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# On finite $p$-groups of supersoluble type

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A finite $p$-group $S$ is said to be of supersoluble type if every fusion system over $S$ is supersoluble. The main aim of this paper is to characterise the finite $p$-groups of supersoluble type. Abelian and metacyclic $p$-groups of supersoluble type are completely described. Furthermore, we show that the Sylow $p$-subgroups of supersoluble type of a finite simple group must be cyclic.



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## 1. Introduction

All groups considered in this paper will be finite.

A saturated fusion system $\mathcal{F}$ over a $p$-group $S$, $p$ a prime, is a category whose objects are the subgroups of $S$, whose morphisms are monomorphisms between subgroups of $S$, and whose morphism sets satisfy certain axioms motivated by properties of conjugacy relations between $p$-subgroups of a group. If $S$ is a Sylow $p$-subgroup of a group $G$, we can associate the saturated fusion system $\mathcal{F}_S(G)$ over $S$, called the *fusion system* of $G$, whose morphisms are those homomorphisms induced by conjugation in $G$. We refer to [1] for a detailed introduction to the theory of saturated fusion systems: this book will be our reference for the notation, terminology and results.

In this paper, we continue the study, started in [14], of supersoluble saturated fusion systems.

**Definition 1** *([14, Definition 1.2]).* Let $S$ be a $p$-group and let $\mathcal{F}$ be a saturated fusion system over $S$. We say that $\mathcal{F}$ is *supersoluble* if there exists a series $1 = S_0 \leq \cdots \leq S_m = S$ of subgroups of $S$ such that $S_i$ is strongly closed in $S$ with respect to $\mathcal{F}$ and $S_{i+1}/S_i$ is cyclic for each $i \in \{0, \ldots, m-1\}$.

It has been shown that supersoluble fusion systems are precisely the fusion systems of supersoluble groups (see [14, Proposition 1.3(b)]).

It is clear that every nilpotent saturated fusion system in the sense of [11] is supersoluble and every supersoluble saturated fusion system is soluble in the sense of [1, Part II, Definition 12.1].

**Definition 2.** A $p$-group $S$ is said to be *of supersoluble type* if for every saturated fusion system $\mathcal{F}$ over $S$, $\mathcal{F}$ is supersoluble.

We characterise the $p$-groups of supersoluble type in the following theorem.

**Theorem A.** *Let $S$ be a $p$-group. Then $S$ is of supersoluble type if and only if $S$ is resistant and every $p'$-subgroup of* $\mathrm{Aut}(S)$ *is abelian of exponent dividing $p-1$.*

We will give several applications of the above characterisation theorem. One of the consequences is the following.

**Corollary 3.** *If $S$ is a $p$-group of supersoluble type, then* $\mathrm{Aut}(S)$ *is soluble.*

Theorem A and Corollary 3 will be proved in the next section.

We then apply this characterisation to describe the abelian and metacyclic $p$-groups of supersoluble type in Theorems B and C. With all these results at hand, we can show that the Sylow $p$-subgroups of supersoluble type of a simple group must be cyclic

(Theorem D), and that the structure of metacyclic Sylow $p$-subgroups of a simple group is quite limited (Theorem 12).

## 2. Proof of Theorem A

Recall that a $p$-group $S$ is called *resistant* if $S$ is normal in every saturated fusion system over $S$ (see [13]).

**Proof of Theorem A.** We prove first the necessity of the condition. Assume that $S$ is of supersoluble type. Let $\mathcal{F}$ be a saturated fusion system over $S$. By [14, Proposition 1.3(b)], there exists a supersoluble group $K$ such that $S$ is a Sylow $p$-subgroup of $K$ and $\mathcal{F} = \mathcal{F}_S(K)$. Without loss of generality, we may assume that $\mathrm{O}_{p'}(K) = 1$. Since $K$ is supersoluble with $\mathrm{O}_{p'}(K) = 1$, we have $S \trianglelefteq K$ and thus $S \trianglelefteq \mathcal{F}_S(K) = \mathcal{F}$. Hence $S$ is resistant.

