1 Generate Matrix C

1.1 Generate Set Ω

Let $|\cdot|$ denotes cardinality and Δ denote symmetric different. Let $m \in \mathbb{Z}^+$, $n = 2^m$, $\Omega = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a set of n element such that $|\alpha_i| \leq |\alpha_{i+1}|$ and $|\alpha_i \Delta \alpha_{i+1}| \leq 2$. Let $\alpha_0 = \{\varnothing\}$. The following pseudo code shows the way generating Ω:

```
let Omega = {{}, {}, {1}, {2}, ..., {m}};
for i in {1, 2, ..., m - 1}:
    let sets = sets in Omega with size equals i
    for set in sets:
        let avail_range = [max(set) + 1, ..., m]
        if avail_range[0] not in Omega[-1]:
            avail_range = avail_range.reverse()
        for element in avail_range:
            Omega.append({set..., element})  // set... means extend the set
```

when $m = 4$, set Ω is shown below:

```
Omega = {set(), set(), {1}, {2}, {3}, {4},
         {1, 4}, {1, 3}, {1, 2}, {2, 4}, {2, 3},
         {3, 4}, {1, 3, 4}, {1, 2, 3}, {1, 2, 4},
         {2, 3, 4}, {1, 2, 3, 4}}
```

1.2 Generate Matrix $A \in \mathcal{A}_n^2$

Let $\mathcal{A}_n^2$ denotes the sets of invertible (-1, 1) matrices of order n. Let matrix $A \in \mathcal{A}_n^2$, with set $\Omega = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ satisfies $|\alpha_i| \leq |\alpha_{i+1}|$ and $|\alpha_i \Delta \alpha_{i+1}| \leq 2$, matrix A can be constructed as follows.
For every $1 \leq i, j \leq n$:

$$a_{ij} = \begin{cases} -1, \ \alpha_j \bigcap (\alpha_{i-1} \bigcup \alpha_i) = \alpha_{i-1} \Delta \alpha_i \ and \ |\alpha_{i-1} \Delta \alpha_i| = 2 \\ (-1)^{|\alpha_{i-1} \bigcap \alpha_j| + 1}, \ \alpha_j \bigcap (\alpha_{i-1} \bigcup \alpha_i) \neq \varnothing \ but \ does \ not \ meet \ the \ condition \ above \\ 1, \ \alpha_j \bigcap (\alpha_{i-1} \bigcup \alpha_i) = \varnothing \end{cases}.$$

With the $Omega$ shown in section 1.1, the matrix $A \in \mathcal{A}_n^2$ constructed is shown below:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1
\end{bmatrix}$$

1.3 Generate Matrix $B \in \mathcal{A}_{n-1}^1$

Let $\mathcal{A}_{n-1}^1$ denotes the sets of invertible (0, 1) matrices of order n. Let matrix $B \in \mathcal{A}_{n-1}^1$. Consider the map Φ which assigns to any matrix $B \in \mathcal{A}_{n-1}^1$ a matrix $\Phi(B) \in \mathcal{A}_{n-1}^1$ in the following way:

$$\Phi(B)=\left(\begin{array}{cc} 1 & 1_{n-1} \\ -1_{n-1}^{T} & 2B-J_{n-1} \end{array}\right).$$

Therefore, we have the following way to construct matrix $B \in \mathcal{A}_{n-1}^1$ with $A=\{\alpha_{ij}\} \in \mathcal{A}_n^2$:

$$B=\frac{1}{2}(J_{n-1}+\{\alpha_{ij}\}_{2\leq i\leq n, 2\leq j\leq n}).$$

Notice that the $A \in \mathcal{A}_n^2$ we constructed above has it's first column as:

$$A_2=\Phi(B)=\left(\begin{array}{c} 1 \\ 1_{n-1}^{T} \end{array}\right),$$

we need a negative first column, which is said $\{\alpha_{ij}\}_{2\leq i\leq n, 2\leq j\leq n} \in \mathcal{A}_n^2$ multiplied with -1.
Therefore we have the relation between matrix $B \in \mathcal{A}_{n-1}^1$ and $A \in \mathcal{A}_n^2$ in the implementation shown as follows:

$$B=\frac{1}{2}(J_{n-1}-\{\alpha_{ij}\}_{2\leq i\leq n, 2\leq j\leq n}).$$

With the matrix $A \in \mathcal{A}_n^2$ constructed above, a constructed matrix B is shown below:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}$$

1.4 Generate and Verify Matrix C

Let S and T be two non-singular matrices of order n_1 and n_2. Define $S \diamond T$ ass follows:

$$\begin{bmatrix}
s_{11} & \dots & s_{1n_1} & 0 & \dots & 0 \\
s_{21} & \dots & s_{2n_1} & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
s_{n_1 1} & \dots & s_{n_1 n_1} & 0 & \dots & 0 \\
0 & 0\dots0 & 1 & t_{11} & \dots & t_{1n_2} \\
0 & 0\dots0 & 0 & t_{21} & \dots & t_{2n_2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0\dots0 & 0 & t_{n_2 1} & \dots & t_{n_2 n_2}
\end{bmatrix}$$

Consider the $(0,1)$ matrix $C=A_1\diamond(A_2\diamond(...(A_{r-1}\diamond A_r))...)$. Let $M=C^{-1}=(m_{ij})$, $\chi(C)=max_{i,j}|m_{ij}|$. Notice that C is a spare $(0,1)$ matrix, as shown in the article, $\chi(C)$ has same order of magnitude as the condition number of matrix C, which can also be used for ill conditioned measurement. The following block shows some result of $\chi(C)$ related to order r.

```
r = 2, order of C = 4, χ(C) = 1.0
r = 3, order of C = 11, χ(C) = 2.0
r = 4, order of C = 26, χ(C) = 260.0
r = 5, order of C = 57, χ(C) = 106491641548.6
```

We can see that as order r grows, $\chi(C)$ grows rapidly.

2 Complexity of Generating Matrix C

To generate set Ω, everytime we only need to take out an generated set and insert one new element inside. Therefore, without cosidering the complexity of set insertion, the ideally time complexity is $O(n)$.

To generate every entry of matrix $A \in \mathcal{A}_n^2$, a visit of three continuous element in set Ω is necessary. Let n be the order of matrix A, without considering the complexity of set accessing, the ideally time complexity is $O(n) + O(n^2) = O(n^2)$.

Generating matrix B is a simple matrix operation, let $n-1$ be the order of B, the time complexity is $O((n-1)^2) + O(n^2) = O(n^2)$.

In order to generate matrix C, we need a series of matrix A_1, A_2, ... , A_r. As shown above, for matrix A_r with order r, the time complexity is $O(r^2)$. Therefore the time complexity generating these matrices is $\sum_{i=1}^{r} O(r^2) = O(n^3)$

Therefore, when r grows, with an $O(n^3)$ time complexity, the time required to generate matrix C grows rapidly.

> TODO: proofs on generation step? Code details?