Mean-variance portfolio notes

Xie Zejian

Contents

Risky Assets

Suppose $\mathbf{r}$ is a $n \times 1$ random vector with mean $\overline{\mathbf{r}}$ and invertible variance $\mathbf{V}$, then the portofolio return is

$$\mathbf{r_p} = \omega' \mathbf{r}$$

hence $E[\mathbf{r_p}] = \omega' \overline{\mathbf{r}}$ and

$$\mathrm{Cov}(\overline{\mathbf{r}}_\mathbf{p}) = \mathrm{cov}(\omega' \mathbf{r}) = \omega' \mathrm{Cov}(\mathbf{r}) \omega = \omega' \mathbf{V} \omega$$

Then the problem is

$$\min \frac{1}{2} \omega' \mathbf{V} \omega \quad s.t. \quad \mathbf{e}' \omega - \mathbf{1}, \omega' \overline{\mathbf{r}} = \overline{r}_p$$

By Lagrangian

$$L = \frac{1}{2}\omega'\mathbf{V}\omega + \lambda(\overline{r}_p - \omega'\overline{\mathbf{r}}) + \gamma(\mathbf{1} - \omega'\mathbf{e})$$

$$\frac{\partial L}{\partial \omega} = \mathbf{V}\omega - \lambda\overline{\mathbf{r}} - \gamma\mathbf{e} = \mathbf{0} \implies \omega^* = \lambda\mathbf{V}^-\overline{\mathbf{r}} + \gamma\mathbf{V}^-\mathbf{e}$$

Hence

$$\begin{aligned}\overline{r}_p &= \overline{\mathbf{r}}'\omega^* = \lambda\overline{\mathbf{r}}'\mathbf{V}^-\overline{\mathbf{r}} + \gamma\overline{\mathbf{r}}'\mathbf{V}^-\mathbf{e} \\ \mathbf{1} &= \mathbf{e}'\omega^* = \lambda\mathbf{e}'\mathbf{V}^-\overline{\mathbf{r}} + \gamma\mathbf{e}'\mathbf{V}^-\mathbf{e}\end{aligned}$$

denoted $\delta = \mathbf{e}'\mathbf{V}^-\mathbf{e}, \alpha = \overline{\mathbf{r}}'\mathbf{V}^-\mathbf{e}, \xi = \overline{\mathbf{r}}'\mathbf{V}^-\overline{\mathbf{r}}$ where $\delta, \xi > 0$ since $\mathbf{V}$ is positive define. Thus we have a linear equations:

$$\begin{bmatrix} \xi & \alpha \\ \alpha & \delta \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} \overline{r}_p \\ 1 \end{bmatrix}$$

Note $\Delta = \delta\xi - \alpha^2 > 0$ since $(\alpha\overline{\mathbf{r}} - \xi\mathbf{e})'\mathbf{V}^-(\alpha\overline{\mathbf{r}} - \xi\mathbf{e}) = \xi(\delta\xi - \alpha^2) > 0$ and thus such equations is consistent.

solve ($\mathbf{A}\mathbf{x} = \mathbf{b} \implies \mathbf{x} = \mathbf{A}^-\mathbf{b}$):

$$\lambda = \frac{\delta\overline{r}_p - \alpha}{\delta\xi - \alpha^2}, \gamma = \frac{\xi - \alpha\overline{r}_p}{\delta\xi - \alpha^2}$$

and

$$\begin{aligned}\omega^* &= \lambda\mathbf{V}^-\overline{\mathbf{r}} + \gamma\mathbf{V}^-\mathbf{e} \\ &= \frac{\delta\overline{r}_p - \alpha}{\delta\xi - \alpha^2}\mathbf{V}^-\overline{\mathbf{r}} + \frac{\xi - \alpha\overline{r}_p}{\delta\xi - \alpha^2}\mathbf{V}^-\mathbf{e} \\ &= a + b\overline{r}_p\end{aligned}$$

where $a = \frac{\xi\mathbf{V}^-\mathbf{e} - \alpha\mathbf{V}^-\overline{\mathbf{r}}}{\delta\xi - \alpha^2}$ and $b = \frac{-\alpha\mathbf{V}^-\mathbf{e} + \delta\mathbf{V}^-\overline{\mathbf{r}}}{\delta\xi - \alpha^2}$

