Mean-variance portfolio notes

Xie Zejian

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$$R_p = \frac{\sum R_i \omega_i W_0}{W_0} = \omega' R$$

$$\mathbf{r_p} = R_p - 1 = \omega' \mathbf{r}$$

where $\mathbf{r}$ is random vector with mean $\overline{\mathbf{r}}$ and variance $\mathbf{V}$.

hence $E[\mathbf{r_p}] = \omega' \overline{\mathbf{r}}$ and

$$\mathrm{Cov}(\mathbf{r}) = \mathrm{cov}(\omega' \mathbf{r}) = \omega' \mathrm{Cov}(\mathbf{r}) \omega = \omega' \mathbf{V} \omega$$

Then the problem is

$$\min \frac{1}{2} \omega' \mathbf{V} \omega \quad s.t. \quad \mathbf{e}' \omega - \mathbf{1}, \omega' \overline{\mathbf{r}} = \overline{r}_p$$

By Lagrangian

$$L = \frac{1}{2} \omega' \mathbf{V} \omega + \lambda (\overline{r}_p - \omega' \overline{\mathbf{r}}) + \gamma (\mathbf{1} - \omega' \mathbf{e})$$

$$\frac{\partial L}{\partial \omega} = \mathbf{V} \omega - \lambda \overline{\mathbf{r}} - \gamma \mathbf{e} = \mathbf{0} \implies \omega^* = \lambda \mathbf{V}^{-} \overline{\mathbf{r}} + \gamma \mathbf{V}^{-} \mathbf{e}$$

Hence

$$\begin{aligned} \overline{r}_p &= \overline{\mathbf{r}}' \omega^* = \lambda \overline{\mathbf{r}}' \mathbf{V}^{-} \overline{\mathbf{r}} + \gamma \overline{\mathbf{r}}' \mathbf{V}^{-} \mathbf{e} \\ \mathbf{1} &= \mathbf{e}' \omega^* = \lambda \mathbf{e}' \mathbf{V}^{-} \overline{\mathbf{r}} + \gamma \mathbf{e}' \mathbf{V}^{-} \mathbf{e} \end{aligned}$$

denoted $\delta = \mathbf{e}'\mathbf{V}^{-}\mathbf{e}, \alpha = \overline{\mathbf{r}}'\mathbf{V}^{-}\mathbf{e}, \xi = \overline{\mathbf{r}}'\mathbf{V}^{-}\overline{\mathbf{r}}$ where $\delta, \xi > 0$ since $\mathbf{V}$ is positive define. Thus we have a linear equations:

$$\begin{bmatrix} \xi & \alpha \\ \alpha & \delta \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} \overline{r}_p \\ 1 \end{bmatrix}$$

Note $\delta\xi - \alpha^2 > 0$ since $(\alpha\overline{\mathbf{r}} - \xi\mathbf{e})'\mathbf{V}^{-}(\alpha\overline{\mathbf{r}} - \xi\mathbf{e}) = \xi(\delta\xi - \alpha^2) > 0$ and thus such equations is consistent.

solve and get

$$\lambda = \frac{\delta\overline{r}_p - \alpha}{\delta\xi - \alpha^2}, \gamma = \frac{\xi - \alpha\overline{r}_p}{\delta\xi - \alpha^2}$$

and

$$\begin{aligned} \omega^* &= \lambda\mathbf{V}^{-}\overline{\mathbf{r}} + \gamma\mathbf{V}^{-}\mathbf{e} \\ &= \frac{\delta\overline{r}_p - \alpha}{\delta\xi - \alpha^2}\mathbf{V}^{-}\overline{\mathbf{r}} + \frac{\xi - \alpha\overline{r}_p}{\delta\xi - \alpha^2}\mathbf{V}^{-}\mathbf{e} \\ &= a + b\overline{r}_p \end{aligned}$$

where $a = \frac{\xi\mathbf{V}^{-}\mathbf{e} - \alpha\mathbf{V}^{-}\overline{\mathbf{r}}}{\delta\xi - \alpha^2}$ and $b = \frac{-\alpha\mathbf{V}^{-}\mathbf{e} + \delta\mathbf{V}^{-}\overline{\mathbf{r}}}{\delta\xi - \alpha^2}$

The minimum variance is given by

$$\sigma_p^2 = \omega'^*\mathbf{V}\omega^* = \omega'^*(\lambda\overline{\mathbf{r}} + \gamma\mathbf{e}) = \lambda\overline{r}_p + \gamma = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}$$

Thus the **global minimum variance**(GMV) portfolio is $\frac{1}{\delta}$ when $\overline{r}_p = \frac{\alpha}{\delta}$.

Meanwhile

$$\begin{aligned} \lambda &= 0 \\ \gamma &= \frac{1}{\delta} \\ \omega_{mv} &= \frac{\mathbf{V}^{-}\mathbf{e}}{\delta} = \frac{\mathbf{V}^{-}\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-}\mathbf{e}} \end{aligned}$$

Geometry

In geometry view

$$\frac{\sigma_p^2}{1/\delta} - \frac{(\overline{r}_p - \frac{\alpha}{\delta})^2}{(\delta\xi - \alpha^2)/\delta^2} = 1$$

it's a hyperbola with center $(0, \alpha/\delta)$ and asymptote $\overline{r}_p = \frac{\alpha}{\delta} \pm \sqrt{\frac{\delta\xi - \alpha^2}{\xi}}\sigma_p$(recall that asymptote is $y = \pm\frac{b}{a}x$ in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$).

