

Mean-variance portfolio notes

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$$R_p = \frac{\sum R_i \omega_i W_0}{W_0} = \omega' R$$

$$\mathbf{r}_p = R_p - 1 = \omega' \mathbf{r}$$

where \mathbf{r} is random vector with mean $\bar{\mathbf{r}}$ and variance \mathbf{V} .

hence $E[\mathbf{r}_p] = \omega' \bar{\mathbf{r}}$ and

$$\text{Cov}(\mathbf{r}) = \text{cov}(\omega' \mathbf{r}) = \omega' \text{Cov}(\mathbf{r}) \omega = \omega' \mathbf{V} \omega$$

Then the problem is

$$\min \frac{1}{2} \omega' \mathbf{V} \omega \quad s.t. \quad \mathbf{e}' \omega - \mathbf{1}, \omega' \bar{\mathbf{r}} = \bar{r}_p$$

By Lagrangian

$$L = \frac{1}{2} \omega' \mathbf{V} \omega + \lambda (\bar{r}_p - \omega' \bar{\mathbf{r}}) + \gamma (\mathbf{1} - \omega' \mathbf{e})$$

$$\frac{\partial L}{\partial \omega} = \mathbf{V} \omega - \lambda \bar{\mathbf{r}} - \gamma \mathbf{e} = \mathbf{0} \implies \omega^* = \lambda \mathbf{V}^{-1} \bar{\mathbf{r}} + \gamma \mathbf{V}^{-1} \mathbf{e}$$

Hence

$$\begin{aligned}\bar{r}_p &= \bar{\mathbf{r}}' \boldsymbol{\omega}^* = \lambda \bar{\mathbf{r}}' \mathbf{V}^- \bar{\mathbf{r}} + \gamma \bar{\mathbf{r}}' \mathbf{V}^- \mathbf{e} \\ \mathbf{1} &= \mathbf{e}' \boldsymbol{\omega}^* = \lambda \mathbf{e}' \mathbf{V}^- \bar{\mathbf{r}} + \gamma \mathbf{e}' \mathbf{V}^- \mathbf{e}\end{aligned}$$

denoted $\delta = \mathbf{e}' \mathbf{V}^- \mathbf{e}$, $\alpha = \bar{\mathbf{r}}' \mathbf{V}^- \mathbf{e}$, $\xi = \bar{\mathbf{r}}' \mathbf{V}^- \bar{\mathbf{r}}$ where $\delta, \xi > 0$ since \mathbf{V} is positive definite. Thus we have a linear equations:

$$\begin{bmatrix} \xi & \alpha \\ \alpha & \delta \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} \bar{r}_p \\ 1 \end{bmatrix}$$

Note $\delta\xi - \alpha^2 > 0$ since $(\alpha\bar{\mathbf{r}} - \xi\mathbf{e})' \mathbf{V}^- (\alpha\bar{\mathbf{r}} - \xi\mathbf{e}) = \xi(\delta\xi - \alpha^2) > 0$ and thus such equations is consistent.

solve and get

$$\lambda = \frac{\delta\bar{r}_p - \alpha}{\delta\xi - \alpha^2}, \gamma = \frac{\xi - \alpha\bar{r}_p}{\delta\xi - \alpha^2}$$

and

$$\begin{aligned}\boldsymbol{\omega}^* &= \lambda \mathbf{V}^- \bar{\mathbf{r}} + \gamma \mathbf{V}^- \mathbf{e} \\ &= \frac{\delta\bar{r}_p - \alpha}{\delta\xi - \alpha^2} \mathbf{V}^- \bar{\mathbf{r}} + \frac{\xi - \alpha\bar{r}_p}{\delta\xi - \alpha^2} \mathbf{V}^- \mathbf{e} \\ &= a + b\bar{r}_p\end{aligned}$$

where $a = \frac{\xi\mathbf{V}^- \mathbf{e} - \alpha\mathbf{V}^- \bar{\mathbf{r}}}{\delta\xi - \alpha^2}$ and $b = \frac{-\alpha\mathbf{V}^- \mathbf{e} + \delta\mathbf{V}^- \bar{\mathbf{r}}}{\delta\xi - \alpha^2}$

The minimum variance is given by

$$\sigma_p^2 = \boldsymbol{\omega}'^* \mathbf{V} \boldsymbol{\omega}^* = \boldsymbol{\omega}'^* (\lambda\bar{\mathbf{r}} + \gamma\mathbf{e}) = \lambda\bar{r}_p + \gamma = \frac{1}{\delta} + \frac{\delta(\bar{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}$$

Thus the **global minimum variance**(GMV) portfolio is $\frac{1}{\delta}$ when $\bar{r}_p = \frac{\alpha}{\delta}$.

