Curved Beams

We will determine the bending stresses developed in a member which is initially curved.

(a) Crane Hook

(b) Chain Link

Here the radius of curvature is of the same order as the dimensions of the cross-sections. The method of analysis was first presented by E. Wrinkler.

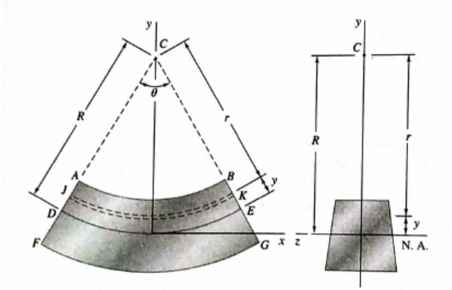
Curved Beams

Assumptions:

- The area of the cross-section is constant along the length of the beam.
- The cross-section is symmetric about the loading plane.
- Cross-sections remain plane after loading.
- All strains are small.
- Material is linear elastic, homogeneous and isotropic.

Curved Beams

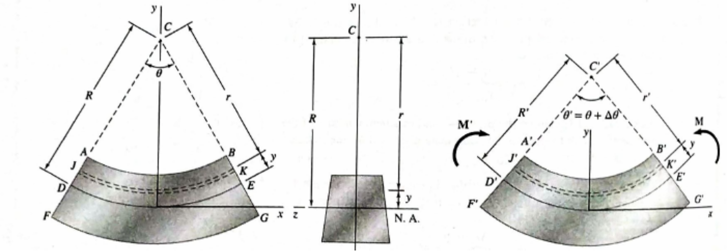
- Consider the undeformed curved beam shown below:



- There are two radii shown in the figure:
  - R: radius of the neutral axis (unknown)
  - r: location of arbitrary point in the cross-section.
- The vertical xy plane intersects the upper and the lower surfaces along the arc of circles AB and FG centered at C.
- The vertical xy plane intersects the neutral surface along the arc of the circle DE.
- Arc JK represents the intersections of the vertical plane with a surface situated at a distance y from the neutral axis.

Curved Beams

- The length of arc JK in the undeformed beam is $l\left(JK\right) = r\theta$.
- Now apply equal and opposite moments and the two ends.



- The length of arc in the JK in the deformed configuration is $l\left(J'K'\right) = r'\left(\theta + \Delta\theta\right)$
- The strain $\epsilon_{\theta\theta}$ in the circumferential direction are given by

$$\begin{aligned}
\epsilon_{\theta\theta} &= \frac{l(J'K') - l(JK)}{l(JK)} \\
&= \frac{r'\left(\theta + \Delta\theta\right) - r\theta}{r\theta} \\
&= \frac{\left(R' - y\right)\left(\theta + \Delta\theta\right) - \left(R - y\right)\theta}{r\theta}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{y\Delta\theta}{r\theta} \\
&= -\frac{\left(R - r\right)\Delta\theta}{r\theta} \\
&= -k\frac{\left(R - r\right)}{r}, \quad k = \Delta\theta/\theta
\end{aligned}$$

Curved Beams

- We have

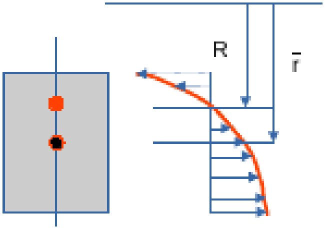
$$\epsilon_{\theta\theta} = -k\frac{(R-r)}{r}, \; k = \Delta\theta/\theta$$

The strain does not vary linearly with distance from the neutral axis.

- For a linear elastic material (assuming all the other stress components are negligible) we get

$$\sigma_{\theta\theta} = E\epsilon_{\theta\theta} = -Ek\frac{(R-r)}{r}$$

The stress does not vary linearly with distance from the neutral axis.



Curved Beams

- To obtain R, the location of the neutral axis, we use the condition that the normal force acting on the cross-section is zero.

