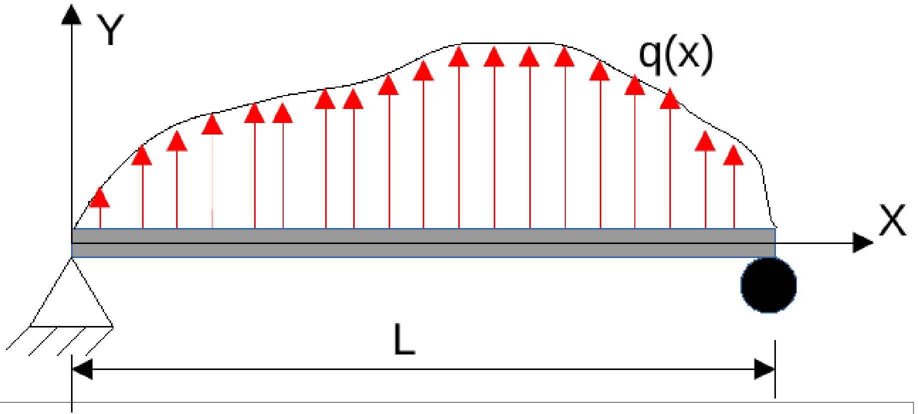
Shaft Design Considerations – Deflection and Rigidity

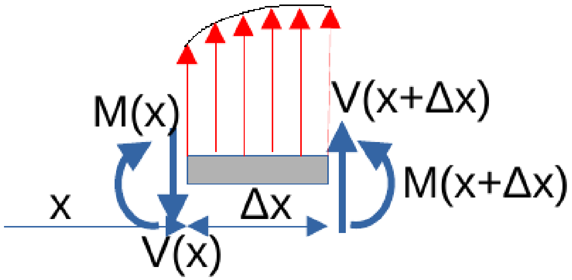
- Deflection of the shaft, both linear and angular, should be checked at gears and bearings.
- Ranges for maximum slopes and transverse deflections of the shaft centerline are given below.
- The allowable transverse deflections for spur gears are dependent on the diametral pitch P, which is the ratio of the number of teeth to the pitch diameter.

Slopes	
Tapered roller	0.0005–0.0012 rad
Cylindrical roller	0.0008–0.0012 rad
Deep-groove ball	0.001–0.003 rad
Spherical ball	0.026–0.052 rad
Self-align ball	0.026–0.052 rad
Uncrowned spur gear	<0.0005 rad

Transverse Deflections	
Spur gears with $P < 10$ teeth/cm	0.25 mm
Spur gears with $11 < P < 19$	0.125 mm
Spur gears with $20 < P < 50$	0.075 mm

Shaft Bending and Deflection





Assumptions:

- straight, <u>prismatic shaft</u>, made with elastic homogeneous material
- Euler-Bernoulli assumptions