Meanwhile

$$\begin{aligned}\lambda &= 0 \\ \gamma &= \frac{1}{\delta} \\ \boldsymbol{\omega}_{mv} &= \frac{\mathbf{V}^- \mathbf{e}}{\delta} = \frac{\mathbf{V}^- \mathbf{e}}{\mathbf{e}' \mathbf{V}^- \mathbf{e}}\end{aligned}$$

Geometry

In geometry view

$$\frac{\sigma_p^2}{1/\delta} - \frac{(\bar{r}_p - \frac{\alpha}{\delta})^2}{(\delta\xi - \alpha^2)/\delta^2} = 1$$

it's a hyperbola with center $(0, \alpha/\delta)$ and asymptote $\bar{r}_p = \frac{\alpha}{\delta} \pm \sqrt{\frac{\delta\xi - \alpha^2}{\delta}}\sigma_p$ (recall that asymptote is $y = \pm \frac{b}{a}x$ in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$).

Two Fund Theorem

The mean-variance frontier can be replicated by any two frontier portfolios and any linear combination of frontier portfolios is on the frontier.

Suppose r_1 and r_2 are any given expected return, note

$$\forall r_3, \exists x \ni xr_1 + (1-x)r_2 = r_3$$

Then the weight combined such way is just what we want

$$\omega_3 = x\omega_1 + (1-x)\omega_2 = x(a + br_1) + (1-x)(a + br_2) = a + br_3$$

If $r_1, r_2 \geq \frac{\alpha}{\delta}$, thus they are on efficient frontier, their convex combine still $\geq \frac{\alpha}{\delta}$

$$cr_1 + (1-c)r_2 = \frac{\alpha}{\delta} + c(r_1 - \frac{\alpha}{\delta}) + (1-c)(r_2 - \frac{\alpha}{\delta}) \geq \frac{\alpha}{\delta}$$

that is, any convex combination of efficient frontier portfolios will be an efficient frontier portfolio.

Decomposition

Suppose covariance between portfolios p and q

$$\text{Cov}(\mathbf{r}_p, \mathbf{r}_q) = \omega_p' \mathbf{V} \omega_q$$

Suppose q on the frontier, then

$$\omega_p' \mathbf{V} \omega_q = \omega_p' (\lambda \bar{\mathbf{r}} + \gamma \mathbf{e})$$

Moreover, if

1. p is GMV portfolio, then

$$\omega'_p(\lambda \bar{\mathbf{r}} + \gamma \mathbf{e}) = \omega'_p\left(\frac{1}{\delta} \mathbf{e}\right) = \frac{1}{\delta}$$

2. p has the same expected return with q , then

$$\omega'_p(\lambda \bar{\mathbf{r}} + \gamma \mathbf{e}) = \lambda \bar{r}_q + \gamma = \sigma_q^2$$

Thus we have

The covariance of the return on the global minimum variance (GMV) portfolio and that on any portfolio (not necessary on the frontier) is always equal to $\sigma_{mv}^2 = \frac{1}{\delta}$.

Apply this we get $\text{Cov}(\mathbf{r}_p - \mathbf{r}_{mv}, \mathbf{r}_{mv}) = \text{Cov}(\mathbf{r}_p, \mathbf{r}_{mv}) - \sigma_{mv}^2 = 0$ immediately. We call $\epsilon = \mathbf{r}_p - \mathbf{r}_{mv}$ **excess return**.

The covariance of the return on a frontier portfolio q and that on any portfolio p (not on the frontier) with the same expected return as q is always equal to the variance of the frontier portfolio q . Formally $E[\mathbf{r}_p] = E[\mathbf{r}_q] \implies \text{Cov}(\mathbf{r}_p, \mathbf{r}_q) = \sigma_q^2$.

