

Graduation Project Report

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Abstract

This report describes the condition number of a matrix, and an equivalent quantity to the condition number when entries of the matrix is specified as $\{0, 1\}$ or $\{-1, 1\}$. Based on this, the implementation of creating an $\{0, 1\}$ matrix is demonstrated, and the equivalent condition number is provided, showing that such kind of $\{0, 1\}$ matrix becomes more ill-conditioned when its order grows. Furthermore, this report discusses the complexity of creating such a matrix, proving and showing with an experiment the exponential time complexity.

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1 Introduction

Let matrix $A \in \mathbb{R}^{n \times n}$ be an invertible matrix, with the *spectral norm* defined as $\|A\|_s = \sup_{x \neq 0} \|Ax\|/|x|$, the *condition number* of A is $c(A) = \|A\|_s \|A^{-1}\|_s$. The *condition number* is a measurement of the sensibility of the equation $Ax = b$ when the right-hand side is changed [1]. If $c(A)$ is large, then A is called *ill-conditioned*.

With such an important property of *condition number*, ill-conditioned matrices are important in numerical algebra, and have been studied extensively by various researchers, such as [2], [3] and [4]. In [5], researchers restricted entries into set $\{0, 1\}$ or $\{-1, 1\}$, denoted by \mathcal{A}_n^1 or \mathcal{A}_n^2 , which they call *anti-Hadamard* matrices, with is of interest in linear algebra, objects in combinatorics and related areas. With such conditions, many quantities are equivalent to the condition number. Let A be a non-singular $(0, 1)$ matrix, $B = A^{-1} = (b_{ij})$, the following quantity as an equivalent condition number is considered in [5]:

$$\chi(A) = \max_{i,j} |b_{ij}| \text{ and } \chi(n) = \max_A \chi(A),$$

which is bounded controlled by n and some absolute positive constant c . Meanwhile, since matrix A only contains 0 and 1, it also stands for a 0/1-polytope space, which is of great interest in geometry.

In this report, we aim to construct an ill-conditioned $(0, 1)$ matrix C satisfied

$$\chi(C) \geq 2^{\frac{1}{2}n \log n - n(2+o(1))}.$$

That is said, matrix C has a controlled lower bound of its *condition number* with respect to n .

As is shown in [1], we started from generating matrix $A \in \mathcal{A}_n^2$, and we generate $B \in \mathcal{A}_n^1$ based on A . Using a series of matrix B with different shapes, we can further concatenate a matrix C with its condition number controlled.

2 Generate Ill-conditioned Matrix C

2.1 Generate Set Ω

To start constructing matrix C with a controlled *condition number*, the set Ω with special restriction is necessary. Let $|\cdot|$ denotes cardinality and Δ denote symmetric different. Let $m \in \mathbb{Z}^+$, $n = 2^m$, set $\Omega = \{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n\}$ with $n + 1$ elements is required to create with a restriction that $|\alpha_i| \leq |\alpha_{i+1}|$ and $|\alpha_i \Delta \alpha_{i+1}| \leq 2$. Shown in [6], Ω is proved to be existed, the only thing considering is the implementational way create such kind of set. The following description shows and implemented way.

Let $\alpha_0 = \{\emptyset\}$. We also have $\alpha_1 = \{\emptyset\}$. Suppose we have $\Omega = \{\{\emptyset\}, \{\emptyset\}, \{1\}, \{2\}, \dots, \{m\}\}$ initially. Considering that every time we only take out all the sets with maximum size in Ω , and insert only one element inside by order. That is, for an existing set $\{1\}$, we insert elements one by one $2, 3, \dots, m$ by order.

With this way of insertion, we only need to consider if the conditions are satisfied between the last old set and the first new set, and if the continuous new sets come from different old sets. For all the sets that come from the same old set, the conditions are satisfied automatically.

Initially $\Omega = \{\{\}, \{\}, \{1\}, \{2\}, \dots, \{m\}\}$, take $\{1\}, \{2\}, \dots, \{m\}$ out, and insert only one element each time. Make sure conditions are satisfied between

1. $\{m\}$ and $\{1, _ \}$;
 2. $\{1, _ \}$ and $\{2, _ \}$, $\{2, _ \}$ and $\{3, _ \}$, ...
-

For the first case above, the only element we can insert is m , which is the first element if we revert the ordered insertion. For the second case, let's say we have set α with size k and β with size $k - 1$, if $\alpha \cup \beta = \alpha$, we can insert any element we wish; if $\alpha \cup \beta \neq \alpha$, we can only insert an element $r \in \alpha$. Since we always insert elements in order or in reversed order, if the first element of the insertion list is not in α , the last element of the insertion list or the first element of the reverted list must be in α .

