Sorting

SFWRENG 2CO3: Data Structures and Algorithms

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Winter 2024

Why sorting

Most computational problems involve *data processing*.

Processing data is typically much simpler if that data is *sorted*.

Example

Finding values: BinarySearch versus LinearSearch.

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Finding values: BinarySearch versus LinearSearch.

The analysis of *sorting* will require universal tools and techniques.
Sort algorithms utilize common design strategies for algorithms.

The power of sorting: The two-sum problem

Problem

Given a list $L[0 \dots N)$ of distinct weights and a target weight w, find all distinct values $v_1, v_2 \in L$ with $w = v_1 + v_2$.

The power of sorting: The two-sum problem

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L:	1	3	7	9	8	4	10	5

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Input: List $L[0 \dots N)$ of N distinct weights, target weight w.

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result := empty bag.
for i := 0 to N − 2 do
    for j := i + 1 to N − 1 do
        if L[i] + L[j] = w then
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Complexity of $\textsc{SimpleTwoSum}$

For a rough estimate, we can count the number of times Line 4 is executed.

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$$\left.\right\} \sum_{j=i+1}^{N-1} 1$$

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$$\sum_{j=i+1}^{N-1} 1 = N - (i + 1)$$

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$$\sum_{i=0}^{N-2} (N - (i+1))$$

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Complexity of SimpleTwoSum

For a rough estimate, we can count the number of times Line 4 is executed.

$$\sum_{i=0}^{N-2} (N - (i+1)) = \sum_{i=0}^{N-2} (N-1) - \sum_{i=0}^{N-2} i$$

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Complexity of SimpleTwoSum

For a rough estimate, we can count the number of times Line 4 is executed.

$$\sum_{i=0}^{N-2} (N - (i + 1)) = \sum_{i=0}^{N-2} (N - 1) - \sum_{i=0}^{N-2} i = (N - 1)^2 - \frac{(N - 2)(N - 1)}{2}$$

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$$\sum_{i=0}^{N-2} (N - (i+1)) = \sum_{i=0}^{N-2} (N-1) - \sum_{i=0}^{N-2} i = (N-1)^2 - \frac{(N-2)(N-1)}{2} = \frac{N(N-1)}{2} \sim N^2.$$

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Given a list $L[0 \ldots N)$ of distinct weights and a target weight w, find all distinct values $v_1, v_2 \in L$ with $w = v_1 + v_2$.

L (sorted):	1	3	4	5	7	8	9	10

Target weight: $w = 11$.

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Algorithm $\textsc{BetterTwoSum}(L, w)$:

Input: *Ordered* list $L[0 \ldots N)$ of N distinct weights, target weight w.

```
result := empty bag.
for i := 0 to N − 2 do
    j := BinarySearch(L, i + 1, N, w − L[i]).
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The power of sorting: The two-sum problem

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Complexity of BetterTwoSum

For a rough estimate, we can count the cost of each BinarySearch call.

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result := empty bag.
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$$\Bigg\} \sum_{i=0}^{N-2} \log_2(N - (i+1))$$

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```

$$\left.\begin{array}{l}\text{lines 2–5}\end{array}\right\} \sum_{i=0}^{N-2} \log_2(N-(i+1))$$

Complexity of $\textsc{BetterTwoSum}$

For a rough *upper bound* estimate, we can count the cost of each $\textsc{BinarySearch}$ call.

$$\sum_{i=0}^{N-2} \log_2(N-(i+1)) \leq \sum_{i=0}^{N-2} \log_2(N)$$

The power of sorting: The two-sum problem

Algorithm BETTERTWOSUM(L, w):

Input: *Ordered* list $L[0 \ldots N)$ of N distinct weights, target weight w.

```
result := empty bag.
for i := 0 to N − 2 do
    j := BINARYSEARCH(L, i + 1, N, w − L[i]).        } log_2(N − (i + 1))
    if j ≠ 'not found' then
        add (L[i], L[j]) to result.
return result.
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$$\left.\begin{array}{l} \text{lines 2–5} \end{array}\right\} \sum_{i=0}^{N-2} \log_2(N - (i+1))$$

Complexity of BETTERTWOSUM

For a rough *upper bound* estimate, we can count the cost of each BINARYSEARCH call.

$$\sum_{i=0}^{N-2} \log_2(N - (i+1)) \leq \sum_{i=0}^{N-2} \log_2(N) = (N-1)\log_2(N).$$

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Complexity of BetterTwoSum

For a rough *lower bound* estimate, we can count the cost of each BinarySearch call.

$$\sum_{i=0}^{N-2} \log_2(N-(i+1)) \geq \sum_{i=0}^{\frac{N}{2}-1} \log_2\left(\frac{N}{2}\right)$$

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$$\sum_{i=0}^{N-2} \log_2(N - (i+1))$$

Complexity of BetterTwoSum

For a rough estimate, we can count the cost of each BinarySearch call.

$$\frac{N}{2}(\log_2(N)-1) \leq \sum_{i=0}^{N-2} \log_2(N-(i+1)) \leq (N-1)\log_2(N).$$

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L (sorted):								
	1	3	4	5	7	8	9	10

Target weight: $w = 11$.

Can we do better than $\sim N \log_2(N)$ if L is ordered?

The power of sorting: The two-sum problem

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find all distinct values $v_1, v_2 \in L$ *with* $w = v_1 + v_2$*.*

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Assume that $(L[i], L[j])$ and $(L[k], L[m])$ are in the output with $i < k$.

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- $L[i] < L[k]$ implies $L[j] > L[m]$.

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- $L[i] + L[j] = w = L[k] + L[m]$.
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We can search from both ends in L: position i as a *lower bound* and j as an *upper bound*.

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We can search from both ends in L: position i as a *lower bound* and j as an *upper bound*.

Algorithm $\textsc{BestTwoSum}(L, w)$:

Input: *Ordered* list $L[0 \ldots N)$ of N distinct weights, target weight w.

$result$:= empty bag.
$i, j := 0, N-1$.
while $i < j$ **do**
    if $L[i] + L[j] = w$ **then**
        add $(L[i], L[j])$ to $result$.
        $i, j := i+1, j-1$.
    else if $L[i] + L[j] < w$ **then**
        $i := i+1$.
    else
        $j := j-1$.
return $result$.

The power of sorting: The two-sum problem

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We can search from both ends in L: position i as a *lower bound* and j as an *upper bound*.

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        $j := j-1$.
return *result*.

$\sim N$

Intermezzo: Correctness of BestTwoSum

Warning

Proving the correctness of BestTwoSum in all details is *tricky*!

Intermezzo: Correctness of BestTwoSum

High-level proof steps

result := empty bag.
$i, j := 0, N - 1$.
while $i < j$ **do**
    if $L[i] + L[j] = w$ **then**
        add $(L[i], L[j])$ to *result*.
        $i, j := i + 1, j - 1$.
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return *result*.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

1. Specify what the *result* should be.

result := empty bag.
$i, j := 0, N - 1$.
while $i < j$ **do**
    if $L[i] + L[j] = w$ **then**
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return *result*.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

1. Specify what the *result* should be.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

result := empty bag.
$i, j := 0, N - 1$.
while $i < j$ **do**
    if $L[i] + L[j] = w$ **then**
        add $(L[i], L[j])$ to *result*.
        $i, j := i + 1, j - 1$.
    else if $L[i] + L[j] < w$ **then**
        $i := i + 1$.
    else
        $j := j - 1$.
return *result*. $result = \mathsf{TS}(L, 0, N - 1)$.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

1. Specify what the *result* should be. The *invariant* must establish this result!

Let $\mathrm{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

```
result := empty bag.
i, j := 0, N − 1.
while i < j do
    if L[i] + L[j] = w then
        add (L[i], L[j]) to result.
        i, j := i + 1, j − 1.
    else if L[i] + L[j] < w then
        i := i + 1.
    else
        j := j − 1.
return result. result = TS(L, 0, N − 1).
```

Intermezzo: Correctness of BestTwoSum

High-level proof steps

2. Specify the *invariant*.

Let $\mathsf{TS}(\mathit{start}, \mathit{end}) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (\mathit{start} \leq i < j \leq \mathit{end})\}$.

result := empty bag.
$i, j := 0, N - 1$.
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    else
        $j := j - 1$.
return result. $\mathit{result} = \mathsf{TS}(L, 0, N - 1)$.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

2. Specify the *invariant*. Look at what you need *after* the loop!

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

$result :=$ empty bag.
$i, j := 0, N - 1$.
while $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ */
  if $L[i] + L[j] = w$ **then**
    add $(L[i], L[j])$ to $result$.
    $i, j := i + 1, j - 1$.
  else if $L[i] + L[j] < w$ **then**
    $i := i + 1$.
  else
    $j := j - 1$.
return $result$. $result = \mathsf{TS}(L, 0, N - 1)$.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

3. Prove the *invariant* right *before the loop*.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: *result* := empty bag.

2: $i, j := 0, N - 1$.

Base case: prove that the invariant holds before the loop.

3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

3. Prove the *invariant* right *before the loop*. Use facts established *before* the loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \le i < j \le end)\}$.

1: $result$:= empty bag.
2: $i, j := 0, N - 1$.
   Known: we have $i = 0$, $j = N - 1$, and $result = \emptyset$ (due to assignments).

   Base case: prove that the invariant holds before the loop.
3: **while** $i < j$ **do** /∗ inv: $result = \mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j)$ ∗/

Intermezzo: Correctness of BestTwoSum

High-level proof steps

3. Prove the *invariant* right *before the loop*. Use facts established *before* the loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: $result$:= empty bag.
2: $i, j := 0, N - 1$.
Known: we have $i = 0$, $j = N - 1$, and $result = \emptyset$ (due to assignments).
Hence, $\mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j) = \emptyset = result$.
Base case: prove that the invariant holds before the loop.
3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j)$ */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

3. Prove the *invariant* right *before the loop*. Use facts established *before* the loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

$result$:= empty bag.
$i, j := 0, N - 1$.
Known: we have $i = 0$, $j = N - 1$, and $result = \emptyset$ (due to assignments).
Hence, $\mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j) = \emptyset = result$.
Base case: the invariant holds before the loop.
while $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j)$ */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

4. Prove that the *invariant* is maintaned by the loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

```
if L[i] + L[j] = w then
    add (L[i], L[j]) to result.
    i, j := i + 1, j − 1.
else if L[i] + L[j] < w then
    i := i + 1.
else
    j := j − 1.
```

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

5. An if-statement introduces a case distinction: prove each branch separately.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

if $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

add $(L[i], L[j])$ to $result$.

$i, j := i + 1, j - 1$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

5. An if-statement introduces a case distinction: prove each branch separately.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

if $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

add $(L[i], L[j])$ to *result*.

$i, j := i + 1, j - 1$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

6. Carry over all facts obtained via the assignments.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

if $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

add $(L[i], L[j])$ to *result*.

$i, j := i + 1, j - 1$.

Known: $result_{\text{new}} = result_{\text{old}} \cup \{(L[i_{\text{old}}], L[j_{\text{old}}])\}$, $i_{\text{new}} = i_{\text{old}} + 1$, $j_{\text{new}} = j_{\text{old}} - 1$,
$result_{\text{old}} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j_{\text{old}})$, and $(L[i_{\text{old}}], L[j_{\text{old}}]) \in \mathsf{TS}(L, i_{\text{old}}, j_{\text{old}})$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

7. Complete the proof for this case using all provided facts.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

if $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

add $(L[i], L[j])$ to *result*.

$i, j := i + 1, j - 1$.

Known: $result_{\text{new}} = result_{\text{old}} \cup \{(L[i_{\text{old}}], L[j_{\text{old}}])\}$, $i_{\text{new}} = i_{\text{old}} + 1$, $j_{\text{new}} = j_{\text{old}} - 1$, $result_{\text{old}} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j_{\text{old}})$, and $(L[i_{\text{old}}], L[j_{\text{old}}]) \in \mathsf{TS}(L, i_{\text{old}}, j_{\text{old}})$.

Need to prove: $result_{\text{new}} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{new}}, j_{\text{new}})$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

7. Complete the proof for this case using all provided facts.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

if $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

add $(L[i], L[j])$ to *result*.

$i, j := i + 1, j - 1$.

Known: $result_{new} = result_{old} \cup \{(L[i_{old}], L[j_{old}])\}$, $i_{new} = i_{old} + 1$, $j_{new} = j_{old} - 1$,
$result_{old} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{old}, j_{old})$, and $(L[i_{old}], L[j_{old}]) \in \mathsf{TS}(L, i_{old}, j_{old})$.

Need to prove: $result_{new} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{new}, j_{new})$.

$$result_{new} = (\mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{old}, j_{old})) \cup \{(L[i_{old}], L[j_{old}])\}.$$

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

7. Complete the proof for this case using all provided facts.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

if $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

add $(L[i], L[j])$ to $result$.

$i, j := i + 1, j - 1$.

Known: $result_{new} = result_{old} \cup \{(L[i_{old}], L[j_{old}])\}$, $i_{new} = i_{old} + 1$, $j_{new} = j_{old} - 1$, $result_{old} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{old}, j_{old})$, and $(L[i_{old}], L[j_{old}]) \in \mathsf{TS}(L, i_{old}, j_{old})$.

Need to prove: $result_{new} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{new}, j_{new})$.

$result_{new} = \mathsf{TS}(L, 0, N-1) \setminus (\mathsf{TS}(L, i_{old}, j_{old}) \setminus \{(L[i_{old}], L[j_{old}])\})$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

7. Complete the proof for this case using all provided facts.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

4: **if** $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

5: add $(L[i], L[j])$ to $result$.

6: $i, j := i + 1, j - 1$.

Known: $result_{new} = result_{old} \cup \{(L[i_{old}], L[j_{old}])\}$, $i_{new} = i_{old} + 1$, $j_{new} = j_{old} - 1$, $result_{old} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{old}, j_{old})$, and $(L[i_{old}], L[j_{old}]) \in \mathsf{TS}(L, i_{old}, j_{old})$.

Need to prove: $result_{new} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{new}, j_{new})$.

$result_{new} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{new}, j_{new})$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

7. Complete the proof for this case using all provided facts.

Let $\mathsf{TS}(\mathit{start}, \mathit{end}) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (\mathit{start} \leq i < j \leq \mathit{end})\}$.

Given: invariant and $i < j \rightarrow \mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

if $L[i] + L[j] = w$ **then**

Given: $\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

add $(L[i], L[j])$ to result.

$i, j := i + 1, j - 1$.

Known: $\mathit{result}_{\mathrm{new}} = \mathit{result}_{\mathrm{old}} \cup \{(L[i_{\mathrm{old}}], L[j_{\mathrm{old}}])\}$, $i_{\mathrm{new}} = i_{\mathrm{old}} + 1$, $j_{\mathrm{new}} = j_{\mathrm{old}} - 1$, $\mathit{result}_{\mathrm{old}} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\mathrm{old}}, j_{\mathrm{old}})$, and $(L[i_{\mathrm{old}}], L[j_{\mathrm{old}}]) \in \mathsf{TS}(L, i_{\mathrm{old}}, j_{\mathrm{old}})$.

Need to prove: $\mathit{result}_{\mathrm{new}} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\mathrm{new}}, j_{\mathrm{new}})$.

$\mathit{result}_{\mathrm{new}} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\mathrm{new}}, j_{\mathrm{new}})$.