The minimum variance is given by

$$\sigma_p^2 = \omega'^*\mathbf{V}\omega^* = \omega'^*(\lambda\overline{\mathbf{r}} + \gamma\mathbf{e}) = \lambda\overline{r}_p + \gamma$$

by some algebra

$$\lambda\overline{r}_p + \gamma = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}$$

Thus the **global minimum variance**(GMV) portfolio is $\frac{1}{\delta}$ when $\overline{r}_p = \frac{\alpha}{\delta}$, meanwhile

$$\begin{aligned}
\lambda &= \frac{\delta \overline{r}_p - \alpha}{\delta\xi - \alpha^2} = 0 \\
\gamma &= \frac{\xi - \alpha \overline{r}_p}{\delta\xi - \alpha^2} = \frac{\xi - \alpha\frac{\alpha}{\delta}}{\delta\xi - \alpha^2} = \frac{1}{\delta} \\
\omega_{mv} &= 0 + \frac{\mathbf{V}^{-}\mathbf{e}}{\delta} = \frac{\mathbf{V}^{-}\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-}\mathbf{e}}
\end{aligned}$$

Geometry

In geometry view, rewrite $\sigma_p^2 = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}$ as

$$\frac{\sigma_p^2}{1/\delta} - \frac{(\overline{r}_p - \frac{\alpha}{\delta})^2}{(\delta\xi - \alpha^2)/\delta^2} = 1$$

it's a hyperbola with center $(0, \alpha/\delta)$ and asymptote $\overline{r}_p = \frac{\alpha}{\delta} \pm \sqrt{\frac{\delta\xi - \alpha^2}{\delta}}\sigma_p$(recall that asymptote is $y = \pm\frac{b}{a}x$ in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$).

Two Fund Theorem

The mean-variance frontier can be replicated by any two frontier portfolios and any linear combination of frontier portfolios is on the frontier.

Suppose r_1 and r_2 are any given expected return, note

$$\forall r_3, \exists x \ni x r_1 + (1-x) r_2 = r_3$$

Then the weight combined such way is just what we want.

$$\omega_3 = x\omega_1 + (1-x)\omega_2 = x(a + br_1) + (1-x)(a + br_2) = a + br_3$$

> Any convex combination of efficient frontier portfolios will be an efficient frontier portfolio.

Proof A portofolio ω is on the efficient froniter iff(if and only if) it's return $\overline{r}_p \geq r_{mv}$ and recall $\omega = a + b\overline{r}_p$, thus there is a bijection between ω_i and r_i. Suppose the return of ω_i is r_i,

$$\omega = \begin{bmatrix} \omega_1 & \cdots & \omega_n \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Firstly, verify the convex combination is a legal portfolio:

$$\mathbf{e}'(\omega\mathbf{c}) = (\mathbf{e}'\omega)\mathbf{c} = \mathbf{e}'\mathbf{c} = 1$$

since $\forall\omega_i \mathbf{e}'\omega_{\mathbf{i}} = 1, \mathbf{e}'\mathbf{c} = 1$. Then it's sufficient to show that $\mathbf{c}'\mathbf{r} \geq r_{mv}$. It's clearly since

$$\mathbf{c}'\mathbf{r} = \sum c_i\omega_i \geq \sum c_i \min(r_i) = \min(r_i) \sum c_i = \min(r_i) \geq r_{mv}$$

Decomposition

Suppose covariance between portfolios p and q

$$\text{Cov}(\mathbf{r_p}, \mathbf{r_q}) = \omega_p' \mathbf{V} \omega_q$$

Suppose q on the frontier, then

$$\omega_p' \mathbf{V} \omega_q = \omega_p'(\lambda\bar{\mathbf{r}} + \gamma\mathbf{e})$$

Moreover, if

1. p is GMV portfolio, then

$$\omega_p'(\lambda\bar{\mathbf{r}} + \gamma\mathbf{e}) = \omega_p'(\frac{1}{\delta}\mathbf{e}) = \frac{1}{\delta}$$

2. p has the same expected return with q, then

$$\omega_p'(\lambda\bar{\mathbf{r}} + \gamma\mathbf{e}) = \lambda\bar{r}_q + \gamma = \sigma_q^2$$

Thus we have

> The covariance of the return on the global minimum variance (GMV) portfolio and that on any portfolio (not necessary on the frontier) is always equal to $\sigma_{mv}^2 = \frac{1}{\delta}$.

