Journal of Pure and Applied Algebra 224 (2020) 106313

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

On a paper of Beltrán and Shao about coprime action

H. Meng [a,b], A. Ballester-Bolinches [c,d]

[a] *Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China*
[b] *Departament de Matemàtiques, Universitat de València, Burjassot, València, Spain*
[c] *Department of Mathematics, Guangdong University of Education, 510310, Guangzhou, People's Republic of China*
[d] *Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain*

ARTICLE INFO

Article history:
Received 14 August 2019
Received in revised form 27 November 2019
Available online 21 January 2020
Communicated by A. Solotar

MSC:
20D10; 20D25

Keywords:
Finite groups
Coprime action
Solubility
p-nilpotency

ABSTRACT

Assume that A and G are finite groups of coprime orders such that A acts on G via automorphisms. Let p be a prime. The following coprime action version of a well-known theorem of Itô about the structure of a minimal non-p-nilpotent groups is proved: if every maximal A-invariant subgroup of G is p-nilpotent, then G is p-soluble. If, moreover, G is not p-nilpotent, then G must be soluble. Some earlier results about coprime action are consequences of this theorem.

1. Introduction

Throughout this paper, all groups are supposed to be finite and we follow standard notation (e.g. [4]).

Let A and G be groups and assume that A acts on G via automorphisms. If we further assume that A and G are coprime orders, many results about G admit versions in which only the A-invariant structure is taken into account. For instance, Beltrán and Shao ([2, Theorem A]) proved that if every maximal A-invariant subgroup of G is nilpotent but G is not nilpotent, then G is soluble, $|G|$ is divisible by two distinct primes and G has a normal Sylow subgroup.

This result is a coprime action version of an important and classical result of Schmidt ([4, Satz III.5.1, Satz III.5.2]) about the structure of a minimal non-nilpotent group, to which it reduces when $G = \mathrm{C}_G(A)$. If $\mathrm{C}_G(A) < G$, then G is soluble by [1, Theorem B]. We should stress that the proof of that result relies on the classification of simple groups.

E-mail addresses: hangyangmenges@gmail.com (H. Meng), Adolfo.Ballester@uv.es (A. Ballester-Bolinches).

https://doi.org/10.1016/j.jpaa.2020.106313
0022-4049/

In this paper, we also study the situation in where a group G is acted by a group A such that $(|G|, |A|) = 1$. Our concern is with local versions of the Beltrán and Shao's theorem.

Let p be a prime. A group G is said to be *p-nilpotent* if G has a normal Hall p'-subgroup. Clearly, p-nilpotency is a local version of nilpotency since a group G is nilpotent if and only if it is p-nilpotent for all primes p.

According to a result of Itô ([4, Satz IV.5.4]), the minimal non-p-nilpotent groups are just the minimal non-nilpotent groups. This significant result has many applications in the structural study of groups.

Our main result is a coprime action version of Itô's theorem and it is an extension of [2, Theorem A].

Theorem A. *Let p be a prime. Assume that a group A acts coprimely on a group G. If every maximal A-invariant subgroup of G is p-nilpotent, then G is p-soluble.*

We highlight that Theorem A is not a mere exercise in generalisation. On one hand, a local version of [1, Theorem B] does not hold in general: if $G = \mathrm{PSL}_2(2^5)$ and A is the group of field automorphisms of G, then $\mathrm{C}_G(A)$ is isomorphic to the symmetric group of degree 3 so that it is 2-nilpotent. On the other hand, if p is odd, the proof of the p-solubility of G does not depend on the classification of simple groups. This is the reason why we consider the case $p = 2$ separately.

Theorem 1. *Let p be an odd prime. Assume that a group A acts coprimely on a group G. If every maximal A-invariant subgroup of G is p-nilpotent, then G is p-soluble.*

Note that a group G is soluble if and only if G is p-soluble for every odd prime p. Hence we have:

Corollary 2 *(see [2, Theorem A]). Let A and G be groups of coprime orders and assume that A acts on G by automorphisms. If every maximal A-invariant subgroup of G is nilpotent, then G is soluble.*

The proof for the case $p = 2$ depends on the classification of simple groups.

Theorem 3. *Assume that a group A acts coprimely on a group G. If every maximal A-invariant subgroup of G is 2-nilpotent, then G is soluble.*

It is interesting to note that an A-version of Itô's theorem follows from Theorems 1 and 3.

Corollary 4. *Let p be a prime. Assume that a group A acts coprimely on a group G. If every maximal A-invariant subgroup of G is p-nilpotent but G is not p-nilpotent, then G has an A-invariant normal Sylow p-subgroup. Moreover, G is of order $p^a q^b$ for some prime $q \neq p$. In particular, G is soluble.*

Proof. We prove that G has an A-invariant normal Sylow p-subgroup by induction on $|G|$. Let N be an A-invariant normal subgroup of G. Then A acts coprimely on G/N and every maximal A-invariant subgroup of G/N is p-nilpotent. If G/N is not p-nilpotent, then the inductive hypothesis applies to this action.

