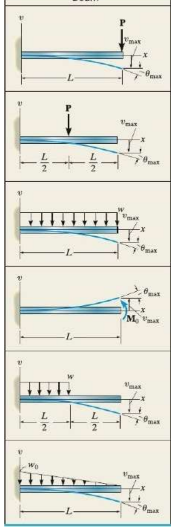
Beam Bending Formulae

Cantilevered Beam Slopes and Deflections



Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \frac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \frac{M_0 L}{EI}$	$v_{max} = \frac{M_0 L^2}{2EI}$	$v = \frac{M_0 x^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right) \quad 0 \le x \le L/2$ $v = \frac{-wL^3}{384EI}(8x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-w_0 L^3}{24EI}$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$v = \frac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$

Constitutive Relations

Constitutive (stress - strain) relations for an **isotropic linear elastic** body in presence of temperature change

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right) + \alpha\Delta T$$

$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right) + \alpha\Delta T$$

$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\left(\sigma_{yy} + \sigma_{xx}\right)\right) + \alpha\Delta T$$

$$\gamma_{xy} = \frac{\sigma_{xy}}{G},\ \gamma_{yz} = \frac{\sigma_{yz}}{G},\ \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

E : Young's modulus, units GPa

G : Shear modulus, units GPa

ν : Poisson's ratio

α : Coefficient of thermal expansion

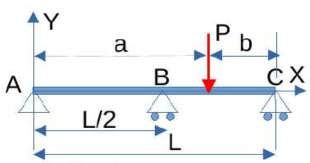
ΔT: Temperature change, units K[-1]

Material	Mass density ρ / Mgm^{-3}	Youngs Modulus E / GNm^{-2}	Poisson Ratio ν	Expansion coeft K^{-1}
Tungsten Carbide	14 – 17	450 – 650	0.22	5×10^{-6}
Silicon Carbide	2.5 – 3.2	450	0.22	4×10^{-6}
Tungsten	13.4	410	0.30	4×10^{-6}
Alumina	3.9	390	0.25	7×10^{-6}
Titanium Carbide	4.9	380	0.19	13×10^{-6}
Silicon Nitride	3.2	320 - 270	0.22	3×10^{-6}
Nickel	8.9	215	0.31	14×10^{-6}
CFRP	1.5-1.6	70 – 200	0.20	2×10^{-6}
Iron	7.9	196	0.30	13×10^{-6}
Low alloy steels	7.8	200 - 210	0.30	15×10^{-6}
Stainless steel	7.5-7.7	190 - 200	0.30	11×10^{-6}
Mild steel	7.8	196	0.30	15×10^{-6}
Copper	8.9	124	0.34	16×10^{-6}
Titanium	4.5	116	0.30	9×10^{-6}
Silicon	2.5-3.2	107	0.22	5×10^{-6}
Silica glass	2.6	94	0.16	0.5×10^{-6}
Aluminum & alloys	2.6-2.9	69-79	0.35	22×10^{-6}
Concrete	2.4-2.5	45-50	0.3	10×10^{-6}
GFRP	1.4-2.2	7-45		10×10^{-6}
Wood, parallel grain	0.4-0.8	9-16	0.2	40×10^{-6}
Polyimides	1.4	3-5	0.1-0.45	40×10^{-6}
Nylon	1.1 – 1.2	2 – 4	0.25	81×10^{-6}
PMMA	1.2	3.4	0.35-0.4	50×10^{-6}
Polycarbonate	1.2 – 1.3	2.6	0.36	65×10^{-6}
Natural Rubbers	0.83-0.91	0.01-0.1	0.49	200×10^{-6}
PVC	1.3-1.6	0.003-0.01	0.41	70×10^{-6}

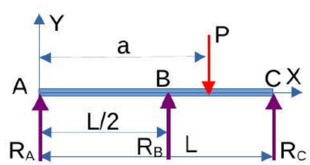
https://solidmechanics.org/Text/Chapter3_2/Chapter3_2.php

Beam Bending and Deflection

Find the reaction at point B



The FBD of the beam is given by

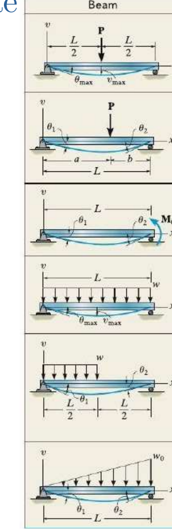


statically indeterminate problem

There are three unknowns and two equilibrium equations and hence it is a statically indeterminate problem.

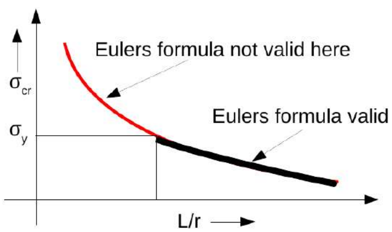
Beam Bending Formulae

Simply Supported Beam Slopes and Deflections



Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v\Big\vert_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \frac{-M_0 L}{6EI}$ $\theta_2 = \frac{M_0 L}{3EI}$	$v_{max} = \frac{-M_0 L^2}{9\sqrt{3}EI}$ at $x = 0.5774L$	$v = \frac{-M_0 x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\Big\vert_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2 x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \frac{-7w_0 L^3}{360EI}$ $\theta_2 = \frac{w_0 L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0 L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0 x}{360EIL}(3x^4 - 10L^2 x^2 + 7L^4)$

Column Buckling



L/r slenderness ratio

$\sigma_{cr} = P_{cr}/A$

r minimum radius of gyration

Validity of Euler's Formula

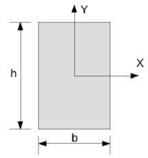
Material	E (GPa)	σ_y (MPa)	$L/r >$
Steel	200	250	89
Pure Aluminum	70	90	87
Al 6061-T6	70	270	51

Radius of Gyration

The radius of gyration, r, relates the moment of inertia of the cross-section, I, to its area, A. The radius of gyration is obtained from

$$I = Ar^2$$

It is the distance from the axis where the the entire area can be assumed to be concentrated so as to give the same moment of inertia. Consider a rectangular cross-section with breadth b and height h and $b < h$.



- We have

$$A = bh,\ I_{xx} = \frac{bh^3}{12},\ I_{yy} = \frac{b^3h}{12}$$

- Now $I = Ar^2$. Therefore

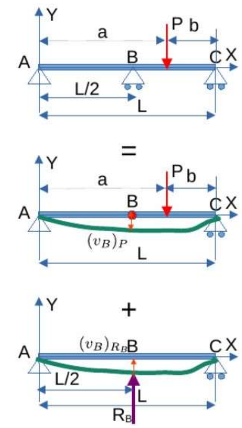
$$r_x = \frac{h}{2\sqrt{3}},\ r_y = \frac{b}{2\sqrt{3}}$$

- Since $b < h$, $r_y < r_x$ and $r_{min} = r_y$.

The radius of gyration can be used to compare the susceptibility of buckling of a column about different axes. In this case the column is more likely to buckle about the $Y-$ axis.

Beam Bending and Deflection

Statically Indeterminate Problem



- Remove R_B to make the problem statically determinate.
- The deflection at point B due to P is

$$(v_B)_P = \frac{Pb}{12EI}\left(\left(\left(L^2 - b^2\right) - \frac{L}{2}\right)^2\right) \downarrow$$

- Now consider the beam with only R_B acting.
- The deflection at B due to R_B is

$$(v_B)_{R_B} = \frac{R_B L^3}{48EI} \uparrow$$

- The net deflection at B is zero due to the support. Hence

$$\frac{Pb}{12EI}\left(\left(\left(L^2 - b^2\right) - \frac{L}{2}\right)^2\right) = \frac{R_B L^3}{48EI}$$

- Solve for R_B

Problem

A tensile specimen with a 12 mm initial diameter and 50 mm gauge length reaches maximum load at 90 kN and fractures at 70 kN. The minimum diameter at fracture is 10 mm. Determine the engineering stress at the maximum load (UTS), the engineering stress at the fracture load and the true stress at the fracture load.

Given data:

$do := 12\ mm \quad Lo := 50\ mm \quad Pm := 90\ kN \quad Pf := 70\ kN \quad df := 10\ mm$

$\sigma u := \frac{Pm}{\frac{\pi}{4}\cdot(do)^2} \quad \sigma u = 7.9577\cdot 10^8\ Pa$ Ultimate tensile stress

$\sigma f := \frac{Pf}{\frac{\pi}{4}\cdot do^2} \quad \sigma f = 6.1894\cdot 10^8\ Pa$ Engineering fracture stress

$\sigma fT := \frac{Pf}{\frac{\pi}{4}\cdot df^2} \quad \sigma fT = 8.9127\cdot 10^8\ Pa$ True fracture stress

Column Buckling

The critical column buckling load, P_{cr}, is estimated using Euler's formula

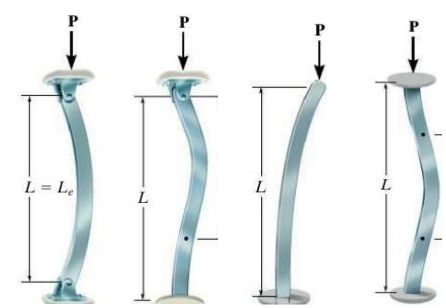
$$P_{cr} = c\pi^2\frac{EI}{L^2} \quad \text{or} \quad P_{cr} = \pi^2\frac{EI}{(kL)^2}$$

where the constants c (and k) depend on the boundary conditions ($k = 1/\sqrt{c}$).

	Boundary conditions	c	k
1.	P-P	1.00	1.00
2.	P-C	2.05	0.70
3.	C-F	0.25	2.00
4.	C-C	4.00	0.50

P-Pinned, C-Clamped, F-Free

Assumptions:
- Column is perfectly straight before loading.
- It is made up homogeneous material.
- The material behaves as a linear elastic material.
- The load is applied through the centroid of the cross section and the column bends in a single plane.



P-P C-F C-F C-C

Maximum Shear Stress Theory or Tresca Yield Criterion

Applicable to Ductile Materials

Yielding occurs when the the maximum shear stress at a point in the part reaches the shear stress that causes the same material to yield when it is subjected to uniaxial tension.

$$\boxed{\tau_{max} \geq \tau_y}$$

- If $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the principal stresses, then the maximum shear stress is given by

$$\tau_{max} = \frac{|\sigma_1 - \sigma_3|}{2}$$

- For the case of uniaxial tension at yielding $\sigma_1 = \sigma_y,\ \sigma_2 = 0,\ \sigma_3 = 0$. Therefore shear stress at yielding is

$$\tau_y = \frac{\sigma_y}{2}$$

- Therefore yielding occurs when

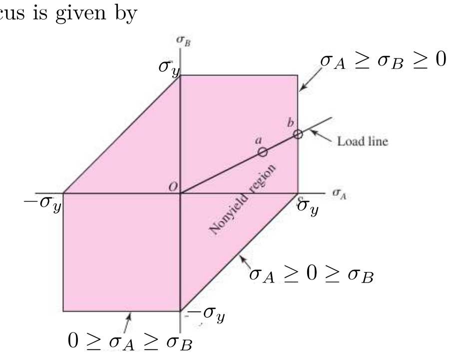
$$\boxed{|\sigma_1 - \sigma_3| \geq \sigma_y}$$

- For design purposes one introduces a factor of safety N and uses the following equation

$$\boxed{|\sigma_1 - \sigma_3| = \frac{\sigma_y}{N}}$$

Maximum Shear Stress Theory for Plane Stress Problems

- Let σ_A and σ_B represent the two in-plane principal stresses and let $\sigma_C = 0$ (out of plane stress component).
- Then the yield locus is given by



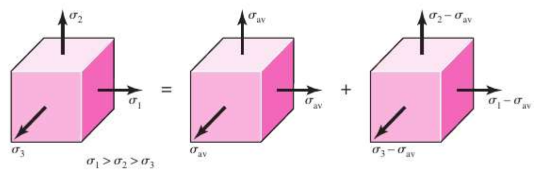
Factor of safety available at point a against yielding

$$n = \frac{l(Ob)}{l(Oa)}$$

Distortion Energy Theory or von Mises-Hencky Theory

Applicable to Ductile Materials

Yielding occurs when the the distortion strain energy per unit volume at a point in a part reaches or exceeds the distortion strain energy per unit volume at yield in the same material when it is subjected to uniaxial tension.



Triaxial state of stress	Hydrostatic component (responsible for volume change)	Deviatoric/Distortional Component (responsible for shape change)

Distortion Energy Theory or von Mises-Hencky Theory

- The distortional strain energy density per unit volume for a general state of stress is

$$U_d = \frac{1}{6G}\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\right]$$

- For the case of uniaxial tension at yielding $\sigma_1 = \sigma_y,\ \sigma_2 = 0,\ \sigma_3 = 0$

$$U_d^{uniaxial} = \frac{1}{6G}\sigma_y^2$$

- Yielding occurs when

$$U_d \geq U_d^{uniaxial}$$

or

$$\boxed{\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \geq \sigma_y}$$

- von Mises or equivalent stress is defined as

$$\sigma_{vm} = \sigma_{eq} = \left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\right]^{1/2}$$

- Then yielding occurs when

$$\boxed{\sigma_{vm} \geq \sigma_y}$$

- For design purposes one introduces a factor of safety N and uses the following equation

$$\boxed{\sigma_{vm} = \frac{\sigma_y}{N}}$$

Distortion Energy Theory or von Mises-Hencky Theory

- The distortion energy theory is also called:
  - The shear-energy theory
  - The octahedral-shear-stress theory
- The von Mises stress terms of the components of the stress tensor referred to a Cartesian coordinate system is given by

$$\sigma_{vm} = \frac{1}{\sqrt{2}}\left[(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+6(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2)\right]^{1/2}$$

- The von Mises stress terms of the components of the stress tensor referred to a Cartesian for case of plane stress is

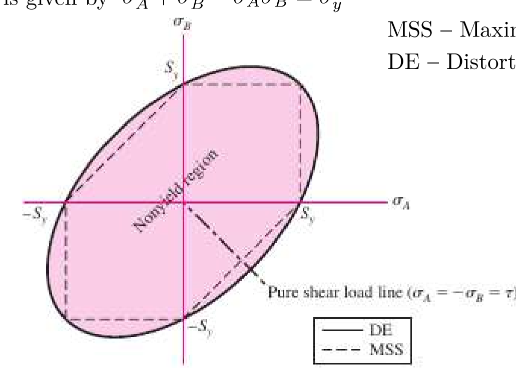
$$\sigma_{vm} = (\sigma_{xx}^2-\sigma_{xx}\sigma_{yy}+\sigma_{yy}^2+3\sigma_{xy})^{1/2}$$

Distortion Energy Theory for Plane Stress Problems

- Let σ_A and σ_B represent the two in-plane principal stresses and let $\sigma_C = 0$ (out of plane stress component).
- Then the yield locus is given by $\sigma_A^2+\sigma_B^2-\sigma_A\sigma_B=\sigma_y^2$

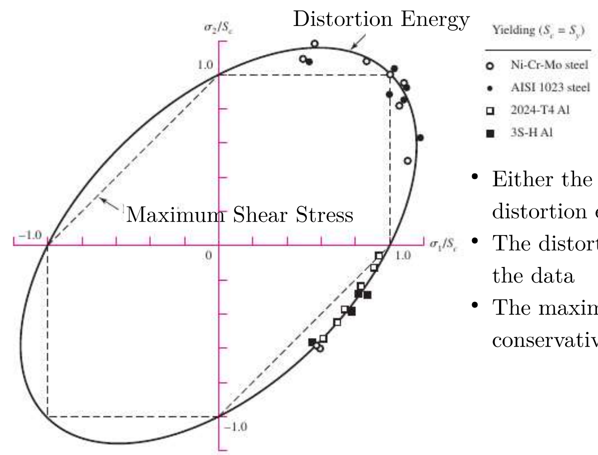
MSS – Maximum shear stress theory

DE – Distortion Energy Theory



Shigley's Mechanical Engineering Design

Comparison of Failure Theories for Ductile Materials



- Either the maximum shear stress theory or the distortion energy theory is acceptable
- The distortion energy theory gives better fit to the data
- The maximum shear stress theory is more conservative

Problem – Ductile Material

A stationary shaft, 50 mm in diameter and made of AISI 1060 hot-rolled steel, is subjected to a maximum bending moment of 3000 Nm and a maximum torque of 2000 Nm. Find the factor of safety corresponding to failure based on the distortion energy theory. The yield stress is 372 MPa.