The following pseudo-code shows the way to generate Ω :

Algorithm 1 Generate Ω

Input: m

Output: Ω

```

1:  $\Omega \leftarrow \{\{\}, \{\}, \{1\}, \{2\}, \dots, \{m\}\}$ 
2: for  $i \leftarrow 1$  to  $m - 1$  do
3:    $\Phi \leftarrow$  sets in  $\Omega$  with size equals  $i$ 
4:   for  $\phi$  in  $\Phi$  do
5:      $\alpha \leftarrow [\max(\phi) + 1, \dots, m]$ 
6:     if  $\alpha$  not in  $\Omega[-1]$  then:
7:        $\alpha \leftarrow \alpha.\text{reverse}()$ 
8:     end if
9:     for  $a$  in  $\alpha$  do
10:       $\Omega \leftarrow \{\Omega..., \{\phi..., a\}\}$ 
11:    end for
12:  end for
13: end for
14: return  $\Omega$ 

```

$\triangleright \phi... \text{ means extend set } \phi$

when $m = 4$, set Ω is shown below:

$\Omega = \{\text{set}(), \text{set}(), \{1\}, \{2\}, \{3\}, \{4\},$
 $\{1, 4\}, \{1, 3\}, \{1, 2\}, \{2, 4\}, \{2, 3\},$
 $\{3, 4\}, \{1, 3, 4\}, \{1, 2, 3\}, \{1, 2, 4\},$
 $\{2, 3, 4\}, \{1, 2, 3, 4\}\}$

2.2 Construct Matrix $A \in \mathcal{A}_n^2$ with Set Ω

Shown in [1], with set $\Omega = \{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n\}$ satisfies $|\alpha_i| \leq |\alpha_{i+1}|$ and $|\alpha_i \Delta \alpha_{i+1}| \leq 2$, matrix A can be constructed as follows such that $\chi(A) = 2^{\frac{1}{2}n \log n - n(1+o(1))}$:

For every $1 \leq i, j \leq n$:

$$a_{ij} = \begin{cases} -1, & \alpha_j \cap (\alpha_{i-1} \cup \alpha_i) = \alpha_{i-1} \Delta \alpha_i \text{ and } |\alpha_{i-1} \Delta \alpha_i| = 2 \\ (-1)^{|\alpha_{i-1} \cap \alpha_j|+1}, & \alpha_j \cap (\alpha_{i-1} \cup \alpha_i) \neq \emptyset \text{ but does not meet the condition above} \\ 1, & \alpha_j \cap (\alpha_{i-1} \cup \alpha_i) = \emptyset \end{cases}.$$

With the Ω shown in section 2.1, matrix $A \in \mathcal{A}_n^2$ constructed is shown below:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

In [1], researchers further discussed some properties of the $\{-1, 1\}$ matrix. Let symmetric Hadamard matrix Q be an n by n matrix given by $q_{ij} = (-1)^{|\alpha_i \cap \alpha_j|}$, that is $Q^2 = nI_n$. There existed a lower triangular matrix L built by Ω satisfying $A = LQ$. In this report, we build a matrix A and show $L = AQ^{-1}$ is a lower triangular matrix.

With matrix A shown above, matrix Q and L is shown below:

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix},$$

$$Q \times Q = \begin{bmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix},$$

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & -0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & -0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & -0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & -0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & -0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.25 & 0.0 & 0.25 & 0.25 & 0.25 & 0.25 & 0.0 & 0.0 & 0.0 & -0.75 & 0.25 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 & 0.25 & 0.25 & 0.0 & 0.25 & 0.0 & 0.25 & 0.25 & -0.75 & 0.25 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.25 & 0.25 & 0.25 & 0.25 & 0.0 & 0.25 & 0.25 & 0.0 & 0.0 & -0.75 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.25 & 0.0 & 0.25 & 0.0 & 0.25 & 0.0 & 0.25 & 0.0 & 0.25 & 0.25 & 0.0 & 0.0 & -0.75 & 0.25 & 0.0 \\ 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & -0.875 & 0.125 \end{bmatrix}.$$