Assume that $O_{p'}(G) \neq 1$. Since G is not p-nilpotent, it follows that $G/O_{p'}(G)$ is not p-nilpotent. The inductive hypothesis yields that $G/O_{p'}(G)$ has an A-invariant normal Sylow p-subgroup. Moreover, by [6, 8.2.3], G has an A-invariant Sylow p-subgroup, P say. Then $PO_{p'}(G)/O_{p'}(G)$ is a normal A-invariant Sylow p-subgroup of $G/O_{p'}(G)$ and so $PO_{p'}(G) \trianglelefteq G$. The Frattini argument implies that $G = O_{p'}(G)\,\mathrm{N}_G(P)$. Since $\mathrm{N}_G(P)$ is an A-invariant non-p-nilpotent subgroup of G, it follows that $\mathrm{N}_G(P) = G$, as desired.

Assume that $O_{p'}(G) = 1$. Since G is p-soluble by Theorems 1 and 3, we have that $O_p(G) \neq 1$. If $G/O_p(G)$ is not p-nilpotent, then $G/O_p(G)$ has an A-invariant normal Sylow p-subgroup by induction. This implies that G has an A-invariant normal Sylow p-subgroup. Consequently, we may assume that $G/O_p(G)$ is p-nilpotent. Let $T/O_p(G) = O_{p'}(G/O_p(G))$. Then T is A-invariant and G/T is a p-group. If $T < G$, then T

is p-nilpotent and $O_{p'}(T) \leq O_{p'}(G) = 1$. Thus T is a p-group and G is a p-group as well. This case is not possible since G is not p-nilpotent. If $T = G$, then $O_p(G)$ is an A-invariant normal Sylow p-subgroup of G.

Let P be an A-invariant normal Sylow p-subgroup of G. Since G is not p-nilpotent, there exists a prime $q \neq p$ and a Sylow q-subgroup Q of G such that P does not centralise Q. By [6, 8.2.3], we may assume that Q is A-invariant. Then PQ is a non-p-nilpotent A-invariant subgroup of G. Therefore $G = PQ$ and the proof of the corollary is complete. □

Note that [2, Theorem A] is now a direct consequence of Corollaries 2 and 4.

Our last result extends [2, Theorem E]. In order to state it we need to recall a known concept.

Let $\preceq$ be a linear ordering of the prime numbers. A group G is called a *Sylow tower group of complexion* $\preceq$ if there exists a series of normal subgroups of G,

$$1 = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

such that G_i/G_{i-1} is a Sylow p_i-subgroup of G/G_{i-1}, where $p_1 \preceq p_2 \ldots \preceq p_n$ is the ordering induced by $\preceq$ on the distinct prime divisors of $|G|$.

Corollary 5. *Let A and G be groups of coprime orders and assume that A acts on G by automorphisms. If every maximal A-invariant subgroup of G is a Sylow tower group of complexion $\preceq$, then G is soluble.*

Proof. Let p be the greatest element with respect the ordering induced by $\preceq$ on the distinct prime divisors of $|G|$. Then every maximal A-invariant subgroup of G is p-nilpotent. By Corollary 4, G is soluble. □

Note that every supersoluble group is a Sylow tower group of completion $\preceq$, where $\preceq$ is the natural reverse linear order of the prime numbers. Therefore we have:

Corollary 6 *([2, Theorem E]). Let A and G be groups of coprime orders and assume that A acts on G by automorphisms. If every maximal A-invariant subgroup of G is supersoluble, then G is soluble.*

2. Proofs of Theorems 1 and 3

Proof of Theorem 1. We suppose that the theorem is false and derive a contradiction. Let (G, A) be a counterexample with G of minimal order. Then obviously $G \neq 1$. Let N be an A-invariant normal subgroup of G. Assume that $1 < N < G$. Then A acts coprimely on N and G/N and every maximal A-invariant subgroup of N and G/N is p-nilpotent. It follows from the minimal choice of G that N and G/N are both p-soluble. We then conclude that G is p-soluble. Thus we can assume that G is characteristically simple and so $G = G_1 \times ... \times G_n$, where all G_i are isomorphic non-abelian simple groups, $1 \leq i \leq n$. Then $\Omega = \{G_1, ..., G_n\}$ is the set of all minimal normal subgroups of G. Let $a \in A$. Then G_i^a is also a minimal normal subgroup of G for all i and $\Omega = \{G_1^a, ..., G_n^a\}$. Therefore the action of A on G induces a natural action of A on Ω. Furthermore, every A-orbit of Ω produces a non-trivial A-invariant normal subgroup of G. Consequently, the action of A on Ω is transitive.