$d := 50\ \mathrm{mm} \quad M := 3000\ \mathrm{N}\cdot 1\ \mathrm{m} \quad T := 2000\ \mathrm{N}\cdot 1\ \mathrm{m} \quad \sigma y := 372\ \mathrm{MPa}$

$$\sigma xx := \frac{\left(M\cdot\frac{d}{2}\right)}{\frac{\pi}{64}\cdot d^4} \qquad \sigma xx = 2.4446\cdot 10^{8}\ \mathrm{Pa}$$

$$\sigma xy := \frac{\left(T\cdot\frac{d}{2}\right)}{\frac{\pi}{32}\cdot d^4} \qquad \sigma xy = 8.1487\cdot 10^{7}\ \mathrm{Pa}$$

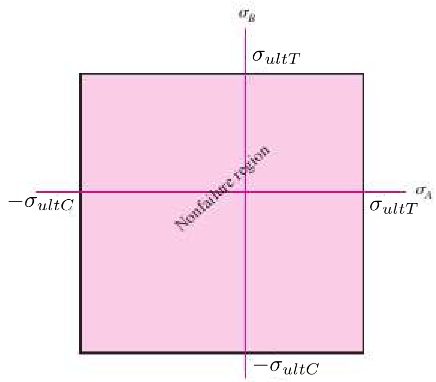
$\sigma yy := 0\ \mathrm{Pa}$

$$\sigma vm := \left(\sigma xx^2+\sigma yy^2-\sigma xx\cdot\sigma yy+3\cdot\sigma xy^2\right)^{\frac{1}{2}} \qquad \sigma vm = 2.8228\cdot 10^{8}\ \mathrm{Pa}$$

$$FOS := \frac{\sigma y}{\sigma vm} \qquad FOS = 1.3178$$

Maximum Normal Stress Theory for Plane Stress Problems

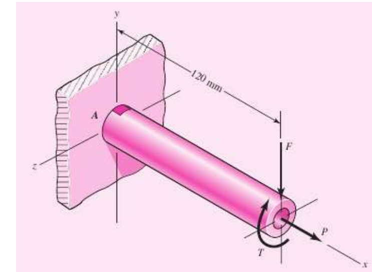
- Let σ_A and σ_B represent the two in-plane principal stresses and let $\sigma_C = 0$ (out of plane stress component). Let $\sigma_A \geq \sigma_B$
- Then the yield locus is



Shigley's Mechanical Engineering Design

Problem – Ductile Material

The cantilevered tube shown in Figure is to be made of 2014 Aluminum alloy treated to obtain a specified minimum yield strength of 276 MPa. We wish to select a stock-size tube from Table given below using a design factor $n_d = 4$. The bending load is F = 1.75 kN, the axial tension is P = 9.0 kN, and the torsion is T = 72 Nm. What is the realized factor of safety? Use both Maximum Shear Stress Theory and Distortion Energy Theory.



Size, mm	m	A	I	k	Z	J
12 × 2	0.490	0.628	0.082	0.361	0.136	0.163
16 × 2	0.687	0.879	0.220	0.500	0.275	0.440
16 × 3	0.956	1.225	0.273	0.472	0.341	0.545
20 × 4	1.569	2.010	0.684	0.583	0.684	1.367
25 × 4	2.060	2.638	1.508	0.756	1.206	3.015
25 × 5	2.452	3.140	1.669	0.729	1.336	3.338
30 × 4	2.550	3.266	2.827	0.930	1.885	5.652
30 × 5	3.065	3.925	3.192	0.901	2.128	6.381
42 × 4	3.727	4.773	8.717	1.351	4.151	17.430
42 × 5	4.536	5.809	10.130	1.320	4.825	20.255
50 × 4	4.512	5.778	15.409	1.632	6.164	30.810
50 × 5	5.517	7.065	18.118	1.601	7.247	36.226

w_a = unit weight of aluminum tubing, lbf/ft
w_s = unit weight of steel tubing, lbf/ft
m = unit mass, kg/m
A = area, in^2 (cm^2)
I = second moment of area, in^4 (cm^4)
J = second polar moment of area, in^4 (cm^4)
k = radius of gyration, in (cm)
Z = section modulus, in^3 (cm^3)
d, t = size (OD) and thickness, in (mm)

Shigley's Mechanical Engineering Design

Choice of Failure Theory

- For **isotropic** material that fail by yielding, use the distortion energy theory
- For **isotropic** material that fail by yielding, use maximum shear stress is more conservative than the distortion energy theory
- For **isotropic** material that fail by brittle fracture, use the maximum normal stress theory.
- For materials that fail by brittle fracture but whose compressive ultimate compressive strength is significantly different from the tensile ultimate strength, use the modified Mohr's theory (not covered in class)

Maximum Normal Stress Theory

Applicable to Brittle Materials (primarily applied to isotropic material)

Failure occurs whenever one of the three principal stresses at a point in a part reaches or exceeds the strength of the material either in tension or compression

- Let $\sigma_1 \geq \sigma_2 \geq \sigma_3$ be the principal stresses at a point
- The material fails whenever

$$\boxed{\sigma_1 \geq \sigma_{ultT},\ \sigma_3 \leq -\sigma_{ultC}}$$

Here σ_{ultT} and σ_{ultC} are the ultimate strength in tension and compression, respectively

- For design purposes one introduces a factor of safety N and uses the following equation

$$\boxed{\sigma_1 = \frac{\sigma_{ultT}}{N},\ \text{or } \sigma_3 = \frac{-\sigma_{ultC}}{N}}$$

Stress Life Approach – Endurance Limits and Fatigue Strengths of Common Metals

Steels $\sigma'_e \approx \begin{cases} 0.5\sigma_{ult} & \sigma_{ult} < 1400 \text{ MPa} \\ 700 \text{ MPa} & \sigma_{ult} \geq 1400 \text{ MPa} \end{cases}$

Cast Iron $\sigma'_e \approx \begin{cases} 0.4\sigma_{ult} & \sigma_{ult} < 400 \text{ MPa} \\ 160 \text{ MPa} & \sigma_{ult} \geq 400 \text{ MPa} \end{cases}$

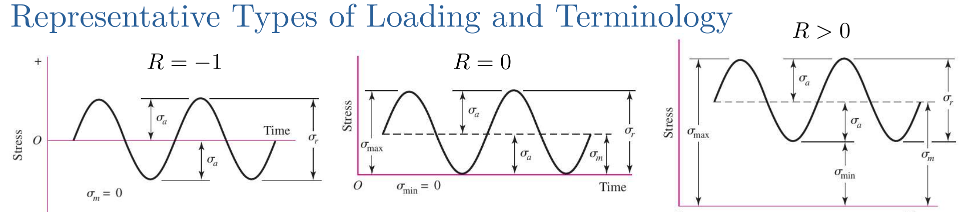
Aluminium $\sigma'_{f@5e8} \approx \begin{cases} 0.4\sigma_{ult} & \sigma_{ult} < 330 \text{ MPa} \\ 130 \text{ MPa} & \sigma_{ult} \geq 300 \text{ MPa} \end{cases}$

Copper $\sigma'_{f@5e8} \approx \begin{cases} 0.4\sigma_{ult} & \sigma_{ult} < 280 \text{ MPa} \\ 100 \text{ MPa} & \sigma_{ult} \geq 280 \text{ MPa} \end{cases}$

σ'_e, σ'_f are the endurance limit or fatigue strength obtained in lab conditions using polished specimens using bending fatigue tests

These are only estimates and are to be use only if the S-N data is unavailable and must be used with caution

Representative Types of Loading and Terminology



Completely reversed stress

Fluctuating stress

Reversed stress

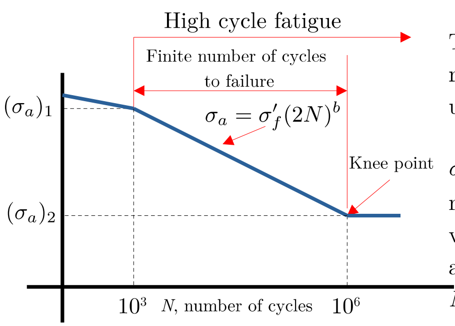
σ_{max}: max stress	σ_{min}: min stress
σ_a: stress amplitude	σ_m: mean stress
σ_r: stress range	

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} \qquad R = \frac{\sigma_{min}}{\sigma_{max}} \quad \text{Stress ratio}$$

$$\sigma_a = \frac{|\sigma_{max} - \sigma_{min}|}{2} \qquad A = \frac{\sigma_a}{\sigma_m} = \frac{1-R}{1+R} \quad \text{Amplitude ratio}$$

$$\sigma_r = \sigma_{max} - \sigma_{min}$$

Estimating Cycles to Failure in the High Cycle Regime



The number of cycles to failure in the high cycle regime up to 10^6 (endurance limit) can be estimated using the Basquin relation

$$\sigma_a = \sigma'_f(2N)^b$$

σ'_f and b are material constants. If they are not readily available, they can be estimated using the values of the alternate stress required for failure at two distinct points, typically at $N_1 = 10^3$ and $N_2 = 10^6$

$$b = \frac{\log\left(\frac{(\sigma_a)_1}{(\sigma_a)_2}\right)}{\log\left(\frac{N_1}{N_2}\right)}, \quad \sigma'_f = \frac{(\sigma_a)_1}{(2N_1)^b} = \frac{(\sigma_a)_2}{(2N_2)^b}$$

$(\sigma_a)_2$ is the endurance limit

$(\sigma_a)_1$ is the stress amplitude that the specimen fails at $N = 10^3$

$(\sigma_a)_1 \approx f\sigma_{ult}$ if no data is available (See next slide for f)

Fatigue Life Methods

Attempt to predict the life in number of cycles to failure, N, for a specific level of loading.

- **Stress life method** – used for high cycle fatigue ($N \geq 10^3$)
  - Based on stress levels only, assumes stresses are within elastic limit
  - Easiest to implement for a wide range of design applications
  - Has ample supporting data (experimental results)
  - Works best when the load amplitudes are predictable and consistent over the life of the part
  - Is the least accurate approach, especially for low-cycle applications.
- **Strain life method** – used for low cycle fatigue ($N \leq 10^3$)
  - Involves detailed analysis of the plastic deformation at localized regions.
  - Gives a good picture of the crack initiation stage
  - In applying this method, several idealizations must be made and hence uncertainties exist in the results. Hence not very widely used.
- **Linear elastic fracture mechanics method**
  - Assumes a crack is already present and detected.
  - It is employed to predict crack growth with respect to stress intensity.
  - Used in conjunction with computer codes and a periodic inspection program.

Problem – No Endurance limit Correction & Zero Mean Stress

Given a 1050 HR (hot rolled) steel, estimate
(a) the rotating-beam endurance limit at 10^6 cycles.
(b) the fatigue strength of a polished rotating-beam specimen corresponding to 10^4 cycles to failure
(c) the expected life of a polished rotating-beam specimen under a completely reversed stress of 380 MPa.

UTS for HR 1050 steel
$sult := 620\ \mathrm{MPa}$ $\quad f := 0.86$

Endurance strength
$s2 := 0.5 \cdot sult = 310\ \mathrm{MPa}$ $\quad N2 := 10^6$

Stress amplitude corresponding to 1000 cycles
$s1 := f \cdot sult = 533.2\ \mathrm{MPa}$ $\quad N1 := 10^3$

To find:
Fatigue strength at $N := 10^4$
Number of cycles to failure at stress amplitude $sp := 380\ \mathrm{MPa}$

$$b := \frac{\log_e\left(\frac{s1}{s2}\right)}{\log_e\left(\frac{N1}{N2}\right)} = -0.0785 \qquad sf := \frac{s2}{(2 \cdot N2)^b} = 968.3945\ \mathrm{MPa}$$

Answers
Fatigue strength at $N = 10000$
$sN := sf \cdot (2 \cdot N)^b = 445.0212\ \mathrm{MPa}$

Number of cycles to failure at stress amplitude $sp = 380\ \mathrm{MPa}$

$$Np := \frac{1}{2} \cdot \left(\frac{sp}{sf}\right)^{\frac{1}{b}} = 74772.5822$$ approximately 74700 cycles

Endurance Limit Modifying Factors

$$\sigma_e = k_a k_b k_c k_d k_e k_f \sigma_e'$$

σ_e = endurance limit of the machine part at the critical location
k_a = surface condition modification factor (highly polishes sample has a higher endurance limit as compared to an unpolished sample)
k_b = size modification factor (smaller sample has larger endurance limit as compared to a larger sample)
k_c = load modification factor (parts subjected to bending have higher endurance as compared to parts subjected to axial loads and torsion)
k_d = temperature modification factor (endurance limit for steel decreases for temperatures > 300° C)
k_e = reliability factor (higher reliability requirements leads to a lower value of k_e)
k_f = miscellaneous-effects modification factor (accounts for reduction in endurance limit due to all other factors – residual stress, corrosion, cyclic frequency)
σ_e' = rotary-beam test specimen endurance limit

Endurance Limit Modifying Factors

Surface Factor k_a

$$k_a = a\sigma_{ult}^b$$

Surface Finish	Factor a S_{ut}, kpsi	Factor a S_{ut}, MPa	Exponent b
Ground	1.34	1.58	−0.085
Machined or cold-drawn	2.70	4.51	−0.265
Hot-rolled	14.4	57.7	−0.718
As-forged	39.9	272.	−0.995

Size Factor k_b
For bending and torsion

$$k_b = \begin{cases} 1.24d^{-0.107} & 2.79 \le d \le 51\,\mathrm{mm} \\ 1.51d^{-0.157} & 51 \le d \le 254\,\mathrm{mm} \end{cases}$$

For non rotating round bar in bending or noncircular c/s see Shigley
For axial loading $k_b = 1$

Load Factor k_c

$$k_c = \begin{cases} 1 & \text{bending} \\ 0.85 & \text{axial} \\ 0.59 & \text{torsion} \end{cases}$$

Temperature Factor k_d

Temperature, °C	S_T/S_{RT}
20	1.000
50	1.010
100	1.020
150	1.025
200	1.020
250	1.000
300	0.975
350	0.943
400	0.900
450	0.843
500	0.768
550	0.672
600	0.549

If σ_e' is known, then

$$k_d = \frac{S_T}{S_{RT}}$$

If σ_e' is unknown, then, calculate it using temperature compensated S_T
And use k_d =1

Endurance Limit Modifying Factors

Reliability Factor k_e

Reliability, %	Transformation Variate z_a	Reliability Factor k_e
50	0	1.000
90	1.288	0.897
95	1.645	0.868
99	2.326	0.814
99.9	3.091	0.753
99.99	3.719	0.702
99.999	4.265	0.659
99.9999	4.753	0.620

$$\boxed{\sigma_e = k_a k_b k_c k_d k_e k_f \sigma_e'}$$

Miscellaneous-Effects Factor k_f

Factors including residual stress, corrosion, cyclic frequency, electrolytic plating, metal spraying, frettage corrosion

Stress Concentration and Notch Sensitivity

- Have looked at the theoretical or geometrical stress concentration factors (K_t, K_{ts}) earlier - it depends on the geometry and loading
- Not all materials are equally sensitive to the presence of geometrically discontinuous
- For materials which are not fully sensitive to the presence of geometric discontinuous, a reduced value of K_t (Kt_s) are used.
- This reduced factor is referred to as fatigue stress concentration factor K_f or K_{fs}

$$\sigma_{max} = K_f \sigma_o \text{ or } \tau_{max} = K_{fs} \tau_o$$

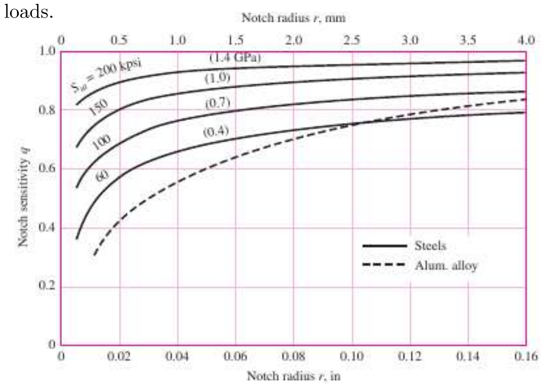
- Notch sensitivity factor is defined as

$$q = \frac{K_f - 1}{K_t - 1} \text{ or } q_{shear} = \frac{K_{fs} - 1}{K_{ts} - 1}$$

- $q = 0$ indicates that the material is insensitive to a notch while $q = 1$ indicates that the material is fully sensitive to the notch

Stress Concentration and Notch Sensitivity

Notch-sensitivity charts for steels and UNS A92024-T wrought aluminum alloys subjected to reversed bending or reversed axial loads.