2.3 Mapping Matrix $A \in \mathcal{A}_n^2$ to $B \in \mathcal{A}_{n-1}^1$

With matrix $A \in \mathcal{A}_n^2$, $\chi(A) = 2^{\frac{1}{2}n \log n - n(1+o(1))}$ being constructed, a mapping can be implemented to generate a matrix $A \in \mathcal{A}_{n-1}^1$ such that $\chi(B) = 2^{\frac{1}{2}n \log n - n(1+o(1))}$. Consider the map Φ which assigns to any matrix $B \in \mathcal{A}_{n-1}^1$ a matrix $\Phi(B) \in \mathcal{A}_n^2$ in the following way:

$$\Phi(B) = \begin{pmatrix} 1 & 1_{n-1} \\ -1_{n-1}^T & 2B - J_{n-1} \end{pmatrix}.$$

We can see that $\Phi(B)$ is a mapping $\mathcal{A}_{n-1}^1 \rightarrow \mathcal{A}_n^2$ with a series of linear operations. Therefore, we have the following reversing way to construct matrix $B \in \mathcal{A}_{n-1}^1$ with $A = \{a_{ij}\} \in \mathcal{A}_n^2$:

$$B = \frac{1}{2}(J_{n-1} + \{a_{ij}\}_{2 \leq i \leq n, 2 \leq j \leq n}).$$

Notice that in the above section, the $A \in \mathcal{A}_n^2$ we constructed has it's first column as:

$$\begin{pmatrix} 1 \\ 1_{n-1}^T \end{pmatrix}.$$

Therefore, we need to negative the first column and have the relation between matrix $B \in \mathcal{A}_{n-1}^1$ and $A \in \mathcal{A}_n^2$ in the implementation shown as follows:

$$B = \frac{1}{2}(J_{n-1} - \{a_{ij}\}_{2 \leq i \leq n, 2 \leq j \leq n}).$$

With the matrix $A \in \mathcal{A}_n^2$ constructed above, a constructed matrix B is shown below:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Shown in [1], matrix B preserves the ill-conditions property, that is $\chi(B) = 2^{\frac{1}{2}n \log n - n(1+o(1))}$.

2.4 Generate and Verify Ill-conditioned Matrix C

Before constructing matrix C , [1] shows a way of concatenating that has a special property. Let S and T be two non-singular matrices of order n_1 and n_2 . Define $S \diamond T$ as follows:

$$R = \begin{bmatrix} s_{11} & \dots & s_{1n_1} & 0 & \dots & 0 \\ s_{21} & \dots & s_{2n_1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{n_1 1} & \dots & s_{n_1 n_1} & 0 & \dots & 0 \\ 0 & 0 \dots 0 & 1 & t_{11} & \dots & t_{1n_2} \\ 0 & 0 \dots 0 & 0 & t_{21} & \dots & t_{2n_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \dots 0 & 0 & t_{n_2 1} & \dots & t_{n_2 n_2} \end{bmatrix}.$$

It's shown that R has the following property:

$$\chi(S \diamond T) \geq \chi(S)\chi(T),$$

Which shows an increased *condition number* leading to the construction of the ill-conditioned C . Consider the $(0, 1)$ matrix $C = A_1 \diamond (A_2 \diamond (\dots (A_{r-1} \diamond A_r) \dots))$ with order $\sum_{i=1}^r n_i = n$, with the definition of the operator \diamond , we have the following conclusion that

$$\chi(C) \geq \prod_{i=1}^r \chi(A_i) > 2^{\frac{1}{2}n \log n - n(2+o(1))}$$

This conclusion describes the lower bound of $\chi(C)$, and we can say matrix C is ill-conditioned concerning n .

Since the concatenation rapidly enlarges the order of the matrix, we could not show the matrix C constructed with all the steps described in previous sections. Instead, we show some results of $\chi(C)$ with C generated in the same steps above. Remind that $C = A_1 \diamond (A_2 \diamond (\dots (A_{r-1} \diamond A_r) \dots))$, where r is the biggest order of all the $\{0, 1\}$ matrices A :

$r = 2$,	order of $C = 4$,	$\chi(C) = 1.0$
$r = 3$,	order of $C = 11$,	$\chi(C) = 2.0$
$r = 4$,	order of $C = 26$,	$\chi(C) = 260.0$
$r = 5$,	order of $C = 57$,	$\chi(C) = 106491641548.6$

With the result above, We can observe that as order r grows, $\chi(C)$ grows rapidly to infinity.