Let $B = \mathrm{N}_A(G_1)$. Then $|A : B| = |\Omega| = n$ and B acts coprimely on G_1. In addition, G_1 is not p-soluble. Let $a_1 = 1, a_2, ..., a_n$ be a transversal of B in A such that $G_1^{a_i} = G_i$ for each i. Assume that X is a maximal B-invariant subgroup of G_1. Then $\Pi_{i=1}^n X^{a_i}$ is an A-invariant proper subgroup of G containing X which is p-nilpotent. Hence X is p-nilpotent and so the pair (G_1, B) satisfies the hypothesis of the theorem. The minimal choice of G yields $n = 1$, that is, $G = G_1$ is a non-abelian simple group.

By [6, 8.2.3], G has an A-invariant Sylow p-subgroup, say P. Then $P \neq 1$ since G is not p-soluble. Let $\mathrm{J}(P) \neq 1$ be the Thompson subgroup of P. Then $\mathrm{Z}(\mathrm{J}(P)) \neq 1$ is an A-invariant subgroup of G which is not normal in G. Therefore $\mathrm{N}_G(Z(J(P)))$ is a proper A-invariant subgroup of G and so $\mathrm{N}_G(Z(J(P)))$

is p-nilpotent. Applying [4, Satz IV.6.2], we conclude that G is p-nilpotent. This is the desired contradiction. $\square$

The proof of Theorem 3 depends on the following two lemmas. The first one can be deduced from [3, Lemma 2.7 and Corollary 2.8].

Lemma 7. *Let G be a non-abelian simple group and let A be an automorphism group of G with $(|A|, |G|) = 1$. Then*

(a) If G is either alternating or sporadic, then $A = 1$.
(b) If G is a group of Lie type and $\mathrm{C}_G(A)$ is solvable, then A is the full group of field automorphisms. Moreover, one of the following cases holds:
  (1) $G \cong PSL_2(2^n)$, $n \geq 2$ and $\mathrm{C}_G(A) \cong PSL_2(2)$;
  (2) $G \cong PSL_2(3^n)$, $n \geq 2$ and $\mathrm{C}_G(A) \cong PSL_2(3)$;
  (3) $G \cong PSU_3(2^{2n})$, $n \geq 2$, and $\mathrm{C}_G(A) \cong PSU_3(4) \cong (C_3 \times C_3) \rtimes Q_8$;
  (4) $G \cong Sz(2^{2n+1})$, $n \geq 1$, and $\mathrm{C}_G(A) \cong Sz(2)$.

The second lemma follows from [4, II, Theorem 8.2], [4, II, Theorem 10.12] and [5, XI, Lemma 3.1].

Lemma 8. *Let G be a non-abelian simple group isomorphic to $PSL_2(2^n)(n \geq 2), PSU_3(2^{2n})(n \geq 2)$ or $Sz(2^{2n+1})(n \geq 1)$ and let P be a* Sylow *2-subgroup of G. Then $\mathrm{N}_G(P)$ is not 2-nilpotent.*

Proof of Theorem 3. If the theorem is false, we can consider a counterexample (G, A) with $|G|$ as small as possible. Arguing as in Theorem 1, we may assume that G is a non-abelian simple group. If $\mathrm{C}_G(A) = G$, then every proper subgroup of G is 2-nilpotent. It follows from Ito's Theorem that G is soluble. Therefore we may assume that $\mathrm{C}_G(A) < G$. Then $\mathrm{C}_G(A)$ is 2-nilpotent. In particular, $\mathrm{C}_G(A)$ is soluble. It follows from Lemma 7 that G is isomorphic to $PSL_2(2^n)(n \geq 2), PSU_3(2^{2n})(n \geq 2)$ or $Sz(2^{2n+1})(n \geq 1)$. Let $P \neq 1$ is an A-invariant Sylow 2-subgroup of G. Since G is simple, we have that $\mathrm{N}_G(P)$ is an A-invariant proper subgroup of G. This contradicts Lemma 8, and the proof is complete. $\square$

Acknowledgements

The first author is supported by Prometeo/2017/057 of Generalitat Valenciana (Valencian Community, Spain). The second author was supported by the predoctoral grant 201606890006 from the China Scholarship Council.

References

[1] Antonio Beltrán, Actions with nilpotent fixed point subgroup, Arch. Math. (Basel) 69 (3) (1997) 177–184.
[2] Antonio Beltrán, Changguo Shao, Restrictions on maximal invariant subgroups implying solvability of finite groups, Ann. Mat. Pura Appl. (4) 198 (2) (2019) 357–366.
[3] Robert Guralnick, Pavel Shumyatsky, Derived subgroups of fixed points, Isr. J. Math. 126 (2001) 345–362.
[4] B. Huppert, Endliche Gruppen I, Grund. Math. Wiss., vol. 134, Springer Verlag, Berlin, Heidelberg, New York, 1967.
[5] B. Huppert, N. Blackburn, Finite Groups III, Grund. Math. Wiss., vol. 243, Springer-Verlag, Berlin, Heidelberg, New York, 1982.
[6] H. Kurzweil, B. Stellmacher, The Theory of Finite Groups. An Introduction, Universitext, Springer-Verlag, New York, 2004.