Induction step: the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

else if $L[i] + L[j] < w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

$i := i + 1$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(\mathit{start}, \mathit{end}) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (\mathit{start} \leq i < j \leq \mathit{end})\}$.

Given: invariant and $i < j \to \mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

Given: $\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

else if $L[i] + L[j] < w$ **then**

Given: $\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

$i := i + 1$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

else if $L[i] + L[j] < w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

$i := i + 1$.

Known: $i_{\text{new}} = i_{\text{old}} + 1$,

$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$, and $(L[i_{\text{old}}], v) \notin \mathsf{TS}(L, i_{\text{old}}, j)$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

else if $L[i] + L[j] < w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

$i := i + 1$.

Known: $i_{\text{new}} = i_{\text{old}} + 1$,
$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$, and $(L[i_{\text{old}}], v) \notin \mathsf{TS}(L, i_{\text{old}}, j)$.

Need to prove: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{new}}, j)$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

else if $L[i] + L[j] < w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

$i := i + 1$.

Known: $i_{\text{new}} = i_{\text{old}} + 1$,

$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$, and $(L[i_{\text{old}}], v) \notin \mathsf{TS}(L, i_{\text{old}}, j)$.

Need to prove: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{new}}, j)$.

$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

7: **else if** $L[i] + L[j] < w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

8: $i := i + 1$.

Known: $i_{new} = i_{old} + 1$,

$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{old}, j)$, and $(L[i_{old}], v) \notin \mathsf{TS}(L, i_{old}, j)$.

Need to prove: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{new}, j)$.

$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{new}, j)$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(\mathit{start}, \mathit{end}) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (\mathit{start} \leq i < j \leq \mathit{end})\}$.

Given: invariant and $i < j \rightarrow \mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

Given: $\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

else if $L[i] + L[j] < w$ **then**

Given: $\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

$i := i + 1$.

Known: $i_{\text{new}} = i_{\text{old}} + 1$,

$\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$, and $(L[i_{\text{old}}], v) \notin \mathsf{TS}(L, i_{\text{old}}, j)$.

Need to prove: $\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{new}}, j)$.

$\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{new}}, j)$.

Induction step: the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

9. Finally, the *else* case of the case distinction (analogous).

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

if $L[i] + L[j] = w$ **then**
  add $(L[i], L[j])$ to *result*.
  $i, j := i + 1, j - 1$.
else if $L[i] + L[j] < w$ **then**
  $i := i + 1$.
else
  $j := j - 1$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

10. The invariant holds!

Let $\mathsf{TS}(\mathit{start}, \mathit{end}) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (\mathit{start} \leq i < j \leq \mathit{end})\}$.

result := empty bag.
$i, j := 0, N - 1$.
while $i < j$ **do** /* inv: $\mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ */
  if $L[i] + L[j] = w$ **then**
    add $(L[i], L[j])$ to result.
    $i, j := i + 1, j - 1$.
  else if $L[i] + L[j] < w$ **then**
    $i := i + 1$.
  else
    $j := j - 1$.
Known: invariant and $\neg(i < j) \rightarrow \mathit{result} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.
return result. $\mathit{result} = \mathsf{TS}(L, 0, N-1)$.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

10. The invariant holds! Do not forget termination of the while-loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \land (start \leq i < j \leq end)\}$.

$result$:= empty bag.
$i, j := 0, N - 1$.
while $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$; bf: $j - i$ */
    if $L[i] + L[j] = w$ **then**
        add $(L[i], L[j])$ to $result$.
        $i, j := i + 1, j - 1$.
    else if $L[i] + L[j] < w$ **then**
        $i := i + 1$.
    else
        $j := j - 1$.
Known: invariant and $\neg(i < j) \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.
return $result$. $result = \mathsf{TS}(L, 0, N-1)$.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

11. Prove the post-condition.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Known: invariant and $\neg(i < j) \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.

return $result$. $result = \mathsf{TS}(L, 0, N-1)$.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

11. Prove the post-condition.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Known: invariant and $\neg(i < j) \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.

By $i \geq j$ and the Definition of TS, we have $\mathsf{TS}(L, i, j) = \emptyset$.

11: **return** $result$. $result = \mathsf{TS}(L, 0, N-1)$.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

11. Prove the post-condition.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Known: invariant and $\neg(i < j) \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.

By $i \geq j$ and the Definition of TS, we have $\mathsf{TS}(L, i, j) = \emptyset$.

Hence, $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j) = \mathsf{TS}(L, 0, N-1) \setminus \emptyset = \mathsf{TS}(L, 0, N-1)$.

11: **return** *result*. $result = \mathsf{TS}(L, 0, N-1)$.

Intermezzo: Correctness of BestTwoSum

Warning

You cannot learn correctness proofs from slides: practice on simple algorithms yourself!

A first sort algorithm: SelectionSort

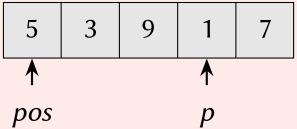
5	3	9	1	7

Algorithm SelectionSort(L)**:**

Input: List $L[0 \dots N)$ of N values.

for $pos := 0$ **to** $N - 2$ **do**
    Find the position p of the *minimum value* in $L[pos \dots N)$.
    Exchange $L[pos]$ and $L[p]$.

A first sort algorithm: SelectionSort



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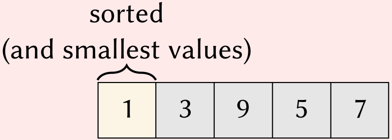

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```
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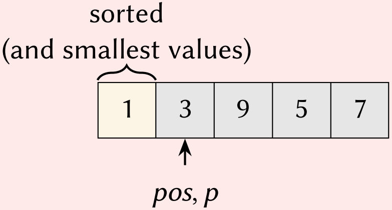
A first sort algorithm: SelectionSort



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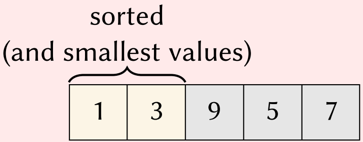
A first sort algorithm: SelectionSort



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for $pos := 0$ **to** $N - 2$ **do**

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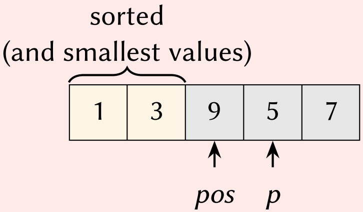
  Exchange $L[pos]$ and $L[p]$.

A first sort algorithm: SelectionSort



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  Exchange $L[pos]$ and $L[p]$.

A first sort algorithm: SELECTIONSORT

sorted
(and smallest values)

1	3	5	9	7

Algorithm SELECTIONSORT(L):

Input: List $L[0 \dots N)$ of N values.

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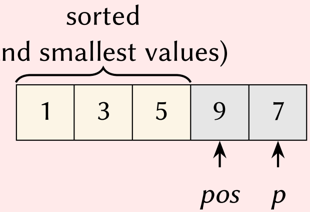
A first sort algorithm: SelectionSort



Algorithm SelectionSort(L)**:**

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2: Find the position p of the *minimum value* in $L[pos \ldots N)$.
3: Exchange $L[pos]$ and $L[p]$.

A first sort algorithm: SelectionSort

$$\overbrace{\boxed{1}\,\boxed{3}\,\boxed{5}\,\boxed{7}\,\boxed{9}}^{\substack{\text{sorted}\\ \text{(and smallest values)}}}$$

Algorithm SelectionSort(L):

Input: List $L[0 \ldots N)$ of N values.

for $pos := 0$ **to** $N - 2$ **do**

  Find the position p of the *minimum value* in $L[pos \ldots N)$.

  Exchange $L[pos]$ and $L[p]$.

Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

A first sort algorithm: SELECTIONSORT

$$\overbrace{\boxed{1}\boxed{3}\boxed{5}\boxed{7}\boxed{9}}^{\substack{\text{sorted} \\ \text{(and smallest values)}}}$$

Algorithm SELECTIONSORT(L):

Input: List $L[0 \dots N)$ of N values.

for $pos := 0$ **to** $N - 2$ **do**
    Find the position p of the *minimum value* in $L[pos \dots N)$. $\longleftarrow$ *?*
    Exchange $L[pos]$ and $L[p]$.

Runtime complexity of SELECTIONSORT

A good estimate: number of comparisons and changes to list values.

A first sort algorithm: SelectionSort

Algorithm SelectionSort(L):

Input: List $L[0 \dots N)$ of N values.

```
1: for pos := 0 to N − 2 do
       Find the position p of the minimum value in L[pos ... N).
2:     p := pos.
3:     for i := pos + 1 to N − 1 do
4:         if L[i] < L[p] then
5:             p := i.
6:     Exchange L[pos] and L[p].
```

Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

A first sort algorithm: SelectionSort

Algorithm SelectionSort(L):

Input: List $L[0\ldots N)$ of N values.

```
for pos := 0 to N − 2 do
    p := pos.
    for i := pos + 1 to N − 1 do
        if L[i] < L[p] then
            p := i.
    Exchange L[pos] and L[p].
```

Comparisons: $N - 1 - pos$.
Changes: 0.

Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

A first sort algorithm: SELECTIONSORT

Algorithm SELECTIONSORT(L):

Input: List $L[0 \dots N)$ of N values.

```
for pos := 0 to N − 2 do
    p := pos.
    for i := pos + 1 to N − 1 do
        if L[i] < L[p] then
            p := i.
    Exchange L[pos] and L[p].
```

Comparisons: $\sum_{pos=0}^{N-2} (N - 1 - pos)$.

Changes: $2(N - 1)$.

Runtime complexity of SELECTIONSORT

A good estimate: number of comparisons and changes to list values.

A first sort algorithm: SELECTIONSORT

Algorithm SELECTIONSORT(L):

Input: List $L[0\ldots N)$ of N values.

1: **for** $pos := 0$ **to** $N-2$ **do**
2: $\quad p := pos$.
3: $\quad$ **for** $i := pos+1$ **to** $N-1$ **do**
4: $\quad\quad$ **if** $L[i] < L[p]$ **then**
5: $\quad\quad\quad p := i$.
6: $\quad$ Exchange $L[pos]$ and $L[p]$.

Comparisons: $\displaystyle\sum_{pos=0}^{N-2}(N-1-pos)$.

Changes: $2(N-1)$.

Runtime complexity of SELECTIONSORT

A good estimate: number of comparisons and changes to list values.

$$\text{Comparisons: } \sum_{pos=0}^{N-2}(N-1-pos)$$

A first sort algorithm: SELECTIONSORT

Algorithm SELECTIONSORT(L):

Input: List $L[0\ldots N)$ of N values.

for $pos := 0$ **to** $N-2$ **do**
    $p := pos$.
    for $i := pos + 1$ **to** $N-1$ **do**
        if $L[i] < L[p]$ **then**
            $p := i$.
    Exchange $L[pos]$ and $L[p]$.

Comparisons: $\sum_{pos=0}^{N-2}(N-1-pos)$.

Changes: $2(N-1)$.

Runtime complexity of SELECTIONSORT

A good estimate: number of comparisons and changes to list values.

$$\text{Comparisons: } \sum_{pos=0}^{N-2}(N-1-pos) = \sum_{j=1}^{N-1} j$$

A first sort algorithm: SelectionSort

Algorithm SelectionSort(L):

Input: List $L[0 \dots N)$ of N values.

```
1: for pos := 0 to N − 2 do
2:     p := pos.
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6:     Exchange L[pos] and L[p].
```

Comparisons: $\sum_{pos=0}^{N-2} (N - 1 - pos)$.

Changes: $2(N - 1)$.

Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

$$\text{Comparisons: } \sum_{pos=0}^{N-2} (N - 1 - pos) = \sum_{j=1}^{N-1} j = \frac{N(N-1)}{2}$$

A first sort algorithm: SelectionSort

Algorithm SelectionSort(L):

Input: List $L[0 \dots N)$ of N values.

for $pos := 0$ **to** $N-2$ **do**
  $p := pos$.
  for $i := pos+1$ **to** $N-1$ **do**
    if $L[i] < L[p]$ **then**
      $p := i$.
  Exchange $L[pos]$ and $L[p]$.

Comparisons: $\sum_{pos=0}^{N-2}(N-1-pos) \sim N^2$.

Changes: $2(N-1) \sim N$.

Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

$$\text{Comparisons: } \sum_{pos=0}^{N-2}(N-1-pos) = \sum_{j=1}^{N-1} j = \frac{N(N-1)}{2} \sim N^2.$$

A first sort algorithm: SelectionSort

Algorithm SelectionSort(L):

Input: List $L[0 \ldots N)$ of N values.

for $pos := 0$ **to** $N - 2$ **do**
    $p := pos$.
    for $i := pos + 1$ **to** $N - 1$ **do**
        if $L[i] < L[p]$ **then**
            $p := i$.
    Exchange $L[pos]$ and $L[p]$.

Correctness of SelectionSort: Some tips

- Rework the for-loops into while loops.
- The inner loop only changes p: prove whatever that loop does separately.
- Include as much information into the invariant of the outer loop.
  What exactly do we know about the values in $L[0 \ldots pos)$.
- A complete proof guarantees that list L keeps all original values!

A second sort algorithm: InsertionSort

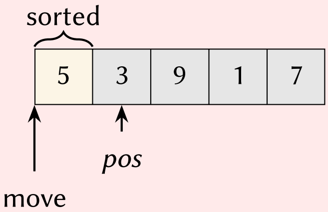
$$\overbrace{\boxed{5}}^{\text{sorted}}\boxed{3}\boxed{9}\boxed{1}\boxed{7}$$

Algorithm InsertionSort(L):

Input: List $L[0 \dots N)$ of N values.

```
for pos := 1 to N − 1 do
    v := L[pos].
    Move all values w ∈ L[0 … pos) with v > w one to the right.
    L[p] := v.
```

A second sort algorithm: InsertionSort



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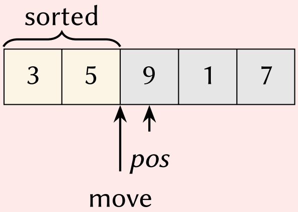

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A second sort algorithm: InsertionSort

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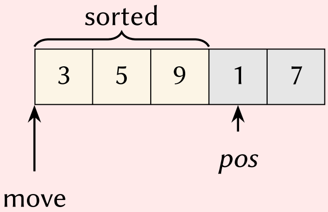

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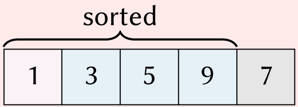
A second sort algorithm: InsertionSort



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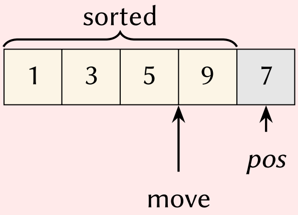
A second sort algorithm: InsertionSort



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A second sort algorithm: InsertionSort

$$\overbrace{\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 7 & 9 \\ \hline\end{array}}^{\text{sorted}}$$

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Input: List $L[0\ldots N)$ of N values.

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A second sort algorithm: InsertionSort

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    $v := L[pos]$.
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    $L[p] := v$.

A second sort algorithm: InsertionSort

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Input: List $L[0 \ldots N)$ of N values.

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    $v := L[pos]$.
    Move all values $w \in L[0 \ldots pos)$ with $v > w$ one to the right.
    $p := pos$.
    while $p > 0$ **and** $v < L[p - 1]$ **do**
        $L[p] := L[p - 1]$.
        $p := p - 1$.
    $L[p] := v$.

Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

Input: List $L[0\ldots N)$ of N values.

for $pos := 1$ **to** $N-1$ **do**
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        $L[p] := L[p-1]$.
        $p := p-1$.
    $L[p] := v$.

Comparisons: $\leq pos$.
Changes: $\leq pos$.

Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

Input: List $L[0 \ldots N)$ of N values.

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for pos := 1 to N − 1 do
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    while p > 0 and v < L[p − 1] do
        L[p] := L[p − 1].
        p := p − 1.
    L[p] := v.
```

Comparisons: $\leq \sum_{pos=1}^{N-1} pos$.

Changes: $\leq \sum_{pos=1}^{N-1} (1 + pos)$.

Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

Input: List $L[0\ldots N)$ of N values.

for $pos := 1$ **to** $N-1$ **do**
  $v := L[pos]$.
  $p := pos$.
  while $p > 0$ **and** $v < L[p-1]$ **do**
    $L[p] := L[p-1]$.
    $p := p - 1$.
  $L[p] := v$.

Comparisons: $\leq \sum_{pos=1}^{N-1} pos = \dfrac{N(N-1)}{2}$.

Changes: $\leq \sum_{pos=1}^{N-1} (1 + pos) = \dfrac{N(N-1)}{2} + N - 1$.

Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

Input: List $L[0 \dots N)$ of N values.

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for pos := 1 to N − 1 do
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        L[p] := L[p − 1].
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    L[p] := v.
```

$$\text{Comparisons: } \leq \sum_{pos=1}^{N-1} pos = \frac{N(N-1)}{2}.$$

$$\text{Changes: } \leq \sum_{pos=1}^{N-1} (1 + pos) = \frac{N(N-1)}{2} + N - 1.$$

Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

When does InsertionSort have $\sim N^2$ comparisons and changes?

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

Input: List $L[0\dots N)$ of N values.

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for pos := 1 to N − 1 do
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Changes: $\leq \sum_{pos=1}^{N-1} (1 + pos) = \frac{N(N-1)}{2} + N - 1$.

Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

When does InsertionSort have $\sim N^2$ comparisons and changes?

Reverse-ordered array: every next array is moved to the start of the list.

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

Input: List $L[0 \ldots N)$ of N values.

```
for pos := 1 to N − 1 do
    v := L[pos].
    p := pos.
    while p > 0 and v < L[p − 1] do
        L[p] := L[p − 1].
        p := p − 1.
    L[p] := v.
```

Comparisons: $\leq \sum_{pos=1}^{N-1} pos = \frac{N(N-1)}{2}$.

Changes: $\leq \sum_{pos=1}^{N-1} (1 + pos) = \frac{N(N-1)}{2} + N - 1$.

Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

When does InsertionSort have less than $\sim N^2$ comparisons and changes?

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

Input: List $L[0 \dots N)$ of N values.

```
for pos := 1 to N − 1 do
    v := L[pos].
    p := pos.
    while p > 0 and v < L[p − 1] do
        L[p] := L[p − 1].
        p := p − 1.
    L[p] := v.
```

Comparisons: $\leq \sum_{pos=1}^{N-1} pos = \frac{N(N-1)}{2}$.

Changes: $\leq \sum_{pos=1}^{N-1} (1 + pos) = \frac{N(N-1)}{2} + N - 1$.

Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

When does InsertionSort have less than $\sim N^2$ comparisons and changes?

Ordered array: $\sim N$ comparisons and changes as every value stays in place.

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

Input: List $L[0\dots N)$ of N values, $L = \mathcal{L}$.

$pos := 1$.
while $pos \neq N$ **do**
    /* inv: $L[0\dots pos)$ is ordered and L holds the same values as $\mathcal{L}$, bf: $N - pos$. */
    $v := L[pos]$.
    $p := pos$.
    while $p > 0$ **and** $v < L[p-1]$ **do**
        /* inv: $F = L[0\dots p)$ is ordered, $S = L[p+1\dots pos+1)$ is ordered, all values in F are smaller than the values in S, all values in S are larger than v, and the values in F, S, $[v]$, and $L[pos+1\dots, N)$ are exactly the values in $\mathcal{L}$., bf: p. */
        $L[p] := L[p-1]$.
        $p := p - 1$.
    $L[p] := v$.
    $pos := pos + 1$.

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):

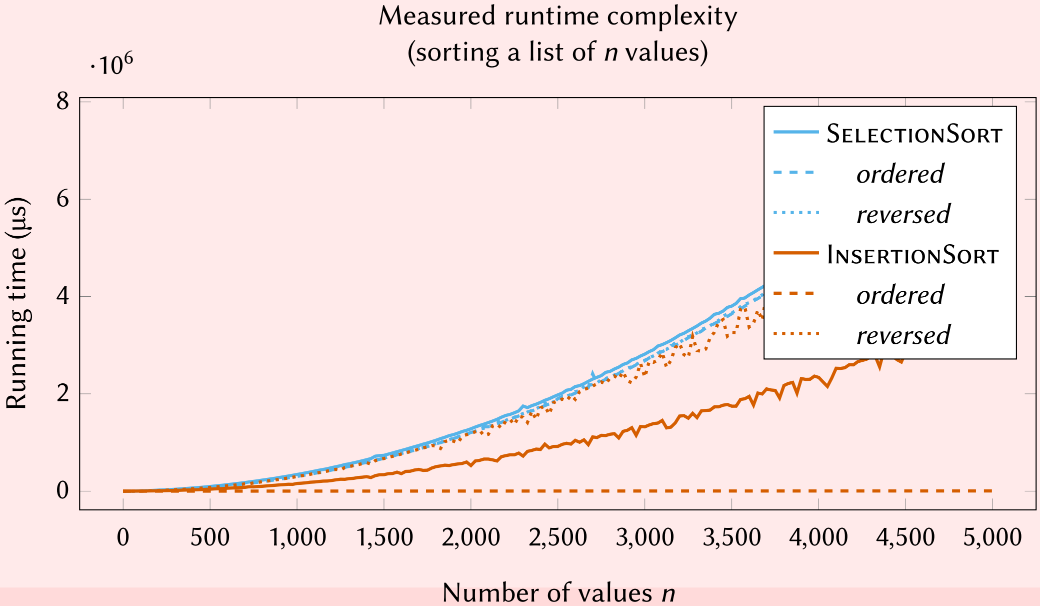
Input: List $L[0 \ldots N)$ of N values, $L = \mathcal{L}$.

$pos := 1$.
while $pos \neq N$ **do**
    /* inv: $L[0 \ldots pos)$ is ordered and L holds the same values as $\mathcal{L}$, bf: $N - pos$. */
    $v := L[pos]$.
    $p := pos$.
    while $p > 0$ **and** $v < L[p-1]$ **do**
        /* inv: $F = L[0 \ldots p)$ is ordered, $S = L[p+1 \ldots pos+1)$ is ordered, all values in F are smaller than the values in S, all values in S are larger than v, and the values in F, S, $[v]$, and $L[pos+1 \ldots, N)$ are exactly the values in $\mathcal{L}$., bf: p. */
        $L[p] := L[p-1]$.
        $p := p - 1$.
    $L[p] := v$.
    $pos := pos + 1$.

A summary of basic sorting

	Comparisons	Changes	Memory
SelectionSort	$\sim N^2$	$\sim N$	~ 1
InsertionSort	$\sim N^2$	$\sim N^2$	~ 1

A summary of basic sorting



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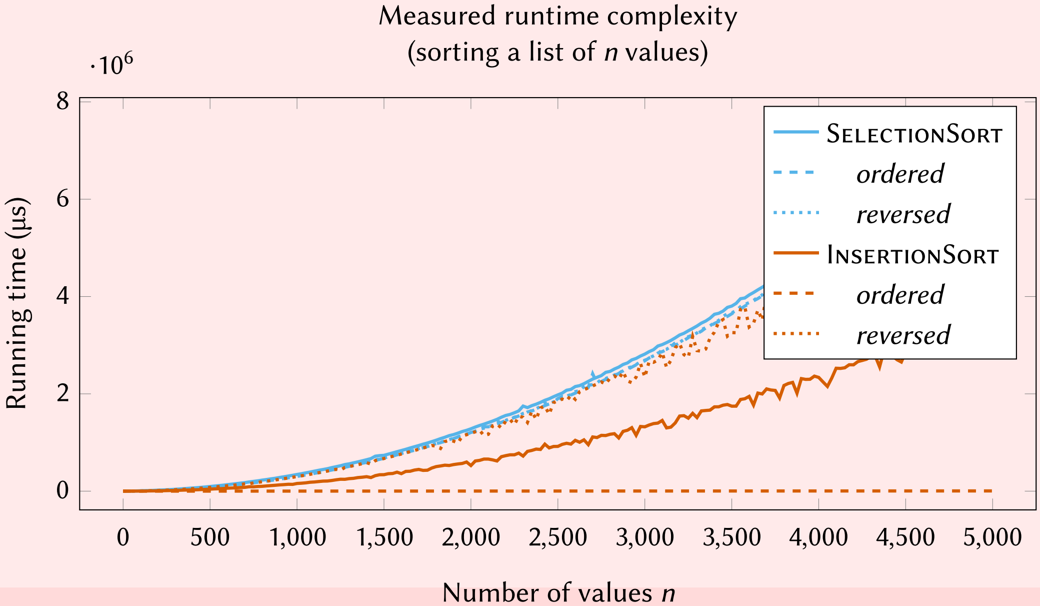

Toward faster sorting

The issue with SelectionSort and InsertionSort

- The algorithms do not perform "global reorderings".
- The algorithms sort one element at a time.
  E.g., small elements at the end of the list are moved to the front one at a time.

MergeSortR: Sorting using divide-and-conquer

Divide-and-conquer

Divide Turn problem into smaller subproblems.

Conquer Solve the smaller subproblems using *recursion*.

Combine Combine the subproblem solutions into a final solution.

MergeSortR: Sorting using divide-and-conquer

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BinarySearchR is a divide-and-conquer algorithm.

MergeSortR: Sorting using divide-and-conquer

Divide-and-conquer

Divide
: Turn problem into smaller subproblems.
Break the list in two halves.

Conquer
: Solve the smaller subproblems using *recursion*.

Combine
: Combine the subproblem solutions into a final solution.

MergeSortR: Sorting using divide-and-conquer

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Sort both halves separately: by recursion, we reach lists with one element.

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MergeSortR: Sorting using divide-and-conquer

Divide-and-conquer

Divide Turn problem into smaller subproblems.
Break the list in two halves.

Conquer Solve the smaller subproblems using *recursion*.
Sort both halves separately: by recursion, we reach lists with one element.

Combine Combine the subproblem solutions into a final solution.
Merge two sorted halves together to obtain the result.

MergeSortR: High-level overview

Algorithm MergeSortR($L[start \ldots end]$):

2	6	3	5	1	4

MergeSortR: High-level overview

Algorithm MergeSortR($L[start \ldots end]$):

if $end - start > 1$ **then**

else return L.

2	6	3	5	1	4

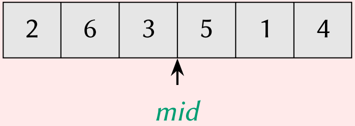
MergeSortR: High-level overview

Algorithm MergeSortR($L[start \ldots end]$):

if $end - start > 1$ **then**

  $mid := (end - start) \operatorname{div} 2.$

else return L.

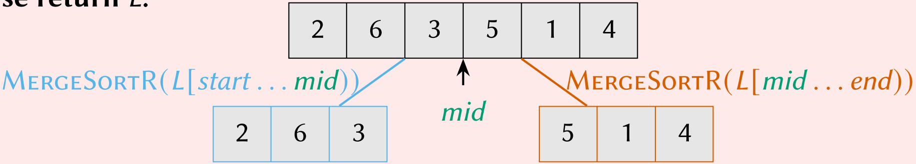


MergeSortR: High-level overview

Algorithm MergeSortR($L[start \ldots end]$):

if $end - start > 1$ **then**
  $mid := (end - start)$ div 2.
  $L_1 :=$ MergeSortR($L[start \ldots mid]$).
  $L_2 :=$ MergeSortR($L[mid \ldots end]$).

else return L.



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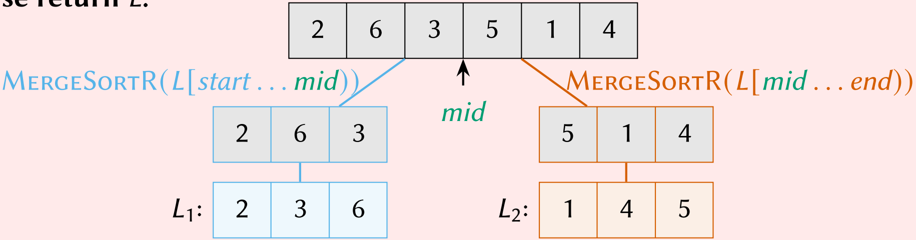
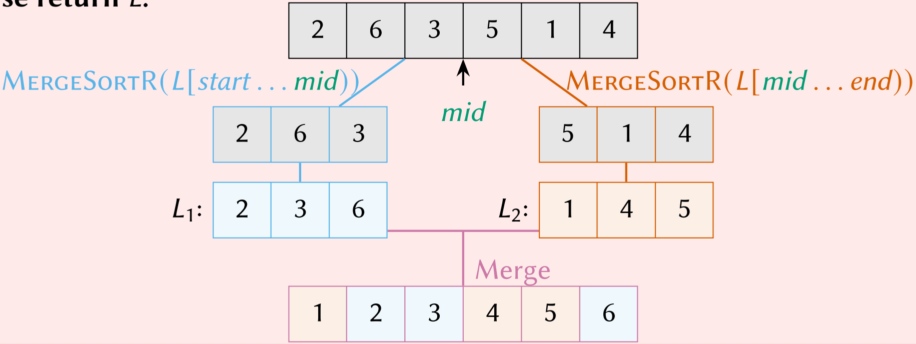


MergeSortR: High-level overview

Algorithm MergeSortR($L[start \ldots end]$):

1: **if** $end - start > 1$ **then**
2: $\quad mid := (end - start) \operatorname{div} 2.$
3: $\quad L_1 :=$ MergeSortR($L[start \ldots mid]$).
4: $\quad L_2 :=$ MergeSortR($L[mid \ldots end]$).
5: $\quad$ **return** Merge(L_1, L_2) (maintain sorted order).
6: **else return** L.



Proof of correctness: $\textsc{MergeSortR}(L[start \ldots end])$ sorts

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Induction hypothesis MergeSortR sorts $0 \leq end - start < n$ values correctly.

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Induction hypothesis $\textsc{MergeSortR}$ sorts $0 \le \mathit{end} - \mathit{start} < n$ values correctly.