3 Complexity Analysis of Generating Matrix C

3.1 Time Complexity of Creating Ω

In this section, we discuss the complexity of constructing matrix C and further show the reason for not being able to generate matrix C with large r . Since we go through all these steps with code, we assume that we have enough memory and focus on the time complexity.

To generate set Ω , every time we only need to take out one generated set and insert one new element inside. The insertion operation is $O(\log n)$, but notice that we always insert an

element larger than the maximum of a set, we can consider it an $O(1)$, without considering the maintenance of the internal tree data structure. Therefore, the most ideal time complexity is $O(n)$, which cannot be reached but is still used.

A more detailed complexity analysis is shown below. In the implementation, there are 2 for-loop with $O(m)$ and at least $O(m)$ time complexity. With all the assignments, set insertion, set reversion operation, and an extra for-loop neglected, a rough time complexity estimation would be $O(m^2)$.

Algorithm 2 Time Complexity of Generating Ω

Input: m

Output: Ω

```

1:  $\Omega \leftarrow \{\{\}, \{\}, \{1\}, \{2\}, \dots, \{m\}\}$ 
2: for  $i \leftarrow 1$  to  $m - 1$  do  $\triangleright O(m)$  loop
3:    $\Phi \leftarrow$  sets in  $\Omega$  with size equals  $i$   $\triangleright O(1)$  with special data structure
4:   for  $\phi$  in  $\Phi$  do  $\triangleright \binom{m}{i} \rightarrow$  at least  $O(m)$ 
5:      $\alpha \leftarrow [\max(\phi) + 1, \dots, m]$   $\triangleright$  the  $\max(\dots)$  of an ordered set is  $O(1)$ 
6:     if  $\alpha$  not in  $\Omega[-1]$  then:
7:        $\alpha \leftarrow \alpha.\text{reverse}()$   $\triangleright$  a set reversion operation
8:     end if
9:     for  $a$  in  $\alpha$  do  $\triangleright m - \max(\phi) - 1$ , more than  $O(1)$ 
10:       $\Omega \leftarrow \{\Omega..., \{\phi..., a\}\}$   $\triangleright$  a set insertion operation
11:    end for
12:  end for
13: end for
14: return  $\Omega$ 

```

3.2 Time Complexity of Creating $\{0, 1\}$ Matrix C

To generate every entry of matrix $A \in \mathcal{A}_n^2$, visits of all elements in A are necessary. Let m stand for the size of set Ω , $n = 2^m$ be the order of matrix A , without considering the complexity of set indexing, the union operation, and the intersection operation needed in every single calculation, the ideal time complexity of construct a matrix $A \in \mathcal{A}_n^2$ is at least $O(m) + O(2^{2m}) = O(2^{2m})$.

Generating matrix $B \in \mathcal{A}_{n-1}^1$ required simple matrix operation. Let $n - 1$ be the order of B , the time complexity of this part is $O((n - 1)^2) = O(n^2) = O(2^{2m})$.

To splice out matrix C , we need a series of matrix A_1, A_2, \dots, A_n with orders $2^1 - 1, 2^2 - 1, \dots, 2^m - 1$. With matrix A_r with order r having $O(r^2)$ time complexity, to create all these matrices, we can estimate a sum up, that is $\sum_{i=1}^m O((2^i - 1)^2) = O(m2^{2m}) = O(m2^{2m}) = O(2^{2m})$. An extra time complexity of matrix assignment is also needed, which is $(\sum_{i=0}^m O(2^m - 1))^2 = O(m^2 2^{2m})$ or $O(2^{2m})$.

Remind that we only consider parts of the essential operation, most of the internal operations including indexing, union, intersection, and more. Therefore, to create the $\{0, 1\}$ Matrix C , we need an exponential time complexity; when m grows, the time required to generate matrix C grows exponentially.

The following figure 1 shows the relation between the time of generating matrix C and size controller m . An exponential growth is shown as expected.