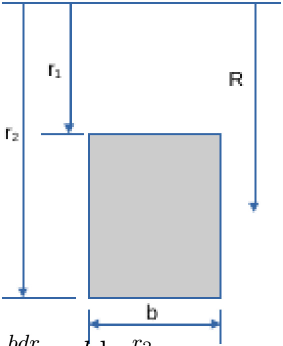
$$\int_A \sigma_{\theta\theta} dA = 0$$

$$\text{or } -\int_A Ek\frac{(R-r)}{r}dA = 0$$

$$\text{or } R\int_A \frac{dA}{r} - \int_A dA = 0$$

$$\text{or } R = \frac{A}{\int_A \frac{dA}{r}}$$

- To find the position of the neutral axis for the rectangular c/s



- We have $A = b(r_2 - r_1)$. Also $\int_A \frac{dA}{r} = \int_{r_1}^{r_2} \frac{bdr}{r} = b\ln\frac{r_2}{r_1}$

- Therefore

$$R = \frac{r_2 - r_1}{\ln\frac{r_2}{r_1}}$$

Curved Beams

The relation between the internal resisting moment and the developed stress is obtained as follows:

- We have

$$\begin{aligned} M &= -\int_A \sigma_{\theta\theta}\,(R-r)\,dA \\ &= kE\int_A \frac{(R-r)^2}{r}dA \\ &= Ek\left(R^2\int_A \frac{dA}{r} - 2R\int_A dA + \int_A r dA\right) \\ &= Ek\left(R^2\cdot\frac{A}{R} - 2RA + \bar{r}A\right) \\ \therefore M &= EkA\,(\bar{r}-R) \qquad \text{(A)} \end{aligned}$$

- We also have

$$\sigma_{\theta\theta} = -Ek\frac{(R-r)}{r} \qquad \text{(B)}$$

Curved Beams

- From Eqns A and B we get

$$\sigma_{\theta\theta} = -\frac{M\,(R-r)}{Ar\,(\bar{r}-R)}$$

- Let $y = R - r$ and $e = \bar{r} - R$. Here e is called the **eccentricity**. The stress can then be written as:

$$\sigma_{\theta\theta} = -\frac{My}{A\,(R-y)\,e}$$

For beams with circular c/s with radius c

$$R = \frac{A}{2\pi(\bar{r} - \sqrt{\bar{r}^2 - c^2})}$$

Curved Beams

Compare the stresses in a 50 mm $\times$ 50 mm square beam subjected to end moment of 2083 Nm in the following three cases: 1. straight beam, 2. curved beam with $\bar{r} = 250$ mm and 3. curved beam with $\bar{r} = 75$ mm.

- For a straight beam:

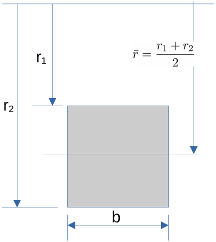
$$\sigma = -\frac{My}{I}$$

- $I = \frac{1}{12}\left(50 \times 10^{-3}\right)\left(50 \times 10^{-3}\right)^3 \; m^4$ and $y = 25 \times 10^{-3} \; m$.

- Therefore $\sigma_T = 100$ MPa and $\sigma_C = -100$ MPa

- For a curved beam

$$\sigma = -\frac{M\left(R - r\right)}{Ar\left(\bar{r} - R\right)}$$



Curved Beams

- For a beam with rectangular c/s

$$R = \frac{r_2 - r_1}{\ln \frac{r_2}{r_1}}$$

- Case 2: $\bar{r} = 250$ mm
We have

$$r_1 = \bar{r} - 25 \times 10^{-3} = 225 \times 10^{-3} m$$
$$r_2 = \bar{r} + 25 \times 10^{-3} = 275 \times 10^{-3} m$$

- Hence $R = 249.164$ mm.