Induction step Consider $\textsc{MergeSortR}$ with $2 \le \mathit{end} - \mathit{start} = n$ values.

start *end*

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start	$\mathit{mid} := (\mathit{end} - \mathit{start})$ div 2	end

$\downarrow \textsc{MergeSortR}(L[\mathit{start} \ldots \mathit{mid}))$ $\quad$ $\downarrow \textsc{MergeSortR}(L[\mathit{mid} \ldots \mathit{end}))$

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Induction step Consider $\textsc{MergeSortR}$ with $2 \leq \mathit{end} - \mathit{start} = n$ values.

start | $\mathit{mid} := (\mathit{end} - \mathit{start}) \operatorname{div} 2$ | end

$\downarrow \textsc{MergeSortR}(L[\mathit{start} \ldots \mathit{mid}))$ $\quad \downarrow \textsc{MergeSortR}(L[\mathit{mid} \ldots \mathit{end}))$

$\boxed{\mathit{start} < \mathit{mid} < \mathit{end}}$

Proof of correctness: MergeSortR($L[start \ldots end)$) sorts

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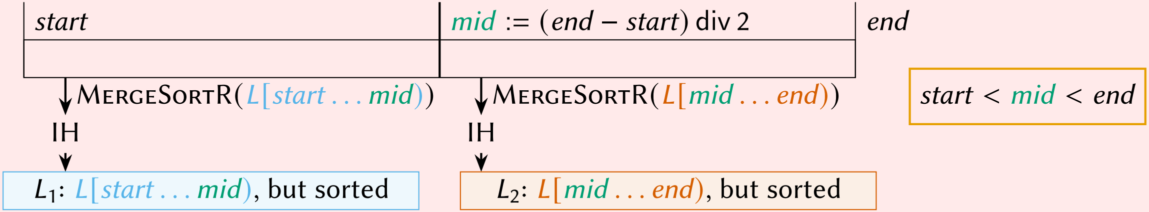

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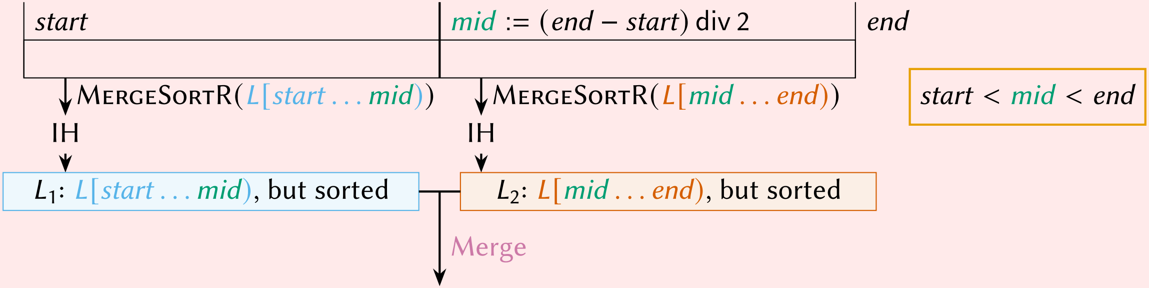

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Induction step Consider MergeSortR with $2 \leq end - start = n$ values.

$start$ | $mid := (end - start) \text{ div } 2$ | end

$\downarrow$ MergeSortR($L[start \ldots mid)$) $\downarrow$ MergeSortR($L[mid \ldots end)$)

IH IH

L_1: $L[start \ldots mid)$, but sorted | L_2: $L[mid \ldots end)$, but sorted

$\boxed{start < mid < end}$

Merge

$\boxed{\text{Assumption: Merge is correct}}$

Values in L_1 ($L[start \ldots mid)$) and L_2 ($L[mid \ldots end)$), sorted

Assumption: Merge is correct

Algorithm $\textsc{Merge}(L_1[0 \ldots N_1), L_2[0 \ldots N_2))$**:**

Input: L_1 and L_2 are sorted.

Assumption: Merge is correct

Algorithm $\textsc{Merge}(L_1[0\ldots N_1), L_2[0\ldots N_2))$**:**

Input: L_1 and L_2 are sorted.

```
R is a new array for N1 + N2 values.

return R.
```

Assumption: Merge is correct

Algorithm $\textsc{Merge}(L_1[0 \ldots N_1), L_2[0 \ldots N_2))$**:**

Input: L_1 and L_2 are sorted.

R is a new array for $N_1 + N_2$ values.

$i_1, i_2 := 0, 0.$

while $i_1 < N_1$ **or** $i_2 < N_2$ **do**

return R.

Assumption: Merge is correct

Algorithm $\textsc{Merge}(L_1[0 \dots N_1), L_2[0 \dots N_2))$:

Input: L_1 and L_2 are sorted.

```
R is a new array for N1 + N2 values.
i1, i2 := 0, 0.
while i1 < N1 or i2 < N2 do
    if i2 = N2 or (i1 < N1 and L1[i1] < L2[i2]) then
        R[i1 + i2] := L1[i1].
        i1 := i1 + 1.
    else
        R[i1 + i2] := L2[i2].
        i2 := i2 + 1.
return R.
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        i1 := i1 + 1.
    else
        R[i1 + i2] := L2[i2].
        i2 := i2 + 1.
return R.
```

L_1:

2	3	6

L_2:

1	4	5

Assumption: Merge is correct

Algorithm $\textsc{Merge}(L_1[0 \ldots N_1), L_2[0 \ldots N_2))$:

Input: L_1 and L_2 are sorted.

R is a new array for $N_1 + N_2$ values.
$i_1, i_2 := 0, 0$.
while $i_1 < N_1$ **or** $i_2 < N_2$ **do**
  if $i_2 = N_2$ **or** $(i_1 < N_1$ **and** $L_1[i_1] < L_2[i_2])$ **then**
    $R[i_1 + i_2] := L_1[i_1]$.
    $i_1 := i_1 + 1$.
  else
    $R[i_1 + i_2] := L_2[i_2]$.
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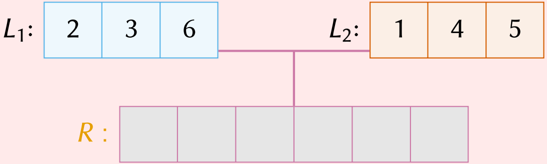
return R.



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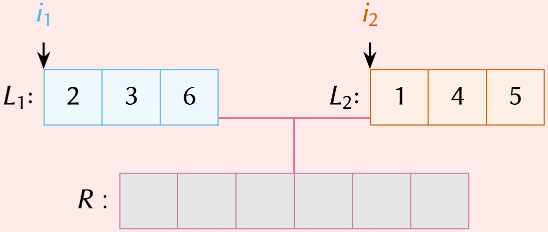

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while $i_1 < N_1$ **or** $i_2 < N_2$ **do**
  if $i_2 = N_2$ **or** $(i_1 < N_1$ **and** $L_1[i_1] < L_2[i_2])$ **then**
    $R[i_1 + i_2] := L_1[i_1].$
    $i_1 := i_1 + 1.$
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    $R[i_1 + i_2] := L_2[i_2].$
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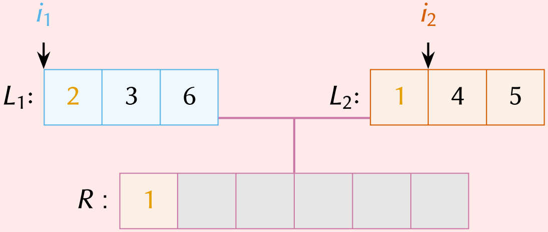

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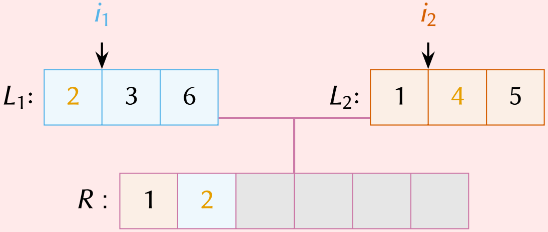

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    else
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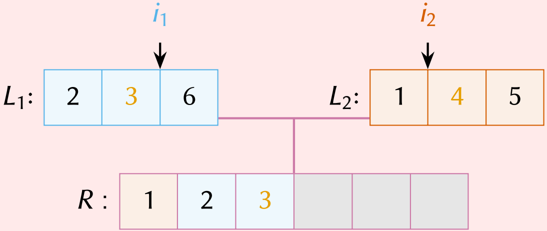
return R.
```



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        i2 := i2 + 1.
return R.
```

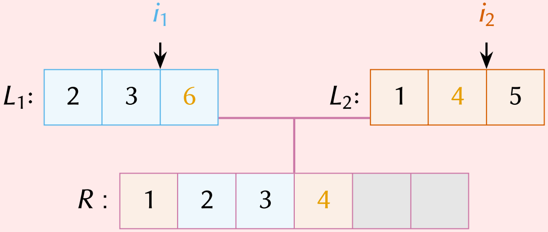

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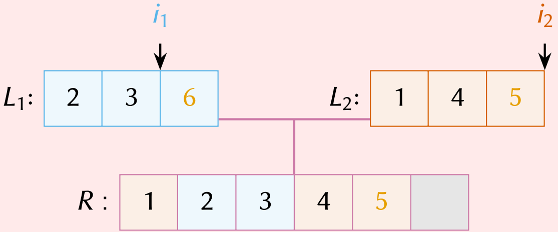