Apply this we get $\text{Cov}(\mathbf{r_p} - \mathbf{r_{mv}}, \mathbf{r_{mv}}) = \text{Cov}(\mathbf{r_p}, \mathbf{r_{mv}}) - \sigma_{mv}^2 = 0$ immediately. We call $\epsilon = \mathbf{r_p} - \mathbf{r_{mv}}$ **excess return**.

> The covariance of the return on a frontier portfolio q and that on any portfolio p (not on the frontier) with the same expected return as q is always equal to the variance of the frontier portfolio q. Formally $E[\mathbf{r_p}] = E[\mathbf{r_q}] \implies \text{Cov}(\mathbf{r_p}, \mathbf{r_q}) = \sigma_q^2$.

It implies that we can decompose $\mathbf{r_p} = \mathbf{r_q} + \epsilon_\mathbf{p}$ where $E[\epsilon] = \mathrm{Cov}(\epsilon_p, \mathbf{r_q}) = 0$

Rewrite the excess return as $\epsilon = b_p \mathbf{r}^*$, then $\mathbf{r_p}$ can be decomposed into

$$\mathbf{r_p} = \mathbf{r_{mv}} + b_p \mathbf{r}^* + \epsilon_p$$

where $\mathbf{r}^*$ is an excess return and $b_p \in \mathbb{R}$.Note $\mathrm{Cov}(\mathbf{r_{mv}}, \epsilon_\mathbf{p}) = \mathrm{Cov}(\mathbf{r_{mv}}, \mathbf{r_p} - \mathbf{r_q}) = 0$ then $\mathrm{Cov}(\mathbf{r}^*, \epsilon_p)$ is also zero. Hence

$$\mathrm{Var}(\mathbf{r_p}) = \overbrace{\underbrace{\sigma_{mv}^2}_{\text{unavoidable risk}} + b_p^2 \mathrm{Var}(\mathbf{r}^*)}^{\text{systematic risk}} + \underbrace{\mathrm{Var}(\epsilon_p)}_{\text{idiot risk}}$$

One can reduce ϵ_p to zero to avoid idiot risk and get a frontier portfolio.

Zero covariance

Continue the discussion of the covariance between p and q, now suppose they are both on frontier:

$$\omega_p' \mathbf{V} \omega_q = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})(\overline{r}_q - \frac{\alpha}{\delta})}{\delta\xi - \alpha^2}$$

Setting this to 0 and slove for $\overline{r}_q$

$$\overline{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)}$$

Then we are ready to show that

$\overline{r}_q$ is equal to the intercept of the tangent line to MVF at $(\overline{r}_p, \sigma_p)$

Suppose the tagent line in $\overline{r}_p$, the slope is

$$\frac{\partial \overline{r}_p}{\partial \sigma_p} = \frac{\delta\xi - \alpha^2}{\delta(\overline{r}_p - \frac{\alpha}{\delta})} \sigma_p$$

(Recall $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \implies \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \implies y' = \frac{b^2 x^2}{a^2 y^2}$)
thus its intercept at $\sigma_p = 0$ is

$$\overline{r}_p - \frac{\delta\xi - \alpha^2}{\delta(\overline{r}_p - \frac{\alpha}{\delta})}\sigma_p^2 = \overline{r}_p - \frac{\delta\xi - \alpha^2}{\delta(\overline{r}_p - \frac{\alpha}{\delta})}(\frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}) = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)} = \overline{r}_q$$

Let $\overline{r}_q = 0$, then

$$\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)} = 0$$

solve for $\overline{r}_p$, we get

$$\overline{r}_p = \frac{\delta\xi - \alpha^2}{\alpha\delta} + \frac{\alpha}{\delta}$$

Substituted in $a + b\overline{r}_p$:

$$\omega_D = -\frac{\mathbf{V}^-\mathbf{e}}{\delta} + \frac{\mathbf{V}^-\overline{\mathbf{r}}}{\alpha} + (a+b)\frac{\alpha}{\delta} = -\frac{\mathbf{V}^-\mathbf{e}}{\delta} + \frac{\mathbf{V}^-\overline{\mathbf{r}}}{\alpha} + \frac{\mathbf{V}^-\mathbf{e}}{\delta} = \frac{\mathbf{V}^-\overline{\mathbf{r}}}{\alpha}$$

That is the tangency portofolio. If $\overline{r}_q > 0$, $\overline{r}_q$ can be interpreted as risk-free asset return in next chapter

