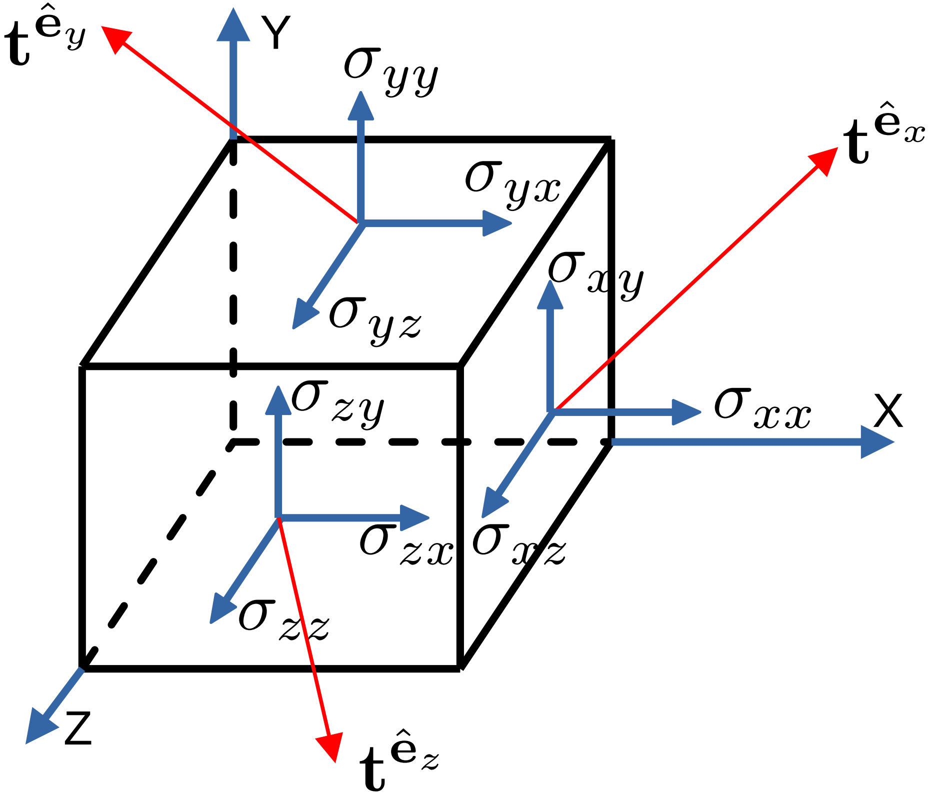
Stress Tensor in Three-Dimensions



$\sigma_{xx},\ \sigma_{yy},\ \sigma_{zz}$ normal stress components

$\sigma_{yz},\ \sigma_{zx},\ \sigma_{xy}$ shear stress components

Cauchy Stress Tensor

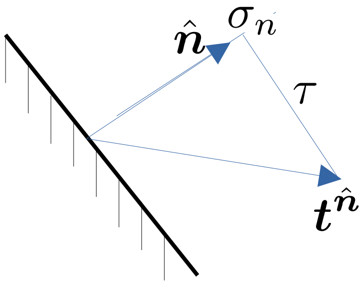
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\sigma_{xy} = \sigma_{yx},\ \sigma_{xz} = \sigma_{zx},\ \sigma_{yz} = \sigma_{zy}$$

Traction vector acting on a plane with normal $\hat{\boldsymbol{n}}$

$$\boldsymbol{t}^{\hat{\boldsymbol{n}}} = \boldsymbol{\sigma}\hat{\boldsymbol{n}} \quad \text{or}$$

$$\begin{bmatrix} t_x^{\hat{\boldsymbol{n}}} \\ t_y^{\hat{\boldsymbol{n}}} \\ t_z^{\hat{\boldsymbol{n}}} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$