To calculate the shaft bending and deflection

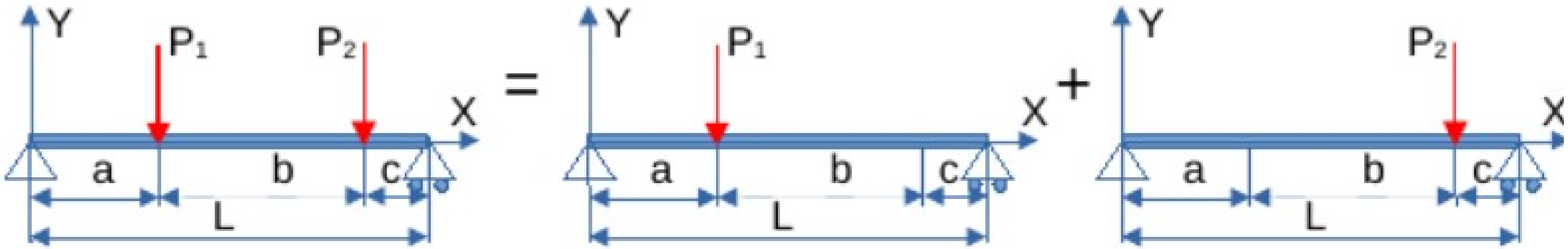
$$EI\frac{d^2v}{dx^2} = M \tag{A}$$

$$EI\frac{d^3v}{dx^3} = \frac{dM}{dx} = -V \tag{B}$$

$$EI\frac{d^4v}{dx^4} = -\frac{dV}{dx} = q \tag{C}$$

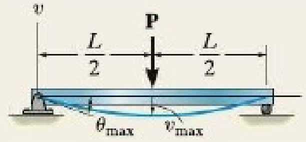
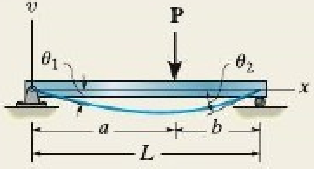
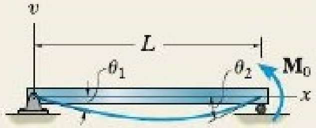
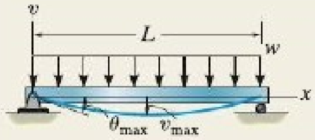
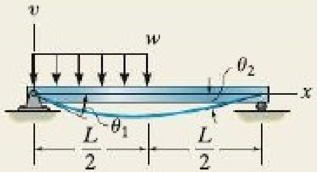
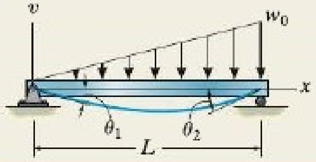
For the case of stepped shafts, these equations have to be solved for each section maintaining continuity of displacements and displacements across the sections. This limits their use in finding deflections of stepped shafts.

Shaft Bending and Deflection - Superposition



Beam Bending Formulae

Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{\max} = \dfrac{-PL^2}{16EI}$	$v_{\max} = \dfrac{-PL^3}{48EI}$	$v = \dfrac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \dfrac{-Pab(L+b)}{6EIL}$ $\theta_2 = \dfrac{Pab(L+a)}{6EIL}$	$v\Big\vert_{x=a} = \dfrac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \dfrac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \dfrac{-M_0L}{6EI}$ $\theta_2 = \dfrac{M_0L}{3EI}$	$v_{\max} = \dfrac{-M_0L^2}{9\sqrt{3}EI}$ at $x = 0.5774L$	$v = \dfrac{-M_0x}{6EIL}(L^2 - x^2)$
	$\theta_{\max} = \dfrac{-wL^3}{24EI}$	$v_{\max} = \dfrac{-5wL^4}{384EI}$	$v = \dfrac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \dfrac{-3wL^3}{128EI}$ $\theta_2 = \dfrac{7wL^3}{384EI}$	$v\Big\vert_{x=L/2} = \dfrac{-5wL^4}{768EI}$ $v_{\max} = -0.006563\dfrac{wL^4}{EI}$ at $x = 0.4598L$	$v = \dfrac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \dfrac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \dfrac{-7w_0L^3}{360EI}$ $\theta_2 = \dfrac{w_0L^3}{45EI}$	$v_{\max} = -0.00652\dfrac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \dfrac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

Salil S. Kulkarni

Shaft Deflection using Castigliano's Second Theorem

Method belongs to a class of method referred to as **Energy Methods**

Strain Energy

- When loads are applied to an elastic body, they deform the body and do external work.
- This external work is stored in the deformed body in form of an 'energy'.
- This stored internal energy is called the strain energy.
- The strain energy is caused by the action of both normal stresses and shear stresses developed in the body due to the presence of the external forces.

Castigliano's Second Theorem

The partial derivative of the total strain energy in a structure with respect to the force applied at any point is equal to the displacement at the point of application of that force, in the direction of the applied force. The strain energy needs to be expressed in terms of the applied loads.

Castigliano's Second theorem is applicable to bodies made of **linear elastic materials**.

Let $P_1, P_2, \ldots, P_n$ denote the forces acting on linear elastic body and let $\delta_1, \delta_2, \ldots, \delta_n$ denote the displacements at the point of application of the forces in the direction of the forces.

The total strain energy can be written as

$$U = U(P_1, P_2, \ldots, P_n)$$

Then from Castigliano's second theorem

$$\delta_k = \frac{\partial U}{\partial P_k},\ k = 1, \ldots, n$$

Expression for Strain Energy for Different Loading Conditions

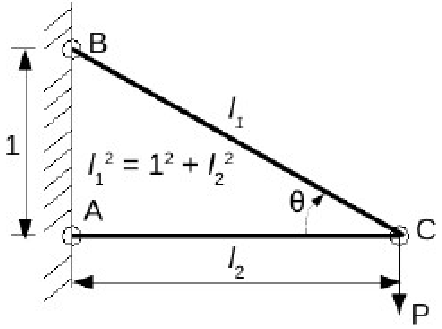
Loading	Slightly varying cross-section	Constant cross-section
Axial	$\int_0^L \frac{P^2}{2EA} \mathrm{d}x$	$\frac{P^2 L}{2EA}$
Torsion	$\int_0^L \frac{T^2}{2GJ} \mathrm{d}z$	$\frac{T^2 L}{2GJ}$
Bending	$\int_0^L \frac{M^2}{2EI} \mathrm{d}x$	$\frac{M^2 L}{2EI}$