Let $H$ be a $p'$-subgroup of $\mathrm{Aut}(S)$. We will show that $H$ is abelian of exponent dividing $p-1$. Set $G = S \rtimes H$, the natural semidirect product of $S$ and $H$. Clearly $\mathrm{C}_G(S) \leq S$ since $\mathrm{C}_H(S) = 1$. Write $\mathcal{F} = \mathcal{F}_S(G)$. As $\mathcal{F}$ is a saturated fusion system over $S$, it follows that $\mathcal{F}$ is supersoluble. By [14, Proposition 1.3(d)], $\mathrm{Aut}_{\mathcal{F}}(S)$ is $p$-closed and a Hall $p'$-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$ is abelian of exponent dividing $p-1$. Note that $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_G(S) \cong \mathrm{N}_G(S)/\mathrm{C}_G(S) = HS/\mathrm{C}_G(S)$. Then $H\mathrm{C}_G(S)/\mathrm{C}_G(S)$ is a Hall $p'$-subgroup of $G/\mathrm{C}_G(S)$ and so $H \cong H\mathrm{C}_G(S)/\mathrm{C}_G(S)$ is abelian of exponent dividing $p-1$.

We prove now the sufficiency of the condition. Assume that $S$ is resistant and every $p'$-subgroup of $\mathrm{Aut}(S)$ is abelian of exponent dividing $p-1$. Let $\mathcal{F}$ be a saturated fusion system over $S$. We shall show that $\mathcal{F}$ is supersoluble. As $S$ is resistant, $S \trianglelefteq \mathcal{F}$ and $\mathcal{F}$ is constrained. Since $S \trianglelefteq \mathcal{F}$, it is clear that $\mathcal{F}$ is the fusion system of a finite group $G = S \rtimes H$ for some $p'$-subgroup $H$ of $\mathrm{Aut}(S)$. It then follows from the assumption that $H$ is abelian of exponent dividing $p-1$. Thus $G$ is soluble and every $p$-chief factor of $G$ is cyclic by [5, Chapter B, Theorem 9.8] and every $p'$-chief factor is central. Consequently, $G$ is supersoluble. By [14, Proposition 1.3(b)], $\mathcal{F} = \mathcal{F}_S(G)$ is supersoluble. We conclude then that $S$ is of supersoluble type. $\square$

**Corollary 4.** *Let $S$ be a 2-group. Then $S$ is of supersoluble type if and only if $S$ is resistant and* $\mathrm{Aut}(S)$ *is a 2-group.*

We can then obtain Corollary 3 by combining Theorem A and the following result.

**Theorem 5.** *If every $p'$-subgroup of a group $G$ is abelian of exponent dividing $p-1$, then $G$ is soluble.*

The proof of Theorem 5 requires the following lemma.

**Lemma 6.** *Let $G = \mathrm{PSL}_2(r^f)$, where $r$ is a prime and $f \geq 1$. Set $u = (r^f - 1)/k$ and $s = (r^f + 1)/k$, where $k = (r^f - 1, 2)$. Then $G$ has two cyclic subgroups $U$ and $S$ of orders $u$ and $s$, respectively. Moreover $\mathrm{N}_G(U)$ is dihedral of order $2u$ and $\mathrm{N}_G(S)$ is dihedral of order $2s$.*

**Proof.** It is a consequence of [9, Kapitel II, Satz 8.3 and 8.4]. □

**Proof of Theorem 5.** Let $G$ be a counterexample of minimal order. Then $p$ is the largest prime dividing the order of $G$. By [5, Chapter I, Section 2], $p \neq 2$, and $p \neq 3$. The minimal choice of $G$ implies that every proper subgroup and every nontrivial epimorphic image of $G$ are soluble. Hence $G$ is a minimal simple group.