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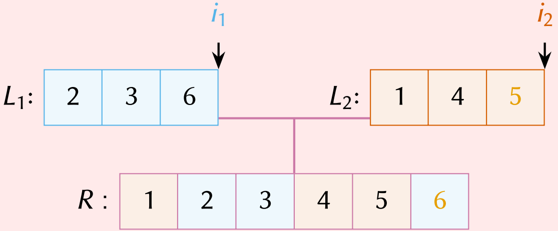
```



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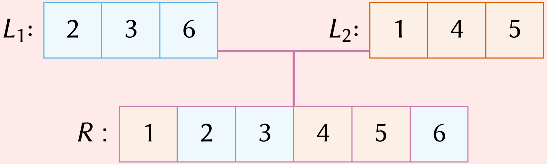

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Algorithm $\textsc{Merge}(L_1[0 \ldots N_1), L_2[0 \ldots N_2))$:

Input: L_1 and L_2 are sorted.

R is a new array for $N_1 + N_2$ values.

$i_1, i_2 := 0, 0.$

while $i_1 < N_1$ **or** $i_2 < N_2$ **do**

    /* inv: $R[0 \ldots i_1 + i_2)$ has all values from $L_1[0 \ldots i_1)$ and $L_2[0 \ldots i_2)$, sorted. */

    /* bf: $(N_1 + N_2) - (i_1 + i_2)$. */

    if $i_2 = N_2$ **or** $(i_1 < N_1$ **and** $L_1[i_1] < L_2[i_2])$ **then**

        $R[i_1 + i_2] := L_1[i_1].$

        $i_1 := i_1 + 1.$

    else

        $R[i_1 + i_2] := L_2[i_2].$

        $i_2 := i_2 + 1.$

return R.

MergeSortR: A complete example

4	6	5	3	2	1	7	9	8

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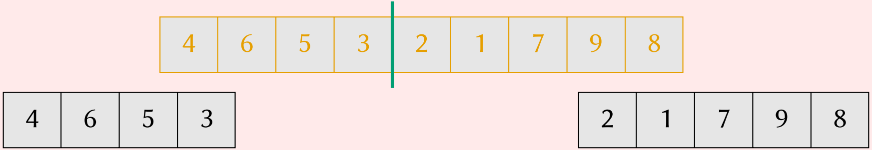

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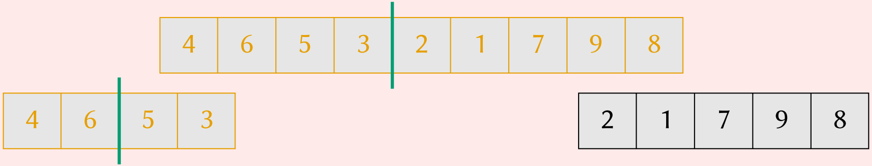
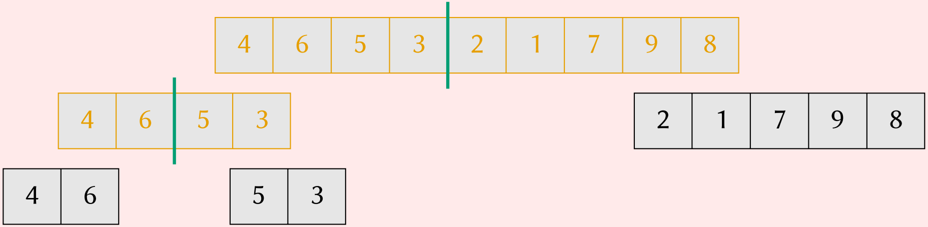


MERGESORTR: A complete example



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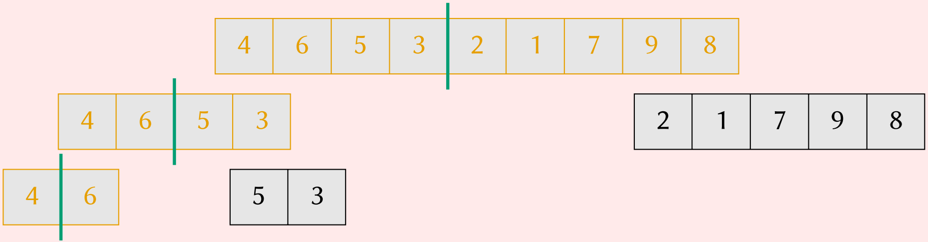
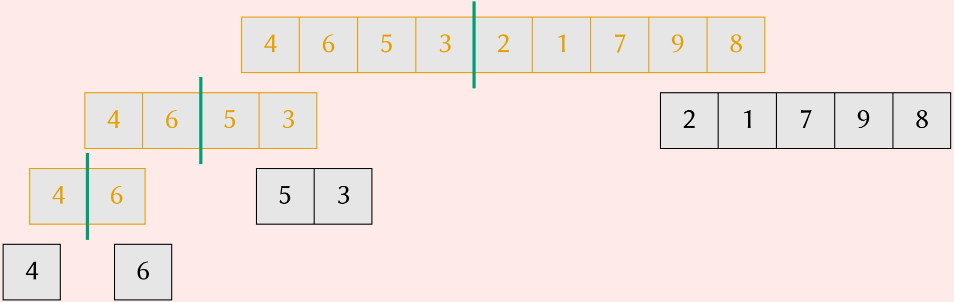
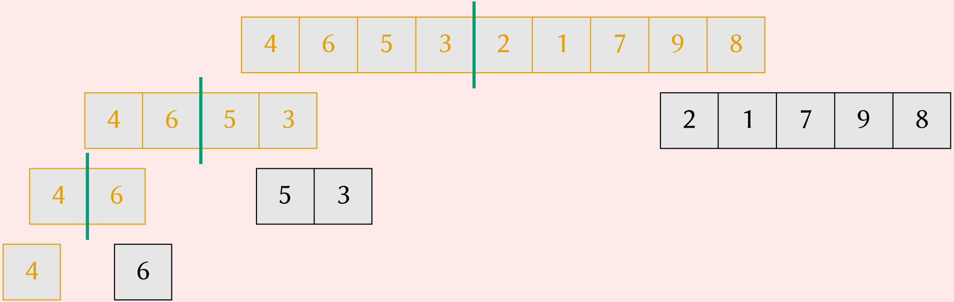


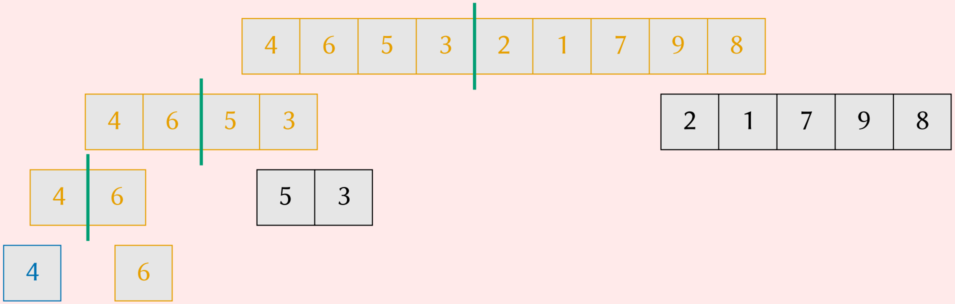
MergeSortR: A complete example



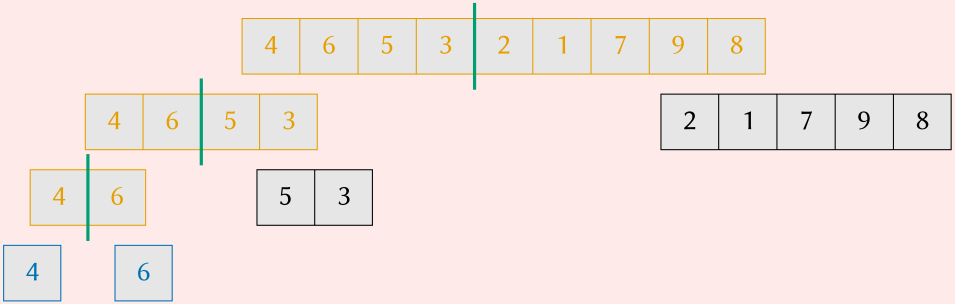
MergeSortR: A complete example



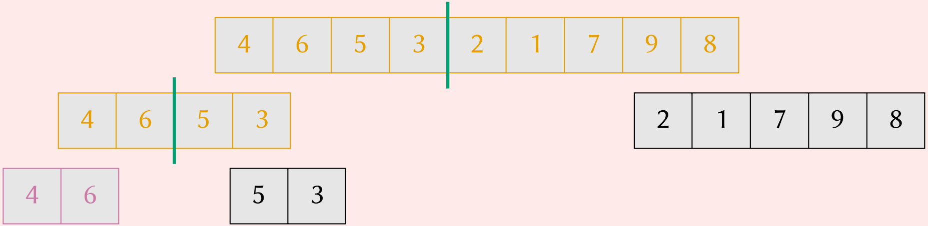
MergeSortR: A complete example



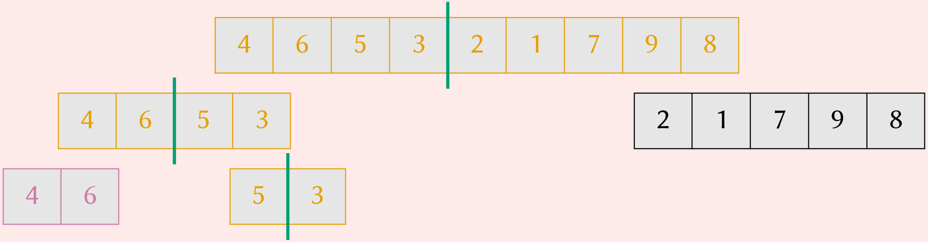
MergeSortR: A complete example



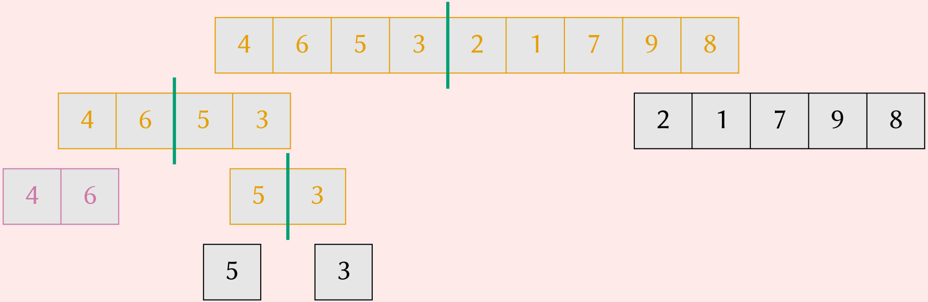
MergeSortR: A complete example



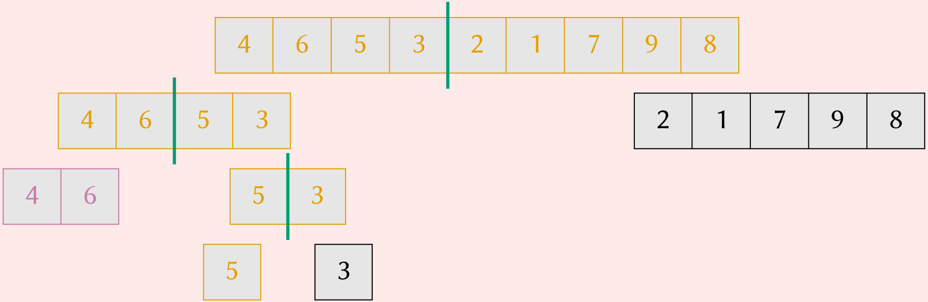
MergeSortR: A complete example



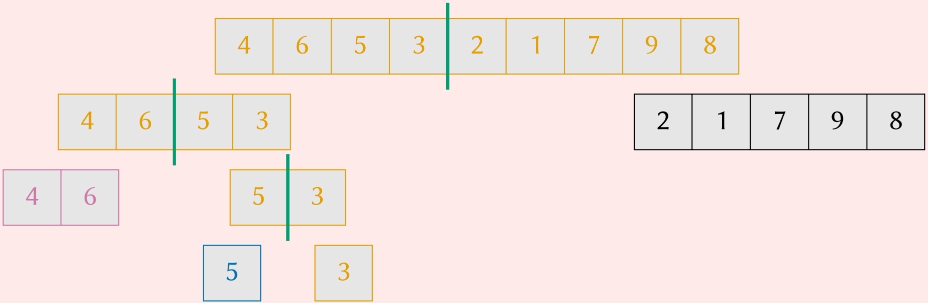
MergeSortR: A complete example



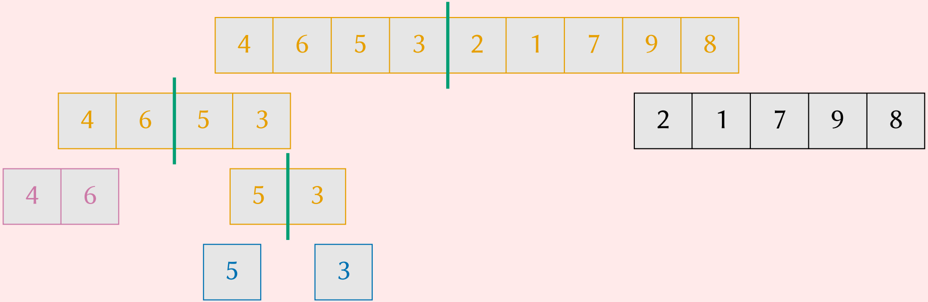
MergeSortR: A complete example



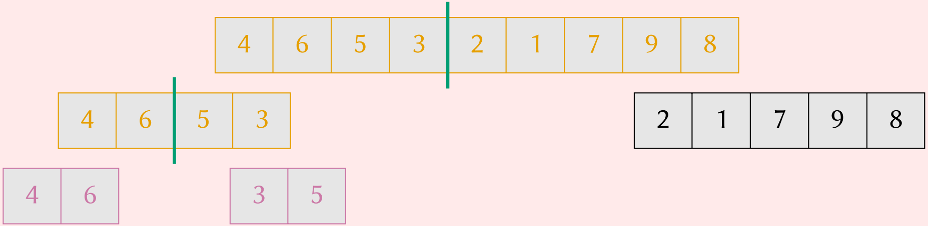
MergeSortR: A complete example



MergeSortR: A complete example



MergeSortR: A complete example



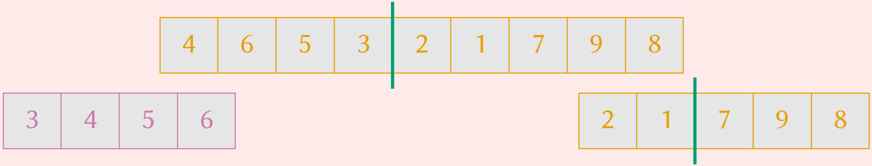
MergeSortR: A complete example

4	6	5	3	2	1	7	9	8

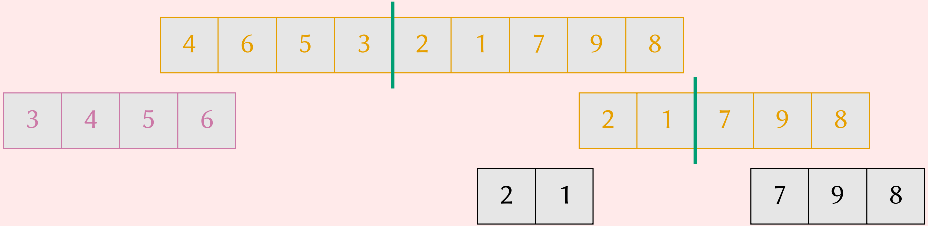
3	4	5	6

2	1	7	9	8

MergeSortR: A complete example



MergeSortR: A complete example



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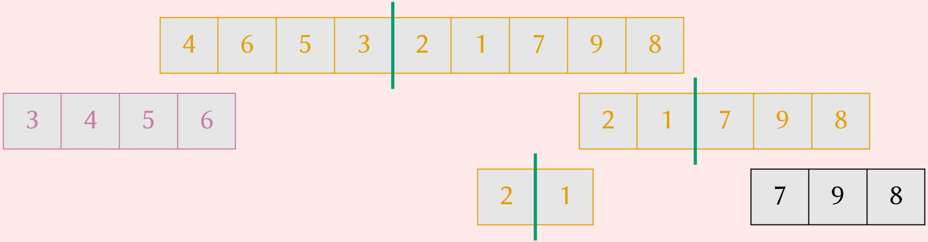
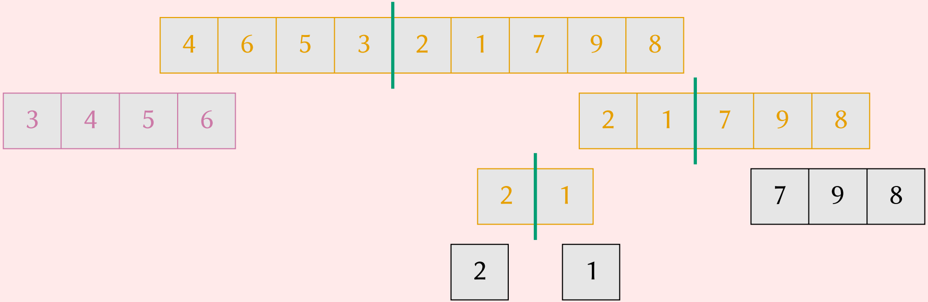


MergeSortR: A complete example



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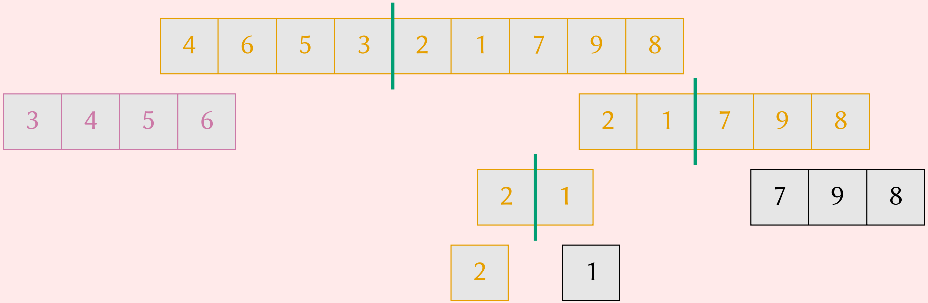

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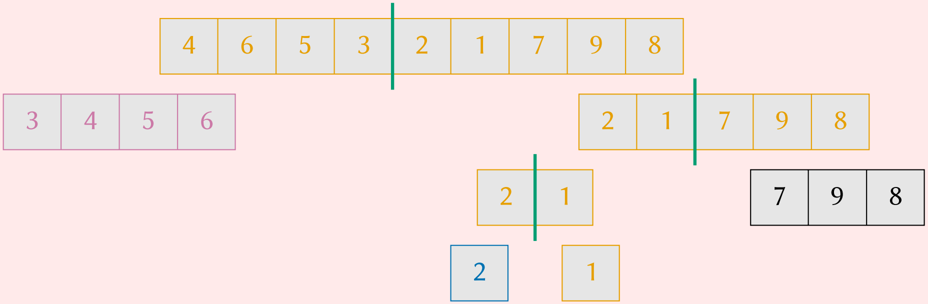
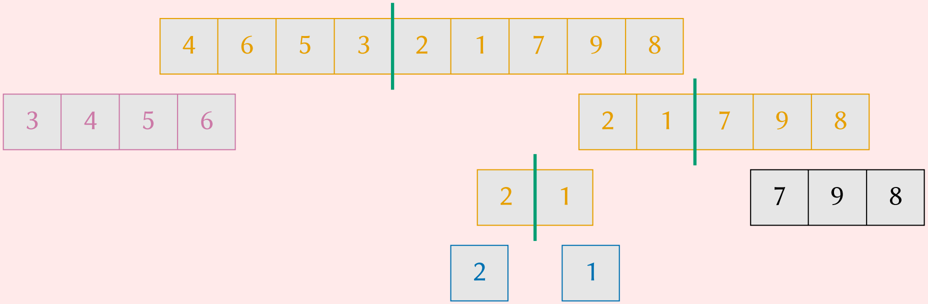


MergeSortR: A complete example



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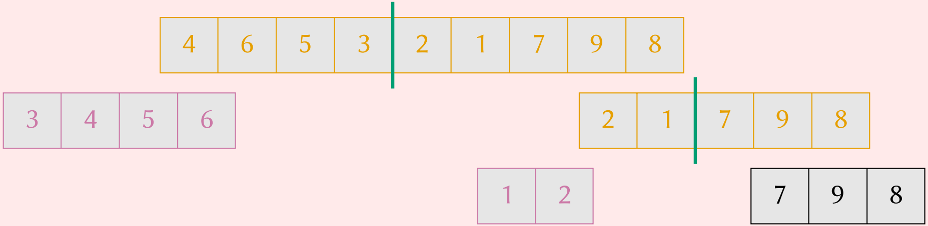

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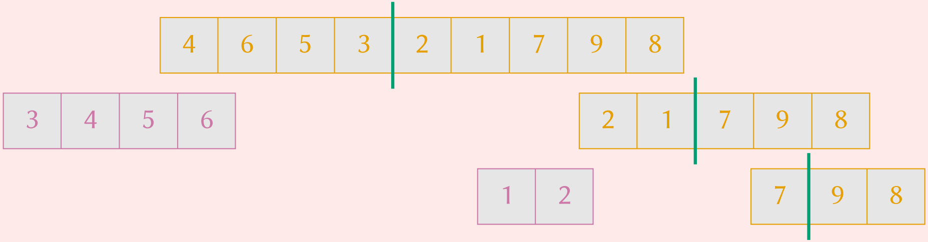

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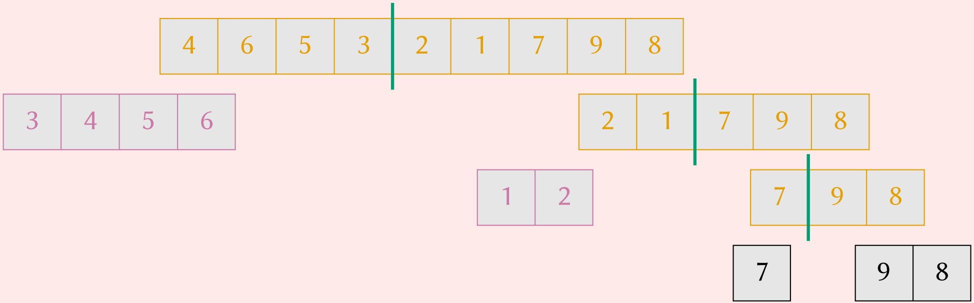

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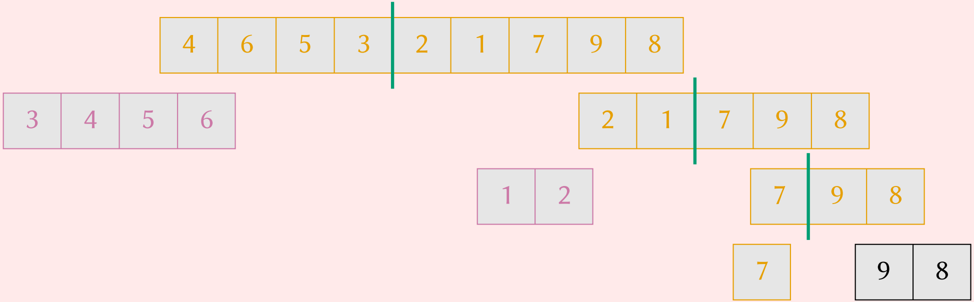

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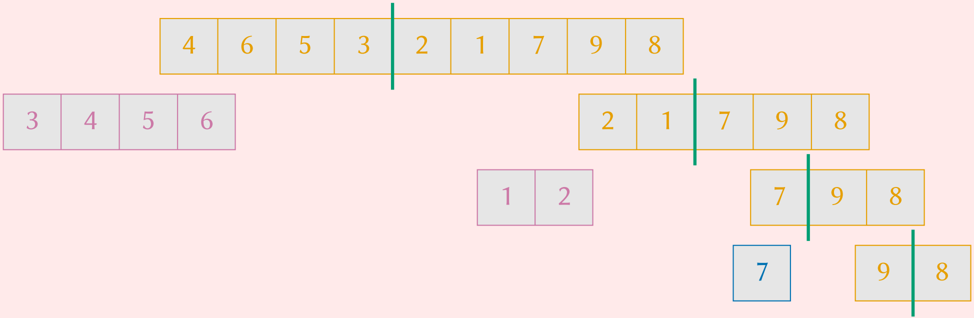

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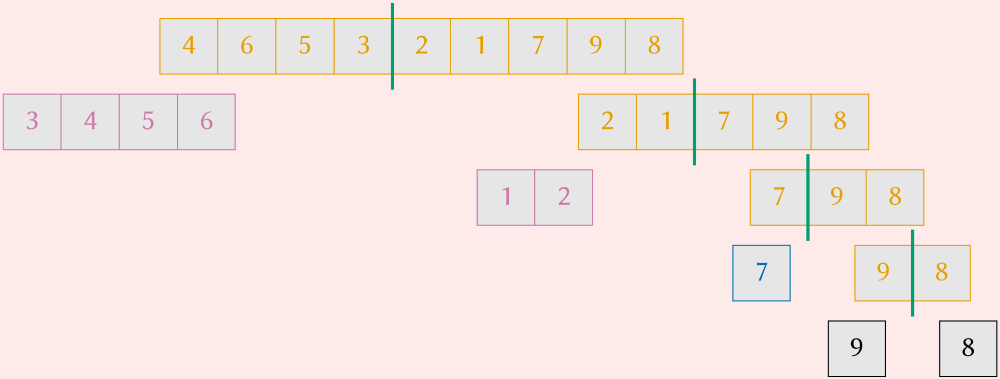

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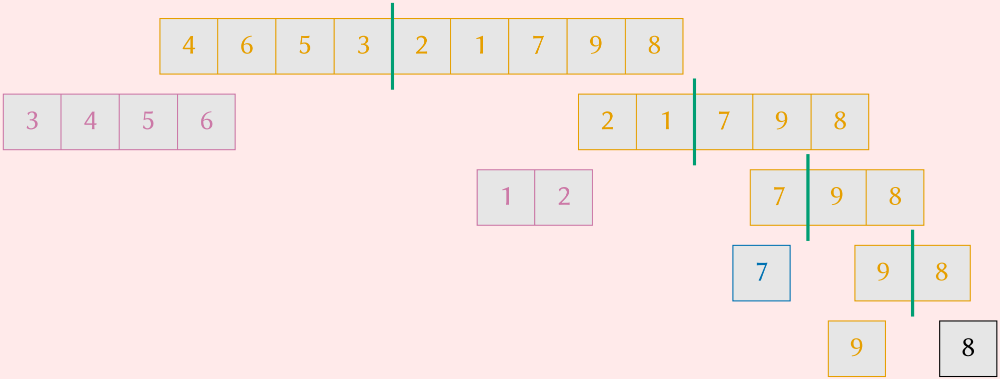
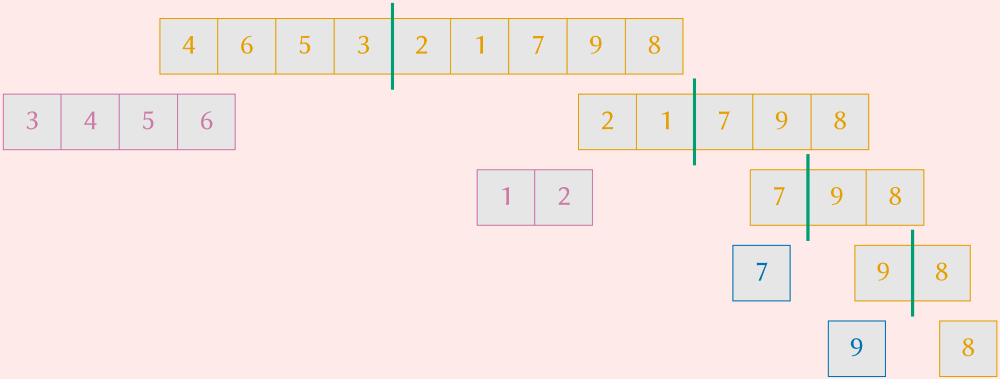
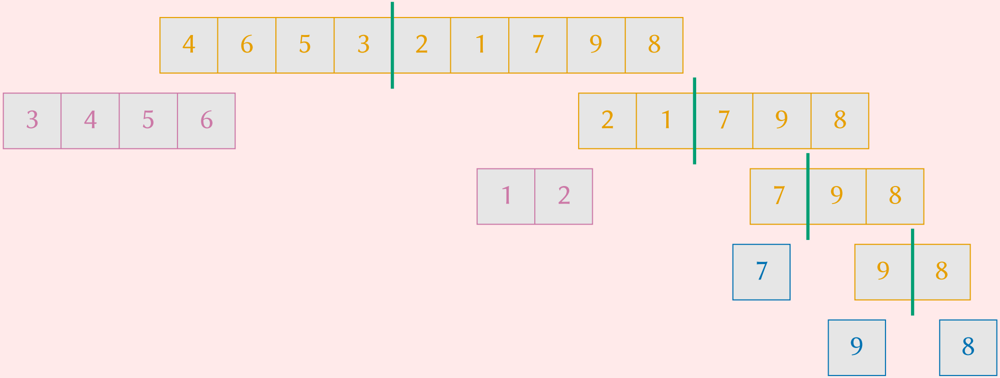


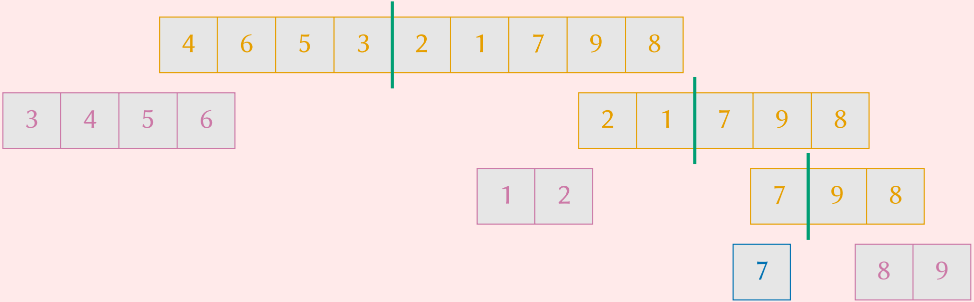
MergeSortR: A complete example



MergeSortR: A complete example



MergeSortR: A complete example



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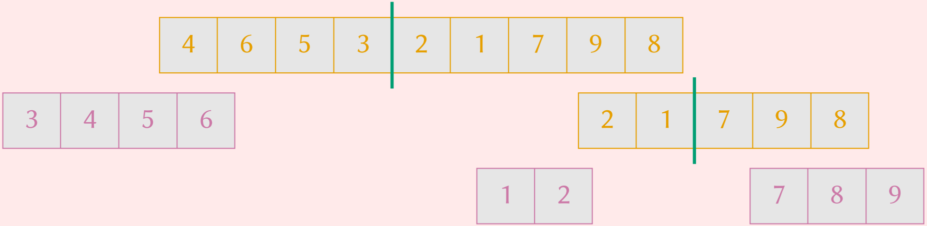

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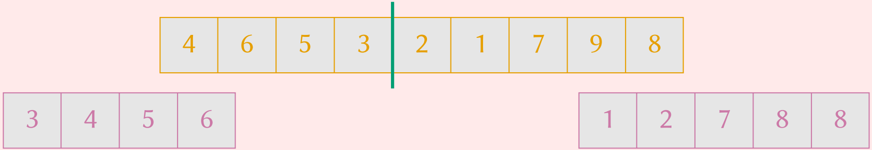

MergeSortR: A complete example

1	2	3	4	5	6	7	8	9

The complexity of MergeSortR

Plan

1. First, determine the complexity of a Merge call.
2. Then we can look at MergeSortR.

The complexity of MergeSortR

Algorithm Merge($L_1[0 \ldots N_1)$, $L_2[0 \ldots N_2)$):

Input: L_1 and L_2 are sorted.

R is a new array for $N_1 + N_2$ values.
$i_1, i_2 := 0, 0.$
while $i_1 < N_1$ **or** $i_2 < N_2$ **do**
    if $i_2 = N_2$ **or** ($i_1 < N_1$ **and** $L_1[i_1] < L_2[i_2]$) **then**
        $R[i_1 + i_2] := L_1[i_1].$
        $i_1 := i_1 + 1.$
    else
        $R[i_1 + i_2] := L_2[i_2].$
        $i_2 := i_2 + 1.$
return R.

The complexity of MergeSortR

Algorithm $\textsc{Merge}(L_1[0\ldots N_1), L_2[0\ldots N_2))$:

Input: L_1 and L_2 are sorted.

R is a new array for $N_1 + N_2$ values.
$i_1, i_2 := 0, 0$.
while $i_1 < N_1$ **or** $i_2 < N_2$ **do**
    if $i_2 = N_2$ **or** ($i_1 < N_1$ **and** $L_1[i_1] < L_2[i_2]$) **then**
        $R[i_1 + i_2] := L_1[i_1]$.
        $i_1 := i_1 + 1$.
    else
        $R[i_1 + i_2] := L_2[i_2]$.
        $i_2 := i_2 + 1$.
return R.

Comparisons: $< N_1 + N_2$.
Changes: $N_1 + N_2$.

The complexity of MergeSortR

Algorithm MergeSortR($L[start \ldots end]$):