If a slender elastic shaft oriented along the X-axis carries a tensile force $P(x)$, a torque $T(x)$ and a bending moment $M(x)$, then using the principle of superposition, the total strain energy in the member is

$$U = \int_0^L \frac{P^2(x)}{2EA} \mathrm{d}x + \int_0^L \frac{T^2(x)}{2GJ} \mathrm{d}x + \int_0^L \frac{M^2(x)}{2EI} \mathrm{d}x$$

Problem

Find the deflection of the point of application of point in the vertical direction using Castigliano's Second Theorem. The material is linear elastic with Young's modulus E.



- This is a statically determinate problem.

- The strain energy in the two member truss is given by

$$U = \sum_{i=1}^{2} \frac{P_i^2 l_i}{2EA_i}$$

$$= \frac{F_{AC}^2 l_2}{2EA_2} + \frac{F_{BC}^2 l_1}{2EA_1}$$

$$U(P) = \frac{1}{2}\frac{P^2 l_2^3}{A_2 E} + \frac{1}{2}\frac{P^2 l_1^3}{A_1 E}$$

- Let δ denote the vertical displacement of point C. From Castigliano's Second theorem

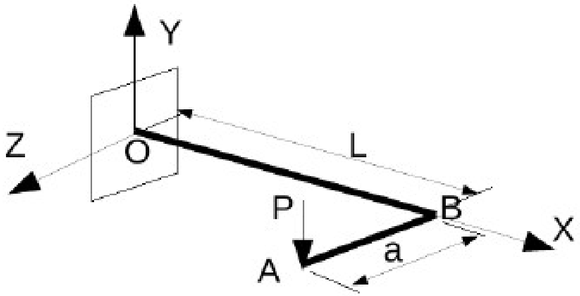
$$\delta = \frac{\partial U}{\partial P}$$

- Therefore

$$\delta = \frac{P l_2^3}{A_2 E} + \frac{P l_1^3}{A_1 E}$$

Problem

Find the vertical deflection at the point of application of the force P using Castigliano's Second Theorem.



- The strain energy in the member is given by

$$U = \frac{P^2a^3}{6EI} + \frac{P^2L^3}{6EI} + \frac{P^2La^2}{2GJ}$$

- Let δ denote the vertical displacement of point C. From Castigliano's Second theorem

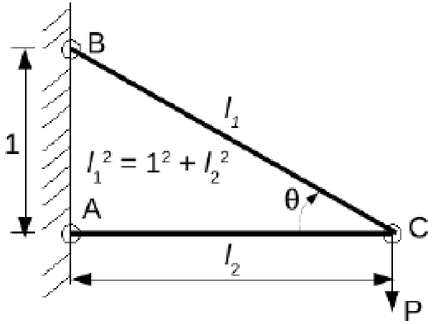
$$\delta = \frac{\partial U}{\partial P}$$

- Therefore

$$\delta = \frac{Pa^3}{3EI} + \frac{PL^3}{3EI} + \frac{PLa^2}{GJ}$$

Problem – Dummy Load Method

Find the deflection of the point of application of point in the horizontal direction. The material is linear elastic with Young's modulus E.



- Castigliano's Second Theorem gives us the displacement in the direction of the load.
- In this problem there is no applied force in the horizontal direction.

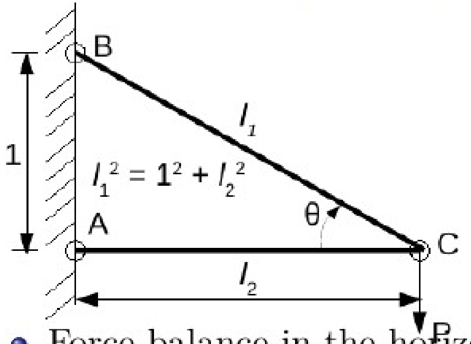
Main Idea of the Dummy Load Method

- We are asked to find the displacement in a direction where there is no applied force.
- Assume a dummy load, Q, acts in the direction in which the displacement is required.
- Express the strain energy of the system in terms of the applied loads and the dummy load Q.
- Find the displacement in the direction of the dummy load using Castigliano's Second Theorem.

