup of H. Moreover, if H has a normal complement in G, H controls G-fusion in itself.

Lemma 3. *Let a group A act coprimely on a group G. Then $\mathrm{C}_G(A)$ controls G-fusion in itself.*

Proof. Let x, y be two elements of $\mathrm{C}_G(A)$ with $x=y^g$ for some $g\in G$ and we will show x, y are $\mathrm{C}_G(A)$-conjugate. Let $\Gamma=GA$ the semidirect product of G and A. It is easy to see that $A, A^{g^{-1}}\subseteq \mathrm{C}_\Gamma(y)$. Note that $A, A^{g^{-1}}$ are both complements of $\mathrm{C}_G(y)$ in $\mathrm{C}_\Gamma(y)$. As $(|G|,|A|)=1, \mathrm{C}_G(y)$ is a normal Hall subgroup of $C_\Gamma(y)$. Hence the Schur–Zassenhaus theorem implies that $A^h = A^{g^{-1}}$ for some $h \in C_G(y)$. Write $c = hg \in N_G(A)$. As $[A, c] \leq A \cap G = 1$, it implies that $c \in C_G(A)$. Thus $y^c = y^{hg} = y^g = x$, as desired. □

Recall that the focal subgroup of H with respect to G is defined by

$$\begin{aligned}\mathrm{Foc}_G(H) =& \langle [x,g] \in H : x \in H, g \in G\rangle \\ =& \langle x^{-1}y : x,y \in H, x,y \text{ are conjugate in } G\rangle.\end{aligned}$$

Clearly, $\mathrm{Foc}_H(H) = H'$, and if K controls G-fusion in H, then $\mathrm{Foc}_G(H) = \mathrm{Foc}_K(H)$. The following theorem about focal subgroups is well-known.

Theorem 4. *[5, 7.3.1, p. 127] Let H be a Hall π-subgroup of a group G. Then*

$$\mathrm{Foc}_G(H) = H \cap G' = H \cap G'(\pi) \text{ and } H/\mathrm{Foc}_G(H) \cong G/G'(\pi),$$

where $G'(\pi) = G'O^\pi(G)$ is the smallest normal subgroup of G for which the factor group is an abelian π-group.

Theorem 5. *Let a group A act coprimely on a group G and let H be a solvable Hall subgroup of G. Suppose that $H \leq C_G(A)$ and H has a normal complement in $C_G(A)$. Then H has a normal complement in G.*

Proof. Using the same arguments in Theorem A **Step (1)–(4)**, we can reduce to the case that G is a non-abelian simple group. By Lemma 3, $C_G(A)$ controls G-fusion in H. As, by hypothesis, H has a normal complement in $C_G(A)$, it implies that H controls $C_G(A)$-fusion in itself. Hence H controls G-fusion in itself and so $H' = \mathrm{Foc}_G(H)$. By Theorem 4, $H/H' \cong G/G'(\pi)$, where $\pi = \pi(H)$. The solvability of H implies that $H/H' \neq 1$. Thus $G'(\pi) \neq G$, contrary to the simplicity of G. The proof is complete. □

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References

[1] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A., Wilson, R. A. (1985). *Atlas of Finite Groups*. Eynsham: Oxford University Press. Maximal subgroups and ordinary characters for simple groups, with computational assistance from J. G. Thackray.

[2] Gorenstein, D., Lyons, R., Solomon, R. (1998). *The Classification of the Finite Simple Groups, Number 3, Mathematical Surveys and Monographs*, Vol. 40. Providence, RI: American Mathematical Society.

[3] Isaacs, I. M. (2008). *Finite Group Theory, Volume 92 of Graduate Studies in Mathematics*. Providence, RI: American Mathematical Society.

[4] Kızmaz, M. Y. (2019). A sufficient condition for fixed points of a coprime action to have a normal complement. *Arch. Math.* 112(1):1–3.

[5] Kurzweil, H., Stellmacher, B. (2004). *The Theory of Finite Groups. Universitext*. New York: Springer-Verlag. An introduction, translated from the 1998 German original.