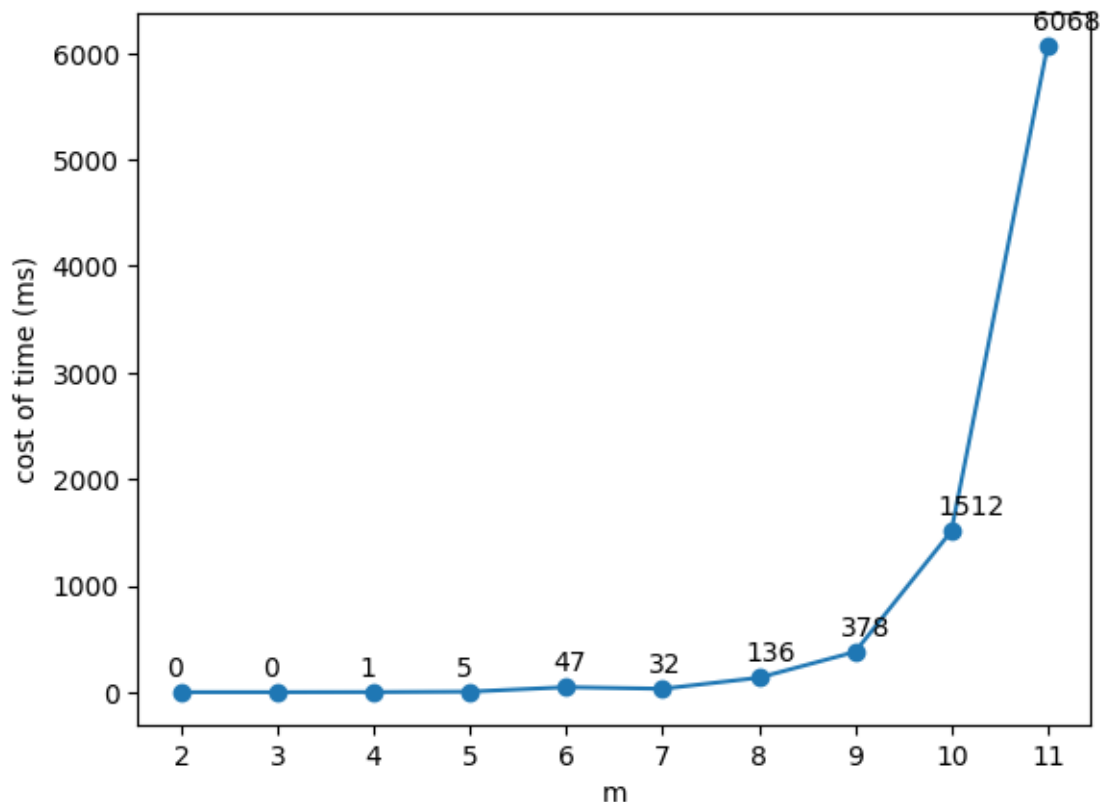


Figure 1: Cost of time in milliseconds related to m

4 Conclusion

This report shows the $\{0,1\}$ matrix C created with specific steps to ensure its equivalent condition number $\chi(C) > 2^{\frac{1}{2}n \log n - n(2+o(1))}$ is ill-conditioned as its order grows. Also, we prove the exponential time complexity of creating such a matrix, showing that it is expensive to create when the order of C grows.

5 Acknowledgements

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References

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Code of Implementation

```
In [ ]: import numpy as np
        from math import log

        # pretty print for np.array
        # from https://stackoverflow.com/questions/53126305/pretty-printing-numpy-ndarrays-using-unicode-characters/53164538#53164538
        def pretty_print(A):
            if A.ndim==1:
                print(A)
            else:
                w = max([len(str(s)) for s in A])
                print(u'\u250c' + u'\u2500' * w + u'\u2510')
                for AA in A:
                    print(' ', end='')
                    print('[', end='')
                    for i, AAA in enumerate(AA[:-1]):
                        w1 = max([len(str(s)) for s in A[:, i]])
                        print(str(AAA) + ' ' * (w1 - len(str(AAA)) + 1), end='')
                        w1 = max([len(str(s)) for s in A[ :, -1]])
                        print(str(AA[-1]) + ' ' * (w1 - len(str(AA[-1]))), end='')
                    print(']')
                print(u'\u2514'+u'\u2500' * w + u'\u2518')

        def compare(left: set, right: set):
            if len(left) == len(right):
                for i in range(len(left)):
                    if left[i] != right[i]:
                        return left[i] < right[i]
            return left < right
```

Generate \mathcal{A}_n^2

Let $|\cdot|$ denotes cardinality and Δ denote symmetric different

```
In [ ]: def Delta(left: set, right: set):
        return left.symmetric_difference(right)

        def Cardi(x: set):
            return len(x)
```