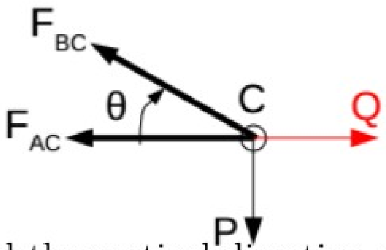
$$\delta_Q = \frac{\partial U}{\partial Q}$$

- Set $Q = 0$ in the expression of δ_Q. This gives the answer to the original problem.

Problem – Dummy Load Method



The FBD of joint C with a dummy load Q is shown below:



- Force balance in the horizontal direction and the vertical direction gives:

$$F_{BC} = \frac{P}{\sin\theta}, \quad F_{AC} = Q - P\cot\theta$$

- The strain energy in the two member truss is given by

$$U = \frac{F_{AC}^2 l_2}{2EA_2} + \frac{F_{BC}^2 l_1}{2EA_1} \text{ or } U(P,Q) = \frac{(Q - Pl_2)^2 l_2}{2EA_2} + \frac{P^2 l_1^3}{2A_1 E}$$

- From Castigliano's Second Theorem

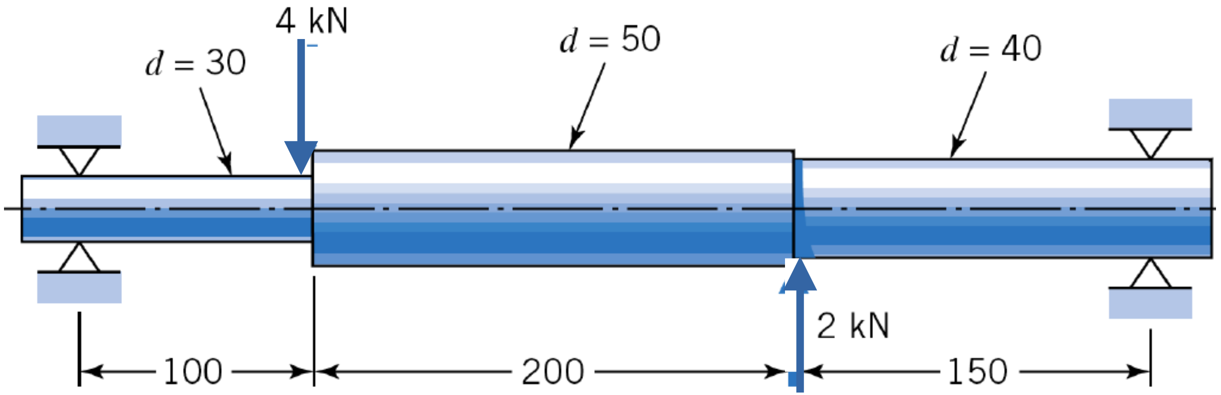
$$\tilde{\delta}_Q = \frac{\partial U}{\partial Q} = \frac{(Q - Pl_2) l_2}{EA_2}$$

- The horizontal displacement, δ_Q in the original problem is then given by

$$\delta_Q = \left(\tilde{\delta}_Q\right)_{Q=0} = -\frac{Pl_2^2}{EA_2}$$

Shaft Deflection using Castigliano's Second Theorem

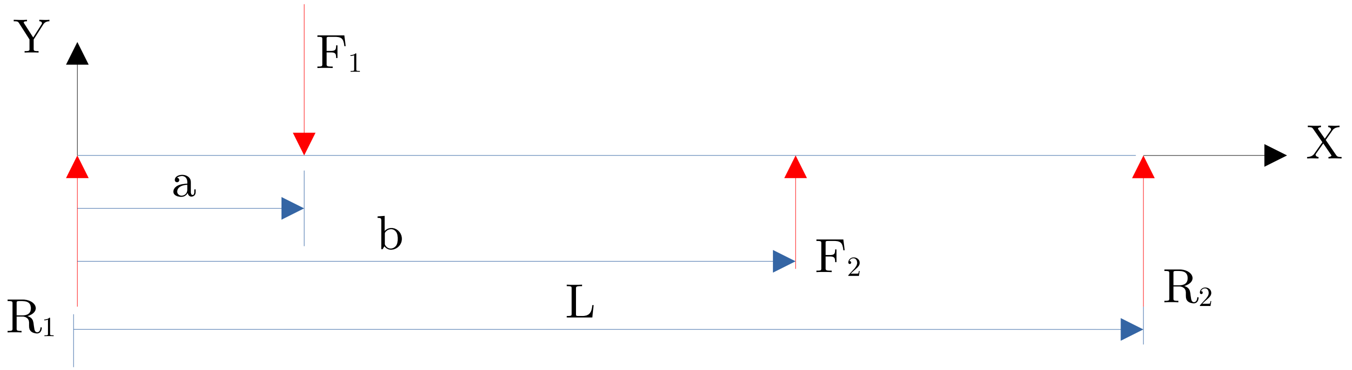
The shaft shown in the figure is made up of low carbon steel with E = 207 GPa. Find the deflection at the point of application of the 4 kN and 2 kN forces