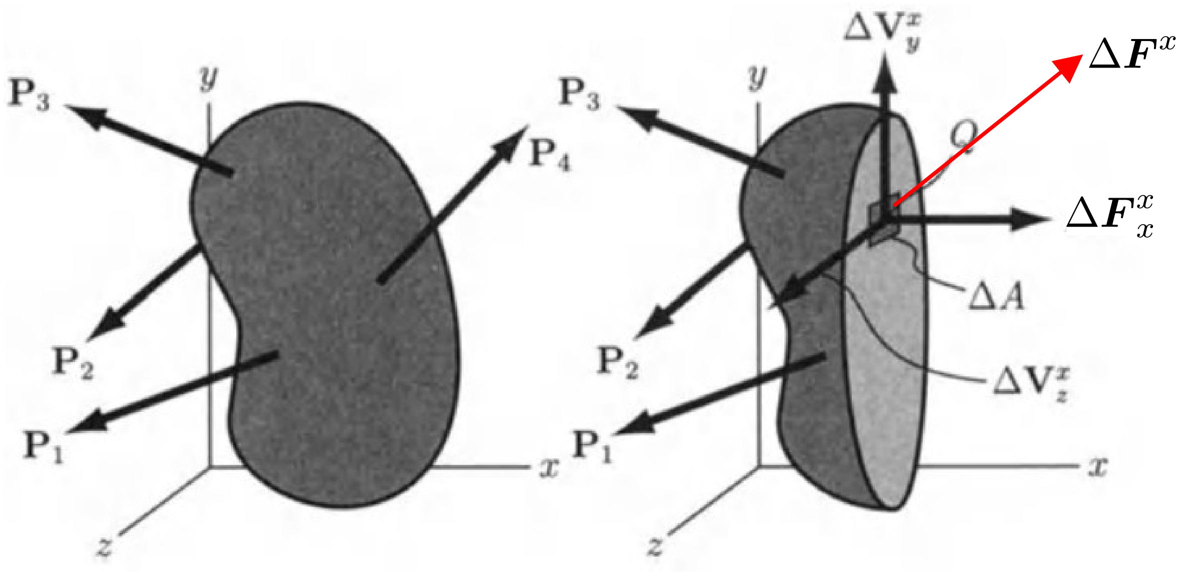
Normal component of the traction vector

$$\sigma_n = \boldsymbol{t}^{\hat{\boldsymbol{n}}} \cdot \hat{\boldsymbol{n}}$$

Shear component of the traction vector

$$\tau^2 = |\boldsymbol{t}^{\hat{\boldsymbol{n}}}|^2 - \sigma_n^2$$

Interpretation of the Stress Components



$$\sigma_{xx} = \lim_{\Delta A \to 0} \frac{\Delta F^x}{\Delta A}, \ \sigma_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_y^x}{\Delta A}, \ \sigma_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V_z^x}{\Delta A}$$

Slaughter, linearized Theory of Elasticity

Stress Tensor in Three-Dimensions

- We have

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}, \ \sigma_{xy}=\sigma_{yx},\ \sigma_{xz}=\sigma_{zx},\ \sigma_{yz}=\sigma_{zy}$$

- The principal stresses are obtained by solving for the following **cubic** equation

$$|\boldsymbol{\sigma}-\lambda\boldsymbol{I}|=\begin{vmatrix}\sigma_{xx}-\lambda & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy}-\lambda & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-\lambda\end{vmatrix}=0$$

or

$$-\lambda^3+I_1\lambda^2-I_2\lambda+I_3=0$$

where

$$I_1=\mathrm{trace}(\boldsymbol{\sigma})=\sigma_{xx}+\sigma_{yy}+\sigma_{zz}$$ First Invariant

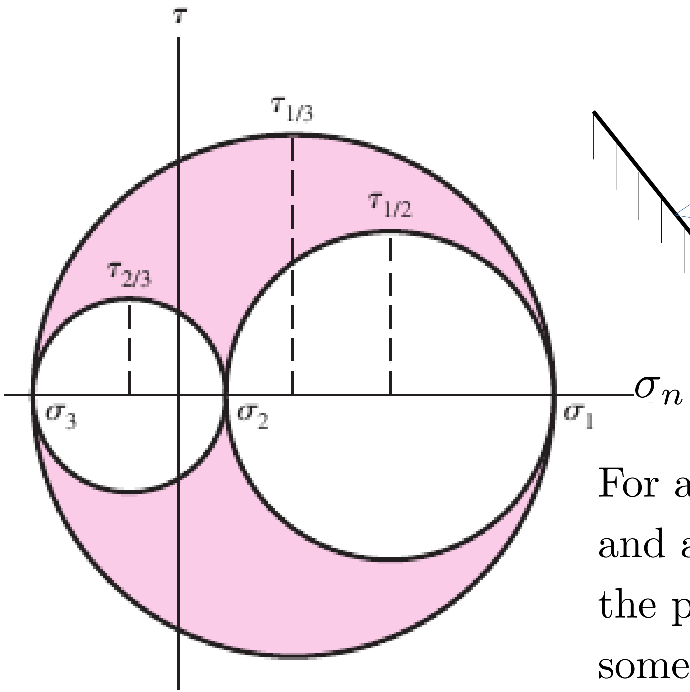
$$I_2=\frac{1}{2}\left[(\mathrm{trace}(\boldsymbol{\sigma}))^2-\mathrm{trace}(\boldsymbol{\sigma}^2)\right]=\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}-(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2)$$ Second Invariant