By a result of Thompson (see [9, Kapitel II, Bemerkung 7.5]), $G$ is isomorphic to one of the following groups:

1. $\mathrm{PSL}_2(q)$, where $q > 3$ is a prime and $5 \nmid q^2 - 1$;
2. $\mathrm{PSL}_2(2^q)$, where $q$ is a prime;
3. $\mathrm{PSL}_2(3^q)$, where $q$ is an odd prime;
4. $\mathrm{PSL}_3(3)$;
5. the Suzuki group $\mathrm{Sz}(2^q)$, where $q$ is an odd prime.

If $G \cong \mathrm{PSL}_3(3)$, then $|G| = 13 \cdot 3^3 \cdot 2^4$ and $p = 13$. Observe that $\mathrm{PSL}_3(3) \cong \mathrm{SL}_3(3)$ has a subgroup isomorphic to $\mathrm{SL}_2(3)$, which is a nonabelian $13'$-group, contrary to assumption. Hence $G$ cannot be isomorphic to $\mathrm{PSL}_3(3)$. If $G \cong \mathrm{Sz}(2^q)$, where $q$ is an odd prime, we can apply [10, Chapter XI, Lemma 3.1(a) and Theorem 3.3] to conclude that $G$ has a nonabelian Sylow 2-subgroup. This contradiction shows that $G$ is not isomorphic to $\mathrm{Sz}(2^q)$ for any odd prime $q$.

Assume that $G = \mathrm{PSL}_2(r^f)$ for some prime $r$ and integer $f$ such that $r^f \geq 4$. Then $|G| = r^f(r^f - 1)(r^f + 1)k^{-1}$, where $k = (2, r^f - 1)$. Observe that $(r^f + 1, r^f - 1) = 1$ or 2. As $p \neq 2$, we can conclude that $r^f - 1$ or $r^f + 1$ is a $p'$-number. Suppose that $r^f = 4$ or 5. By [9, Kapitel II, Satz 6.14], $G \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$ is isomorphic to the alternating group of degree 5 which has a nonabelian $p'$-subgroup isomorphic to the alternating group of degree 4. This contradiction yields $r^f \geq 6$. Then $2(r^f + 1/k) > 2(r^f - 1/k) > 4$; by Lemma 6, $G$ has dihedral $p'$-subgroups. This final contradiction proves the theorem. □

## 3. Abelian and metacyclic $p$-groups of supersoluble type

Our aim in this section is to characterise the abelian and metacyclic $p$-groups of supersoluble type. Such characterizations will be used later in Section 4 to investigate the structure of Sylow $p$-subgroups of supersoluble type of finite simple groups and the structure of metacyclic Sylow $p$-subgroups of finite simple groups.

We need two preliminary lemmas. The first one is elementary.

**Lemma 7.** *Let $P$ be a group isomorphic to $C_{p^n} \times C_{p^n}$ for some positive integer $n$. Then* $\mathrm{Aut}(P)$ *has a quotient isomorphic to* $\mathrm{GL}_2(p)$*.*

The second lemma is a consequence of Theorem A.

**Lemma 8.** *Let $S$ be a resistant $p$-group. Assume that $S$ has a series of characteristic subgroups $\Phi(S) = D_0 \le D_1 \le \cdots \le D_n = S$ such that $D_i/D_{i-1}$ is cyclic for each $0 < i \le n$. Then $S$ is of supersoluble type.*

**Proof.** Without loss of generality, we may assume that $D_i/D_{i-1}$ is of order $p$ for $i = 1, \ldots, n$. Let $H$ be a $p'$-subgroup of $\mathrm{Aut}(S)$, and let $H^*$ be the smallest normal subgroup of $H$ such that $H/H^*$ is an abelian group of exponent dividing $p-1$. We want to show that $H^* = 1$. Let $1 \le i \le n$. Then $H/\mathrm{C}_H(D_i/D_{i-1})$ is isomorphic to a subgroup of $\mathrm{Aut}(D_i/D_{i-1}) \cong \mathrm{Aut}(C_p)$ and so $H/\mathrm{C}_H(D_i/D_{i-1})$ is abelian of exponent dividing $p-1$. Thus $H^* \le \mathrm{C}_H(D_i/D_{i-1})$ for all $i$. Therefore $H^*$ stabilises the chain $S = D_n \ge D_{n-1} \ge \cdots \ge D_0 = \Phi(S)$. By [5, Chapter I, Lemma 1.5], $H^*$ acts trivially on $D = S/\Phi(S)$. It follows from [5, Chapter I, Proposition 1.7] that $H^* = 1$. Consequently, $S$ is of supersoluble type by Theorem A. □