It implies that we can decompose $\mathbf{r}_p = \mathbf{r}_q + \epsilon_p$ where $E[\epsilon] = \text{Cov}(\epsilon_p, \mathbf{r}_q) = 0$

Rewrite the excess return as $\epsilon = b_p \mathbf{r}^*$, then \mathbf{r}_p can be decomposed into

$$\mathbf{r}_p = \mathbf{r}_{mv} + b_p \mathbf{r}^* + \epsilon_p$$

where \mathbf{r}^* is an excess return and $b_p \in \mathbb{R}$. Note $\text{Cov}(\mathbf{r}_{mv}, \epsilon_p) = \text{Cov}(\mathbf{r}_{mv}, \mathbf{r}_p - \mathbf{r}_q) = 0$ then $\text{Cov}(\mathbf{r}^*, \epsilon_p)$ is also zero. Hence

$$\text{Var}(\mathbf{r}_p) = \underbrace{\sigma_{mv}^2}_{\text{unavoidable risk}} + \underbrace{b_p^2 \text{Var}(\mathbf{r}^*)}_{\text{systematic risk}} + \underbrace{\text{Var}(\epsilon_p)}_{\text{idiot risk}}$$

One can reduce ϵ_p to zero to avoid idiot risk and get a frontier portfolio.

Zero covariance

Recall the covariance between p and q , now suppose they are both on frontier:

$$\omega'_p \mathbf{V} \omega_q = \frac{1}{\delta} + \frac{\delta(\bar{r}_p - \frac{\alpha}{\delta})(\bar{r}_q - \frac{\alpha}{\delta})}{\delta \xi - \alpha^2}$$

Setting this to 0 and solve for \bar{r}_q

$$\bar{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)}$$

Suppose the tangent line in \bar{r}_p , which slope is

$$\frac{\partial \bar{r}_p}{\partial \sigma_p} = \frac{\delta\xi - \alpha^2}{\delta(\bar{r}_p - \frac{\alpha}{\delta})} \sigma_p$$

(Recall $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \implies \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \implies y' = \frac{b^2 x^2}{a^2 y^2}$)

hence its intercept at $\sigma_p = 0$ is

$$\bar{r}_p - \frac{\delta\xi - \alpha^2}{\delta(\bar{r}_p - \frac{\alpha}{\delta})} \sigma_p^2 = \bar{r}_p - \frac{\delta\xi - \alpha^2}{\delta(\bar{r}_p - \frac{\alpha}{\delta})} \left(\frac{1}{\delta} + \frac{\delta(\bar{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2} \right) = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)} = \bar{r}_q$$

Let $\bar{r}_q = 0$, then

$$\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)} = 0$$

solve for \bar{r}_p , we get

$$\bar{r}_p = \frac{\delta\xi - \alpha^2}{\alpha\delta} + \frac{\alpha}{\delta}$$

Substituted in $a + b\bar{r}_p$:

$$\omega_D = -\frac{\mathbf{V}^-\mathbf{e}}{\delta} + \frac{\mathbf{V}^-\bar{\mathbf{r}}}{\alpha} + (a+b)\frac{\alpha}{\delta} = -\frac{\mathbf{V}^-\mathbf{e}}{\delta} + \frac{\mathbf{V}^-\bar{\mathbf{r}}}{\alpha} + \frac{\mathbf{V}^-\mathbf{e}}{\delta} = \frac{\mathbf{V}^-\bar{\mathbf{r}}}{\alpha}$$

Risk-free asset

Suppose we have a riskless asset with return r_f , and we assign ω_0 weight on it. Then the portfolio choice problem becomes

$$\min_{\omega, \omega_0} \frac{1}{2} \omega' \mathbf{V} \omega \quad s.t. \quad \mathbf{e}' \omega + \omega_0 = 1, \omega' \bar{\mathbf{r}} + \omega_0 r_f = \bar{r}_p$$

substitute $\omega_0 = 1 - \mathbf{e}' \omega$, then

$$\omega' \bar{\mathbf{r}} + (1 - \mathbf{e}' \omega) r_f = \bar{r}_p \implies \omega' (\bar{\mathbf{r}} - r_f \mathbf{e}) + r_f = \bar{r}_p$$