Let $n = 2^m$, Ω be a set of m element such that $|\alpha_i| \leq |\alpha_{i+1}|$ and $|\alpha_i \Delta \alpha_{i+1}| \leq 2$
 Let $\alpha_0 = \{\emptyset\}$, now generate Ω

utils func for checking $|\alpha_i| \leq |\alpha_{i+1}|$ and $|\alpha_i \Delta \alpha_{i+1}| \leq 2$

```
In [ ]: def legal(alpha_i: set, alpha_i_1: set):
        return Cardi(alpha_i) <= Cardi(alpha_i_1) and Cardi(Delta(alpha_i, alpha_i_1)) <= 2

        def assert_omega(ome, _m, skip=0):
            for i in range(0 + skip, len(ome) - 1):
                if not legal(ome[i], ome[i + 1]):
                    print(ome[i], ome[i + 1])
                assert(legal(ome[i], ome[i + 1]))
            assert(len(ome) == 2**_m + 1)
```

```
In [ ]: from itertools import chain, combinations

        # generate Omega
        def insert_one_element(initial_sets: list, super_end: int) -> list:
            res = []
            for initial_set in initial_sets:
                avail_range = list(range(max(initial_set) + 1, super_end + 1))
                if len(avail_range) == 0:
                    continue
                if len(res) == 0 and avail_range[0] not in initial_sets[-1]:
                    avail_range.reverse()
                elif len(res) != 0 and avail_range[0] not in res[-1]:
                    avail_range.reverse()
                for avail_ele in avail_range:
                    if avail_ele in initial_set:
                        break
                    cur_res = initial_set.copy()
                    cur_res.add(avail_ele)
                    res.append(cur_res)

            return res

        def create_Omega(_m: int):
            initial_sets = [{i + 1} for i in range(_m)]
            grouped_res = [initial_sets]
            for _ in range(1, _m):
                grouped_res.append(insert_one_element(grouped_res[-1], _m))

            res = [set(), set()]
            for single_res in grouped_res:
                res.extend(single_res)
            ### the follow code is for verification
            s = list(range(1, _m + 1))
            unordered = list(chain.from_iterable(combinations(s, r) for r in range(len(s)+1)))
            for ele in unordered:
                assert(set(ele) in res)
            assert(len(unordered) == len(res) - 1)
            assert_omega(res, _m)
            ### verification ends
            return res
```



```
[16 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ]
[0 16 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ]
[0 0 16 0 0 0 0 0 0 0 0 0 0 0 0 0 ]
[0 0 0 16 0 0 0 0 0 0 0 0 0 0 0 0 ]
[0 0 0 0 16 0 0 0 0 0 0 0 0 0 0 0 ]
[0 0 0 0 0 16 0 0 0 0 0 0 0 0 0 0 ]
[0 0 0 0 0 0 16 0 0 0 0 0 0 0 0 0 ]
[0 0 0 0 0 0 0 16 0 0 0 0 0 0 0 0 ]
[0 0 0 0 0 0 0 0 16 0 0 0 0 0 0 0 ]
[0 0 0 0 0 0 0 0 0 16 0 0 0 0 0 0 ]
[0 0 0 0 0 0 0 0 0 0 16 0 0 0 0 0 ]
[0 0 0 0 0 0 0 0 0 0 0 16 0 0 0 0 ]
[0 0 0 0 0 0 0 0 0 0 0 0 16 0 0 0 ]
[0 0 0 0 0 0 0 0 0 0 0 0 0 16 0 0 ]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 16 0 ]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 16]
```

And verify matrix $L = A_Q^{-1}$ is a lower triangular matrix

```
In [ ]: pretty_print(np.array(np.matrix(A_mat) * np.matrix(Q_mat).I))
```

```
[1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.5  0.5  0.0  0.0  -0.5  0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  0.0  0.0  0.5  0.5  -0.5  0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  0.0  0.5  0.5  0.0  0.0  -0.5  0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  0.5  0.0  0.0  0.5  0.0  0.0  -0.5  0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  0.0  0.0  0.5  0.5  0.0  0.0  0.0  -0.5  0.5  0.0  0.0  0.0  0.0  0.0  0.0 ]
[0.0  0.0  0.5  0.0  0.5  0.0  0.0  0.0  0.0  -0.5  0.5  0.0  0.0  0.0  0.0  0.0 ]
[0.25 0.25 0.0 0.25 0.25 0.25 0.25 0.0 0.0 0.0 -0.75 0.25 0.0 0.0 0.0 0.0 ]
[0.0 0.0 0.25 0.0 0.25 0.25 0.25 0.0 0.25 0.25 -0.75 0.25 0.0 0.0 0.0 0.0 ]
[0.0 0.0 0.0 0.25 0.25 0.25 0.25 0.0 0.25 0.25 0.0 0.0 -0.75 0.25 0.0 0.0 ]
[0.0 0.25 0.0 0.25 0.0 0.25 0.0 0.25 0.0 0.25 0.25 0.0 0.0 -0.75 0.25 0.0 ]
[0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125 -0.875 0.125]
```