Risk-free asset

Suppose we have a riskless asset with return r_f, and we assign ω_0 weight on it. Then the portfolio choice problem becomes

$$\min_{\omega,\omega_0} \frac{1}{2}\omega'\mathbf{V}\omega \quad s.t. \quad \mathbf{e}'\omega + \omega_0 = 1, \omega'\overline{\mathbf{r}} + \omega_0 r_f = \overline{r}_p$$

substitute $\omega_0 = 1 - \mathbf{e}'\omega$, then

$$\omega'\overline{\mathbf{r}} + (1 - \mathbf{e}'\omega)r_f = \overline{r}_p \implies \omega'(\overline{\mathbf{r}} - r_f\mathbf{e}) + r_f = \overline{r}_p$$

The problem is

$$\min_{\omega,\omega_0} \frac{1}{2}\omega'\mathbf{V}\omega \quad s.t. \quad \omega'(\overline{\mathbf{r}} - r_f\mathbf{e}) + r_f = \overline{r}_p$$

Again by the Lagrangian:

$$L = \frac{1}{2}\omega'\mathbf{V}\omega + \lambda(\overline{r}_p - \omega'(\overline{\mathbf{r}} - r_f\mathbf{e}) - r_f)$$

$$\frac{\partial L}{\partial \omega} = \mathbf{V}\omega - \lambda(\overline{\mathbf{r}} - r_f\mathbf{e}) = 0 \implies \omega^* = \lambda\mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e})$$

$$\overline{r}_p - {\omega^*}'(\overline{\mathbf{r}} - r_f\mathbf{e}) - r_f = 0 \implies \lambda = \frac{\overline{r}_p - r_f}{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e})}$$

$$\sigma_p^2 = \omega'\mathbf{V}\omega = \lambda^2(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^-\mathbf{V}\mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e}) = \frac{(\overline{r}_p - r_f)^2}{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e})}$$

In geometry view, the frontier degenerate into two crossing line:

$$\overline{r}_p = r_f \pm \sqrt{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e})}\sigma_p$$

One Fund Theorem

Substitue λ in the expression of ω^*:

$$\omega^* = \frac{(\overline{r}_p - r_f)}{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e})}\mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e})$$

We denote $c = \frac{(\overline{r}_p - r_f)}{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e})}$(since it's a scalar) and $\tilde{\omega} = \mathbf{V}^-(\overline{\mathbf{r}} - r_f\mathbf{e})$ then we can write

$$\omega^* = c\tilde{\omega}$$

That is so called one fund theorem

> When $r_f \neq r_{mv}$ any minimal-variance frontier portfolio is a combination of the tangency portfolio (with risk assets only) and the riskless asset

Normalized $\tilde{\omega}(\frac{\tilde{\omega}}{\mathbf{e}'\tilde{\omega}})$ is the tangecy portfolio, i.e. $\omega_D = \frac{\tilde{\omega}}{\mathbf{e}'\tilde{\omega}}$, the reason is showing below.

Now we prove the degenerated frontier is tangent to the the origin frontier, that is, the hyperbola $\frac{\sigma_p^2}{1/\delta} - \frac{(\overline{r}_p - \frac{\alpha}{\delta})^2}{(\delta\xi - \alpha^2)/\delta^2} = 1$. Assume they do tangent and the tangent point is $(\sigma_p, \overline{r}_p)$

Recall the polar of (x_0, y_0) w.r.t. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$ and the slope is

$$\frac{b^2 x_0}{a^2 y_0} = \sqrt{\frac{b^4 x_0^2}{a^4 y_0^2}} = \sqrt{\frac{b^2(a^2b^2 + a^2y_0^2)}{a^4 y_0^2}}$$

Then since the tangent line through $(0, r_f)$:

$$-\frac{(\overline{r}_p - \frac{\alpha}{\delta})(r_f - \frac{\alpha}{\delta})}{\Delta/\delta^2} = 1$$

solved for $\overline{r}_p$:

$$\overline{r}_p = \frac{-\alpha^2 + r_f\alpha\delta - \Delta}{\delta(-\alpha + r_f\delta)} = \frac{\xi - r_f\alpha}{\alpha - r_f\delta}$$