The problem is

$$\min_{\omega, \omega_0} \frac{1}{2} \omega' \mathbf{V} \omega \quad s.t. \quad \omega' (\bar{\mathbf{r}} - r_f \mathbf{e}) + r_f = \bar{r}_p$$

Again by the Lagrangian:

$$L = \frac{1}{2} \omega' \mathbf{V} \omega + \lambda (\bar{r}_p - \omega' (\bar{\mathbf{r}} - r_f \mathbf{e}) - r_f)$$

$$\frac{\partial L}{\partial \omega} = \mathbf{V} \omega - \lambda (\bar{\mathbf{r}} - r_f \mathbf{e}) = 0 \implies \omega^* = \lambda \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e})$$

$$\bar{r}_p - \omega^{*'} (\bar{\mathbf{r}} - r_f \mathbf{e}) - r_f = 0 \implies \lambda = \frac{\bar{r}_p - r_f}{(\bar{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e})}$$

$$\sigma_p^2 = \omega^{*'} \mathbf{V} \omega^* = \lambda^2 (\bar{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^{-1} \mathbf{V} \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e}) = \frac{(\bar{r}_p - r_f)^2}{(\bar{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e})}$$

In geometry view, the frontier degenerate into two crossing line:

$$\bar{r}_p = r_f \pm \sqrt{(\bar{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e})} \sigma_p$$

One Fund Theorem

Substitutue λ in the expression of ω^* :

$$\omega^* = \frac{(\bar{r}_p - r_f)}{(\bar{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e})} \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e})$$

We denote $c = \frac{(\bar{r}_p - r_f)}{(\bar{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e})}$ (since it's a scalar) and $\tilde{\omega} = \mathbf{V}^{-1} (\bar{\mathbf{r}} - r_f \mathbf{e})$ then we can write

$$\omega^* = c \tilde{\omega}$$

That is so called one fund theorem

When $r_f \neq r_{mv}$ any minimal-variance frontier portfolio is a combination of the tangency portfolio (with risk assets only) and the riskless asset

Normalized $\tilde{\omega}(\frac{\tilde{\omega}}{\mathbf{e}'\tilde{\omega}})$ is the tangency portfolio, i.e. $\omega_D = \frac{\tilde{\omega}}{\mathbf{e}'\tilde{\omega}}$, the reason is showing below.

Now we prove the degenerated frontier is tangent to the the origin frontier, that is, the hyperbola $\frac{\sigma_p^2}{1/\delta} - \frac{(\bar{r}_p - \frac{\alpha}{\delta})^2}{(\delta\xi - \alpha^2)/\delta^2} = 1$. Assume they do tangent and the tangent point is (σ_p, \bar{r}_p)

Recall the polar of (x_0, y_0) w.r.t. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$ and the slope is

$$\frac{b^2x_0}{a^2y_0} = \sqrt{\frac{b^4x_0^2}{a^4y_0^2}} = \sqrt{\frac{b^2(a^2b^2 + a^2y_0^2)}{a^4y_0^2}}$$

Then since the tangent line through $(0, r_f)$:

$$-\frac{(\bar{r}_p - \frac{\alpha}{\delta})(r_f - \frac{\alpha}{\delta})}{\Delta/\delta^2} = 1$$

solved for \bar{r}_p :

$$\bar{r}_p = \frac{-\alpha^2 + r_f\alpha\delta - \Delta}{\delta(-\alpha + r_f\delta)} = \frac{\xi - r_f\alpha}{\alpha - r_f\delta}$$

the square of slope is

$$\begin{aligned} \frac{\Delta \left(\frac{\Delta}{\delta^3} + \frac{(y_0 - \frac{\alpha}{\delta})^2}{\delta} \right)}{(y_0 - \frac{\alpha}{\delta})^2} &= \frac{\Delta (\alpha^2 + \Delta + \delta^2 y_0^2 - 2\alpha\delta y_0)}{\delta(\alpha - \delta y_0^2)} \\ &= \frac{\Delta \left(\alpha^2 + \Delta - \frac{2\alpha(-\alpha^2 - \Delta + \alpha\delta r_f)}{\delta r_f - \alpha} + \frac{(-\alpha^2 - \Delta + \alpha\delta r_f)^2}{(\delta r_f - \alpha)^2} \right)}{\delta \left(\alpha - \frac{-\alpha^2 - \Delta + \alpha\delta r_f}{\delta r_f - \alpha} \right)^2} \\ &= \frac{\alpha^2 + \Delta + \delta^2 r_f^2 - 2\alpha\delta r_f}{\delta} \\ &= \xi + \delta r_f^2 - 2\alpha r_f \end{aligned}$$