Two Fund Theorem

The mean-variance frontier can be replicated by any two frontier portfolios and any linear combination of frontier portfolios is on the frontier.

Suppose r_1 and r_2 are any given expected return, note

$$\forall r_3, \exists x \ni xr_1 + (1-x)r_2 = r_3$$

Then the weight combined such way is just what we want

$$\omega_3 = x\omega_1 + (1-x)\omega_2 = x(a + br_1) + (1-x)(a + br_2) = a + br_3$$

If $r_1, r_2 \geq \frac{\alpha}{\delta}$, thus they are on efficient frontier, their convex combine still $\geq \frac{\alpha}{\delta}$

$$cr_1 + (1-c)r_2 = \frac{\alpha}{\delta} + c(r_1 - \frac{\alpha}{\delta}) + (1-c)(r_2 - \frac{\alpha}{\delta}) \geq \frac{\alpha}{\delta}$$

that is, any convex combination of efficient frontier portfolios will be an efficient frontier portfolio.

Decomposition

Suppose covariance between portfolios p and q

$$\text{Cov}(\mathbf{r_p}, \mathbf{r_q}) = \omega_p' \mathbf{V} \omega_q$$

Suppose q on the frontier, then

$$\omega_p' \mathbf{V} \omega_q = \omega_p'(\lambda \overline{\mathbf{r}} + \gamma \mathbf{e})$$

Moreover, if

1. p is GMV portfolio, then

$$\omega_p'(\lambda \overline{\mathbf{r}} + \gamma \mathbf{e}) = \omega_p'(\frac{1}{\delta}\mathbf{e}) = \frac{1}{\delta}$$

2. p has the same expected return with q, then

$$\omega_p'(\lambda \overline{\mathbf{r}} + \gamma \mathbf{e}) = \lambda \overline{r}_q + \gamma = \sigma_q^2$$

Thus we have

> The covariance of the return on the global minimum variance (GMV) portfolio and that on any portfolio (not necessary on the frontier) is always equal to $\sigma_{mv}^2 = \frac{1}{\delta}$.

Apply this we get $\mathrm{Cov}(\mathbf{r_p} - \mathbf{r_{mv}}, \mathbf{r_{mv}}) = \mathrm{Cov}(\mathbf{r_p}, \mathbf{r_{mv}}) - \sigma_{mv}^2 = 0$ immediately. We call $\epsilon = \mathbf{r_p} - \mathbf{r_{mv}}$ **excess return**.

> The covariance of the return on a frontier portfolio q and that on any portfolio p (not on the frontier) with the same expected return as q is always equal to the variance of the frontier portfolio q. Formally $E[\mathbf{r_p}] = E[\mathbf{r_q}] \implies \mathrm{Cov}(\mathbf{r_p}, \mathbf{r_q}) = \sigma_q^2$.

It implies that we can decompose $\mathbf{r_p} = \mathbf{r_q} + \mathbf{\epsilon_p}$ where $E[\epsilon] = \mathrm{Cov}(\epsilon_p, \mathbf{r_q}) = 0$

Rewrite the excess return as $\epsilon = b_p \mathbf{r}^*$, then $\mathbf{r_p}$ can be decomposed into

$$\mathbf{r_p} = \mathbf{r_{mv}} + b_p \mathbf{r}^* + \epsilon_p$$

where $\mathbf{r}^*$ is an excess return and $b_p \in \mathbb{R}$.Note $\mathrm{Cov}(\mathbf{r_{mv}}, \mathbf{\epsilon_p}) = \mathrm{Cov}(\mathbf{r_{mv}}, \mathbf{r_p} - \mathbf{r_q}) = 0$ then $\mathrm{Cov}(\mathbf{r}^*, \epsilon_p)$ is also zero. Hence

$$\mathrm{Var}(\mathbf{r_p}) = \overbrace{\underbrace{\sigma_{mv}^2}_{\text{unavoidable risk}} + b_p^2 \mathrm{Var}(\mathbf{r}^*)}^{\text{systematic risk}} + \underbrace{\mathrm{Var}(\epsilon_p)}_{\text{idiot risk}}$$

One can reduce ϵ_p to zero to avoid idiot risk and get a frontier portfolio.

Zero covariance

Recall the covariance between p and q, now suppose they are both on frontier:

$$\omega_p' \mathbf{V} \omega_q = \frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})(\overline{r}_q - \frac{\alpha}{\delta})}{\delta\xi - \alpha^2}$$

Setting this to 0 and slove for $\overline{r}_q$

$$\overline{r}_q = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)}$$

Suppose the tagent line in $\overline{r}_p$, which slope is

$$\frac{\partial \overline{r}_p}{\partial \sigma_p} = \frac{\delta\xi - \alpha^2}{\delta(\overline{r}_p - \frac{\alpha}{\delta})} \sigma_p$$

(Recall $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \implies \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \implies y' = \frac{b^2x^2}{a^2y^2}$)
hence its intercept at $\sigma_p = 0$ is

$$\overline{r}_p - \frac{\delta\xi - \alpha^2}{\delta(\overline{r}_p - \frac{\alpha}{\delta})}\sigma_p^2 = \overline{r}_p - \frac{\delta\xi - \alpha^2}{\delta(\overline{r}_p - \frac{\alpha}{\delta})}(\frac{1}{\delta} + \frac{\delta(\overline{r}_p - \frac{\alpha}{\delta})^2}{\delta\xi - \alpha^2}) = \frac{\alpha}{\delta} - \frac{\delta\xi - \alpha^2}{\delta^2(\overline{r}_p - \alpha/\delta)} = \overline{r}_q$$