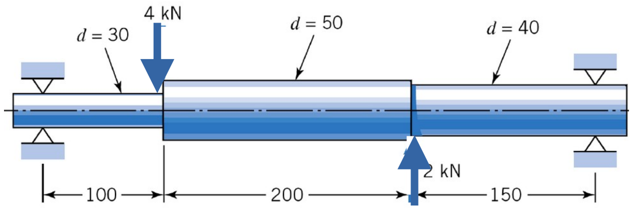
From force balance and moment balance:

$$R_1 = \frac{(L-a)F_1 - (L-b)F_2}{L}$$

$$R_2 = \frac{aF_1 - bF_2}{L}$$



Shaft Deflection using Castigliano's Second Theorem



For $0 < x < a$,
$V = -R_1$, $M(x) = R_1 x$
For $a < x < b$,
$V = F_1 - R_1$, $M(x) = R_1 x - F_1(x-a)$
For $b < x < L$,
$V = F_1 - R_1 - F_2$, $M(x) = R_1 x - F_1(x-a) + F_2(x-b)$

Let

$$M_1 = R_1 x$$
$$M_2 = R_1 x - F_1(x-a)$$
$$M_3 = R_1 x - F_1(x-a) + F_2(x-b)$$

The total strain energy (neglecting shear deformation) is given by

$$U(F_1, F_2) = \int_o^a \frac{M_1^2}{2EI_1} dx + \int_a^b \frac{M_2^2}{2EI_2} dx + \int_b^L \frac{M_3^2}{2EI_3} dx$$

Shaft Deflection using Castigliano's Second Theorem

Using Castigliano's Second Theorem, the vertical displacement at the point of application of F_1 is

$$\delta_1 = \frac{\partial U}{\partial F_1} = \int_o^a \frac{M_1}{EI_1}\frac{\partial M_1}{\partial F_1}dx + \int_a^b \frac{M_2}{EI_2}\frac{\partial M_2}{\partial F_1}dx + \int_b^L \frac{M_3}{EI_3}\frac{\partial M_3}{\partial F_1}dx$$

Similarly, the vertical displacement at the point of application of F_2 is

$$\delta_2 = \frac{\partial U}{\partial F_2} = \int_o^a \frac{M_1}{EI_1}\frac{\partial M_1}{\partial F_2}dx + \int_a^b \frac{M_2}{EI_2}\frac{\partial M_2}{\partial F_2}dx + \int_b^L \frac{M_3}{EI_3}\frac{\partial M_3}{\partial F_2}dx$$

Substituting the problem parameters, we get $\delta_1 = 0.092$ mm and $\delta_2 = -0.033$ mm

In the direction opposite to F_2

In case we take the the transverse shear into account

$$U(F_1, F_2) = \int_o^a \frac{M_1^2}{2EI_1}dx + \int_a^b \frac{M_2^2}{2EI_2}dx + \int_b^L \frac{M_3^2}{2EI_3}dx + \int_o^a \frac{f_s V_1^2}{2GA_1}dx + \int_a^b \frac{f_s V_2^2}{2GA_2}dx + \int_b^L \frac{f_s V_3^2}{2GA_3}dx$$

f_s is the correction factor

$f_s = 1.11$ for circular c/s

Dynamic Behaviour of Shafts

- Lateral Vibrations
- Whirling of Shafts
- Torsional Vibrations

Lateral Vibrations

The natural frequency for an elastic system on mass m and stiffness k can be estimated using

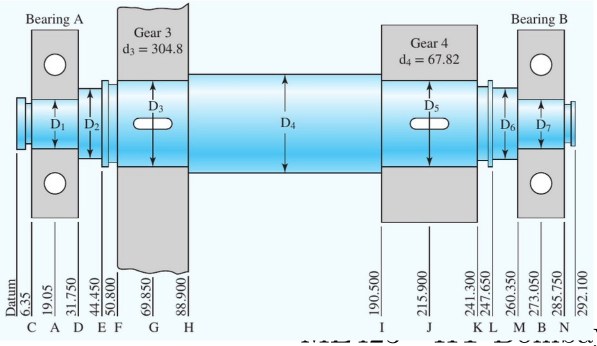
$$\omega_n = \sqrt{\frac{k}{m}}$$

The influence of various parameters which includes shaft diameter D, length L, material with Young's modulus E and density ρ is expressed as