For a given material, geometric discontinuity and loading determine K_t and q. Using the equation for the notch sensitivity find K_f.

i.e

$$K_f = 1 + q(K_t - 1)$$

The fatigue stress-concentration factor is then used as a multiplier of the nominal stress. We will use to modify both the mean stress component and the alternating stress component

Problem – Zero Mean Stress & Endurance Limit Correction

UTS for HR 1050 steel

$sult := 620\ \text{MPa}$ $f := 0.86$

Endurance strength

$s2 := 0.5 \cdot sult = 310\ \text{MPa}$ $N2 := 10^6$

Correction Factors

Calculation of surface factor

$a := 4.51$ $bsurF := -0.265$

$$ka := a \cdot \left(\frac{sult}{1\ \text{MPa}}\right)^{bsurF} = 0.8207$$

$kb := 1$ No size effect for axial loading

$kc := 0.85$ Load factor

$kd := 1$ Temperature effect

$ke := 0.814$ reliability factor

$kf := 1$ Miscellaneous factor

$s2C := ka \cdot kb \cdot kc \cdot kd \cdot ke \cdot kf \cdot s2 = 176.0345\ \text{MPa}$

$s1 := f \cdot sult = 533.2\ \text{MPa}$ $N1 := 10^3$

Problem: Find the fatigue strength at $N := 10000$

Number of cycles to fail at $sp := 380\ \text{MPa}$

$$b := \frac{\log_e\left(\frac{s1}{s2C}\right)}{\log_e\left(\frac{N1}{N2}\right)} = -0.1604 \qquad sf := \frac{s2C}{(2 \cdot N2)^b} = 1804.9991\ \text{MPa}$$

Answers

Fatigue strength at $N = 10000$

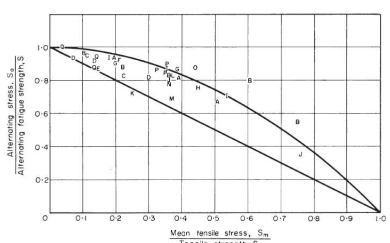
$$sN := sf \cdot (2 \cdot N)^b = 368.5185\ \text{MPa}$$

Number of cycles to fail at $sp = 380\ \text{MPa}$

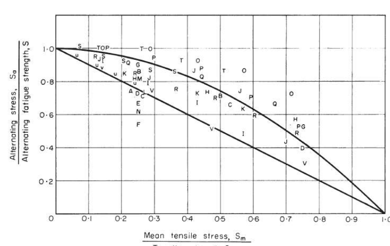
$$Np := \frac{1}{2} \cdot \left(\frac{sp}{sf}\right)^{\frac{1}{b}} = 8259.3729$$

Approximately 8200 cycles

Effect of Tensile Mean Stress on the Fatigue Life



Effects of Mean Stress on Fatigue Strength of Steels based on 10^7 to 10^8 Cycles

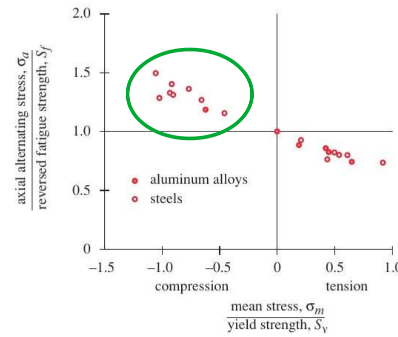


Effects of Mean Stress on Fatigue Strength of Aluminum based on 5 x 10^8 Cycles

- A parabola called the Gerber line, can be fitted to the data with reasonable accuracy.
- A straight line connecting the fatigue strength with the ultimate strength, called the modified Goodman line, is a reasonable fit to the lower envelope of the data.

Norton, Machine Design, An Integrated Approach

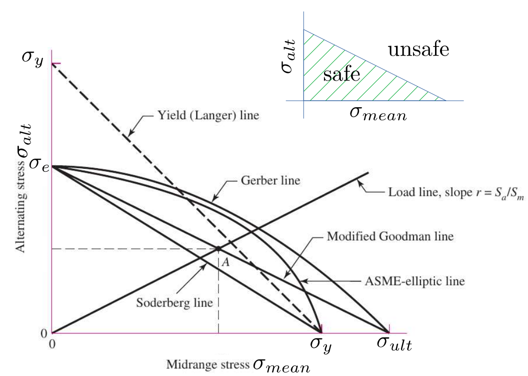
Effect of Mean Stress on the Fatigue Life



- Compressive mean stresses have a beneficial effect
- Tensile mean stresses have a detrimental effect
- To reduce the effects of alternating tensile stresses one deliberately introduce mean compressive stresses
- One way to do this is to create **residual compressive stress** in the material in regions where large alternating components are expected

Norton, Machine Design, An Integrated Approach

Fatigue Criteria in Presence of Mean Stress



Soderberg line $\dfrac{\sigma_a}{\sigma_e} + \dfrac{\sigma_m}{\sigma_y} = 1$

Modified Goodman line $\dfrac{\sigma_a}{\sigma_e} + \dfrac{\sigma_m}{\sigma_{ult}} = 1$

Gerber line $\dfrac{\sigma_a}{\sigma_e} + \left(\dfrac{\sigma_m}{\sigma_{ult}}\right)^2 = 1$

.SME-elliptic line $\left(\dfrac{\sigma_a}{\sigma_e}\right)^2 + \left(\dfrac{\sigma_m}{\sigma_y}\right)^2 = 1$

Yield (Langer) line $\sigma_a + \sigma_m = \sigma_y$

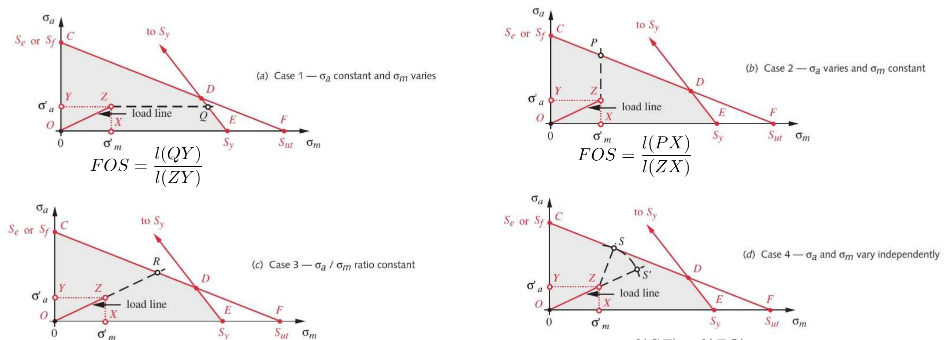
Smith-Watson-Topper Criterion

$$\sigma_e = \sqrt{\sigma_{max}\sigma_a} = \sqrt{(\sigma_m + \sigma_a)\sigma_a}$$

σ_e is the modified/corrected endurance limit

If N is the factor of safety then replace σ_m and σ_a with $N\sigma_m$ and $N\sigma_a$, respectively

Factor of Safety (FOS) in the Presence of Mean Stress using augmented modified Goodman diagram



(a) $FOS = \dfrac{l(QY)}{l(ZY)}$

(b) $FOS = \dfrac{l(PX)}{l(ZX)}$

(c) $FOS = \dfrac{l(RO)}{l(ZO)}$

(d) $FOS = \dfrac{l(SZ) + l(ZO)}{l(ZO)}$

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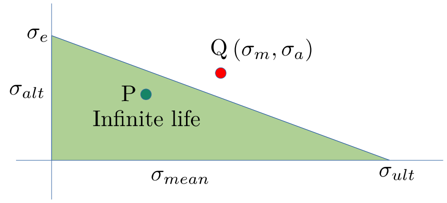
Finite Life in Presence of Mean Stress



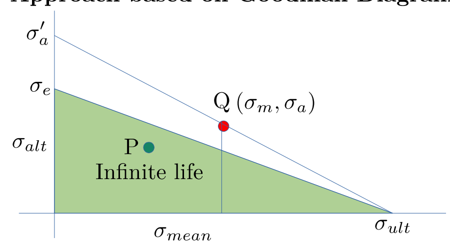
- At point P, the part has infinite life
- At point Q, the part will fail before 10^6 cycles (finite life)
- Want to estimate the number of cycles when it fails
- Data is generally available for completely reversed loading
- Procedure: Find the equivalent complete reversed stress amplitude σ'_a and then use the Basquin equation to estimate the number of cycles to failure

$$\sigma'_a = \sigma'_f (2N_f)^b$$

Finite Life in Presence of Mean Stress

Multiple approaches available to estimate the equivalent fully reversed stress amplitude, σ_a' corresponding to (σ_m, σ_a)

Approach based on Goodman Diagram



From similar triangles

$$\frac{\sigma_a'}{\sigma_{ult}} = \frac{\sigma_a}{(\sigma_{ult} - \sigma_m)}$$

Therefore

$$\boxed{\sigma_a' = \frac{\sigma_a}{\left(1 - \frac{\sigma_m}{\sigma_{ult}}\right)}}$$

Approach based on Smith-Watson-Topper (SWT) criterion

$$\boxed{\sigma_a' = \sqrt{\sigma_{max}\sigma_a} = \sqrt{(\sigma_m + \sigma_a)\sigma_a}}$$

- Approach based on Goodman diagram is widely used but is overly conservative and inaccurate
- SWT is reasonably accurate and **does not require any material properties**

Problem – Non Zero Mean Stress

A 1050 hot-rolled steel bar has been **machined** to a **diameter of 2.5 cm**. It is to be placed in **reversed axial loading** for 10000 cycles to failure at room temperature.
Using ASTM minimum properties, and a **reliability of 99 percent**, estimate the endurance limit, and the number of cycles to fail at $\sigma_a = 160$ MPa, $\sigma_m = 100$ MPa using
1. approach based on the Goodman diagram
2. Approach based on the SWT criterion

Problem – Endurance Limit Correction & Non Zero Mean Stress

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$sult := 620\ \text{MPa}$ $\quad f := 0.86$

Endurance strength

$s2 := 0.5 \cdot sult = 310\ \text{MPa}$ $\quad N2 := 10^6$

Correction Factors

Calculation of surface factor

$a := 4.51$ $\quad bsurF := -0.265$

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$kb := 1$ No size effect for axial loading

$kc := 0.85$ Load factor

$kd := 1$ Temperature effect

$ke := 0.814$ reliability factor

$kf := 1$ Miscellaneous factor

$s2C := ka \cdot kb \cdot kc \cdot kd \cdot ke \cdot kf \cdot s2 = 176.0345\ \text{MPa}$ Endurance limit

$s1 := f \cdot sult = 533.2\ \text{MPa}$ $\quad N1 := 10^3$

Find Number of cycles to fail at $\quad sa := 160\ \text{MPa}$ $\quad sm := 100\ \text{MPa}$

Constants for the Basquin curve

$$b := \frac{\log_e\left(\frac{s1}{s2C}\right)}{\log_e\left(\frac{N1}{N2}\right)} = -0.1604 \qquad sf := \frac{s2C}{(2 \cdot N2)^b} = 1804.9991\ \text{MPa}$$

Answers

equivalent fully reversed stress amplitude

$$sap := \frac{sa}{\left(1 - \frac{sm}{sult}\right)} = 1.9077 \cdot 10^8\ \text{Pa}$$

$$Nf := \frac{1}{2} \cdot \left(\frac{sap}{sf}\right)^{\frac{1}{b}} = 6.0589 \cdot 10^5$$

Number of cycles to fail is approximately 605000

$saswt := \sqrt{(sa + sm) \cdot sm} = 161.2452\ \text{MPa}$

Equivalent stress amplitude based on SWT criteria 161.2 MPa. Less than the Endurance limit – does not fail

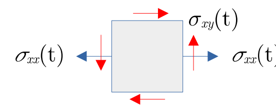
Combination of Loading Modes (Multiaxial Loading)

In general loading can be classified as
- Completely reversing simple loads (zero mean, but only one mode of loading)
- Fluctuating simple loads (nonzero mean, but only one mode of loading)
- Combinations of loading modes (axial, torsion, bending, e.g. rotating shaft subject to a static bending moment and carrying a torque)

Complications:
- The loading components can be at same/different frequencies, same/different phases
- As a result, in general, the direction of principal stresses change with time and damage accumulates in different planes

Classification of combined (multiaxial) loading:
- **Simple multiaxial loading**
  loading where direction of principal stresses do not change with time
- **Complex multiaxial**
  loading where direction of principal stresses change with time

Combination of Loading Modes (Multiaxial Loading)



Simple Multiaxial Loading
(same frequency, same phase)

$\sigma_{xx} = a_o \sin 2\pi t$

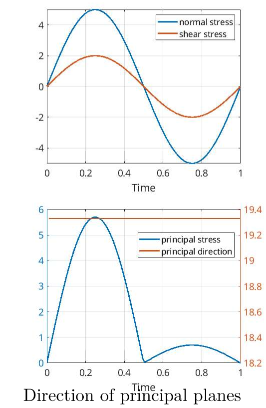
$\sigma_{xy} = b_o \sin 2\pi t$

Complex Multiaxial Loading
(same frequency, different phase)

$\sigma_{xx} = a_o \sin 2\pi t$

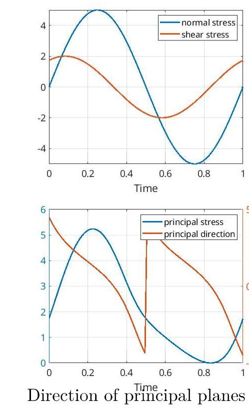
$\sigma_{xy} = b_o \sin(2\pi t + \pi/3)$

Simple Multiaxial Loading



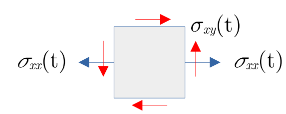
Direction of principal planes do not change

Complex Multiaxial Loading



Direction of principal planes changes

Combination of Loading Modes (Multiaxial Loading)

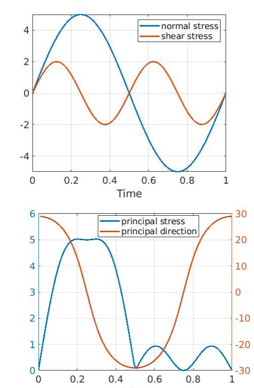


Complex Multiaxial Loading
(different frequencies)

$\sigma_{xx} = a_o \sin 2\pi t$

$\sigma_{xy} = b_o \sin 4\pi t$

Complex Multiaxial Loading

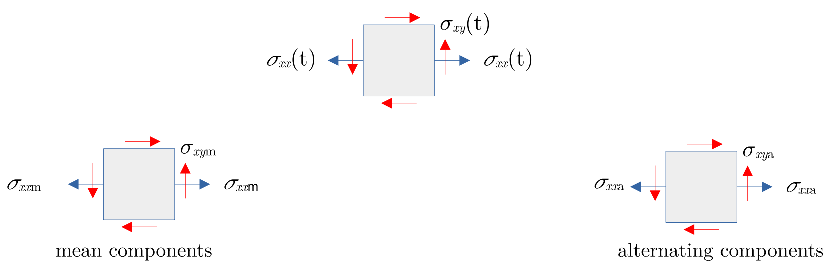


Direction of principal planes changes

Combination of Loading Modes (Multiaxial Loading)