the square of slope is

$$\begin{aligned}
\frac{\Delta\left(\frac{\Delta}{\delta^3} + \frac{\left(y_0 - \frac{\alpha}{\delta}\right)^2}{\delta}\right)}{\left(y_0 - \frac{\alpha}{\delta}\right)^2} &= \frac{\Delta\left(\alpha^2 + \Delta + \delta^2 y_0^2 - 2\alpha\delta y_0\right)}{\delta(\alpha - \delta y_0^2}\\
&= \frac{\Delta\left(\alpha^2 + \Delta - \frac{2\alpha\left(-\alpha^2 - \Delta + \alpha\delta r_f\right)}{\delta r_f - \alpha} + \frac{\left(-\alpha^2 - \Delta + \alpha\delta r_f\right)^2}{(\delta r_f - \alpha)^2}\right)}{\delta\left(\alpha - \frac{-\alpha^2 - \Delta + \alpha\delta r_f}{\delta r_f - \alpha}\right)^2}\\
&= \frac{\alpha^2 + \Delta + \delta^2 r_f^2 - 2\alpha\delta r_f}{\delta}\\
&= \xi + \delta r_f^2 - 2\alpha r_f
\end{aligned}$$

Which is equal to

$$(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e}) = \xi + \delta r_f^2 - 2\alpha r_f$$

Hence our assumption is correct. Consider the tangency portfolio:

$$\overline{r}_p = \frac{\xi - r_f\alpha}{\alpha - r_f\delta} = \frac{\Delta/\delta^2}{r_{mv} - r_f} + r_{mv}$$

If $r_f = \frac{\alpha}{\delta} = r_{mv}$, the tangency doesn't exist and the frontier becomes asymptotes. If $r_f > r_{mv}$, the tangency is in the lower straight line and vice versa.

The weight is

$$\omega^* = a + b\overline{r}_p = \frac{\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})}{\alpha - \delta r_f} = \frac{\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})} = \frac{\tilde{\omega}}{\mathbf{e}'\tilde{\omega}}$$

That is why we called $\tilde{\omega}$ tangency portfolio. Recall the result in zero covariance, for any portofolio $\overline{r}_p > r_{mv}$, we can find $r_f = \overline{r}_q$ with zero covariance with $\overline{r}_p$ to make $\overline{r}_p$ be a tangency portfolio.

Sharpe ratio

The shrpe ratio is defined by

$$S_p = \frac{\overline{r}_p - r_f}{\sigma_p} = \frac{\omega'(\overline{\mathbf{r}} - r_f\mathbf{e})}{\sqrt{\omega'\mathbf{V}\omega}}$$

Which can be interpreted as a measure of **expected excess return per unit of risk**.

To maximize S_p, suppose

$$\frac{\partial S_p}{\partial \omega} = 0$$

Let $\mathbf{r} = \overline{\mathbf{r}} - r_f\mathbf{e}$

$$\phi : w \mapsto \begin{bmatrix} \omega'\mathbf{r} \\ \omega'\mathbf{V}\omega \end{bmatrix}, \quad h(x,y) := \frac{x}{y^{1/2}}$$

Then $S_p = h \circ \phi(w)$, and thus

$$\begin{aligned} \frac{\partial S_p}{\partial \omega} &= \frac{\partial h}{\partial \phi}\frac{\partial \phi}{\partial \omega} \\ &= \begin{bmatrix} \frac{1}{(\omega'\mathbf{V}\omega)^{1/2}} & -\frac{\omega'\mathbf{r}}{2(\omega'\mathbf{V}\omega)^{3/2}} \end{bmatrix} \begin{bmatrix} \mathbf{r}' \\ 2\omega'\mathbf{V} \end{bmatrix} \\ &= \frac{\omega'\mathbf{V}\omega\mathbf{r}' - \omega'\mathbf{r}\omega'\mathbf{V}}{(\omega'\mathbf{V}\omega)^{3/2}} \end{aligned}$$

Setting to zero,

$$\omega'\mathbf{V}\omega\mathbf{r} - \omega'\mathbf{r}\mathbf{V}\omega = \mathbf{0} \implies \omega = \frac{\omega'\mathbf{V}\omega}{\omega'\mathbf{r}}\mathbf{V}^{-}\mathbf{r}$$

Note the scale of ω is independent to S_p. If we assume $\mathbf{e}'\omega = 1$ additionally, then

$$\omega = \frac{\mathbf{V}^{-}\mathbf{r}}{\mathbf{e}'\mathbf{V}^{-}\mathbf{r}} = \frac{\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})} = \frac{\tilde{\omega}}{\mathbf{e}'\tilde{\omega}} = \omega_D$$

Remark

1. The maximun sharpe ration is the slope of frontier $\sqrt{(\overline{\mathbf{r}} - r_f\mathbf{e})'\mathbf{V}^{-}(\overline{\mathbf{r}} - r_f\mathbf{e})}$.
2. ω_D is the only maxima on the frontier without risk-free asset. However, every portfolio on the frontier with a risk-free asset has the maximal sharpe ratio by one fund theorem($\omega^* = c\tilde{\omega}$) if $r_f > r_{mv}$. (Otherwise ω_D is on the lower straight line and become a minima).