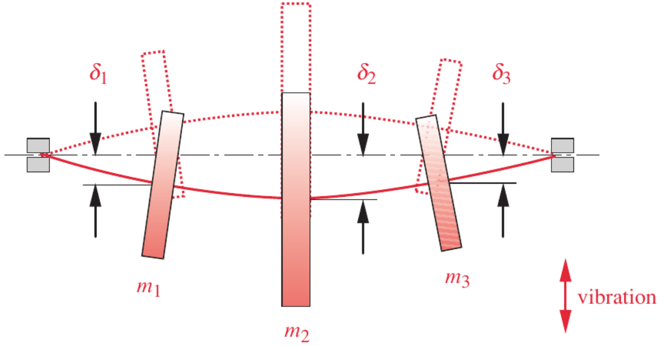
$$\omega_n \propto \frac{D}{L^2}\sqrt{\frac{E}{\rho}}$$

An external periodic force is necessary for these vibrations to occur.

Finite Element Method is frequently used to obtain accurate estimates of the natural frequencies of the vibration. One performs a "Modal Analysis".



Lateral Vibrations



Rayleigh's Method:

The lowest natural frequency of the shaft corresponding to lateral vibrations can be estimated using the Rayleigh Method as follows

$$\omega_1^2 = \frac{g\sum_{i=1}^{N} m_i\delta_i}{\sum_{i=1}^{N} m_i\delta_i^2}$$

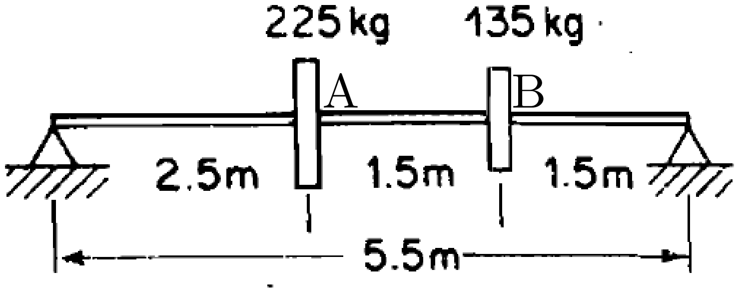
Here $\delta_1, \delta_2, \ldots, \delta_N$ are the static deflections of the shaft due to masses $m_1, m_2, \ldots, m_N$ mounted on the shaft. It is assumed that the mass of the shaft is negligible as compared to the other mass

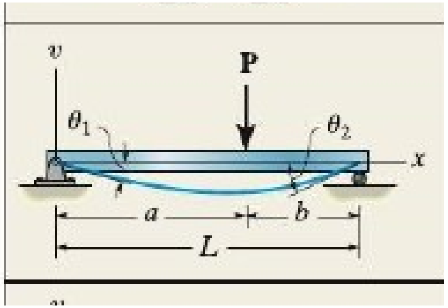
The deflections $\delta_1, \delta_2, \ldots, \delta_N$ are always taken to be positive

The Rayleigh method always overestimates the lowest natural frequency $(\omega_1)_{\text{rayleigh}} \geq (\omega_1)_{\text{actual}}$

Lateral Vibrations

Estimate the lowest natural frequency of the shaft in lateral vibration of the system shown below. The mass of the shaft can be neglected.





$$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$$

$$0 \le x \le a$$

$$\omega_1^2 = \frac{g\sum_{i=1}^{2} m_i\delta_i}{\sum_{i=1}^{2} m_i\delta_i^2}$$

$$\delta_{AA} = \frac{7524.72}{EI}\text{m},\ \delta_{BA} = \frac{5455.42}{EI}\text{m}$$

$$\delta_{AB} = \frac{3273.25}{EI}\text{m},\ \delta_{BB} = \frac{2889.49}{EI}\text{m}$$