Approach:

- To analyse static failures we used the idea of von Mises stress to combine various stress components to come up with a single number characterizing the state of stress at a point.
- Here too we will use the von Mises stress to find the equivalent alternating component and the equivalent mean components



Combination of Loading Modes (Multiaxial Loading)

Biaxial state of stress

$$\sigma_{vma} = (\sigma_{xxa}^2 - \sigma_{xxa}\sigma_{yya} + \sigma_{yya}^2 + 3\sigma_{xya})^{1/2}$$ Equivalent alternating stress

$$\sigma_{vmm} = (\sigma_{xxm}^2 - \sigma_{xxm}\sigma_{yym} + \sigma_{yym}^2 + 3\sigma_{xym})^{1/2}$$ Equivalent mean stress

Here

$$\sigma_{xxa} = \frac{(\sigma_{xx})_{max} - (\sigma_{xx})_{min}}{2},\ \sigma_{xxm} = \frac{(\sigma_{xx})_{max} + (\sigma_{xx})_{min}}{2} \qquad \sigma_{yya} = \frac{(\sigma_{yy})_{max} - (\sigma_{yy})_{min}}{2},\ \sigma_{yym} = \frac{(\sigma_{yy})_{max} + (\sigma_{yy})_{min}}{2}$$

$$\sigma_{xya} = \frac{(\sigma_{xy})_{max} - (\sigma_{xy})_{min}}{2},\ \sigma_{xym} = \frac{(\sigma_{xy})_{max} + (\sigma_{xy})_{min}}{2}$$

Triaxial state of stress

$$\sigma_{vma} = \frac{1}{\sqrt{2}}\left[(\sigma_{xxa} - \sigma_{yya})^2 + (\sigma_{yya} - \sigma_{zza})^2 + (\sigma_{zza} - \sigma_{xxa})^2 + 6(\sigma_{xya}^2 + \sigma_{yza}^2 + \sigma_{zxa}^2)\right]^{1/2}$$

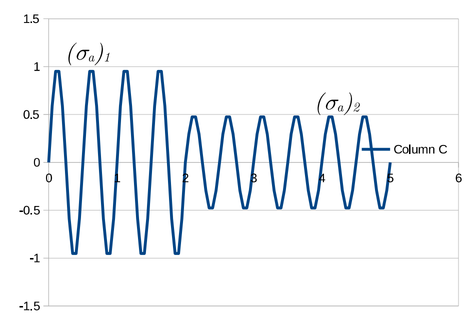
$$\sigma_{vmm} = \frac{1}{\sqrt{2}}\left[(\sigma_{xxm} - \sigma_{yym})^2 + (\sigma_{yym} - \sigma_{zzm})^2 + (\sigma_{zzm} - \sigma_{xxm})^2 + 6(\sigma_{xym}^2 + \sigma_{yzm}^2 + \sigma_{zxm}^2)\right]^{1/2}$$

Steps for Multiaxial Loading

- Theoretical/geometric stress concentration factors and notch sensitivity will be different for different loadings. Calculate the corresponding fatigue stress concentration factors
- Generate two stress elements—one for the alternating stresses and one for the mean stresses.
- Apply the appropriate fatigue stress concentration factors to each of the stresses; that is, apply $(K_f)_{\text{bending}}$ for the bending stresses, $(K_{fs})_{\text{torsion}}$ for the torsional stresses, and $(K_f)_{\text{axial}}$ for the axial stresses.
- Calculate an equivalent von Mises stress for each of these two stress elements, σ_a' and σ_m'
- Select a fatigue failure criterion.
- For the endurance limit, σ_e, the only correction factors that are affected by multiple load types are the size factor k_b and the load factor k_c.
- Find k_b, calculate it for each load and select the lowest one.
- The only load factor to be applied is corresponding to the axial load if it happens to be the dominant mode of loading. The load factor corresponding to the torsion load should not
- be applied as the that mode of loading is already considered while calculating the von Mises stress.

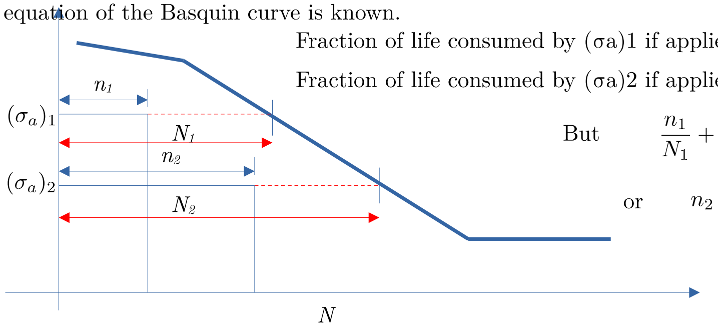
Cumulative Damage

A machine part is subject to n_1 (given) fully reversed stress cycles with amplitude $(\sigma_a)_1$ (given). It is then subject to n_2 (not known) fully reversed stress cycles of amplitude $(\sigma_a)_2$ (given) before it eventually fails. Find n_2. Assume $(\sigma_a)_1 > \sigma_e$, $(\sigma_a)_1 > \sigma_e$. Assume that the equation of the Basquin curve is known.



Cumulative Damage – Palmergren – Miner Theory

A machine part is subject to n_1 (given) fully reversed stress cycles with amplitude $(\sigma_a)_1$ (given). It is then subject to n_2 (not known) fully reversed stress cycles of amplitude $(\sigma_a)_2$ (given) before it eventually fails. Find n_2. Assume $(\sigma_a)_1 > \sigma_e$, $(\sigma_a)_1 > \sigma_e$. Assume that the equation of the Basquin curve is known.



Fraction of life consumed by (σa)1 if applied alone: n1/N1

Fraction of life consumed by (σa)2 if applied alone: n2/N2 (unknown)

But $$\frac{n_1}{N_1} + \frac{n_2}{N_2} = 1$$

or $$n_2 = N_2\left(1 - \frac{n_1}{N_1}\right)$$

Cumulative Damage – Palmergren-Miner Theory

In general, if fully reversed stress cycles with amplitude $(\sigma_a)_1,\ (\sigma_a)_2, \ldots, (\sigma_a)_k$ act for $n_1, n_2, \ldots, n_k$ cycles with $\sum_{i=1}^{k} n_i = n_{total},\ n_{total}$ total number of cycles to failure

then $$\sum_{i=1}^{k} \frac{n_i}{N_i} = 1$$

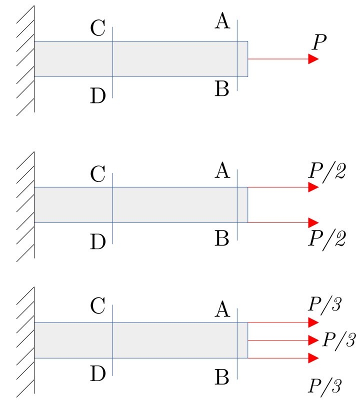
and $$\sum_{i=1}^{k} \frac{\alpha_i}{N_i} = \frac{1}{n_{total}},\ \sum_{i=1}^{k} \alpha_i = 1,\ \alpha_i = \frac{n_i}{n_{total}}$$

$\alpha_1, \alpha_2, \ldots, \alpha_k$ are the fraction of the total life spent at $(\sigma_a)_1,\ (\sigma_a)_2, \ldots, (\sigma_a)_k$

Limitations of the Miner Rule:

Does not take into account the sequence in which the stress cycles are applied

Saint Venant's Principle



In each of the three cases the resultant load is the same

Saint Venant's principle states that the displacement, strain and stress distributions caused by statically equivalent force distributions in parts of the body which are sufficiently far from the loading parts are approximately the same

Accordingly the displacement, strain and stress distributions at section CD will be approximately the same in the three cases

The displacement, strain and stress distributions at section AB will be different in the three cases

Design Factor and Factor of Safety

A solid circular rod undergoes a bending moment $M = 100$ Nm. Assuming that the yield strength of the material is 170 MPa and a **design factor of 2.5**, determine the minimum diameter of the rod. From the available sizes, choose an appropriate sized rod and determine the **factor of safety**.

Given: $M = 100$ Nm, $\sigma_y = 170$ MPa, $n_d = 2.5$
To find: diameter of the circular rod and the factor of safety
For a rod of diameter d, the maximum bending stress is given by

$$\sigma_{max0} = \frac{Md/2}{I} = \frac{Md/2}{\frac{\pi}{64}d^4} = \frac{32M}{\pi d^3}$$

Using the design factor, the component is to be designed for σ_{max} given by

$$\sigma_{max} = n_d \sigma_{max0}$$

The minimum diameter, d_{min}, required to withstand σ_{max} is obtained by solving

$$\sigma_y = \sigma_{max}$$

From the catalog select $d_s > d_{min}$ which is closest to d_{min}. The factor of safety is then given by

$$\text{FOS} = \frac{\sigma_y}{32M/(\pi d_s^3)}$$

Design Factors to be used for Ductile Materials

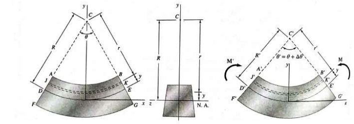
1.25 to 2.0	Design of structures under static load for which there is a high level of confidence in all design data
2.0 to 2.5	Design of machine elements under dynamic loading with average confidence in all design data
2.5 to 4.0	Design of static structures or machine element under dynamic loading with uncertainty about loads, material properties, environment
4.0 and higher	Design of static structures or machine element under dynamic loading with uncertainty about some combinations of loads, material properties, environment

Design Factors to be used for Brittle Materials

3.0 to 4.0	Design of structures under static load for which there is a high level of confidence in all design data
4.0 to 8.0	Design of static structures or machine element under dynamic loading with uncertainty about loads, material properties, environment

Curved Beams

- The length of arc JK in the undeformed beam is $l\,(JK) = r\theta$.
- Now apply equal and opposite moments and the two ends.



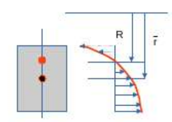
- The length of arc in the JK in the deformed configuration is $l\,(J'K') = r'\,(\theta + \Delta\theta)$
- The strain $\epsilon_{\theta\theta}$ in the circumferential direction are given by

$$\epsilon_{\theta\theta} = \frac{l(J'K') - l(JK)}{l(JK)} = \frac{r'\,(\theta + \Delta\theta) - r\theta}{r\theta} = \frac{\left(R' - y\right)(\theta + \Delta\theta) - (R - y)\,\theta}{r\theta}$$

$$= -\frac{y\Delta\theta}{r\theta} = -\frac{(R - r)\,\Delta\theta}{r\theta} = -k\frac{(R - r)}{r},\ k = \Delta\theta/\theta$$

Curved Beams

- We have
$$\epsilon_{\theta\theta} = -k\frac{(R - r)}{r},\ k = \Delta\theta/\theta$$
The strain does not vary linearly with distance from the neutral axis.
- For a linear elastic material (assuming all the other stress components are negligible) we get
$$\sigma_{\theta\theta} = E\epsilon_{\theta\theta} = -Ek\frac{(R - r)}{r}$$
The stress does not vary linearly with distance from the neutral axis.

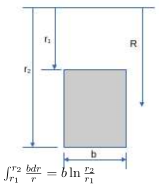


Curved Beams

- To obtain R, the location of the neutral axis, we use the condition that the normal force acting on the cross-section is zero.

$$\int_A \sigma_{\theta\theta} dA = 0$$
$$\text{or } -\int_A Ek\frac{(R - r)}{r}dA = 0$$
$$\text{or } R\int_A \frac{dA}{r} - \int_A dA = 0$$
$$\text{or } R = \frac{A}{\int_A \frac{dA}{r}}$$

- To find the position of the neutral axis for the rectangular c/s



- We have $A = b(r_2 - r_1)$. Also $\int_A \frac{dA}{r} = \int_{r_1}^{r_2} \frac{bdr}{r} = b\ln\frac{r_2}{r_1}$
- Therefore
$$R = \frac{r_2 - r_1}{\ln\frac{r_2}{r_1}}$$

Curved Beams

The relation between the internal resisting moment and the developed stress is obtained as follows:

- We have
$$\begin{aligned} M &= -\int_A \sigma_{\theta\theta}\,(R - r)\,dA \\ &= kE\int_A \frac{(R - r)^2}{r}dA \\ &= Ek\left(R^2\int_A \frac{dA}{r} - 2R\int_A dA + \int_A rdA\right) \\ &= Ek\left(R^2 \cdot \frac{A}{R} - 2RA + \bar{r}A\right) \\ \therefore M &= EkA\,(\bar{r} - R) \qquad \text{(A)} \end{aligned}$$
- We also have
$$\sigma_{\theta\theta} = -Ek\frac{(R - r)}{r} \qquad \text{(B)}$$

Curved Beams

- From Eqns A and B we get

$$\sigma_{\theta\theta} = -\frac{M\,(R-r)}{Ar\,(\bar{r}-R)}$$

- Let $y = R - r$ and $e = \bar{r} - R$. Here e is called the **eccentricity**. The stress can then be written as:

$$\sigma_{\theta\theta} = -\frac{My}{A\,(R-y)\,e}$$

For beams with circular c/s with radius c

$$R = \frac{A}{2\pi(\bar{r} - \sqrt{\bar{r}^2 - c^2})}$$

Curved Beams

Compare the stresses in a 50 mm × 50 mm square beam subjected to end moment of 2083 Nm in the following three cases: 1. straight beam, 2. curved beam with $\bar{r} = 250$ mm and 3. curved beam with $\bar{r} = 75$ mm.

- For a straight beam:

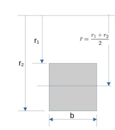
$$\sigma = -\frac{My}{I}$$

- $I = \frac{1}{12}\left(50 \times 10^{-3}\right)\left(50 \times 10^{-3}\right)^3\ m^4$ and $y = 25 \times 10^{-3}\ m$.

- Therefore $\sigma_T = 100$ MPa and $\sigma_C = -100$ MPa

- For a curved beam

$$\sigma = -\frac{M\,(R-r)}{Ar\,(\bar{r}-R)}$$



Curved Beams

- For a beam with rectangular c/s

$$R = \frac{r_2 - r_1}{\ln\frac{r_2}{r_1}}$$

- Case 2: $\bar{r} = 250$ mm
We have

$$r_1 = \bar{r} - 25 \times 10^{-3} = 225 \times 10^{-3} m$$
$$r_2 = \bar{r} + 25 \times 10^{-3} = 275 \times 10^{-3} m$$

- Hence $R = 249.164$ mm.

- Therefore $\sigma_T = 93.7$ MPa and $\sigma_C = -107.1$ MPa

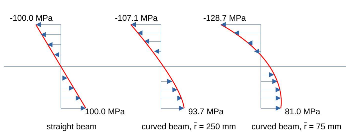
- Case 3: $\bar{r} = 75$ mm
We have

$$r_1 = \bar{r} - 25 \times 10^{-3} = 50 \times 10^{-3} m$$
$$r_2 = \bar{r} + 25 \times 10^{-3} = 100 \times 10^{-3} m$$

- Hence $R = 72.134$ mm.

- Therefore $\sigma_T = 81.0$ MPa and $\sigma_C = -128.7$ MPa

Curved Beams



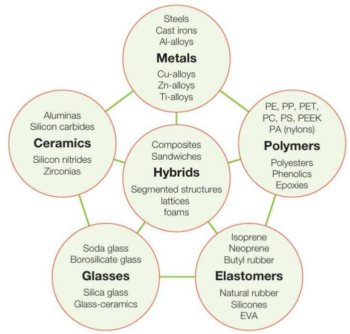
- Let h denote the depth of the beam. Here $h = 50$ mm. As the ratio $\bar{r}/h$ increases, the results of the curved beam approach those of the straight beam.