Generate \mathcal{A}_{n-1}^1

Let $B \in \mathcal{A}_{n-1}^1, \Phi(B) \in \mathcal{A}_n^2$:

$$\Phi(B) = \begin{pmatrix} 1 & 1_{n-1} \\ -1_{n-1}^T & 2B - J_{n-1} \end{pmatrix}$$

notice that the $A_2 \in \mathcal{A}_n^2$ we have has its first column as:

$$A_2 = \Phi(B) = \begin{pmatrix} 1 \\ 1_{n-1}^T \end{pmatrix}$$

we need to multiply $\{\alpha_{ij}\}_{2 \leq i \leq n, 2 \leq j \leq n} \in \mathcal{A}_n^2$ with -1
so we can generate matrix $A_1 \in \mathcal{A}_{n-1}^1$ with $A_2 = \{\alpha_{ij}\} \in \mathcal{A}_n^2$ using the following relation:

$$A_1 = \frac{1}{2}(J_{n-1} - \{\alpha_{ij}\}_{2 \leq i \leq n, 2 \leq j \leq n})$$

```
In [ ]: def create_A_1_mat(A_2_mat):
        A_1_mat = (np.ones(A_2_mat.shape[0] - 1) - A_2_mat[1:, 1:]) * 0.5
        assert((A_1_mat.max() == 1 and A_1_mat.min() == 0) if A_1_mat.shape[0] > 1 else True)
        return A_1_mat
```

```
In [ ]: # test
        pretty_print(create_A_1_mat(A_mat).astype(int))
```

```
[1 0 0 0 1 1 1 0 0 0 1 1 1 0 1]
[0 1 0 0 0 0 1 1 1 0 0 1 1 1 1]
[0 0 1 0 0 1 0 0 1 1 1 1 0 1 1]
[0 0 0 1 1 0 0 1 0 1 1 1 0 1 1]
[1 0 0 0 0 1 1 0 0 0 0 1 0 0 0]
[0 0 1 0 1 0 0 0 1 1 1 0 1 1 1]
[0 1 0 0 0 1 0 1 1 0 1 1 0 1 1]
[0 0 0 1 1 0 1 0 0 1 1 1 1 0 1]
[0 0 1 0 0 1 0 1 1 0 1 1 1 1]
[0 0 0 1 1 0 0 1 1 0 0 1 1 1 1]
[1 0 0 0 0 1 0 0 1 1 0 0 1 1]
[0 1 0 0 1 1 0 1 0 1 0 1 1 1 0]
[0 0 0 1 0 1 1 0 1 1 1 0 1 1 0]
[0 0 1 0 1 1 1 1 0 0 1 1 0 1 0]
[1 0 0 0 0 0 1 1 1 1 1 1 0 0]
```

```
In [ ]: def funcX(A):
        return np.absolute(np.matrix(A).I).max()

        def get_o_1(X, _n):
            return (0.5 * _n * log(_n, 2) - _n - log(X, 2)) / _n
```

Verify $\chi(A) = 2^{\frac{1}{2}n \log n - n(1+o(1))}$

with $\chi(A) = \max_{i,j} |A^{-1}|$

```
In [ ]: for m in range(2, 9):
        omega = create_Omega(m)
        A_2_mat = create_A(omega, m)
        X_A_2 = funcX(A_2_mat)
        n = A_2_mat.shape[0]
        o_1 = get_o_1(X_A_2, n)
        print("m = {m_term}, n = {n_term}, \chi(A_2) = {X_A_2_term}, o(1) term = {o_1_term:.3f}".format(m_term=m, n_term=n, X_A_2_term=X_A_2, o_1_term=o_1))
```

```

m = 2, n = 4,  $\chi(A_2) = 0.5$ , o(1) term = 0.250
m = 3, n = 8,  $\chi(A_2) = 1.0$ , o(1) term = 0.500
m = 4, n = 16,  $\chi(A_2) = 131.500000000000142$ , o(1) term = 0.560
m = 5, n = 32,  $\chi(A_2) = 2247877518.8656816$ , o(1) term = 0.529
m = 6, n = 64,  $\chi(A_2) = 3344658001622888.0$ , o(1) term = 1.194
m = 7, n = 128,  $\chi(A_2) = 1596000892827338.0$ , o(1) term = 2.105
m = 8, n = 256,  $\chi(A_2) = 1231531590891313.5$ , o(1) term = 2.804