δ_{ij} deflection at point i due to load at point j

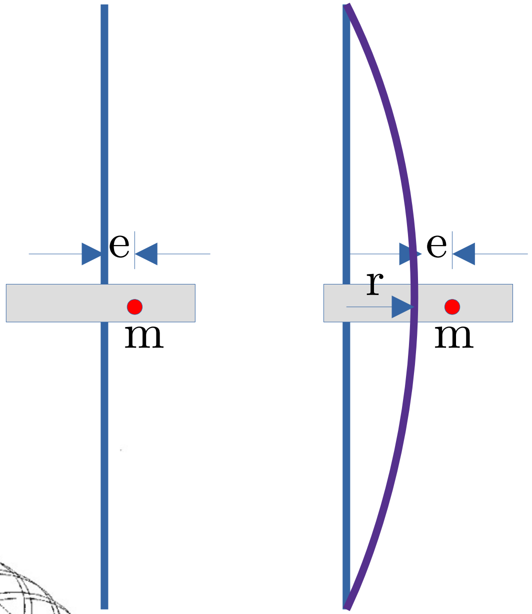
$$\delta_A = \delta_{AA} + \delta_{BA} = \frac{10798.0}{EI}\text{m}$$

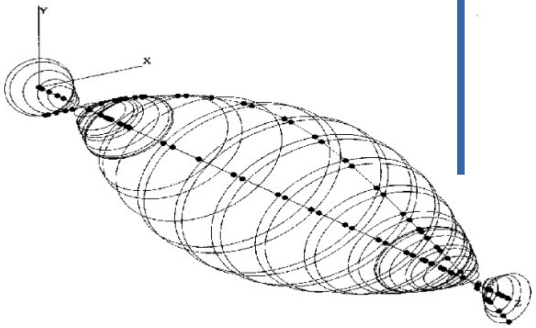
$$\delta_B = \delta_{AB} + \delta_{BB} = \frac{8344.9}{EI}\text{m}$$

$$\omega_1 = \frac{0.0312}{\sqrt{EI}}\ \text{rad/s}$$

Whirling of Shafts

Shaft whirl is a self-exited phenomenon caused due to an unbalanced rotating mass.





Balance of forces

$$m(r+e)\omega^2 - kr = 0$$

Centrifugal force Elastic restoring force

$$r = \frac{e\omega^2}{\frac{k}{m} - \omega^2} = \frac{e(\omega/\omega_n)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

As the rotational speed approaches the first **natural frequency of lateral vibration**, the deflection of the shaft increases and becomes theoretically infinite when they coincide

Critical Speeds: The shaft rotation speed or the spin speed which coincide with one of the natural frequencies of the shaft are referred to as the critical speeds.

Torsional Vibrations

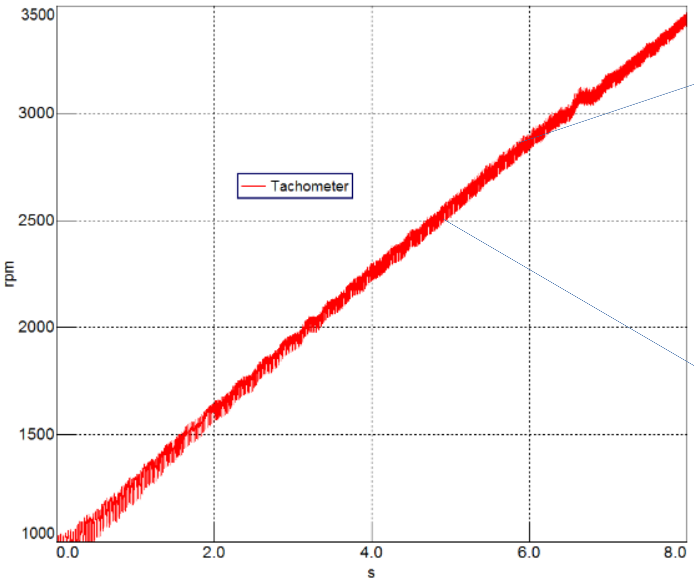


Figure 1: RPM vs time curve of a 4 cylinder engine run-up.