Selection of Materials in Mechanical Design

Factors to Consider While Selecting a Material

- Nature of loading on the component
- Expected stress state at the critical locations
- Allowable deformations at the critical locations
- Interfaces with other components of the product
- Environment in which the component is expected to operate.
- Physical size and weight of the component
- Aesthetics expected of the component/product
- Cost target of the component/product
- Availability of anticipated manufacturing processes

Material Families



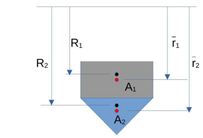
http://www.matweb.com

Primary Reference

Ashby, Material Selection in Mechanical Design

Curved Beams

Find R of the c/s made of two areas A_1 and A_2 shown below



- We have

$$R = \frac{A}{\int_A \frac{dA}{r}}$$

- Now

$$\int_A \frac{dA}{r} = \int_{A_1} \frac{dA}{r} + \int_{A_2} \frac{dA}{r}$$

- Also,

$$R_1 = \frac{A_1}{\int_{A_1} \frac{dA}{r}}, \; R_2 = \frac{A_2}{\int_{A_2} \frac{dA}{r}}$$

- Therefore

$$\int_A \frac{dA}{r} = \frac{A_1}{R_1} + \frac{A_2}{R_2}$$

- Hence

$$R = \frac{A}{\frac{A_1}{R_1} + \frac{A_2}{R_2}} = \frac{A_1 + A_2}{\frac{A_1}{R_1} + \frac{A_2}{R_2}}$$

- In general, for a c/s made up of n areas, $A_i, i = 1, 2, \ldots, n$

$$R = \frac{\sum_{i=1}^{n} A_i}{\sum_{i=1}^{n} \left(\frac{A_i}{R_i}\right)}$$

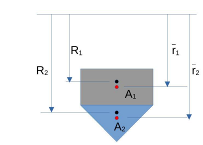
Design Limiting Material Properties

Class	Property	Symbol and units	
General	Density	ρ	(kg/m³ or Mg/m³)
	Price	C_m	($/kg)
Mechanical	Elastic moduli (Young's, shear, bulk)	E, G, K	(GPa)
	Yield strength	σ_y	(MPa)
	Ultimate strength	σ_u	(MPa)
	Compressive strength	σ_c	(MPa)
	Failure strength	σ_f	(MPa)
	Hardness	H	(Vickers)
	Elongation	ε	(–)
	Fatigue endurance limit	σ_e	(MPa)
	Fracture toughness	K_{1C}	(MPa.m$^{1/2}$)
	Toughness	G_{1C}	(kJ/m²)
	Loss coefficient (damping capacity)	η	(–)
Thermal	Melting point	T_m	(C or K)
	Glass temperature	T_g	(C or K)
	Maximum service temperature	T_{max}	(C or K)
	Minimum service temperature	T_{max}	(C or K)
	Thermal conductivity	λ	(W/m.K)
	Specific heat	C_p	(J/kg.K)
	Thermal expansion coefficient	α	(K^{-1})
	Thermal shock resistance	ΔT_s	(C or K)
Electrical	Electrical resistivity	ρ_e	(Ω.m or μΩ.cm)
	Dielectric constant	ε_d	(–)
	Breakdown potential	V_b	(10^6 V/m)
	Power factor	P	(–)
Optical	Optical, transparent, translucent, opaque	Yes/No	
	Refractive index	n	(–)
Eco-properties	Energy/kg to extract material	E_f	(MJ/kg)
	CO_2/kg to extract material	CO_2	(kg/kg)
Environmental resistance	Oxidation rates	Very low, low, average,	
	Corrosion rates	high, very high	
	Wear rate constant	K_A	MPa^{-1}

Ashby, Material Selection in Mechanical Design

Curved Beams

Find $\bar{r}$ of the c/s made of two areas A_1 and A_2 shown below



- We have

$$\bar{r} = \frac{\int_A r dA}{A}$$

- Now

$$\int_A r dA = \int_{A_1} r dA + \int_{A_2} r dA$$

- Also,

$$\bar{r}_1 = \frac{\int_{A_1} r dA}{A_1}, \; \bar{r}_2 = \frac{\int_{A_2} r dA}{A_2}$$

- Therefore

$$\bar{r} = \frac{\bar{r}_1 A_1 + \bar{r}_2 A_2}{A} = \frac{\bar{r}_1 A_1 + \bar{r}_2 A_2}{A_1 + A_2}$$

- In general, for a c/s made up of n areas, $A_i, i = 1, 2, \ldots, n$

$$\bar{r} = \frac{\sum_{i=1}^{n} \bar{r}_i A_i}{\sum_{i=1}^{n} A_i}$$

Important Thermal Properties of Engineering Materials

- Coefficient of Thermal Expansion (K^{-1}) – The thermal strain per degree of temperature change is measured by the coefficient of thermal expansion

- Thermal Conductivity (W/m.K) – indicates the ability to conduct/transfer heat at steady state

- Thermal Diffusivity (m^2/s) . It is given by $\alpha = \lambda/\rho C_p$. The distance x the heat diffuses in time t is given by $x \approx \sqrt{2\alpha t}$

- Heat capacity or Specific heat (units J/kg.K) It is the energy required to heat 1 kg of a material by 1 K. The measurement is usually done at constant pressure for solids

Important Mechanical Properties of Engineering Materials

- Ductility: It is the ability to undergo inelastic deformation before failure. It is measured in terms of elongation (tensile strain at failure)

- Elastic Modulus (GPa) – measure of stiffness

- Resilience: The ability of a material to absorb energy per unit volume without permanent deformation is called its resilience U_R (also called modulus of resilience) and is equal to the area under the stress-strain curve up to the elastic limit

- Toughness: The ability of a material to absorb energy per unit volume without fracture is called its toughness U_T (also called modulus of toughness) and is equal to the area under the stress-strain curve up to the fracture point

- Fracture toughness($MPa\sqrt{m}$) It is is a material property that defines its ability of a material to resist crack growth and is denoted by K_c

AISI/SAE Designation of Steel Alloys

Type	AISI/SAE Series	Principal Alloying Elements
Carbon Steels		
Plain	10xx	Carbon
Free-cutting	11xx	Carbon plus Sulphur (resulphurized)
Alloy Steels		
Manganese	13xx	1.75% Manganese
	15xx	1.00 to 1.65% Manganese
Nickel	23xx	3.50% Nickel
	25xx	5.00% Nickel
Nickel-Chrome	31xx	1.25% Nickel and 0.65 or 0.80% Chromium
	33xx	3.50% Nickel and 1.55% Chromium
Molybdenum	40xx	0.25% Molybdenum
	44xx	0.40 or 0.52% Molybdenum
Chrome-Moly	41xx	0.95% Chromium and 0.20% Molybdenum
Nickel-Chrome-Moly	43xx	1.82% Nickel, 0.50 or 0.80% Chromium, and 0.25% Molybdenum
	47xx	1.45% Nickel, 0.45% Chromium, and 0.20 or 0.35% Molybdenum
Nickel-Moly	46xx	0.82 or 1.82% Nickel and 0.25% Molybdenum
	48xx	3.50% Nickel and 0.25% Molybdenum
Chrome	50xx	0.27 to 0.65% Chromium
	51xx	0.80 to 1.05% Chromium
	52xx	1.45% Chromium
Chrome-Vanadium	61xx	0.60 to 0.95% Chromium and 0.10 to 0.15% Vanadium minimum

AISI – American Iron and Steel Institute
SAE – Society of Automotive Engineers

Last two digits indicate the % of carbon present.
e.g. 1040 – plain carbon steel with 0.40 % carbon (0.37% - 0.43%

Important Mechanical Properties of Engineering Materials

- Hardness – the resistance of a material to penetration by a pointed tool (Rockwell hardness, Brinell hardness). For steel: $\sigma_{ult} = 3.40 H_B$ (MPa)

- Machinability – related to the ease with which the material can be machined to a good surface finish with a reasonable tool life

- Creep – progressive elongation experienced by materials when subjected to high stress (less than yield strength) at high temperatures. (T > 0.3 Tm)

Material Index

- **Material index**: group of material properties which govern some aspect of performance of the component
- The material is selected by maximizing/minimizing the **material index**

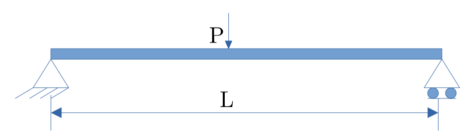
Function, objective, and constraints	Index
Tie, minimum weight, stiffness prescribed	$\frac{E}{\rho}$
Beam, minimum weight, stiffness prescribed	$\frac{E^{1/2}}{\rho}$
Beam, minimum weight, strength prescribed	$\frac{\sigma_y^{2/3}}{\rho}$
Beam, minimum cost, stiffness prescribed	$\frac{E^{1/2}}{C_m\rho}$
Beam, minimum cost, strength prescribed	$\frac{\sigma_y^{2/3}}{C_m\rho}$
Column, minimum cost, buckling load prescribed	$\frac{E^{1/2}}{C_m\rho}$
Spring, minimum weight for given energy storage	$\frac{\sigma_y^2}{E\rho}$
Thermal Insulation, minimum cost, heat flux prescribed	$\frac{1}{\lambda C_p\rho}$
Electromagnet, maximum field, temperature rise prescribed	$\frac{C_p\rho}{\rho_e}$

ρ = density; E = Young's modulus; σ_y = elastic limit; C_m = cost/kg; λ = thermal conductivity; ρ_e = electrical resistivity; C_p = specific heat.

Ashby, Material Selection in Mechanical Design

Example

Choose a material for a simply supported light stiff beam with a square c/s. The stiffness of the beam should be at least S^*



Material	ρ(kg/m^3)	E(GPa)	σ_y(MPa)
1020 Steel	7850	205	320
6061 Al	2700	70	120
Ti-6Al-4V	4400	115	950
GFRP	1750	28	300

Function: The beam supports a load P
Constraint: stiffness should be at least S^*
Objective: minimize the mass of the beam
Free variable: material and c/s area A

$$\delta = \frac{PL^3}{48EI}, \; I = \frac{b^4}{12}$$

$$\text{Stiffness} \quad S = \frac{P}{\delta} = \frac{48EI}{L^3} = \frac{4EA^2}{L^3}$$

Example

Mass $m = \rho A L$

Stiffness $S = \frac{4EA^2}{L^3}$

$S \geq S^*$ constraint

$$\frac{4EA^2}{L^3} \geq S^*$$

$$\frac{4Em^2}{L^5\rho^2} \geq S^*$$ Substitute for the area A in terms of mass m

or $$m \geq \left(\frac{1}{2}\sqrt{S^*}\right)\left(L^{5/2}\right)\left(\frac{\rho}{E^{1/2}}\right)$$

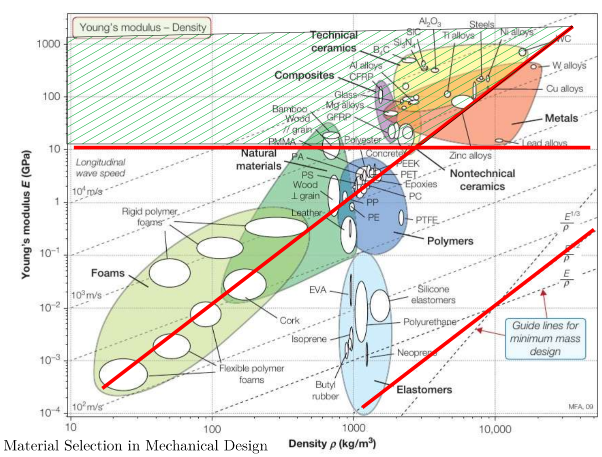
Functional parameters, **G**eometric parameters, **M**aterial properties

$m \geq f_1(F) f_2(G) f_3(M)$ separable form

To minimize the mass we therefore need to maximize the material index $\frac{E^{1/2}}{\rho}$

Material	ρ(kg/m^3)	E(GPa)	σ_y(MPa)	$E^{1/2}/\rho$
1020 Steel	7850	205	320	58
6061 Al	2700	70	120	98
Ti-6Al-4V	4400	115	950	77
GFRP	1750	28	300	96

Material Property Chart



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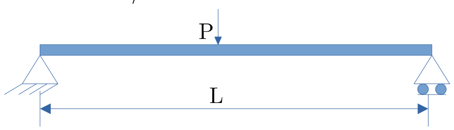
Example – Single Objective Multiple Constraints

Choose a material for a simply supported light beam with a square c/s with max stress being less than the failure strength **and** stiffness being at least S^*. Here L = 1m, P = 100 kN, $S^* = 3\text{x}10^7$ N/m



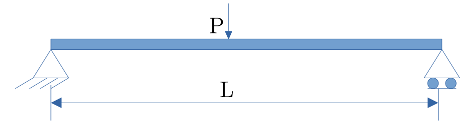
$$m \geq \left(\frac{1}{2}\sqrt{S^*}\right)\left(L^{5/2}\right)\left(\frac{\rho}{E^{1/2}}\right) \quad \text{stiffness}$$

$$m \geq \left((3P)^{2/3}\right)\left(L^{5/3}\right)\left(\frac{\rho}{\sigma_y^{2/3}}\right) \quad \text{strength}$$

Material	ρ (kg/m^3)	E(GPa)	σ_y(MPa)
1020 Steel	7850	205	320
6061 Al	2700	70	120
Ti-6Al-4V	4400	115	950
GFRP	1750	28	300

Example – Single Objective Single Constraint

Choose a material for a simply supported light beam with a square c/s with max stress being less than the failure strength



Constraint: max stress should be at least σ_f^*
Objective: minimize the mass of the beam
Free variable: material and c/s area A

$$I = \frac{b^4}{12},\ M_{max} = \frac{PL}{4}$$

$$\sigma = -\frac{My}{I},\ \sigma_{max} = \frac{3PL}{b^3}$$

Material	ρ (kg/m^3)	E(GPa)	σ_y(MPa)
1020 Steel	7850	205	320
6061 Al	2700	70	120
Ti-6Al-4V	4400	115	950
GFRP	1750	28	300

Example – Single Objective Multiple Constraints

Material	ρ (kg/m^3)	E(GPa)	σ_y(MPa)	m1 [S*](kg)	m2 [σ_y](kg)	max(m1,m2)
1020 Steel	7850	205	320	47	75	75
6061 Al	2700	70	120	28	50	50
Ti-6Al-4V	4400	115	950	36	20	36
GFRP	1750	28	300	29	18	**29**

min(max(m1,m2))=29

Example of a min-max problem

Example – Single Objective Single Constraint

Mass $m = \rho AL$

Max stress $\sigma_{max} = \frac{3PL}{b^3}$

$$\sigma_{max} \leq \sigma_y \quad \text{constraint}$$

$$\frac{3PL}{b^3} \leq \sigma_y$$

$$\frac{3PL}{\left(\frac{m}{\rho L}\right)^{3/2}} \leq \sigma_y \quad \text{Substitute for } b \text{ in terms of mass } m$$

or

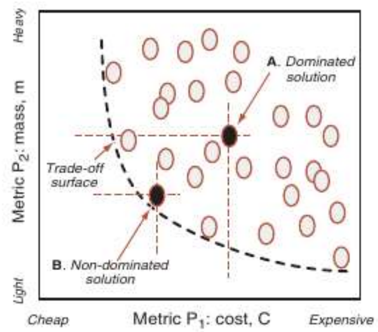
$$m \geq \left((3P)^{2/3}\right)\left(L^{5/3}\right)\left(\frac{\rho}{\sigma_y^{2/3}}\right)$$

To minimize the mass we therefore need to maximize the material index $\frac{\sigma_y^{2/3}}{\rho}$

Material	ρ(kg/m^3)	E(GPa)	σ_y(MPa)	$\sigma_y^{2/3}/\rho$
1020 Steel	7850	205	320	60
6061 Al	2700	70	120	90
Ti-6Al-4V	4400	115	950	219
GFRP	1750	28	300	256

Multi-Objective Problems

Problem: Identify a material which minimizes both mass (performance metric P1) and cost (performance metric P2) while also meeting a set of constraints



- Each bubble represents a material – does not violate any constraints
- The materials that minimize P1 do not minimize P2 and vice versa
- Materials such as that at A, are far from optimal — all the materials in the box attached have lower values of both P1 and P2 – Dominated Solutions
- Materials like those at B have the characteristic that no other materials exists with lower values of both P1 and P2. Non-dominated solutions.
  The line or surface on which the non-dominated solutions lie is called the optimal trade-off surface or Pareto surface
- Different methods exists for solving such projections