**Theorem B.** *Let $S$ be an abelian $p$-group of type $(m_1, \ldots, m_t)$. Then $S$ is of supersoluble type if and only if $m_1, \ldots, m_t$ are all distinct.*

**Proof.** We can assume, arguing by contradiction, that $S$ is of supersoluble type and $m_1, \ldots, m_t$ are not all distinct. Without loss of generality we may suppose that $m_1 = m_2 = n$. Then $S = P \times H$, where $P, H \le S$ and $P \cong C_{p^n} \times C_{p^n}$. By Lemma 7, $\mathrm{Aut}(P)$ has a quotient isomorphic to $\mathrm{GL}_2(p)$. Observe that $\mathrm{Aut}(P) \times \mathrm{Aut}(H)$ is a subgroup of $\mathrm{Aut}(S)$. Thus $\mathrm{Aut}(S)$ has a section isomorphic to $\mathrm{GL}_2(p)$. Suppose that $p = 2$. Since $\mathrm{GL}_2(2) \cong S_3$, it follows that $\mathrm{Aut}(S)$ is not a 2-group. Hence $p > 2$ by Corollary 4. If $p$ is odd, the Sylow 2-subgroups of $\mathrm{SL}_2(p)$ are nonabelian by [9, Kapitel II, Hauptsatz 8.27]. This contradicts Theorem A. Consequently, $m_1, \ldots, m_t$ are distinct.

Assume that $m_1, \ldots, m_t$ are all distinct and $m_1 < m_2 < \cdots < m_t$. We shall show that $S$ is of supersoluble type. By [1, Part I, Corollary 4.7], $S$ is resistant. Let $D_i = \Omega_i(S)\Phi(S)$, where $\Omega_i(S)$ is the subgroup generated by all elements of $S$ of order dividing $p^i$. Then there exists a positive integer $n$ such that

$$\Phi(S) = D_0 \le D_1 \le \cdots \le D_n = S. \tag{1}$$

Then (1) is a characteristic series of $S$ such that $D_i/D_{i-1}$ is cyclic for each $0 < i \le n$. By Lemma 8, $S$ is of supersoluble type. □

**Theorem C.** *Let $S$ be a metacyclic $p$-group. Then $S$ is of supersoluble type if and only if $S$ is none of the following groups:*

1. *the abelian group* $C_{p^n} \times C_{p^n}$ *for some positive integer* $n$*,*
2. *dihedral, semidihedral or generalised quaternion if* $p = 2$*.*

**Proof.** First assume that $p = 2$ and let $S$ be a metacyclic 2-group. Applying [4, Theorems 1.1] and Corollary 4, we have that $S$ is of supersoluble type if and only if $S$ is none of the groups listed in the statement of the theorem.

Now assume that $p$ is odd. If $S$ is an abelian $p$-group, then by Theorem B, $S$ is of supersoluble type if and only if $S$ is not isomorphic to $C_{p^n} \times C_{p^n}$ for any positive integer $n$.

Suppose that $S$ is a nonabelian metacyclic $p$-group. We prove that $S$ is of supersolvable type. By [13, Proposition 5.4], $S$ is resistant. Since $S'$ is a nontrivial cyclic subgroup of $S$, we can apply [9, Kapitel III, Satz 10.2(c)] to conclude that $S$ is regular.