```
if end − start > 1 then
    mid := (end − start) div 2.
    L1 := MergeSortR(L[start ... mid]).
    L2 := MergeSortR(L[mid ... end]).
    return Merge(L1, L2). N comparisons and changes.
else return L.
```

The complexity of MergeSortR

Algorithm MergeSortR($L[start \dots end]$):

1: **if** $end - start > 1$ **then**
2: $\quad mid := (end - start) \operatorname{div} 2.$
3: $\quad L_1 :=$ MergeSortR($L[start \dots mid]$).
4: $\quad L_2 :=$ MergeSortR($L[mid \dots end]$).
5: $\quad$ **return** Merge(L_1, L_2). N comparisons and changes.
6: **else return** L.

} Recursive case.

} Base case.

The complexity of MergeSortR

Algorithm MergeSortR($L[start \dots end]$):

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if end − start > 1 then
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else return L.
```

Recursive case.

Base case.

The runtime complexity of MergeSortR(L, $start$, end) with $N = end - start$ is

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + T\left(\left\lceil \frac{N}{2} \right\rceil\right) + N & \text{if } N > 1. \end{cases}$$

The complexity of MergeSortR

Algorithm MergeSortR($L[start \dots end]$):

if $end - start > 1$ **then**
    $mid := (end - start) \operatorname{div} 2$.
    $L_1 :=$ MergeSortR($L[start \dots mid]$).
    $L_2 :=$ MergeSortR($L[mid \dots end]$).
    return Merge(L_1, L_2). N comparisons and changes.
else return L.

Recursive case.

Base case.

The runtime complexity of MergeSortR(L, $start$, end) with $N = end - start$ is

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

How can we determine that $T(N) = f(N)$ for a closed-form $f(N)$?

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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How can we determine that $T(N) = f(N)$ for a closed-form $f(N)$?

We can use induction!

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

How can we determine that $T(N) = f(N)$ for a closed-form $f(N)$?

We can use induction!?

We need to know $f(N)$ to formalize an induction hypothesis!