Indifference curve

If the utility function of investor is negative exponential, then the optimal portfolio is still tangency portfolio. Suppose its utility is

$$U(W) = -e^{-bW}$$

and its initial wealth is 1. Then

$$W = r_p = (1 - \mathbf{e}'\omega)r_f + \omega'\mathbf{r} = r_f + \omega'\mathbf{r}$$

To maximize its utility expection

$$E[U(W)] = E[U(r_p)] = E[-e^{-b(r_f+\omega'\mathbf{r})}]$$

where $\mathbf{r} = \tilde{\mathbf{r}} - r_f\mathbf{e} \sim N(\bar{\mathbf{r}} - r_f\mathbf{e}, \mathbf{V})$. It's sufficent to maximize

$$E[e^{(-b\omega)'\mathbf{r}}] = \exp\{(-b\omega)'E(\mathbf{r}) + b^2\omega'\mathbf{V}\omega/2\}$$

then

$$\frac{\partial(-b\omega)'E(\mathbf{r}) + b^2\omega'\mathbf{V}\omega/2}{\partial\omega} = -bE(\mathbf{r}) + b^2\mathbf{V}\omega = 0$$

hence

$$\omega = \frac{\mathbf{V}^- E(\mathbf{r})}{b} = \frac{\mathbf{V}^-(\bar{\mathbf{r}} - r_f\mathbf{e})}{b}$$

CAPM

Beta representation

Recall the tangency portfolio is $\omega_D = \frac{\mathbf{V}^-(\bar{\mathbf{r}}-r_f\mathbf{e})}{\mathbf{e}'\mathbf{V}^-(\bar{\mathbf{r}}-r_f\mathbf{e})}$. Write $\omega_D = m\mathbf{V}^-(\bar{\mathbf{r}} - r_f\mathbf{e})$ where $m = \frac{1}{\mathbf{e}'\mathbf{V}^-(\bar{\mathbf{r}}-r_f\mathbf{e})}$, then we have

$$\bar{r} - r_f\mathbf{e} = \frac{1}{m}\mathbf{V}\omega_D$$

Note $\mathrm{Cov}(\mathbf{r}, \omega'\mathbf{r}) = \mathbf{V}\omega$ and

$$\sigma_D^2 = \omega_D'\mathbf{V}\omega_D = m\omega_D'(\bar{\mathbf{r}} - r_f\mathbf{e}) = mr_D - mr_f$$

we have

$$\overline{\mathbf{r}} - r_f\mathbf{e} = \frac{r_D - r_f}{\sigma_D^2}\mathrm{Cov}(\mathbf{r}, r_D)$$

Denote $\frac{\mathrm{Cov}(\mathbf{r},r_D)}{\sigma_D^2} = \beta_D$, we have

$$\overline{\mathbf{r}} - r_f\mathbf{e} = \beta_D(r_D - r_f)$$

Similar results also holds for any portfolio $\overline{r}_p$ in the MVF:

$$\overline{\mathbf{r}} - \overline{r}_q\mathbf{e} = \beta_p(\overline{r}_p - \overline{r}_q)$$

It's clear in the view of every portfolio $\overline{r}_p$ is also a tangency portfolio by selecting proper r_f. One can also check it in a dirty way:

Proof Suppose r_p and r_q both in the MVF without risk-free asset, recall

$$\omega_p'\mathbf{V}\omega_q = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})(\overline{r}_q - \frac{\alpha}{\delta})}{\delta\xi - \alpha^2}$$