Assume that the exponent of $S$ is $p^m$. Since $S$ is regular, we can apply [9, Kapitel III, Hauptsatz 10.5(b)] to conclude that

$$\mho_{m-1}(S) = \langle x^{p^{m-1}} : x \in S \rangle = \{x^{p^{m-1}} : x \in S\}.$$

Moreover, by [9, Kapitel III, Satz 10.6], $\mho_{m-1}(S)$ is elementary abelian. Since $\mho_{m-1}(S)$ is metacyclic, we have that $\mho_{m-1}(S) \cong C_p$ or $\mho_{m-1}(S) \cong C_p \times C_p$.

Suppose that $\mho_{m-1}(S) \cong C_p$. By [9, Kapitel III, Satz 10.7 (a)], we have that $|S/\Omega_{m-1}(S)| = |\mho_{m-1}(S)| = p$. Since $S$ is 2-generated, $|S/\Phi(S)| = p^2$. Hence $\Phi(S) \leq \Omega_{m-1}(S) \leq S$ is a characteristic series of $S$ with cyclic factors. Applying Lemma 8, we conclude that $S$ is of supersoluble type.

Suppose that $\mho_{m-1}(S) \cong C_p \times C_p$. Since $S'$ is a nontrivial cyclic subgroup of $S$, we have $\mho_{m-1}(S)$ is not contained in $S'$. Moreover $\mho_{m-1}(S) \cap S' \neq 1$, because otherwise $\mho_{m-1}(S)S' = \mho_{m-1}(S) \times S' \cong (C_p \times C_p) \times C_{p^t}$, $t > 0$, would not be metacyclic. Hence $|\mho_{m-1}(S) \cap S'| = p$.

Let $D$ be the subgroup of $S$ such that $\Omega_{m-1}(S/S') = D/S'$. Clearly $D$ is a characteristic subgroup of $G$, and

$$\begin{aligned}|S : D| &= |S/S' : D/S'| = |S/S' : \Omega_{m-1}(S/S')| = |\mho_{m-1}(S/S')| \\ &= |\mho_{m-1}(S)S'/S'| = |\mho_{m-1}(S) : \mho_{m-1}(S) \cap S'| = p.\end{aligned}$$

It follows that $\Phi(S) \leq D \leq S$ is a characteristic series of $S$ with $|S/D| = |D/\Phi(S)| = p$. By Lemma 8, $S$ is of supersoluble type. □

## 4. Simple groups with Sylow $p$-subgroups of supersoluble type

The aim of this section is to determine the Sylow $p$-subgroups of simple groups that are of supersoluble type. As an application we also determine the metacyclic Sylow $p$-subgroups of simple groups. This is achieved in the last two theorems in the section and requires some preliminary results. The first lemma is well known.

**Lemma 9.** *If $p$ is an odd prime, then the group* $\mathrm{SL}_2(p)$ *has an element of order* $p+1$.

**Lemma 10.** *If $S$ is an extraspecial group of order $p^3$ and exponent $p$, with $p$ an odd prime, then $S$ is not of supersoluble type.*

**Proof.** It is enough to prove the existence of a $p'$-automorphism of $S$ with order not dividing $p-1$. Since every $p'$-automorphism of the elementary abelian quotient of $S$ lifts to $S$, Lemma 9 yields that $\mathrm{Aut}(S)$ has an element of order divisible by $p+1$. Since $p+1$ is not a divisor of $p-1$, the result follows as a consequence of Theorem A. □

**Lemma 11.** *The Sylow $p$-subgroups of $G = \mathrm{PSU}_3(q)$ for $q$ a power of the prime $p$ are not of supersoluble type.*