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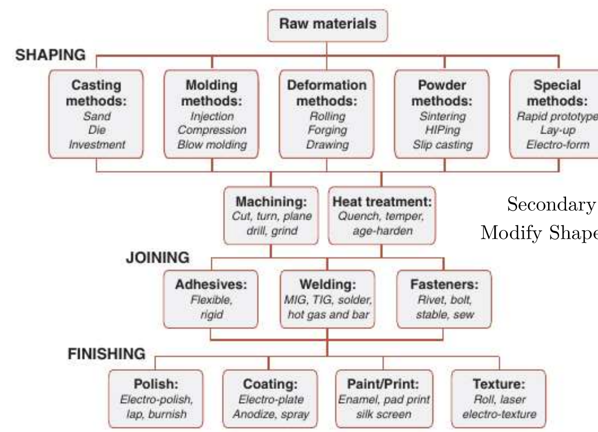
Types of Processes



Primary processes
Create Shapes

Secondary processes
Modify Shapes/Properties

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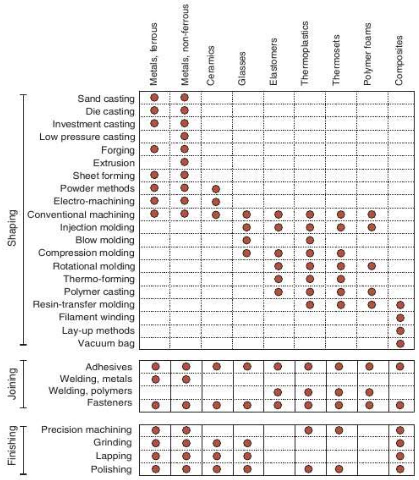
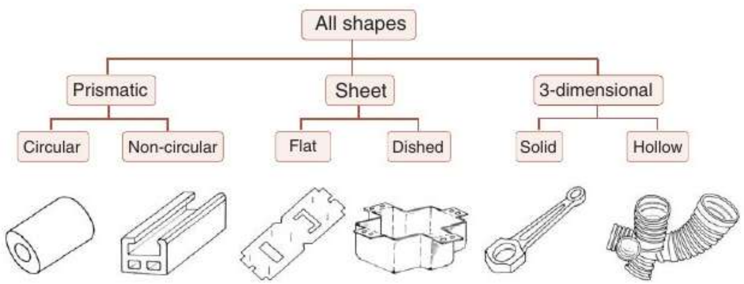
The Process–Material Matrix



A red dot indicates that the pair are compatible

Ashby, Material Selection in Mechanical Design

Types of Shapes



Ashby, Material Selection in Mechanical Design

Shaft Design Considerations – Static Strength

- The maximum shear stress is then given by

$$\tau_{max} = \frac{\sigma_A - \sigma_B}{2} = \left[\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

- The equivalent or the von Mises stress is then given by

$$\sigma_{eq} = (\sigma_A^2 + \sigma_B^2 - \sigma_A\sigma_B)^{1/2} = (\sigma_{xx}^2 + 3\tau_{xy}^2)^{1/2}$$

- Substituting the expression for the stress components:

$$\tau_{max} = \frac{2}{\pi d^3}\sqrt{(Fd + 8M)^2 + (8T)^2} \qquad \sigma_{eq} = \frac{4}{\pi d^3}\sqrt{(Fd+8M)^2 + 48T^2}$$

- If N is the factor of safety and σ_y is the yield strength of the material in tension, then the diameter d is chosen such that

$$\frac{\sigma_y}{2N} = \frac{2}{\pi d^3}\sqrt{(Fd + 8M)^2 + (8T)^2}$$

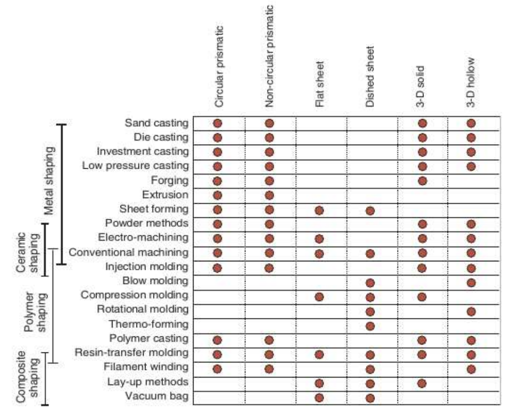
(Tresca criterion)

$$\frac{\sigma_y}{N} = \frac{4}{\pi d^3}\sqrt{(Fd+8M)^2 + 48T^2}$$

(Distortion Energy criterion)

If $F = 0$, it is easier to solve these equations for d

The Process–Shape Matrix



A red dot indicates that the pair are compatible

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Shaft Design Considerations – Fatigue Strength

- For a point on the surface of a circular solid shaft of diameter d located at a critical location with fluctuating bending moment (mean component M_m, alternating component M_a) and fluctuating torque (mean component T_m, alternating component T_a) [axial loading is ignored]

$$\sigma_m = K_f\frac{32M_m}{\pi d^3}, \quad \sigma_a = K_f\frac{32M_a}{\pi d^3}$$

$$\tau_m = K_{fs}\frac{16T_m}{\pi d^3}, \quad \tau_a = K_{fs}\frac{16T_a}{\pi d^3}$$

K_f and K_{fs} are the fatigue stress concentration factors at the critical location

- The mean and alternating von Mises (equivalent) stresses are given by

$$\sigma_{eqm} = (\sigma_m^2 + 3\tau_m^2)^{1/2} \qquad \sigma_{eqa} = (\sigma_a^2 + 3\tau_a^2)^{1/2}$$

or

$$\sigma_{eqm} = \left[\left(K_f\frac{32M_m}{\pi d^3}\right)^2 + 3\left(K_{fs}\frac{16T_m}{\pi d^3}\right)^2\right]^{1/2} \qquad \sigma_{eqa} = \left[\left(K_f\frac{32M_a}{\pi d^3}\right)^2 + 3\left(K_{fs}\frac{16T_a}{\pi d^3}\right)^2\right]^{1/2}$$

Shaft Design Considerations – Static Strength

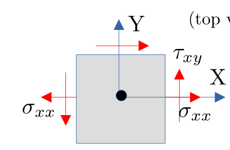
- Though a shaft is under dynamic loading (fatigue), a static analysis gives a preliminary estimate of the dimensions of the required dimensions.
- For a point on the surface of a circular solid shaft of diameter d located at a critical location with constant bending moment M, constant torque T and constant axial force F (x-axis along the axis, y - horizontal)

$$\sigma_{xx} = \frac{32M}{\pi d^3} + \frac{4F}{\pi d^2}$$

(the second component can be either positive or negative (tension/compression)

$$\tau_{xy} = \frac{16T}{\pi d^3}$$

(stress concentration effects are neglected for ductile material)



- Since the point is assumed to be on an unloaded surface, it is in the state of plane stress.

- The three principal stresses are given by:

$$\sigma_A, \sigma_B = \frac{\sigma_{xx}}{2} \pm \left[\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}, \quad \sigma_C = 0$$

Shaft Design Considerations – Fatigue Strength

or

$$\sigma_{eqm} = \frac{1}{\pi d^3}\left[K_f^2(32M_m)^2 + 3K_{fs}^2(16T_m)^2\right]^{1/2} \qquad \sigma_{eqa} = \frac{1}{\pi d^3}\left[K_f^2(32M_a)^2 + 3K_{fs}^2(16T_a)^2\right]^{1/2}$$

- If N is the factor of safety (given), the Goodman criterion can be written as

$$\frac{\sigma_{eqa}}{\sigma_e} + \frac{\sigma_{eqm}}{\sigma_{ult}} = \frac{1}{N}$$

- Substituting the expressions for σ_{eqa} and σ_{eqm} obtained earlier one gets an equation for d.
- On solving the equation, we get the estimate of the minimum diameter which needs to be used.
- One can also use the the other fatigue criterion – Gerber, etc

Shaft Design Considerations – Fatigue Strength

- In addition one also needs to check for static failure in the first cycle itself – for the estimated d, we calculate the maximum equivalent (von Mises) stress

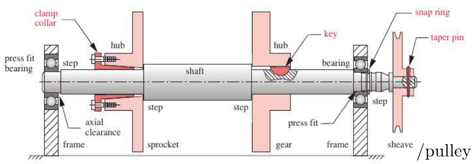
$$\sigma_{eqmax} = \left((\sigma_m + \sigma_a)^2 + 3(\tau_m + \tau_a)^2\right)^{1/2}$$

and compare it with $\dfrac{\sigma_y}{N}$

- Another class of problems consists of given d, estimate the factor safety, N. This is typically easier to solve.

Shaft Design Considerations – Fatigue Strength

- **Estimates of the Stress Concentration Factors**
  The design of the shaft is greatly influenced by the presence of geometric features such as shoulders, holes, keyways which lead to stress concentration.



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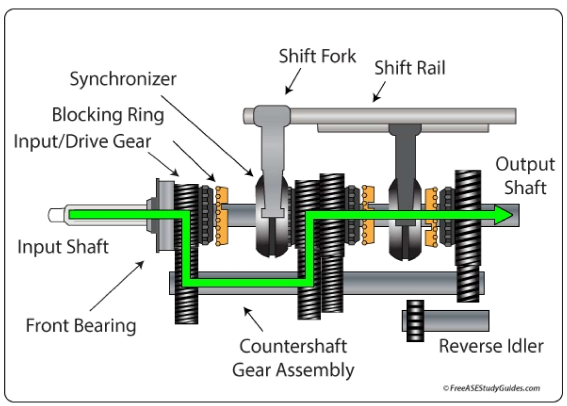
Stress concentration factors depend on the dimensions of the shaft which is not available. Hence estimates of the these factors are needed and are given by

	Bending	Torsional	Axial
Shoulder fillet—sharp ($r/d = 0.02$)	2.7	2.2	3.0
Shoulder fillet—well rounded ($r/d = 0.1$)	1.7	1.5	1.9
End-mill keyseat ($r/d = 0.02$)	2.14	3.0	—
Sled runner keyseat	1.7	—	—
Retaining ring groove	5.0	3.0	5.0

What is a Countershaft



Countershaft: It is a secondary shaft, which is driven by the main shaft and from which The power is supplied to a machine component. Often, the countershaft is driven from the main shaft by means of a pair of spur or helical gears and thus rotates 'counter' to the direction of the main shaft.

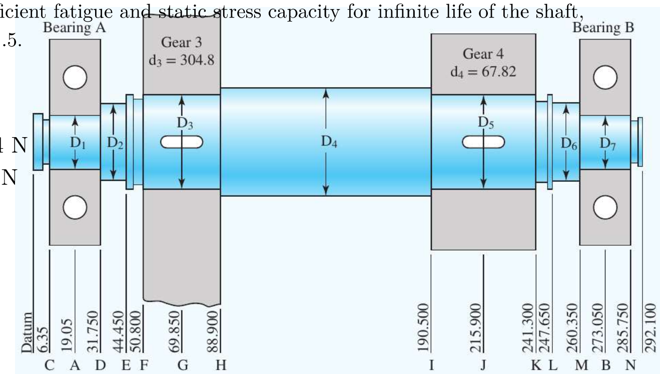
https://www.freeasestudyguides.com/manual-transmission-countershafts.html

Problem – Fatigue Strength

The general layout and axial dimensions of the countershaft carrying two spur gears has been is shown in Figure. The gears and bearings are located and supported by shoulders, and held in place by retaining rings. The gears transmit torque through keys. Gears have been specified as shown, allowing the tangential and radial forces transmitted through the gears to the shaft are given. Choose appropriate material. Determine $D_1,\ldots,D_7$, based on providing sufficient fatigue and static stress capacity for infinite life of the shaft, with minimum safety factors of 1.5.

$W^t_{23} = 2401$ N $\quad W^t_{54} = 10814$ N

$W^r_{23} = 876$ N $\quad W^r_{54} = 3937$ N



Budinas, Shigley'S Mechanical Engineering Design (SIE), 9[th] Ed (pg 374-378)

Solution

Perform free body diagram analysis to get reaction forces at the bearings.

$R_{Az} = 511.54$ N

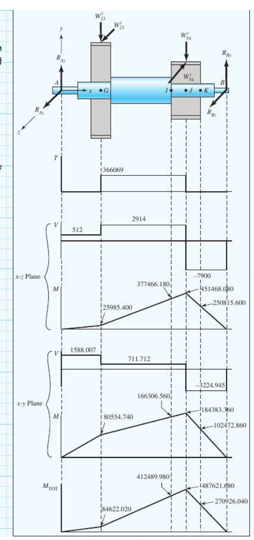
$R_{Ay} = 1586.67$ N

$R_{Bz} = 7900.00$ N

$R_{By} = 3226.28$ N

From ΣM_x, find the torque in the shaft between the gears, $T = W^t_{23}(d_3/2) = 2402(304.8/2) = 366069.07$ N · mm.

Generate shear-moment diagrams for two planes.



Combine orthogonal planes as vectors to get total moments, e.g., at J, $\sqrt{451468^2 + 184383^2} = 487622$ N · mm.

Start with point I, where the bending moment is high, there is a stress concentration at the shoulder, and the torque is present.

At I, $M_a = 412489.98$ N · mm, $T_m = 366069.07$ N · mm, $M_m = T_a = 0$

Assume generous fillet radius for gear at I.

From Table 7-1, estimate $K_t = 1.7$, $K_{ts} = 1.5$. For quick, conservative first pass, assume $K_f = K_t$, $K_{fs} = K_{ts}$.

Choose inexpensive steel, 1020 CD, with $S_{ut} = 470$ MPa. For S_e,

Equation (6–18) $\quad k_a = aS_{ut}^b = 3.04(470)^{-0.217} = 0.8$

Guess $k_b = 0.9$. Check later when d is known.

$$k_c = k_d = k_e = 1$$

Equation (6–17) $\quad S_e = (0.8)(0.9)(0.5)(470) = 169.18$ MPa

For first estimate of the small diameter at the shoulder at point I, use the DE-Goodman criterion of Equation (7-8). This criterion is good for the initial design, since it is simple and conservative. With $M_m = T_a = 0$, Equations (7-6) and (7-8) reduce to

$$d = \left\{\frac{16n}{\pi}\left(\frac{2(K_f M_a)}{S_e} + \frac{[3(K_{fs}T_m)^2]^{1/2}}{S_{ut}}\right)\right\}^{1/3}$$

$$d = \left\{\frac{16(1.5)}{\pi}\left(\frac{2(1.7)(412489.98)}{169.18} + \frac{\{3[(1.5)(366069.07)]^2\}^{1/2}}{470}\right)\right\}^{1/3}$$

$$d = 42.35 \text{ mm}$$

All estimates have probably been conservative, so select the next standard size below 1.69 in and check, $d = 41.275$ mm.

A typical D/d ratio for support at a shoulder is $D/d = 1.2$, thus, $D = 1.2(41.275) = 49.5300$ mm. Increase to $D = 50.8$ mm. A nominal 2-in. cold-drawn shaft diameter can be used. Check if estimates were acceptable.

$$D/d = 50.8/41.275 = 1.23$$

Assume fillet radius $r = d/10 = 4.1275$ mm, $\quad r/d = 0.1$

$K_t = 1.6$ (Fig. A–15–9), $q = 0.82$ (Fig. 6–26)

Equation (6–32) $\quad K_f = 1 + 0.82(1.6 - 1) = 1.49$

$K_{ts} = 1.35$ (Fig. A–15–8), $q_s = 0.85$ (Fig. 6–27)

$K_{fs} = 1 + 0.85(1.35 - 1) = 1.30$

$k_a = 0.801$ (no change)

Equation (6–19) $\quad k_b = \left(\frac{41.275}{7.62}\right)^{-0.107} = 0.835$

$S_e = (0.801)(0.835)(0.5)(470.000) = 157.12$ MPa

Equation (7–4) $\quad \sigma'_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.49)412489.98(1000)}{\pi(41.275)^3} = 89149.95$ kPa

Equation (7–5) $\quad \sigma'_m = \left[3\left(\frac{16K_{fs}T_m}{\pi d^3}\right)^2\right]^{1/2} = \frac{\sqrt{3}(16)(1.30)366069.07(1000)}{\pi(41.275)^3} = 59585.40$ kPa

Draw the FBD and calculate the reaction forces.