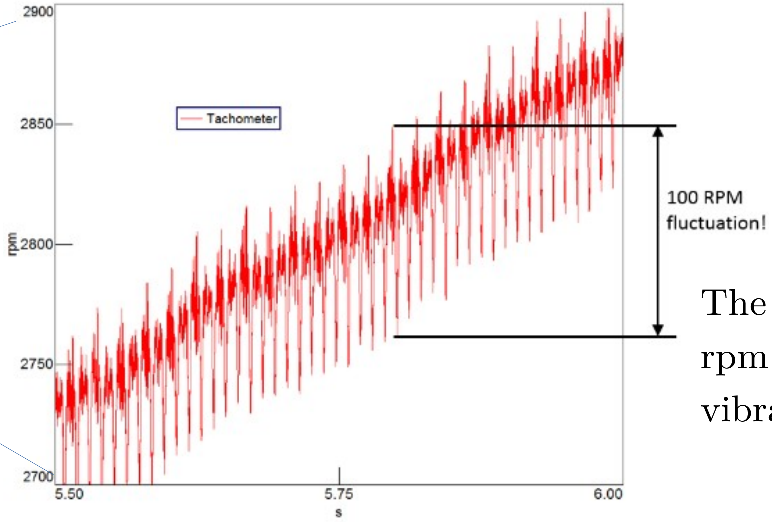


Figure 2: There is a 100 RPM fluctuation in rotational speed!

The fluctuations in the rpm is caused by torsional vibrations

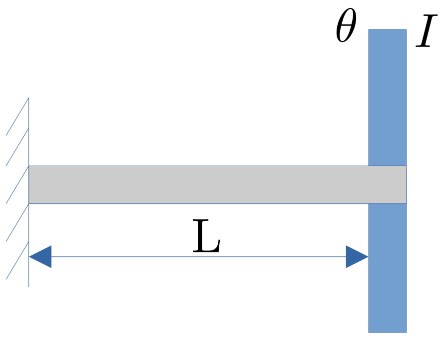
https://community.sw.siemens.com/s/article/torsional-vibration-what-is-it

Have a look at the following video on torsional vibrations

https://youtu.be/AXSc3_Pal8s?list=PL5e0AcdojuT5uPgr2yLzMmGSE3Qscfy3

Torsional Vibrations

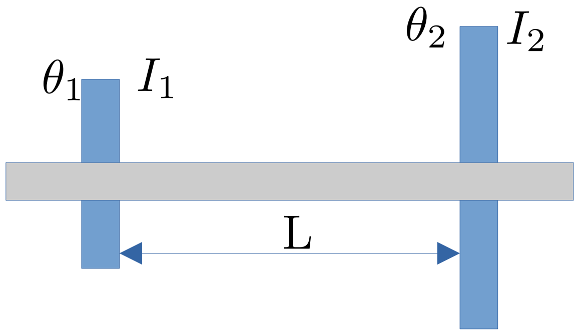
Single Degree of Freedom System



$$I\ddot{\theta} = -k_t\theta, \quad \boxed{k_t = \frac{GJ}{L}}$$ Torsional stiffness

$$\omega_n = \sqrt{\frac{k_t}{I}} = \sqrt{\frac{GJ}{IL}}$$

Two Degree of Freedom System
Two gears/pulleys on a rotating shaft



$$I\ddot{\theta}_1 = k_t(\theta_2 - \theta_1), \quad k_t = \frac{GJ}{L}$$

$$I\ddot{\theta}_2 = -k_t(\theta_2 - \theta_1)$$

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_t & -k_t \\ -k_t & k_t \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Need to solve an Eigenvalue problem to obtain the natural frequencies

Rigid body mode $\omega_1^2 = 0$, $\boxed{\omega_2^2 = k_t \frac{I_1 + I_2}{I_1 I_2}}$ Natural frequency

Shaft Design Considerations – Shaft Layout

Axial Layout of Components (primary aim is reduce deflections)

- Shafts should be kept short to minimize bending moments and deflections
- It is best to support load-carrying components between bearings in order to minimize deflections.
- The length of the cantilever part of the shaft should be kept short to minimize the deflection.
- Only two bearings should be used in most cases for ease of alignment.
- Load-bearing components should be placed near the bearings to minimize the bending moment at the locations that will likely have stress concentrations and to minimize the deflection at the load-carrying components.
- The components must be accurately located on the shaft to line up with other mating components, and provision must be made to securely hold the components in position.
- The primary means of locating the components is to position them against a shoulder of the shaft. A shoulder also provides a solid support to minimize deflection and vibration of the component.

General Principles for Shaft Design

1. Keep shafts as short as possible, with bearings close to the applied loads. This reduces deflections and bending moments and increases critical speeds.
2. Place necessary stress raisers away from highly stressed shaft regions if possible. If not possible, use generous radii and good surface finishes. Consider local surface-strengthening processes (as shot peening or cold rolling).
3. Use inexpensive steels for deflection-critical shafts, as all steels have essentially the same elastic modulus.
4. When weight is critical, consider hollow shafts. For example, propeller shafts on rear-wheel-drive cars are made of tubing in order to obtain the low-weight–stiffness ratio needed to keep critical speeds above the operating range.

End