**Proof.** As in [9, Kapitel II, Satz 10.12], we consider $\mathrm{GU}_3(q)$ as the group of matrices $\mathsf{M} \in \mathrm{GL}_3(q^2)$ such that $\mathsf{M}^{\varphi}\mathsf{J}\mathsf{M} = \mathsf{J}$, where

$$\mathsf{J} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and $\varphi$ acts on each entry of the matrix as the field automorphism $x \longmapsto x^q$. Then the Sylow $p$-subgroups of $\mathrm{PSU}_3(q)$ are isomorphic to the Sylow $p$-subgroups of $\mathrm{GU}_3(q)$, and there is a Sylow $p$-subgroup $U$ of $\mathrm{GU}_3(q)$ composed of matrices of the form

$$\mathsf{M}(c,d) = \begin{bmatrix} 1 & c & d \\ 0 & 1 & -c^q \\ 0 & 0 & 1 \end{bmatrix},$$

where $d \in \mathrm{GF}(q^2)$ and $c \in \mathrm{GF}(q^2)$ satisfy $cc^q = -(d+d^q)$ and multiplication given by $\mathsf{M}(c,d)\mathsf{M}(e,f) = \mathsf{M}(c+e, d+f-ce^q)$. Let $U$ be the set composed of all these matrices with $c$, $d \in \mathrm{GF}(q^2)$ and $cc^q = -(d+d^q)$.

Let $\zeta$ be a generator of the multiplicative group of $\mathrm{GF}(q) \subseteq \mathrm{GF}(q^2)$. Then the matrix

$$\mathsf{D} = \begin{bmatrix} \zeta^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{bmatrix}$$

is an element of $\mathrm{GU}_3(q)$ that induces by conjugation an automorphism $\delta$ of $\mathrm{GU}_3(q)$ such that $\mathsf{D}^{-1}\mathsf{M}(c,d)\mathsf{D} = \mathsf{M}(\zeta c, \zeta^2 d)$, and since $\zeta^q = \zeta$, $(\zeta c)(\zeta c)^q = \zeta^2 cc^q = -\zeta^2(d+d^q) = -(\zeta^2 d + (\zeta^2 d)^q)$ and $\mathsf{M}(\zeta c, \zeta^2 d) \in U$. We note that this automorphism has order $q-1$, because if $\xi$ is a generator of the multiplicative group of $\mathrm{GF}(q^2)$, then $\mathsf{A} = \mathsf{M}(1, -\xi/(\xi + \xi^q)) \in U$ and $\mathsf{A}^{\delta^t} = \mathsf{M}(\zeta^r, -\zeta^{2r}\xi/(\xi+\xi^q))$. Hence $\mathsf{A}^{\delta^t} = \mathsf{A}$ if and only if $q-1 \mid t$, and so the order of $\delta$ is divisible by $q-1$. If $U$ is of supersoluble type, then by Theorem A we have that $q = p$, a prime number.

Suppose that $G = \mathrm{PSU}_3(p)$, with $p$ prime, then $p > 2$ since $\mathrm{PSU}_3(2)$ is soluble. Let $\mathsf{A} = \mathsf{M}(1, -\xi/(\xi+\xi^p))$, $\mathsf{B} = \mathsf{M}(\xi, -\xi^2\xi^p/(\xi+\xi^p))$, where $\xi$ is a generator of $\mathrm{GF}(p^2)^\times$. Since $\xi - \xi^p \neq 0$, these elements do not commute, because $\mathsf{AB} = \mathsf{M}(\xi+1, -\xi-\xi^2\xi^q/(\xi+\xi^q))$ and $\mathsf{BA} = \mathsf{M}(\xi+1, -\xi^q-\xi^2\xi^q/(\xi+\xi^q))$. Moreover, the elements of $U$ have order $p$, since $\mathsf{M}(c,d)^r = \mathsf{M}(rc, rd-(r(r-1)/2)cc^p)$. We conclude that $U$ is an extraspecial group of order $p^3$ and exponent $p$. By Lemma 10, this case is also ruled out. □

**Theorem D.** *Let $S$ be a Sylow $p$-subgroup of a finite simple group $G$. If $S$ is of supersoluble type, then $S$ is cyclic.*