Which is equal to

$$(\bar{\mathbf{r}} - r_f \mathbf{e})' \mathbf{V}^- (\bar{\mathbf{r}} - r_f \mathbf{e}) = \xi + \delta r_f^2 - 2\alpha r_f$$

Hence our assumption is correct. Consider the tangency portfolio:

$$\bar{r}_p = \frac{\xi - r_f\alpha}{\alpha - r_f\delta} = \frac{\Delta/\delta^2}{r_{mv} - r_f} + r_{mv}$$

If $r_f = \frac{\alpha}{\delta} = r_{mv}$, the tangent point doesn't exist and the frontier becomes asymptotes. If $r_f > r_{mv}$, then the tangent point is in the lower straight line and vice versa.

The weight is

$$\omega^* = a + b\bar{r}_p = \frac{\mathbf{V}^-(\bar{\mathbf{r}} - r_f \mathbf{e})}{\alpha - \delta r_f} = \frac{\mathbf{V}^-(\bar{\mathbf{r}} - r_f \mathbf{e})}{\mathbf{e}' \mathbf{V}^-(\bar{\mathbf{r}} - r_f \mathbf{e})} = \frac{\tilde{\omega}}{\mathbf{e}' \tilde{\omega}}$$

That is why we called $\tilde{\omega}$ tangency portfolio.

Sharpe ratio

The sharpe ratio is defined by

$$S_p = \frac{\omega'(\bar{\mathbf{r}} - r_f \mathbf{e})}{\sqrt{\omega' \mathbf{V} \omega}}$$

Which can be interpreted as a measure of **expected excess return per unit of risk**.

To maximize S_p , suppose

$$\frac{\partial S_p}{\partial \omega} = 0$$

Let $\mathbf{r} := \bar{\mathbf{r}} - r_f \mathbf{e}$

$$\phi : w \mapsto \begin{bmatrix} \omega^T \mathbf{r} \\ \omega' \mathbf{V} \omega \end{bmatrix}, \quad h(x, y) := \frac{x}{y^{1/2}}$$

Then $S_p = h \circ \phi(w)$, and thus

$$\begin{aligned} \frac{\partial S_p}{\partial \omega} &= \frac{\partial h}{\partial \phi} \frac{\partial \phi}{\partial \omega} \\ &= \nabla h(\phi(\omega)) \nabla \phi(\omega) \\ &= \begin{bmatrix} \frac{1}{(\omega' \mathbf{V} \omega)^{1/2}} & -\frac{\omega' \mathbf{r}}{2(\omega' \mathbf{V} \omega)^{3/2}} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ 2\mathbf{V} \omega \end{bmatrix} \\ &= \frac{\omega' \mathbf{V} \omega \mathbf{r} - \omega' \mathbf{r} \mathbf{V} \omega}{(\omega' \mathbf{V} \omega)^{3/2}} \end{aligned}$$

Setting to zero,

$$\omega' \mathbf{V} \omega \mathbf{r} - \omega' \mathbf{r} \mathbf{V} \omega = \mathbf{0} \implies \omega = \frac{\omega' \mathbf{V} \omega}{\omega' \mathbf{r}} \mathbf{V}^- \mathbf{r}$$

Note the scale of ω is independent to S_p . If we assume $\mathbf{e}'\omega = 1$ additionally, then

$$\omega = \frac{\mathbf{V}^-\mathbf{r}}{\mathbf{e}'\mathbf{V}^-\mathbf{r}} = \frac{\mathbf{V}^-(\bar{r} - r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^-(\bar{r} - r_f\mathbf{e})} = \frac{\tilde{\omega}}{\mathbf{e}'\tilde{\omega}} = \omega_D$$

ω_D is the only maxima on the frontier without risk-free asset. However, every portfolio on the frontier with a risk-free asset has the maximal sharpe ratio by one fund theorem($\omega^* = c\tilde{\omega}$) if $r_f > r_{mv}$. Otherwise, ω_D is on the lower straight line and become a minima.