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Recurrence tree for $T(N)$

N

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Recurrence tree for $T(N)$

N

Number	Cost
$1 = 2^0$	N

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Recurrence tree for $T(N)$

$$\begin{tikzcd}[column sep=small] & N \arrow[dl] \arrow[dr] & \\ \frac{N}{2} & & \frac{N}{2} \end{tikzcd}$$

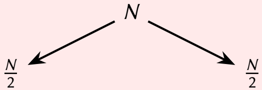
Number	Cost
$1 = 2^0$	N

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Recurrence tree for $T(N)$



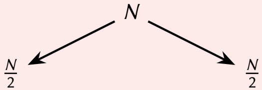
Number	Cost
$1 = 2^0$	N
$2 = 2^1$	$\frac{N}{2}$

The complexity of MERGESORTR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Recurrence tree for $T(N)$



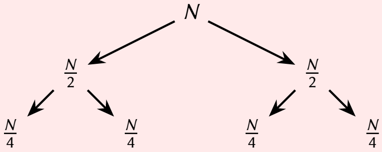
Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Recurrence tree for $T(N)$



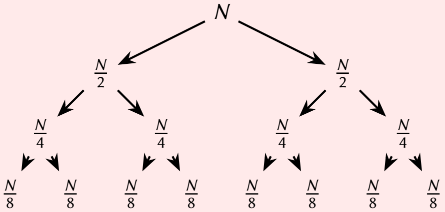
Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Recurrence tree for $T(N)$



Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$
$8 = 2^3$	$\frac{N}{8}$	$8\frac{N}{8} = N$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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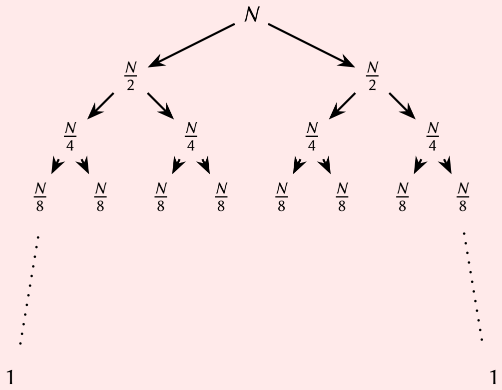
Recurrence tree for $T(N)$



Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$
$8 = 2^3$	$\frac{N}{8}$	$8\frac{N}{8} = N$
$\vdots$	$\vdots$	$\vdots$
2^i	$\frac{N}{2^i}$	$2^i\frac{N}{2^i} = N$
$\vdots$	$\vdots$	$\vdots$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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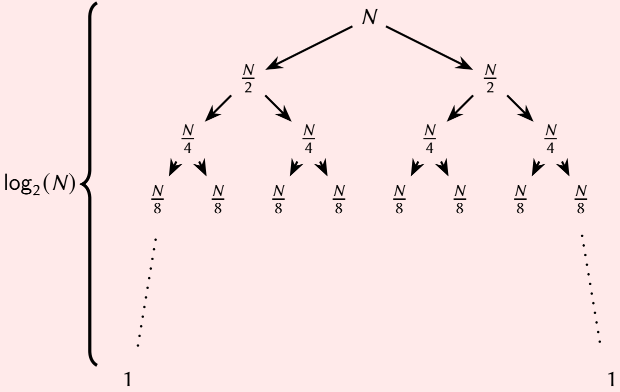
Recurrence tree for $T(N)$



Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$
$8 = 2^3$	$\frac{N}{8}$	$8\frac{N}{8} = N$
$\vdots$	$\vdots$	$\vdots$
2^i	$\frac{N}{2^i}$	$2^i\frac{N}{2^i} = N$
$\vdots$	$\vdots$	$\vdots$

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$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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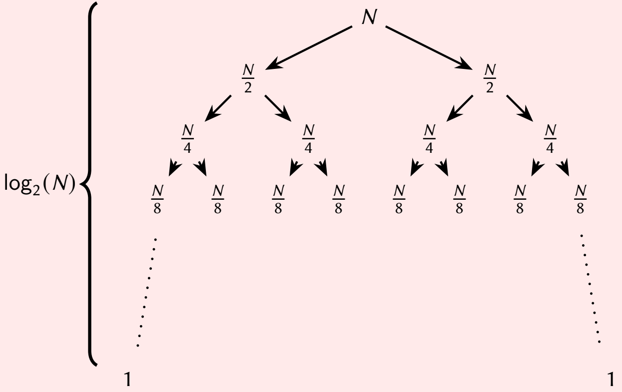
Recurrence tree for $T(N)$



Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$
$8 = 2^3$	$\frac{N}{8}$	$8\frac{N}{8} = N$
$\vdots$	$\vdots$	$\vdots$
2^i	$\frac{N}{2^i}$	$2^i\frac{N}{2^i} = N$
$\vdots$	$\vdots$	$\vdots$ $+$
		$\sim N\log_2(N)$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Can do without a power-of-two assumption?

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Can do without a power-of-two assumption?

For any N, we have $2^{\lfloor \log_2(N) \rfloor} \leq N \leq 2^{\lceil \log_2(N) \rceil}$.

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Can do without a power-of-two assumption?

For any N, we have $2^{\lfloor \log_2(N) \rfloor} \leq N \leq 2^{\lceil \log_2(N) \rceil}$.

The assumption provides lower and upper bounds that are off by a small factor

$\longrightarrow$ Typically good enough to understand the complexity of your code.

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + T\left(\left\lceil \frac{N}{2} \right\rceil\right) + N & \text{if } N > 1. \end{cases}$$

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This induction becomes messy due to terms $\left\lfloor \frac{N}{2} \right\rfloor$ and $\left\lceil \frac{N}{2} \right\rceil$.

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$$\log_2(2+1) - 1 = \log_2(2) + (\log_2(3) - \log_2(2)) - 1 \approx 1 + (1.6 - 1) - 1) = \log_2(2) - 0.4.$$

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For *big enough* values of i and c_2, $i > B$, this is certainly true!

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For *big enough* values of i and c_2, $i > B$, this is certainly true!

Trick: make sure we always have big values of i.

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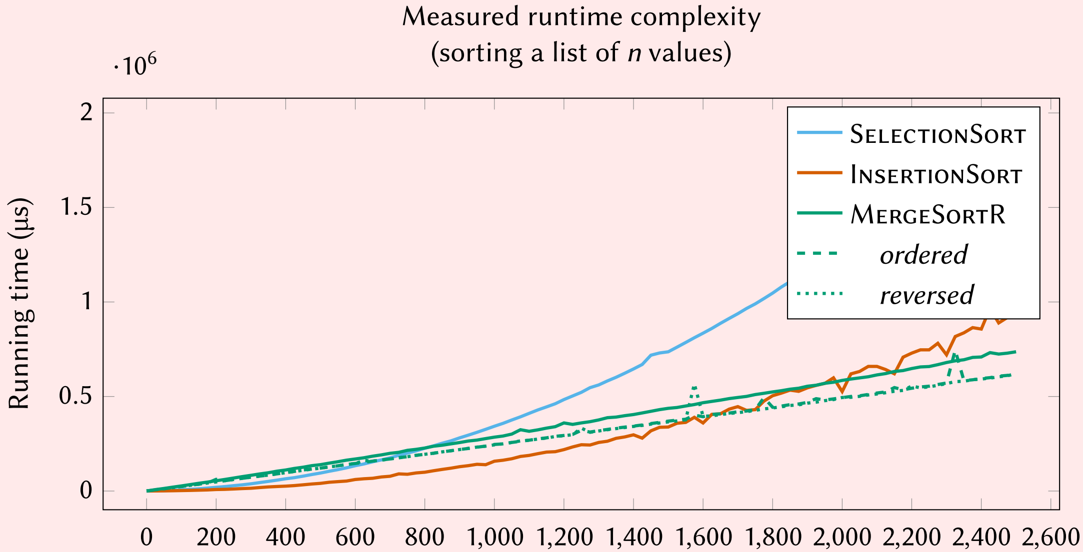
There are also *standard solutions* that you can use: the Master Theorem.

The performance of MergeSortR



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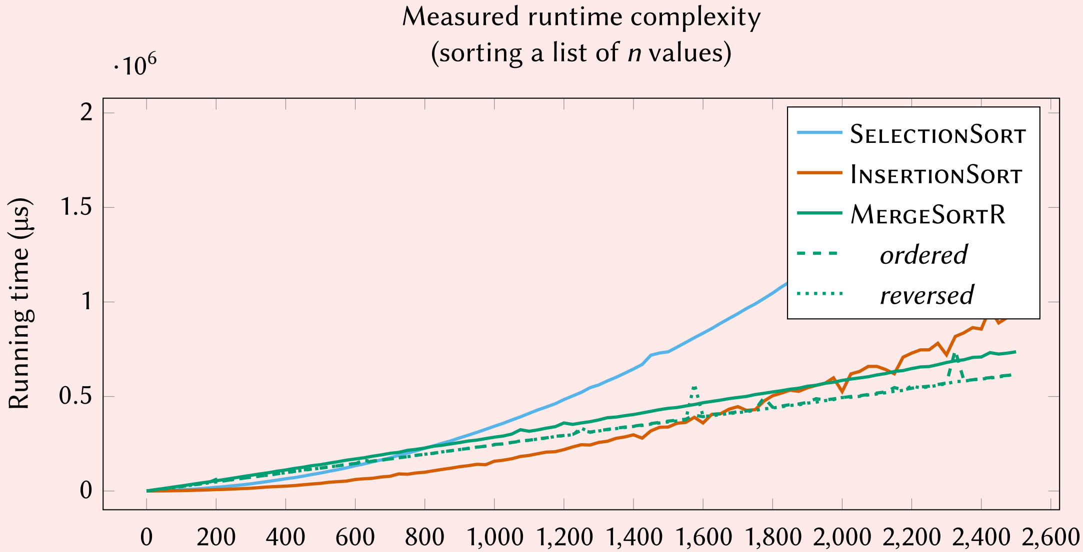

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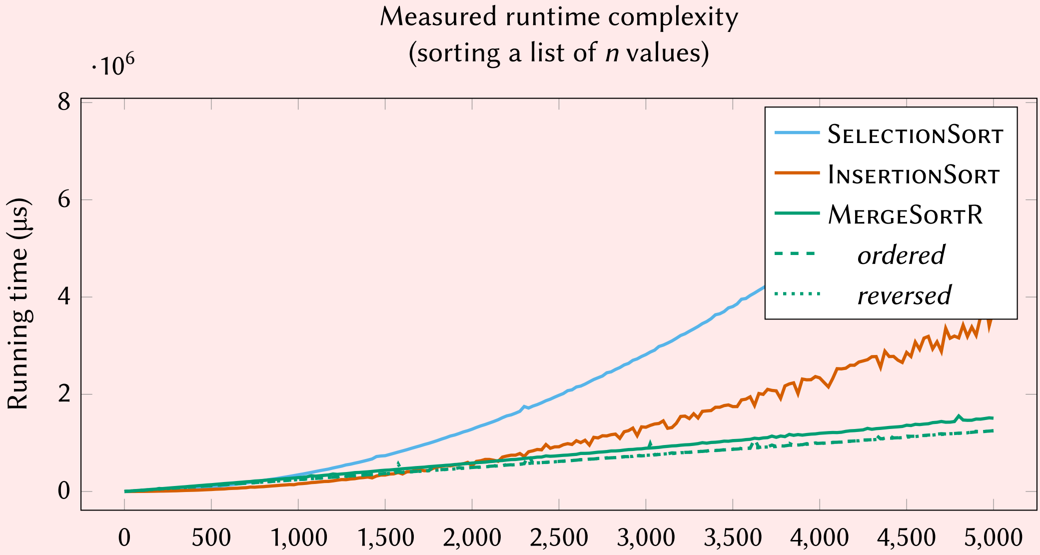

The performance of MergeSortR

MergeSortR should be much better than SelectionSort and InsertionSort:
Especially on big lists.

Concern: MergeSortR has big constants.

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- A lot of recursive calls that only get us to arrays of size one.

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Especially on big lists.

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Can we finetune MergeSortR to reduce these constants?

- Each Merge makes new arrays.
  Idea: make a single target array to merge into.
- A lot of recursive calls that only get us to arrays of size one.
  Idea: switch from top-down (big-to-small arrays) to bottom-up (small-to-big arrays),
  we can do so using a loop instead of recursion!

MergeSort: A better MergeSortR

Algorithm MergeSort($L[0 \dots N)$):

R is a new array for N values.
$sl := 1$. The current *sorted length* of blocks in L.
while $sl \leq N$ **do**
    $i := 0$.
    while $i < N$ **do**
        Conceptually: Merge $L[i \dots i + sl)$ and $L[i + sl \dots i + 2sl)$ into $R[i \dots i + 2sl)$.

        $i := i + 2sl$.
    $sl := 2sl$.
    Switch the role of L and R.

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        Careful: N does not have to be a multiple of $2sl$.
        MergeInto(L, i, $\min(i+sl, N)$, $\min(i+2sl, N)$, R).
        $i := i + 2sl$.
    $sl := 2sl$.
    Switch the role of L and R.

MergeSort: A better MergeSortR

Algorithm $\textsc{MergeInto}(S[0\ldots N), \mathit{start}, \mathit{mid}, \mathit{end}, T[0\ldots N))$**:**

Input: $0 \leq \mathit{start} \leq \mathit{mid} \leq \mathit{end} \leq N$ and $S[\mathit{start}\ldots\mathit{mid})$ and $S[\mathit{mid}\ldots\mathit{end})$ are sorted.

```
i_1, i_2 := start, mid.
while i_1 < mid or i_2 < end do
    if i_2 = end or (i_1 < mid and S[i_1] < S[i_2]) then
        T[i_1 + i_2] := S[i_1].
        i_1 := i_1 + 1.
    else
        T[i_1 + i_2] := S[i_2].
        i_2 := i_2 + 1.
```

MergeSort: A better MergeSortR

4	6	5	3	2	1	7	9	8

MergeSort: A better MergeSortR

$sl = 1$, L:

4	6	5	3	2	1	7	9	8

Conceptually:

4	6	5	3	2	1	7	9	8

MergeSort: A better MergeSortR

$sl = 2$, L:

4	6	3	5	1	2	7	9	8

Conceptually:

4	6		3	5		1	2		7	9		8

MergeSort: A better MergeSortR

$sl = 4$, L:

3	4	5	6	1	2	7	9	8

Conceptually:

3	4	5	6

1	2	7	9

8

MergeSort: A better MergeSortR

$sl = 8$, L:

1	2	3	4	5	6	7	9	8

Conceptually:

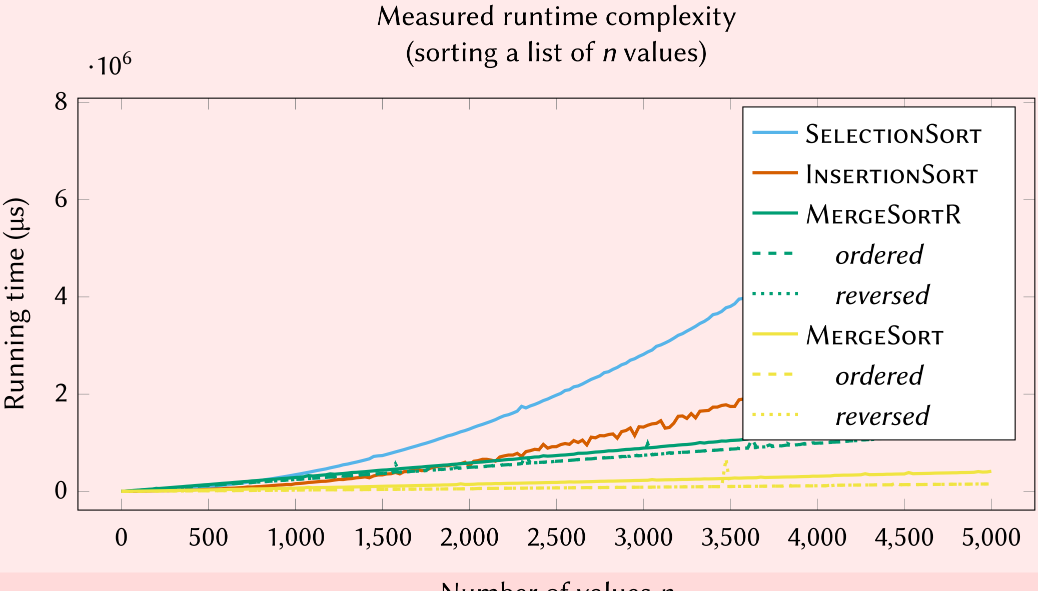
1	2	3	4	5	6	7	9		8

MergeSort: A better MergeSortR

$sl = 16$, L:

1	2	3	4	5	6	7	8	9

MergeSort: A better MergeSortR



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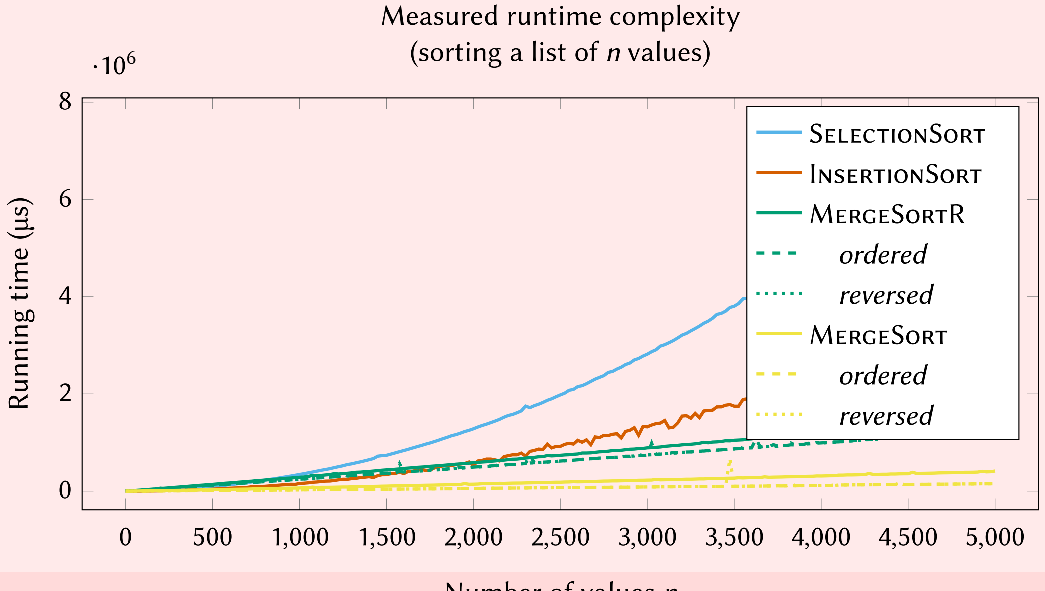

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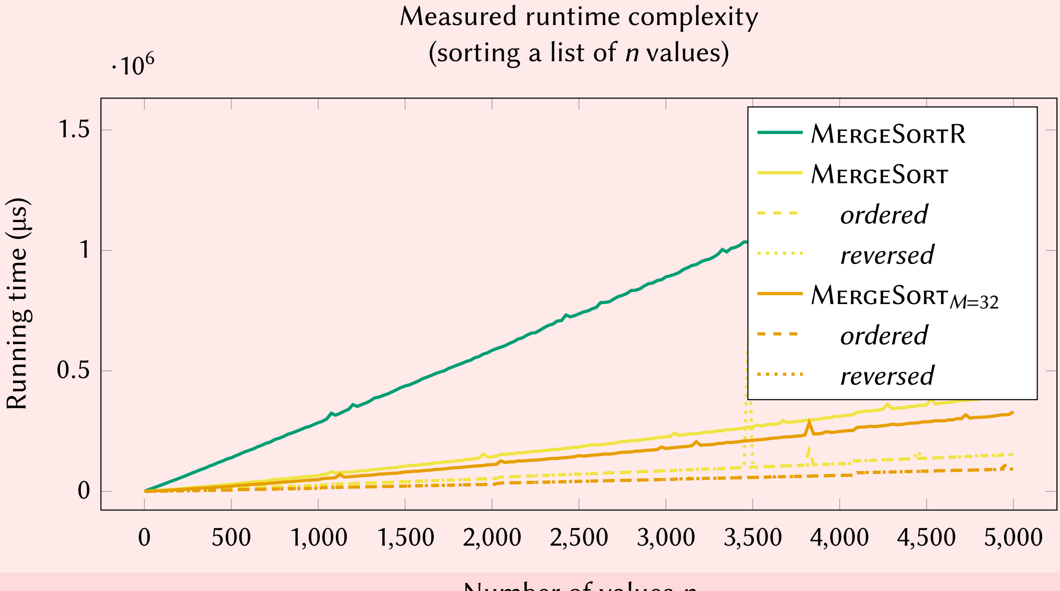

Final notes on MergeSort

- Runtime complexity: $\sim N\log_2(N)$ comparisons and changes;
- Memory complexity: $\sim N$ (for merging).

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- Memory complexity: $\sim N$ (for merging).

The power of MergeSort

The MergeSort algorithm is at the basis of many large-scale sort algortihms:

- multi-threaded sorting (GiB),
- sorting data on external memory (GiB–TiB),
- sorting data in a cluster (TiB–PiB).

Final notes on MergeSort

- Runtime complexity: $\sim N \log_2(N)$ comparisons and changes;
- Memory complexity: $\sim N$ (for merging).

The power of MergeSort

The MergeSort algorithm is at the basis of many large-scale sort algortihms:

- multi-threaded sorting (GiB),
- sorting data on external memory (GiB–TiB),
- sorting data in a cluster (TiB–PiB).

The power of Merge

The Merge algorithm is flexible: you can easily change it to

- compute the *union* (without duplicates) of two sorted list;
- compute the *intersection* of two sorted list;
- compute the *difference* of two sorted list;
- compute a *join* of two tables (if sorted on the join attributes).

Final notes on MergeSort

	C++	Java
MergeSort	`std::stable_sort`	`java.util.Arrays.sort` (usually)
Merge	`std::merge`	

Merge-like	`std::set_union`
	`std::set_intersection`
	`std::set_difference`
	`std::set_symmetric_difference`	

(related)	`std::inplace_merge`	

Intermezzo: Recurrence trees

In a recurrence tree

- nodes labeled N represent a *function call* with “input size N”;
- the children of a node represent *recursive calls*;
- per node, we can determine *the work* within that call (besides recursion);
- per depth, we can determine the *total work for that depth*;
- by *summing over all depths*: the total complexity.

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We already saw two examples: BinarySearchR and MergeSortR.

Intermezzo: Recurrence trees

Example: the *Fibonacci numbers*

$$fib(N) = \begin{cases} 1 & \text{if } N = 1 \text{ or } N = 2; \\ fib(N-1) + fib(N-2) & \text{if } N > 2. \end{cases}$$

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Prove that $fib(N) \leq 2^N$

Simplication: $fib(i-2) \leq fib(i-1)$.

	Number	Cost	Total
N	$1 = 2^0$	1	$1 \cdot 1 = 1$

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 & & & N \arrow[dll] \arrow[drr] & & & \\
 & N-1 \arrow[dl] \arrow[dr] & & & & N-1 \arrow[dl] \arrow[dr] & \\
N-2 & & N-2 & & N-2 & & N-2
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Number	Cost	Total
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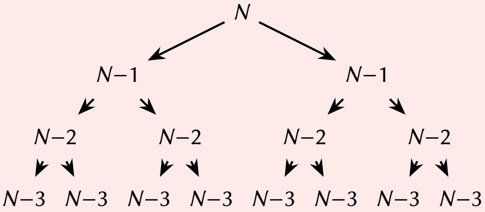

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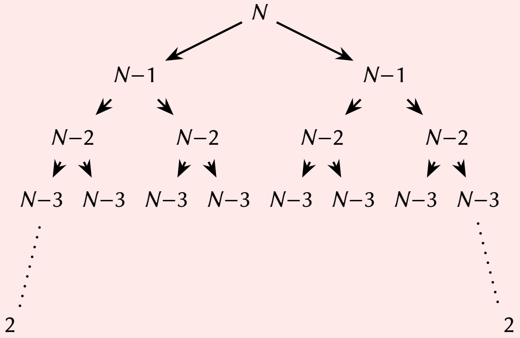

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$\vdots$	$\vdots$	$\vdots$
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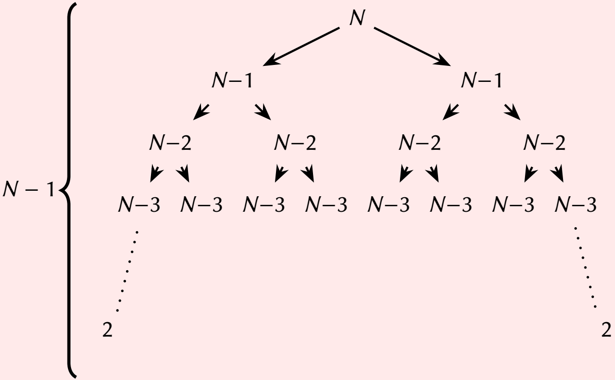

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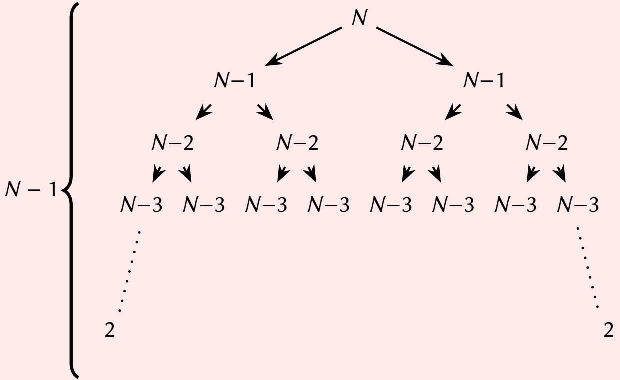

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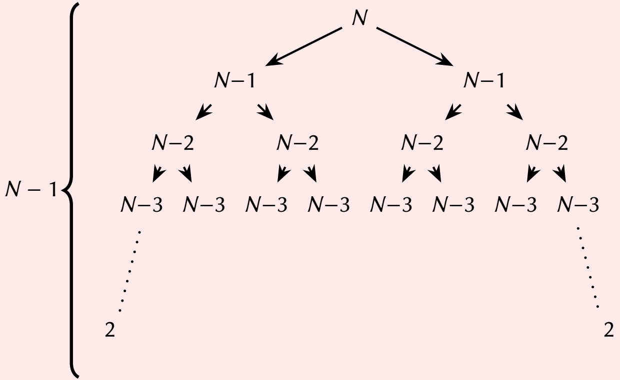

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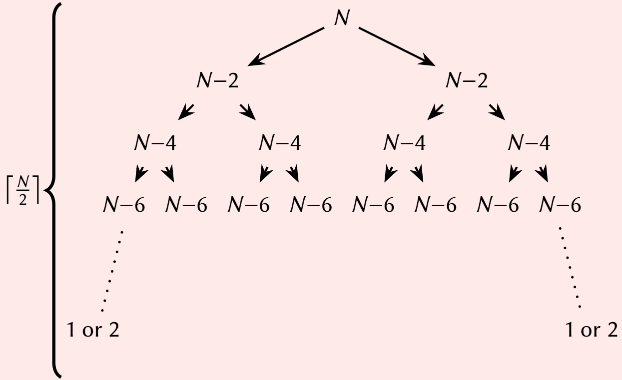
Prove that $2^{\lceil \frac{N}{2} \rceil} \leq fib(N)$

Simplication: $fib(i-1) \geq fib(i-2)$.



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$$\sum_{i=0}^{\lceil \frac{N}{2} \rceil} 2^i = 2^{\lceil \frac{N}{2} \rceil + 1} - 1$$

Intermezzo: Recurrence trees

Example: the *Fibonacci numbers*

$$fib(N) = \begin{cases} 1 & \text{if } N = 1 \text{ or } N = 2; \\ fib(N-1) + fib(N-2) & \text{if } N > 2. \end{cases}$$

Via recurrence trees, we have proven that:

$$2^{\lceil \frac{N}{2} \rceil} \leq fib(N) \leq 2^N.$$

Intermezzo: The Master Theorem

Let $T(N)$ be a *recurrence* of the form

$$T(N) = \begin{cases} \textit{constant} & \text{if } \textit{base case}; \\ aT\left(\frac{N}{b}\right) + f(N) & \text{if } \textit{recursive case}, \end{cases}$$

with $a \geq 1$, $b > 1$, and we can read $\frac{N}{b}$ also as $\left\lceil \frac{N}{b} \right\rceil$ or $\left\lfloor \frac{N}{b} \right\rfloor$.

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1. if $f(N) = \mathcal{O}N^{\log_b(a-\epsilon)}$ with $\epsilon > 0$, then $T(N) \sim N^{\log_b(a)}$.
2. if $f(N) = \Theta N^{\log_b(a)} \log^k(N)$ with $k \geq 0$, then $T(N) \sim N^{\log_b(a)} \log^{k+1}(N)$.
3. if $f(N) = \Omega N^{\log_b(a+\epsilon)}$ with $\epsilon > 0$ and $af\left(\frac{N}{b}\right) \leq cf(N)$ for a $c < 1$ (for large N), then $T(N) \sim f(N)$.

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Someone else has already proved this—so we can reuse the result!

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Example: Runtime complexity of BinarySearchR

$$T(N) = \begin{cases} 4 & \text{if } N = 1; \\ T\left(\frac{N}{2}\right) + 8 & \text{if } N > 1. \end{cases}$$

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We have $a = 1$, $b = 2$, $f(N) = 8 \sim 1 = N^{\log_2(1)}$.

Case 2 yields: $T(N) \sim N^{\log_2(1)} \log^1(N) = \log(N)$.

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Example: Runtime complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + T\left(\left\lceil \frac{N}{2} \right\rceil\right) + N & \text{if } N > 1. \end{cases}$$

Intermezzo: The Master Theorem

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We have $a = 2$, $b = 2$, $f(N) = N \sim N = N^{\log_2(2)}$.

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Example: Runtime complexity of MergeSortR

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A third example

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ 7T\left(\left\lfloor \frac{N}{4} \right\rfloor\right) + N & \text{if } N > 1. \end{cases}$$

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A third example

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We have $a = 7$, $b = 4$, $f(N) = N = \mathcal{O}N^{\log_4(7)-\epsilon}$.

Intermezzo: The Master Theorem

Let $T(N)$ be a *recurrence* of the form

$$T(N) = \begin{cases} \textit{constant} & \text{if } \textit{base case}; \\ aT\left(\frac{N}{b}\right) + f(N) & \text{if } \textit{recursive case}, \end{cases}$$

with $a \geq 1$, $b > 1$, and we can read $\frac{N}{b}$ also as $\left\lceil \frac{N}{b} \right\rceil$ or $\left\lfloor \frac{N}{b} \right\rfloor$. We have the following

1. if $f(N) = \mathcal{O}N^{\log_b(a-\epsilon)}$ with $\epsilon > 0$, then $T(N) \sim N^{\log_b(a)}$.
2. if $f(N) = \Theta N^{\log_b(a)} \log^k(N)$ with $k \geq 0$, then $T(N) \sim N^{\log_b(a)} \log^{k+1}(N)$.
3. if $f(N) = \Omega N^{\log_b(a+\epsilon)}$ with $\epsilon > 0$ and $af\left(\frac{N}{b}\right) \leq cf(N)$ for a $c < 1$ (for large N), then $T(N) \sim f(N)$.

A third example

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Case 1 yields: $T(N) \sim N^{\log_4(7)} \approx N^{1.40367\ldots}$.

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A fourth example

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ 2T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + N^3 & \text{if } N > 1. \end{cases}$$

Intermezzo: The Master Theorem

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We have $a = 2$, $b = 2$, $f(N) = N^3 = \Omega N^{\log_2(2)+\epsilon}$.

Intermezzo: The Master Theorem

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We have $a = 2$, $b = 2$, $f(N) = N^3 = \Omega N^{\log_2(2)+\epsilon}$.

Case 3 yields: $T(N) \sim N^3$.

Intermezzo: The Master Theorem

Let $T(N)$ be a *recurrence* of the form

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with $a \geq 1$, $b > 1$, and we can read $\frac{N}{b}$ also as $\left\lceil \frac{N}{b} \right\rceil$ or $\left\lfloor \frac{N}{b} \right\rfloor$. We have the following

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Feel free to use the Master Theorem, we will provide a copy during the final exam.