**Proof.** Since $S$ is resistant by Theorem A, we have that $S \trianglelefteq \mathcal{F}_S(G)$. Then $G$ is a $p$-Goldschmidt group (see [1, Part II, Definition 12.9]). According to results of Foote and Flores and Foote ([7,6], see also [1, Part II, Theorem 12.10]), $G$ is a $p$-Goldschmidt group if and only if one of the following conditions holds:

1. $S$ is abelian.
2. $G$ is of Lie type in characteristic $p$ of Lie rank 1.
3. $p = 5$ and $G \cong \mathrm{McL}$.
4. $p = 11$ and $G \cong \mathrm{J}_4$.
5. $p = 3$ and $G \cong \mathrm{J}_2$.
6. $p = 5$ and $G \cong \mathrm{HS}$, $\mathrm{Co}_2$, or $\mathrm{Co}_3$.
7. $p = 3$ and $G \cong \mathrm{G}_2(q)$ for some prime power $q$ prime to 3 such that $q$ is not congruent to $\pm 1$ modulo 9.
8. $p = 3$ and $G \cong \mathrm{J}_3$.

First assume that $S$ is not abelian. In Cases 3–7, according to the Atlas [3], the Sylow $p$-subgroup is extraspecial of order $p^3$ and exponent $p$. These cases are ruled out by Lemma 10. In Case 8, if $p = 3$ and $G = \mathrm{J}_3$, then $|S| = 3^5$ and we can check with GAP [8] that $\mathrm{Aut}(S)$ is a $\{2,3\}$-group whose Sylow 2-subgroup is isomorphic to a semidihedral group $\mathrm{QD}_{16}$ of order 16, therefore the Sylow 3-subgroup of $\mathrm{J}_3$ is not of supersoluble type by Theorem A. Consequently we can assume $G$ is of Lie type in characteristic $p$ and $G$ has Lie rank 1. Since the Sylow $p$-subgroups of $\mathrm{PSL}_2(q)$ for $q = p^f$ are isomorphic to the multiplicative group of the field $\mathrm{GF}(q)$, that is abelian, we have that $G$ is either isomorphic to $\mathrm{PSU}_3(q)$ for $q = p^f$ a prime power, or to a Suzuki group $\mathrm{Sz}(2^{2m+1})$ for $p = 2$, or to a Ree group ${}^2\mathrm{G}_2(3^{2m+1})$ for $p = 3$. By Lemma 11, the Sylow $p$-subgroups of $\mathrm{PSU}_3(q)$ are not of supersoluble type. In the Suzuki and Ree cases, the field automorphism $x \mapsto x^p$ induces an automorphism of the Sylow subgroup $S$ of order $2m+1 \geq 3$, that cannot be a divisor of $p-1$ and thus $S$ is not of supersoluble type by Theorem A.

Therefore we can suppose that $S$ is abelian. Assume that $S$ is of type $(m_1, \ldots, m_t)$. Now, by Theorem B, we know that $m_1, \ldots, m_t$ are all distinct. Moreover, it is shown in

[12] that $S$ is isomorphic to a direct product of copies of a cyclic group. Hence $S$ must be cyclic. This completes the proof of the theorem. $\square$

By combining Theorem C, Theorem D, and [2, Theorem 1], we can determine the structure the metacyclic Sylow $p$-subgroups of finite simple groups.

**Theorem 12.** *Let $S$ be a Sylow $p$-subgroup of a finite simple group $G$. If $S$ is metacyclic, then $S$ is one of the following:*

1. $C_{p^n} \times C_{p^n}$ *for some positive integer* $n$*,*
2. *cyclic if* $p \neq 2$*,*
3. *dihedral or semidihedral if* $p = 2$*.*

**Remark 13.** It is clear that the classes of metacyclic $p$-group listed in Theorem 12 do occur as Sylow $p$-subgroups of some finite simple groups. For instance, $A_7$, the alternating group of degree 7, has dihedral Sylow 2-subgroups, has cyclic Sylow 7-subgroups, and has elementary Sylow 3-subgroups of order 9. And the linear group $\mathrm{PSL}_3(7)$ has semidihedral Sylow 2-subgroups.

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