If the covariance is 0, we have

$$\overline{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)}$$

Then

$$\begin{aligned}
\overline{\mathbf{r}} - \overline{r}_q\mathbf{e} &= \overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e} \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\delta^2(\overline{r}_p - \alpha/\delta))(\overline{\mathbf{r}} - (\frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)})\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta(\overline{r}_p - \alpha/\delta) - (\delta\xi - \alpha^2))\mathbf{e}) \\
&= \frac{1}{\delta^2(\overline{r}_p - \alpha/\delta)}(\overline{\mathbf{r}}(\delta^2(\overline{r}_p - \alpha/\delta)) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e} \\
&= \frac{(\delta^2\overline{r}_p\overline{\mathbf{r}} - \alpha\delta\overline{\mathbf{r}}) - (\alpha\delta\overline{r}_p - \delta\xi)\mathbf{e}}{\delta^2(\overline{r}_p - \alpha/\delta)} \\
&= \frac{(\delta\overline{r}_p - \alpha)\overline{\mathbf{r}} - (\alpha\overline{r}_p - \xi)\mathbf{e}}{\delta(\overline{r}_p - \alpha/\delta)}
\end{aligned}$$

On the other hand:

$$
\begin{aligned}
\beta_p &= \frac{\mathbf{V}\omega_{\mathbf{p}}}{\omega_p'\mathbf{V}\omega_p} \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\lambda_p\bar{\mathbf{r}} + \gamma\mathbf{e}) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{\xi\mathbf{e} - \alpha\bar{\mathbf{r}}}{\delta\xi - \alpha^2} + \frac{-\alpha\mathbf{e} + \delta\bar{\mathbf{r}}}{\delta\xi - \alpha^2}\bar{r}_p) \\
&= \frac{1}{\omega_p'\mathbf{V}\omega_p}(\frac{(\delta\bar{r}_p - \alpha)\bar{\mathbf{r}} - (\alpha\bar{r}_p - \xi)\mathbf{e}}{\Delta})
\end{aligned}
$$

Then it's remain to show that

$$(\bar{r}_p - \bar{r}_q)\delta(\bar{r}_p - \alpha/\delta) = \omega'\mathbf{V}\omega\Delta$$

It's clear since

$$\omega'\mathbf{V}\omega\Delta = \sigma_p^2\Delta = \frac{\Delta}{\delta} + \delta(\bar{r}_p - \frac{\alpha}{\delta})^2$$

and

$$
\begin{aligned}
(\bar{r}_p - \bar{r}_q)\delta(\bar{r}_p - \alpha/\delta) &= ((\bar{r}_p - \frac{\alpha}{\delta}) + \frac{\delta\xi - \alpha^2}{\delta^2(\bar{r}_p - \alpha/\delta)})\delta(\bar{r}_p - \alpha/\delta) \\
&= \frac{\Delta}{\delta} + \delta(\bar{r}_p - \frac{\alpha}{\delta})^2. \blacksquare
\end{aligned}
$$

CAPM

In capital market equilibrium, the market portfolio is tangecy portfolio $r_D = r_m$, then

$$\bar{\mathbf{r}} - r_f\mathbf{e} = \beta_m(r_m - r_f)$$

where

$$\beta_m = \begin{bmatrix} \frac{\mathrm{Cov}(r_1, r_m)}{\sigma_m^2} \\ \frac{\mathrm{Cov}(r_2, r_m)}{\sigma_m^2} \\ \cdots \\ \frac{\mathrm{Cov}(r_n, r_m)}{\sigma_m^2} \end{bmatrix}$$

this equation is called **Sharpe-Lintner CAPM**. $r_m - r_f$ is called **market risk premium** and $\frac{r_m - r_f}{\sigma_m}$ is called **market sharpe ration**. Translate it from vector form, we get the **Security Market Line**:

$$r_i - r_f = \frac{\mathrm{Cov}(r_i, r_m)}{\sigma_m^2}(r_m - r_f) = \beta_{i,m}(r_m - r_f)$$

Variance decomposition

Now consider both r_i and r_m is random variable, let ϵ be a random vector with zero expection and zero covariance with r_i and r_m, then

$$\mathbf{r} - r_f\mathbf{e} = (r_m - r_f)\beta_m + \epsilon$$

thus

$$\begin{aligned}
\mathrm{Var}(\mathbf{r}) &= \mathrm{Var}(\mathbf{r} - r_f\mathbf{e}) \\
&= \mathrm{Var}(\beta_m(r_m - r_f)) + \mathrm{Var}(\epsilon) \\
&= \mathrm{Var}(r_m\beta_m) + \mathrm{Var}(\epsilon) \\
&= \beta_m\beta_m'\sigma_m^2 + \mathrm{Var}(\epsilon)
\end{aligned}$$