大数据分析

Linear Algebra
Preliminaries

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Vectors

- **x** = $(x_1, x_2, \ldots, x_d) \in R^d$ (each x_i is a component)
  - A point in d-dimensional space
- Norm or magnitude $\|\mathbf{x}\| = (x^T x)^{1/2} = (x_1^2 + x_2^2 + \ldots + x_d^2)^{½}$
  - Length of the vector (Pythagorean theorem)
- Zero vector (norm zero), unit vector (norm one)
- Inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \ldots x_d y_d$
  - Result is a scalar
  - $\|\mathbf{x}\| = (\langle \mathbf{x}, \mathbf{x} \rangle)^{1/2}$
  - $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ implies $\mathbf{x} \perp \mathbf{y}$

Vector spaces

- Space where vectors live
- Formally, a collection of vectors which is closed under linear combination
  - If {**x**, **y**} are in the space, so is a**x**+b**y** for any scalars a, b ϵ R
  - Should always contain zero vector
- Examples: {0}, R^d, the line x = 3y in R^2

Span and basis

- A set of vectors is said to span a vector space if one can write any vector in the vector space as a linear combination of the set
- $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ span the space $\{\sum a_i\mathbf{x}_i \mid a_i \epsilon R\}$
- This set is called the basis set
- Examples
  - The vectors {(0,1), (1,0)} span R^2
  - {(1, 1)} spans x=y which is a subspace of R^2
  - The vector {(0,1), (0,1), (1,1)} also span R^2

Linear independence and orthonormality

- Linear independence – a notion to remove redundancy in the basis
  - $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ are linearly independent iff the only solution to $\sum a_i \mathbf{x}_i = 0$ is $a_1 = a_2 = \ldots = a_n = 0$.
  - Cannot express any vector $\mathbf{x}_i$ as a linear combination of the others
- Dimensionality of a vector space is the maximum number of linearly independent basis vectors
- Orthonormal basis
  - $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is orthonormal basis if $< \mathbf{x}_i, \mathbf{x}_j >= 1$ if i=j and 0 otherwise
  - Coordinate axes for the vector space
- Example: The basis {(0, 1), (1,1)} for R^2 is linear independent but not orthonormal.

Matrices

- Operator which transforms vectors from one vector space to another
  - $\mathbf{y} = A\mathbf{x}$
- The operator is linear, that is

$$A(a\mathbf{x} + b\mathbf{y}) = a(A\mathbf{x}) + b(A\mathbf{y})$$

- The result of applying the operator is a linear combination of the column vectors
  - Thus, $A\mathbf{x} = \mathbf{b}$ has an exact solution iff $\mathbf{b}$ is in the column space of A
- Eigen vectors of A are the special vectors are the special vectors $\mathbf{x}$ which satisfy

$$A\mathbf{x} = \lambda\mathbf{x} \text{ for some } \lambda$$

  - λ is called the eigen value and $\mathbf{x}$ is the eigen vector

- How do we visualize the transformation geometrically?

Visualizing the matrix operator – special cases

- Identity matrix
  - Square matrix with diagonal elements 1 and non-diagonal elements 0
  - The transformed vector A**x** is same **x**
- Diagonal matrix
  - Square matrix with non-diagonal elements 0
  - i^{th} component in A**x** is a scaled version of x_i (scaling = A_{ii})
- Orthonormal (or rotation) matrix
  - Matrix whose columns $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ are such that $< \mathbf{a}_i , \mathbf{a}_j >= 1$ if i=j and 0 otherwise. That is, $A^TA = I$
  - Rotates the vector
  - Preserves norms $\|A\mathbf{x}\| = \|\mathbf{x}\|$ (why?)

Thank you!