Draw the torque, shear force and bending moment diagrams (BMD).

Draw the resultant BMD.

Identify the critically stressed locations (guess).

Start with the one you think is the most critical (I) in the present solution.

Estimate the fatigue stress concentration factors at that location.

Choose the material.

Find the corrected endurance limit.

Find the diameter using the Goodman criterion and the given FOS.

Using Goodman criterion

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{89149.95/157.12}{1000} + \frac{59585.40/470}{1000} = 0.696$$

$$n_f = 1.44$$

Note that we could have used Equation (7-7) directly.

Check yielding.

$$n_y = \frac{S_y}{\sigma'_{max}} > \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{392997.900}{89149.9464 + 59585.4043} = 2.64$$

Also check this diameter at the end of the keyway, just to the right of point I, and at the groove at point K. From moment diagram, estimate M at end of keyway to be $M = 423675$ N · mm.

Assume the radius at the bottom of the keyway will be the standard $r/d = 0.02$, $r = 0.02d = 0.02(41.275) = 0.8255$ mm.

$K_t = 2.14$ (Table 7–1), $q = 0.65$ (Figure 6–26)

$K_f = 1 + 0.65(2.14 - 1) = 1.74$

$K_{ts} = 3.0$ (Table 7–1), $q_s = 0.71$ (Figure 6–27)

$K_{fs} = 1 + 0.71(3 - 1) = 2.42$

$$\sigma'_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.74)(423675 \text{ N} \cdot \text{mm})}{\pi(41.275)^3} = 106849.00 \text{ kPa}$$

$$\sigma'_m = \sqrt{3}(16)\frac{K_{fs}T_m}{\pi d^3} = \frac{\sqrt{3}(16)(2.42)(366069.07)}{\pi(41.275)^3} = 111134.24 \text{ kPa}$$

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{106849.00}{157(1000)} + \frac{111134.2416}{470(1000)} = 0.919$$

$$n_f = 1.09$$

The keyway turns out to be more critical than the shoulder. We can either increase the diameter or use a higher strength material. Unless the deflection analysis shows a need for larger diameters, let us choose to increase the strength. We started with a very low strength and can afford to increase it some to avoid larger sizes. Try 1050 CD with $S_{ut} = 690$ MPa.

Recalculate factors affected by S_{ut}, i.e., $k_a \rightarrow S_e$; $q \rightarrow K_f \rightarrow \sigma'_a$

$k_a = 3.04(690)^{-0.217} = 0.736$, $\quad S_e = 0.736(0.835)(0.5)(690) = 211.91$ MPa

$q = 0.72$, $K_f = 1 + 0.72(2.14 - 1) = 1.82$

$$\sigma'_a = \frac{32(1.82)(423675)}{\pi(41.275)^3} = 111746 \text{ kPa}$$

$$\frac{1}{n_f} = \frac{111746}{211.91(1000)} + \frac{111134.2416}{690(1000)} = 0.689$$

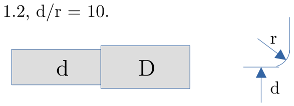
$$n_f = 1.45$$

This is slightly under the goal for the design factor to be 1.5. If we round to 2 significant figures, which is actually more realistic for fatigue, then we get 1.5. If the situation calls for a more conservative decision, a higher strength material can be selected.

Check at the groove at K, since K_t for flat-bottomed grooves are often very high. From the torque diagram, note that no torque is present at the groove. From the moment diagram, $M_a = 270926$ N · mm,

Check for static failure at the chosen location. If there is adequate FOS against static failure, find the exact local dimensions at the location

Typical D/d = 1.2, d/r = 10.



Repeat the strength calculations at the location using the exact dimensions.

If the FOS is acceptable, proceed to the next critically stressed location

Keyway to the right of location I:.

Radius at the bottom of the keyway is typically r/d' = 0.02.

Calculate r and use it to find the fatigue stress concentration factors.

Then calculate the FOS. In the present case, the FOS is not sufficient at the base of the keyway – can change either change D or the material.

For strength calculations, one can go with a better material if one has started with the most basis materials in the first place. Once this location is made safe, proceed to to the next critical locations.

Continue till all the selected critical locations have an adequate FOS.

$M_m = T_a = T_m = 0$. To quickly check if this location is potentially critical, just use $K_f = K_t = 5.0$ as an estimate, from Table 7-1.

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(5)(270926(1000))}{\pi(41.275)^3} = 196227 \text{ kPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{211.91(1000)}{196227} = 1.08$$

This is low. We will look up data for a specific retaining ring to obtain K_t more accurately. With a quick online search of a retaining ring specification using the website www.globalspec.com, appropriate groove specifications for a retaining ring for a shaft diameter of 41.275 mm are obtained as follows: width, $a = 1.727$ mm; depth, $t = 1.219$ mm; and corner radius at bottom of groove, $r = 0.254$ mm. From Figure A-15-16, with $r/t = 0.254/1.219 = 0.208$, and $a/t = 1.727/1.219 = 1.42$

$K_t = 4.3$, $q = 0.65$ (Figure 6–26)

$K_f = 1 + 0.65(4.3 - 1) = 3.15$

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(3.15)(270926)}{\pi(41.275)^3} = 123427 \text{ kPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{211(1000)}{123427} = 1.71$$

Quickly check if point M might be critical. Only bending is present, and the moment is small, but the diameter is small and the stress concentration is high for a sharp fillet required for a bearing. From the moment diagram, $M_a = 108347.82$ N · mm, and $M_m = T_m = T_a = 0$.

Estimate $K_t = 2.7$ from Table 7-1, $d = 25.4$ mm, and fillet radius r to fit a typical bearing.

$r/d = 0.02$, $r = 0.02(25.4) = 0.5080$

$q = 0.7$ (Figure 6–26)

$K_f = 1 + (0.7)(2.7 - 1) = 2.19$

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(2.19)(108347.82)}{\pi(25.4)^3} = 147490.23 \text{ kPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{211.91(1000)}{147490.23} = 1.44$$

This location is more critical than perhaps anticipated. The estimate for stress concentration is likely on the high side, so we will choose to continue and recheck after a specific bearing is selected.

With the diameters specified for the critical locations, fill in trial values for the rest of the diameters, taking into account typical shoulder heights for bearing and gear support.

$D_1 = D_7 = 25.4$ mm

$D_2 = D_6 = 35.56$ mm

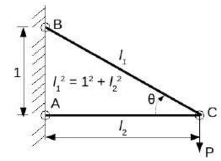
$D_3 = D_5 = 41.275$ mm

$D_4 = 50.8$ mm

The bending moments are much less on the left end of the shaft, so D_1, D_2, and D_3 could be smaller. However, unless weight is an issue, there is little advantage to requiring more material removal. Also, the extra rigidity may be needed to keep deflections small.

Problem

Find the deflection of the point of application of point in the vertical direction using Castigliano's Second Theorem. The material is linear elastic with Young's modulus E.



- This is a statically determinate problem.

- The strain energy in the two member truss is given by

$$U = \sum_{i=1}^{2} \frac{P_i^2 l_i}{2EA_i}$$
$$= \frac{F_{AC}^2 l_2}{2EA_2} + \frac{F_{BC}^2 l_1}{2EA_1}$$
$$U(P) = \frac{1}{2}\frac{P^2 l_2^3}{A_2 E} + \frac{1}{2}\frac{P^2 l_1^3}{A_1 E}$$

- Let δ denote the vertical displacement of point C. From Castigliano's Second theorem

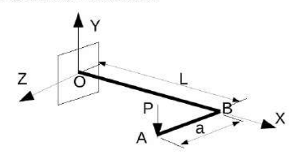
$$\delta = \frac{\partial U}{\partial P}$$

- Therefore

$$\delta = \frac{P l_2^3}{A_2 E} + \frac{P l_1^3}{A_1 E}$$

Problem

Find the vertical deflection at the point of application of the force P using Castigliano's Second Theorem.



- The strain energy in the member is given by

$$U = \frac{P^2 a^3}{6EI} + \frac{P^2 L^3}{6EI} + \frac{P^2 L a^2}{2GJ}$$

- Let δ denote the vertical displacement of point C. From Castigliano's Second theorem

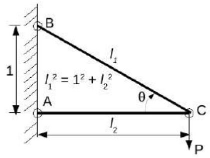
$$\delta = \frac{\partial U}{\partial P}$$

- Therefore

$$\delta = \frac{P a^3}{3EI} + \frac{P L^3}{3EI} + \frac{P L a^2}{GJ}$$

Problem – Dummy Load Method

Find the deflection of the point of application of point in the horizontal direction. The material is linear elastic with Young's modulus E.



- Castigliano's Second Theorem gives us the displacement in the direction of the load.

- In this problem there is no applied force in the horizontal direction.

Main Idea of the Dummy Load Method

- We are asked to find the displacement in a direction where there is no applied force.

- Assume a dummy load, Q, acts in the direction in which the displacement is required.

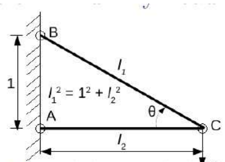
- Express the strain energy of the system in terms of the applied loads and the dummy load Q.

- Find the displacement in the direction of the dummy load using Castigliano's Second Theorem.

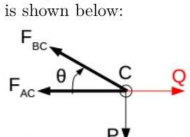
$$\delta_Q = \frac{\partial U}{\partial Q}$$

- Set $Q = 0$ in the expression of δ_Q. This gives the answer to the original problem.

Problem – Dummy Load Method



The FBD of joint C with a dummy load Q is shown below:



- Force balance in the horizontal direction and the vertical direction gives:

$$F_{BC} = \frac{P}{\sin\theta}, \quad F_{AC} = Q - P\cot\theta$$

- The strain energy in the two member truss is given by

$$U = \frac{F_{AC}^2 l_2}{2EA_2} + \frac{F_{BC}^2 l_1}{2EA_1} \text{ or } U(P,Q) = \frac{(Q - Pl_2)^2 l_2}{2EA_2} + \frac{P^2 l_1^3}{2A_1 E}$$

- From Castigliano's Second Theorem

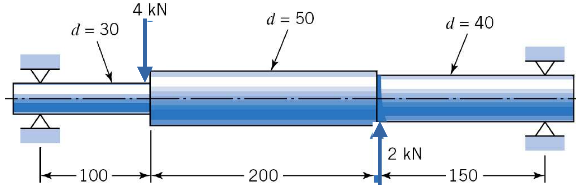
$$\tilde{\delta}_Q = \frac{\partial U}{\partial Q} = \frac{(Q - Pl_2)\, l_2}{EA_2}$$

- The horizontal displacement, δ_Q in the original problem is then given by

$$\delta_Q = \left(\tilde{\delta}_Q\right)_{Q=0} = -\frac{P l_2^2}{EA_2}$$

Shaft Deflection using Castigliano's Second Theorem

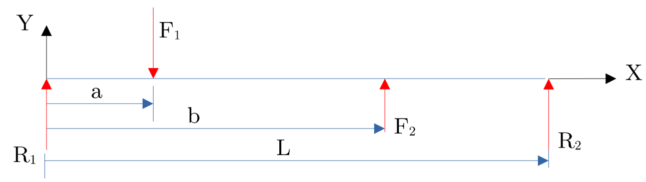
The shaft shown in the figure is made up of low carbon steel with E = 207 GPa. Find the deflection at the point of application of the 4 kN and 2 kN forces



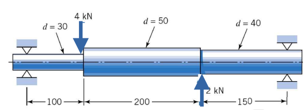
From force balance and moment balance:

$$R_1 = \frac{(L-a)F_1 - (L-b)F_2}{L}$$

$$R_2 = \frac{aF_1 - bF_2}{L}$$



Shaft Deflection using Castigliano's Second Theorem



For $0 < x < a$,
$V = -R_1,\ M(x) = R_1 x$
For $a < x < b$,
$V = F_1 - R_1,\ M(x) = R_1 x - F_1(x-a)$
For $b < x < L$,
$V = F_1 - R_1 - F_2,\ M(x) = R_1 x - F_1(x-a) + F_2(x-b)$

Let

$$M_1 = R_1 x$$
$$M_2 = R_1 x - F_1(x-a)$$
$$M_3 = R_1 x - F_1(x-a) + F_2(x-b)$$

The total strain energy (neglecting shear deformation) is given by

$$U(F_1, F_2) = \int_o^a \frac{M_1^2}{2EI_1}dx + \int_a^b \frac{M_2^2}{2EI_2}dx + \int_b^L \frac{M_3^2}{2EI_3}dx$$

Shaft Deflection using Castigliano's Second Theorem

Using Castigliano's Second Theorem, the vertical displacement at the point of application of F_1 is

$$\delta_1 = \frac{\partial U}{\partial F_1} = \int_o^a \frac{M_1}{EI_1}\frac{\partial M_1}{\partial F_1}dx + \int_a^b \frac{M_2}{EI_2}\frac{\partial M_2}{\partial F_1}dx + \int_b^L \frac{M_3}{EI_3}\frac{\partial M_3}{\partial F_1}dx$$

Similarly, the vertical displacement at the point of application of F_2 is

$$\delta_2 = \frac{\partial U}{\partial F_2} = \int_o^a \frac{M_1}{EI_1}\frac{\partial M_1}{\partial F_2}dx + \int_a^b \frac{M_2}{EI_2}\frac{\partial M_2}{\partial F_2}dx + \int_b^L \frac{M_3}{EI_3}\frac{\partial M_3}{\partial F_2}dx$$

Substituting the problem parameters, we get $\delta_1 = 0.092$ mm and $\delta_2 = -0.033$ mm

In the direction opposite to F_2

In case we take the the transverse shear into account

$$U(F_1, F_2) = \int_o^a \frac{M_1^2}{2EI_1}dx + \int_a^b \frac{M_2^2}{2EI_2}dx + \int_b^L \frac{M_3^2}{2EI_3}dx + \int_o^a \frac{f_s V_1^2}{2GA_1}dx + \int_a^b \frac{f_s V_2^2}{2GA_2}dx + \int_b^L \frac{f_s V_3^2}{2GA_3}dx$$

f_s is the correction factor
$f_s = 1.11$ for circular c/s

Lateral Vibrations

The natural frequency for an elastic system on mass m and stiffness k can be estimated using

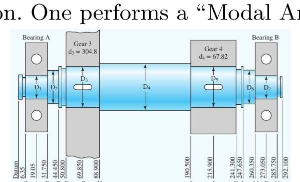
$$\omega_n = \sqrt{\frac{k}{m}}$$

The influence of various parameters which includes shaft diameter D, length L, material with Young's modulus E and density ρ is expressed as

$$\omega_n \propto \frac{D}{L^2}\sqrt{\frac{E}{\rho}}$$

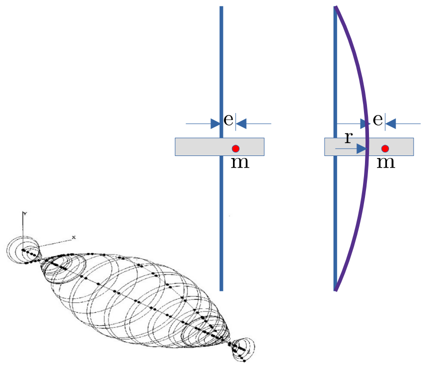
An external periodic force is necessary for these vibrations to occur.

Finite Element Method is frequently used to obtain accurate estimates of the natural frequencies of the vibration. One performs a "Modal Analysis".



Whirling of Shafts

Shaft whirl is a self-exited phenomenon caused due to an unbalanced rotating mass.



Balance of forces

$$m(r+e)\omega^2 - kr = 0$$

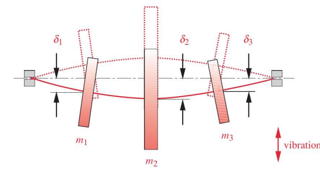
Centrifugal force   Elastic restoring force

$$r = \frac{e\omega^2}{\frac{k}{m}-\omega^2} = \frac{e(\omega/\omega_n)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

As the rotational speed approaches the first **natural frequency of lateral vibration**, the deflection of the shaft increases and becomes theoretically infinite when they coincide

Critical Speeds: The shaft rotation speed or the spin speed which coincide with one of the natural frequencies of the shaft are referred to as the critical speeds.