- Therefore $\sigma_T = 93.7$ MPa
and $\sigma_C = -107.1$ MPa

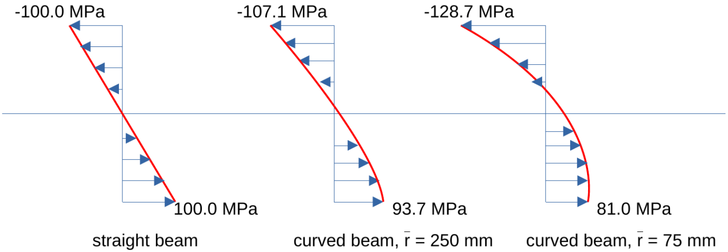
- Case 3: $\bar{r} = 75$ mm
We have

$$r_1 = \bar{r} - 25 \times 10^{-3} = 50 \times 10^{-3} m$$
$$r_2 = \bar{r} + 25 \times 10^{-3} = 100 \times 10^{-3} m$$

- Hence $R = 72.134$ mm.

- Therefore $\sigma_T = 81.0$ MPa
and $\sigma_C = -128.7$ MPa

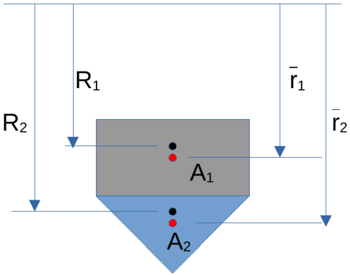
Curved Beams



- Let h denote the depth of the beam. Here $h = 50$ mm. As the ratio $\bar{r}/h$ increases, the results of the curved beam approach those of the straight beam.

Curved Beams

Find R of the c/s made of two areas A_1 and A_2 shown below



- We have

$$R = \frac{A}{\int_A \frac{dA}{r}}$$

- Now

$$\int_A \frac{dA}{r} = \int_{A_1} \frac{dA}{r} + \int_{A_2} \frac{dA}{r}$$

- Also,

$$R_1 = \frac{A_1}{\int_{A_1} \frac{dA}{r}}, \; R_2 = \frac{A_2}{\int_{A_2} \frac{dA}{r}}$$

- Therefore

$$\int_A \frac{dA}{r} = \frac{A_1}{R_1} + \frac{A_2}{R_2}$$

- Hence

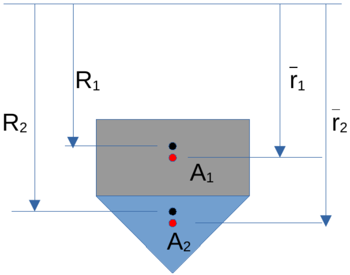
$$R = \frac{A}{\frac{A_1}{R_1} + \frac{A_2}{R_2}} = \frac{A_1 + A_2}{\frac{A_1}{R_1} + \frac{A_2}{R_2}}$$

- In general, for a c/s made up of n areas, $A_i, i = 1, 2, \ldots, n$

$$R = \frac{\sum_{i=1}^{n} A_i}{\sum_{i=1}^{n} \left(\frac{A_i}{R_i}\right)}$$

Curved Beams

Find $\bar{r}$ of the c/s made of two areas A_1 and A_2 shown below



- We have

$$\bar{r} = \frac{\int_A r dA}{A}$$

- Now

$$\int_A r dA = \int_{A_1} r dA + \int_{A_2} r dA$$

- Also,

$$\bar{r}_1 = \frac{\int_{A_1} r dA}{A_1}, \ \bar{r}_2 = \frac{\int_{A_2} r dA}{A_2}$$

- Therefore

$$\bar{r} = \frac{\bar{r}_1 A_1 + \bar{r}_2 A_2}{A} = \frac{\bar{r}_1 A_1 + \bar{r}_2 A_2}{A_1 + A_2}$$

- In general, for a c/s made up of n areas, $A_i, i = 1, 2, \ldots, n$

$$\bar{r} = \frac{\sum_{i=1}^{n} \bar{r}_i A_i}{\sum_{i=1}^{n} A_i}$$