```

Verify $\chi(A') = 2^{\frac{1}{2}n \log n - n(1+o(1))}$

```

In [ ]: for m in range(2, 7):
        omega = create_Omega(m)
        A_mat = create_A(omega, m)
        A_1_mat = create_A_1_mat(A_mat)
        X_A = funcX(A_1_mat)
        n = A_1_mat.shape[0] + 1
        o_1 = get_o_1(X_A, n)
        print("m = {m_term}, n = {n_term},  $\chi(A) = \{X_A\_term\}$ , o(1) term = {o_1_term:.3f}".format(m_term=m, n_term=n, X_A_term=X_A, o_1_term=o_1))

m = 2, n = 4,  $\chi(A) = 1.0$ , o(1) term = 0.000
m = 3, n = 8,  $\chi(A) = 2.0$ , o(1) term = 0.375
m = 4, n = 16,  $\chi(A) = 260.0000000000009$ , o(1) term = 0.499
m = 5, n = 32,  $\chi(A) = 4495480119.287977$ , o(1) term = 0.498
m = 6, n = 64,  $\chi(A) = 4212068385296025.0$ , o(1) term = 1.189

```

Generate matrix $C = A_1 \diamond (A_2 \diamond (\dots (A_{r-1} \diamond A_r) \dots))$

```

In [ ]: import time
import matplotlib.pyplot as plt

def rectangle_operator(S, T):
    top_right = np.zeros((S.shape[0], T.shape[1]), dtype=int)
    btn_left = np.zeros((T.shape[0], S.shape[1]), dtype=int)
    if btn_left.shape[1] > 0:
        btn_left[0, -1] = 1
    return np.asarray(np.bmat([[S, top_right], [btn_left, T]]))

def generate_C(r: int):
    res = None
    for m in range(1, r + 1):
        omega = create_Omega(m)
        A_mat = create_A(omega, m)
        A_1_mat = create_A_1_mat(A_mat)
        res = A_1_mat if res is None else rectangle_operator(res, A_1_mat)
    return res

range_list = list(range(2, 12))
time_list = []

for r in range_list:
    start = time.time()
    C_mat = generate_C(r)
    # pretty_print(C_mat.astype(int))
    X_C = funcX(C_mat)
    n = C_mat.shape[0]
    o_1 = get_o_1(X_C, n)
    end = time.time()
    time_list.append((end - start) * 1000)
    print("r = {r_term}, max(n) = {n_term},  $\chi(C) = \{X_C\_term\}$ , o(1) >= {o_1_term:.3f}".format(r_term=r, n_term=n, X_C_term=X_C, o_1_term=o_1))

plt.plot(range_list, time_list, marker='o')
plt.xlabel("r")
plt.xticks(range_list)
plt.ylabel("cost of time (ms)")
for x, y in zip(range_list, time_list):
    plt.text(x-0.15, y + 150, "%0.0f" % y)
plt.show()

r = 2, max(n) = 4,  $\chi(C) = 1.0$ , o(1) >= 0.000
r = 3, max(n) = 11,  $\chi(C) = 2.0$ , o(1) >= 0.639
r = 4, max(n) = 26,  $\chi(C) = 260.000000000000443$ , o(1) >= 1.042
r = 5, max(n) = 57,  $\chi(C) = 106491641548.59639$ , o(1) >= 1.274
r = 6, max(n) = 120,  $\chi(C) = 7.186183766512361e+24$ , o(1) >= 1.765
r = 7, max(n) = 247,  $\chi(C) = 8.981448629516596e+37$ , o(1) >= 2.464
r = 8, max(n) = 502,  $\chi(C) = 1.9679934451880236e+52$ , o(1) >= 3.140
r = 9, max(n) = 1013,  $\chi(C) = 3.962544275444787e+64$ , o(1) >= 3.780
r = 10, max(n) = 2036,  $\chi(C) = 3.4399058744893378e+75$ , o(1) >= 4.373
r = 11, max(n) = 4083,  $\chi(C) = 3.2109048892035864e+86$ , o(1) >= 4.927

```