Lateral Vibrations



Rayleigh's Method:

The lowest natural frequency of the shaft corresponding to lateral vibrations can be estimated using the Rayleigh Method as follows

$$\omega_1^2 = \frac{g\sum_{i=1}^{N} m_i\delta_i}{\sum_{i=1}^{N} m_i\delta_i^2}$$

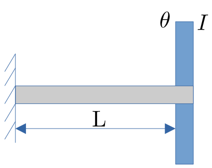
Here $\delta_1, \delta_2, \ldots, \delta_N$ are the static deflections of the shaft due to masses $m_1, m_2, \ldots, m_N$ mounted on the shaft. It is assumed that the mass of the shaft is negligible as compared to the other mass

The deflections $\delta_1, \delta_2, \ldots, \delta_N$ are always taken to be positive

The Rayleigh method always overestimates the lowest natural frequency $(\omega_1)_{\text{rayleigh}} \geq (\omega_1)_{\text{actual}}$

Torsional Vibrations

Single Degree of Freedom System

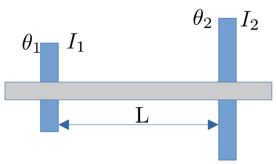


$$I\ddot{\theta} = -k_t\theta, \ \boxed{k_t = \frac{GJ}{L}} \quad \text{Torsional stiffness}$$

$$\omega_n = \sqrt{\frac{k_t}{I}} = \sqrt{\frac{GJ}{IL}}$$

Two Degree of Freedom System

Two gears/pulleys on a rotating shaft



$$I\ddot{\theta}_1 = k_t(\theta_2-\theta_1), \ k_t = \frac{GJ}{L}$$

$$I\ddot{\theta}_2 = -k_t(\theta_2-\theta_1)$$

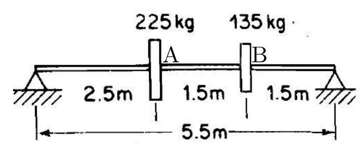
$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_t & -k_t \\ -k_t & k_t \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

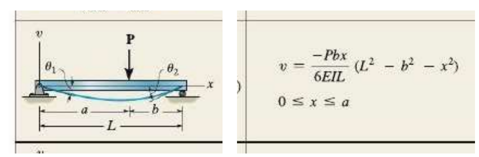
Need to solve an Eigenvalue problem to obtain the natural frequencies

Rigid body mode $\omega_1^2 = 0$, $\boxed{\omega_2^2 = k_t\frac{I_1+I_2}{I_1I_2}}$ Natural frequency

Lateral Vibrations

Estimate the lowest natural frequency of the shaft in lateral vibration of the system shown below. The mass of the shaft can be neglected.





$$\omega_1^2 = \frac{g\sum_{i=1}^{2} m_i\delta_i}{\sum_{i=1}^{2} m_i\delta_i^2}$$

$$\delta_{AA} = \frac{7524.72}{EI}\text{m}, \ \delta_{BA} = \frac{5455.42}{EI}\text{m}$$

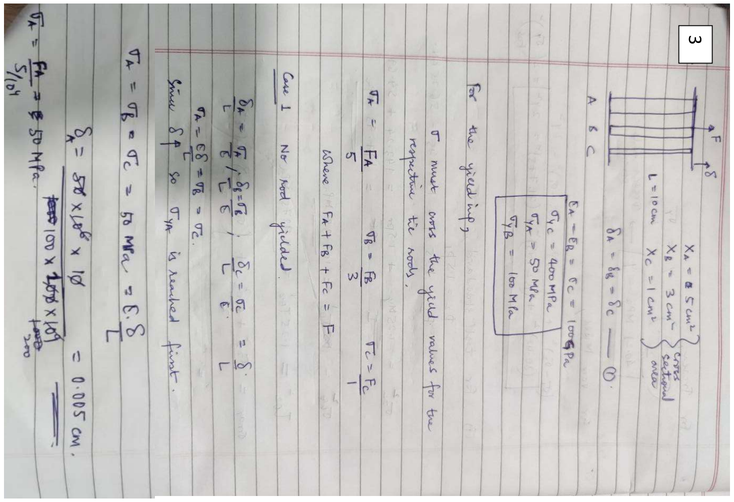
$$\delta_{AB} = \frac{3273.25}{EI}\text{m}, \ \delta_{BB} = \frac{2889.49}{EI}\text{m}$$

δ_{ij} deflection at point i due to load at point j

$$\delta_A = \delta_{AA} + \delta_{BA} = \frac{10798.0}{EI}\text{m}$$

$$\delta_B = \delta_{AB} + \delta_{BB} = \frac{8344.9}{EI}\text{m}$$

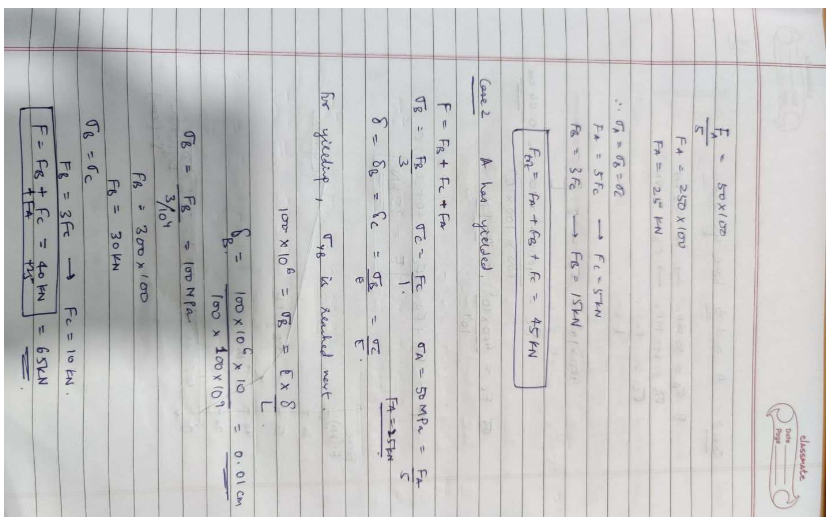
$$\omega_1 = \frac{0.0312}{\sqrt{EI}} \text{ rad/s}$$



Shaft Design Considerations – Shaft Layout

Axial Layout of Components (primary aim is reduce deflections)

- Shafts should be kept short to minimize bending moments and deflections
- It is best to support load-carrying components between bearings in order to minimize deflections.
- The length of the cantilever part of the shaft should be kept short to minimize the deflection.
- Only two bearings should be used in most cases for ease of alignment.
- Load-bearing components should be placed near the bearings to minimize the bending moment at the locations that will likely have stress concentrations and to minimize the deflection at the load-carrying components.
- The components must be accurately located on the shaft to line up with other mating components, and provision must be made to securely hold the components in position.
- The primary means of locating the components is to position them against a shoulder of the shaft. A shoulder also provides a solid support to minimize deflection and vibration of the component.



General Principles for Shaft Design

1. Keep shafts as short as possible, with bearings close to the applied loads. This reduces deflections and bending moments and increases critical speeds.
2. Place necessary stress raisers away from highly stressed shaft regions if possible. If not possible, use generous radii and good surface finishes. Consider local surface-strengthening processes (as shot peening or cold rolling).
3. Use inexpensive steels for deflection-critical shafts, as all steels have essentially the same elastic modulus.
4. When weight is critical, consider hollow shafts. For example, propeller shafts on rear-wheel-drive cars are made of tubing in order to obtain the low-weight–stiffness ratio needed to keep critical speeds above the operating range.

classmate

Date

Page

Case 3   A and B have yielded.

$\sigma_A = 50\,MPa \longrightarrow F_A = 25\,kN$

$\sigma_B = 100\,MPa \longrightarrow F_B = 30\,kN.$

$\sigma_c = \dfrac{F_c}{l}$

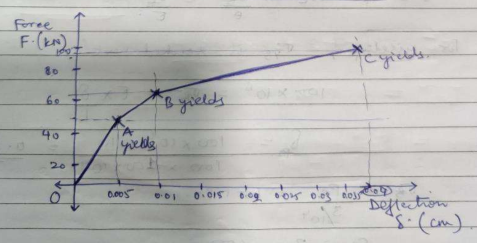
$400\times10^6 = \sigma_c = E\cdot\dfrac{\delta}{L}$

$$\boxed{\delta_c = \frac{400\times10^6\times10}{100\times100\times10^9} \approx \underline{0.04\ cm}}$$

$F_c = \dfrac{400\times10^6}{10^4} = \underline{40\,kN}$

$F = F_A + F_B + F_c = 40+30+25$

$$\boxed{F = 95\,kN}$$



Given

$l = 750;$
$t = 6;$
$d_o = 250;$
$d_i = 250 - 12;$
$W = 45\,000;$
$p = 3.5;$
$r = 125;$
$T = W\,r;$
$M = W\,l;$
$Ib = \frac{\pi}{64}\left(d_o^4 - d_i^4\right)$
$Jt = \frac{\pi}{32}\left(d_o^4 - d_i^4\right)$

$10\,901\,676\,\pi$

$21\,803\,352\,\pi$

Polar moment of Inertia

$Jt = \frac{\pi}{32}\left(d_o^4 - d_i^4\right)$

$21\,803\,352\,\pi$

Second Area moment of Inertia

$Ib = \frac{\pi}{64}\left(d_o^4 - d_i^4\right)$

In[=]:= N[10 901 676 π]

Out[=]= 3.42486×10^7

Two most critical points are A and B, Let the axis be zz and hoop be θθ

At A there will be bending stress and axial stress due to pressure

$\sigma zzb = \frac{M\,r}{Ib}$
$\sigma zzp = \frac{p\,r}{2\,t}$
$\sigma\theta\theta = \frac{p\,r}{t}$
$\tau z\theta = \frac{T\,r}{Jt}$

$\frac{351\,562\,500}{908\,473\,\pi}$

36.4583

72.9167

$\frac{29\,296\,875}{908\,473\,\pi}$

$N\left[\frac{351\,562\,500}{908\,473\,\pi}\right]$

123.18

$N\left[\frac{29\,296\,875}{908\,473\,\pi}\right]$

10.265

$\sigma zz = \sigma zzb + \sigma zzp$

159.638

$$\sigma 1 = \frac{\sigma zz + \sigma\theta\theta}{2} + \sqrt{\left(\frac{\sigma zz - \sigma\theta\theta}{2}\right)^2 + \tau z\theta^2}$$

$$\sigma 2 = \frac{\sigma zz + \sigma\theta\theta}{2} - \sqrt{\left(\frac{\sigma zz - \sigma\theta\theta}{2}\right)^2 + \tau z\theta^2}$$

$\sigma 3 = 0$

160.837

71.7182

0

$$\sigma vmA = \frac{1}{\sqrt{2}}\left(\sqrt{(\sigma 1 - \sigma 2)^2 + (\sigma 2 - \sigma 3)^2 + (\sigma 3 - \sigma 1)^2}\right)$$

139.56

Clear["Global`*"]

At B,

$\sigma zzb = 0$
$\sigma zzp = \frac{p\,r}{2\,t}$
$\sigma\theta\theta = \frac{p\,r}{t}$
$\tau z\theta = \frac{T\,r}{Jt}$

0

36.4583

72.9167

$\frac{29\,296\,875}{908\,473\,\pi}$

$\sigma zz = \sigma zzb + \sigma zzp$

$$\sigma 1 = \frac{\sigma zz + \sigma\theta\theta}{2} + \sqrt{\left(\frac{\sigma zz - \sigma\theta\theta}{2}\right)^2 + \tau z\theta^2}$$

$$\sigma 2 = \frac{\sigma zz + \sigma\theta\theta}{2} - \sqrt{\left(\frac{\sigma zz - \sigma\theta\theta}{2}\right)^2 + \tau z\theta^2}$$

$\sigma 3 = 0$

75.6081

33.7669

0

$$\sigma vmB = \frac{1}{\sqrt{2}}\left(\sqrt{(\sigma 1 - \sigma 2)^2 + (\sigma 2 - \sigma 3)^2 + (\sigma 3 - \sigma 1)^2}\right)$$

65.602907971277`

At C in the bottom,

$\sigma zzb = -\frac{M\,r}{Ib}$
$\sigma zzp = \frac{p\,r}{2\,t}$
$\sigma\theta\theta = \frac{p\,r}{t}$
$\tau z\theta = \frac{T\,r}{Jt}$

$-\frac{351\,562\,500}{908\,473\,\pi}$

36.4583

72.9167

$\frac{29\,296\,875}{908\,473\,\pi}$

$N\left[-\frac{351\,562\,500}{908\,473\,\pi}\right]$

-123.18

$\sigma zz = \sigma zzb + \sigma zzp$

-86.7218

$$\sigma 1 = \frac{\sigma zz + \sigma\theta\theta}{2} + \sqrt{\left(\frac{\sigma zz - \sigma\theta\theta}{2}\right)^2 + \tau z\theta^2}$$

$$\sigma 2 = \frac{\sigma zz + \sigma\theta\theta}{2} - \sqrt{\left(\frac{\sigma zz - \sigma\theta\theta}{2}\right)^2 + \tau z\theta^2}$$

σ3 = 0

73.574

-87.3791

0

$$\sigma vmB = \frac{1}{\sqrt{2}}\left(\sqrt{(\sigma 1 - \sigma 2)^2 + (\sigma 2 - \sigma 3)^2 + (\sigma 3 - \sigma 1)^2}\right)$$

139.56

Most critical location is A The vonMises is 139.56 and the Tresca gives 160 MPa

If the hydrostatic tests 1.5 p and load is 1.25 W

l = 750;
t = 6 ;
d_o = 250;
d_i = 250 - 12;
W = 45 000;
p = 1.5 × 3.5;
r = 125;
T = 1.25 W r;
M = 1.25 W l;

$$Ib = \frac{\pi}{64}\left(d_o^4 - d_i^4\right)$$

$$Jt = \frac{\pi}{32}\left(d_o^4 - d_i^4\right)$$

10 901 676 π

21 803 352 π

$$\sigma zzb = \frac{M\,r}{Ib}$$

$$\sigma zzp = \frac{p\,r}{2\,t}$$

$$\sigma\theta\theta = \frac{p\,r}{t}$$

$$\tau z\theta = \frac{T\,r}{Jt}$$

153.975

54.6875

109.375

12.8313

σzz = σzzb + σzzp

208.663

$$\sigma 1 = \frac{\sigma zz + \sigma\theta\theta}{2} + \sqrt{\left(\frac{\sigma zz - \sigma\theta\theta}{2}\right)^2 + \tau z\theta^2}$$

$$\sigma 2 = \frac{\sigma zz + \sigma\theta\theta}{2} - \sqrt{\left(\frac{\sigma zz - \sigma\theta\theta}{2}\right)^2 + \tau z\theta^2}$$

σ3 = 0

210.294

107.744

0

$$\sigma vmA = \frac{1}{\sqrt{2}}\left(\sqrt{(\sigma 1 - \sigma 2)^2 + (\sigma 2 - \sigma 3)^2 + (\sigma 3 - \sigma 1)^2}\right)$$

182.139

The factor of safety is 182.13 / 139.56 = 1.30

In[=]:= $$\sigma = \frac{4\,F\,x}{\pi\left(\frac{a\,x}{2\,l} + \frac{a}{2}\right)^3}$$

Out[=]= $$\frac{4\,F\,x}{\pi\left(\frac{a}{2} + \frac{a\,x}{2\,l}\right)^3}$$

In[=]:= D[σ , x]

Out[=]= $$-\frac{6\,a\,F\,x}{l\,\pi\left(\frac{a}{2} + \frac{a\,x}{2\,l}\right)^4} + \frac{4\,F}{\pi\left(\frac{a}{2} + \frac{a\,x}{2\,l}\right)^3}$$

In[=]:= Solve[D[σ , x] == 0, x]

Out[=]= $$\left\{\left\{x \to \frac{l}{2}\right\}\right\}$$

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