$$I_3=\det(\boldsymbol{\sigma})$$ Third Invariant

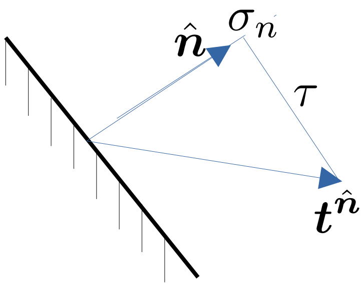
The three invariants do not change on stress transformation

Stress Tensor in Three-Dimensions

- Let $\sigma_1 \geq \sigma_2 \geq \sigma_3$ denote the three principal stresses (roots of the cubic equation).



Three-Dimensional Mohr's Circle



For a given state of stress at a point, and a chosen plane passing through the point, the (σ_n, τ) point lies somewhere in the shaded region bounded by the three circles

Principal shear stresses

$$\tau_{1/2} = \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\tau_{2/3} = \frac{|\sigma_2 - \sigma_3|}{2}$$

$$\boxed{\tau_{max} = \tau_{1/3} = \frac{|\sigma_1 - \sigma_3|}{2}}$$

Budynas RG, Nisbett JK. Shigley's mechanical engineering design. New York: McGraw-Hill.

Equilibrium Equations

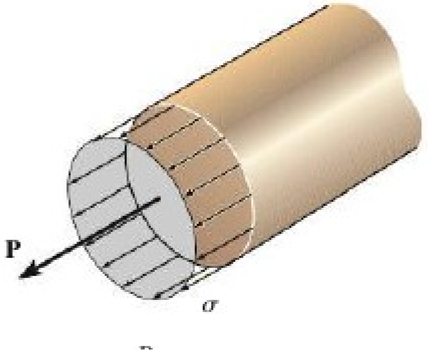
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = 0$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

$b_x, \ b_y, \ b_z$: Components of the body force per unit volume in the X, Y and Z directions, respectively.

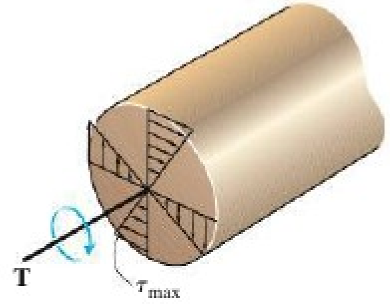
- Have to be satisfied at every point inside the body.
- Set of three coupled first order partial differential equations in 6 unknowns.
- In general cannot be solved without taking into account the strain displacement relation and constitutive (stress strain) relation.
- Have to be solved taking into account the prescribed boundary conditions.

Types of resisting loads and the corresponding stress distributions

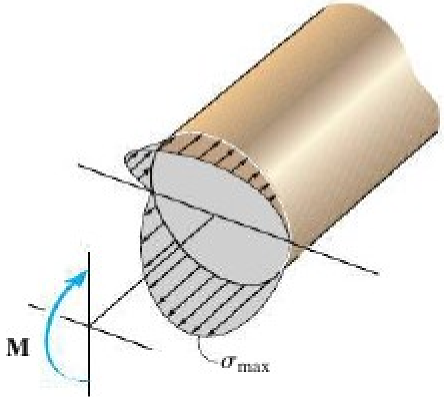
Normal Load



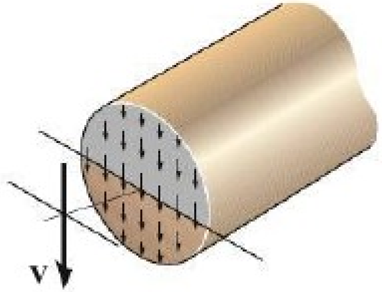
Torsional Load



Bending Load

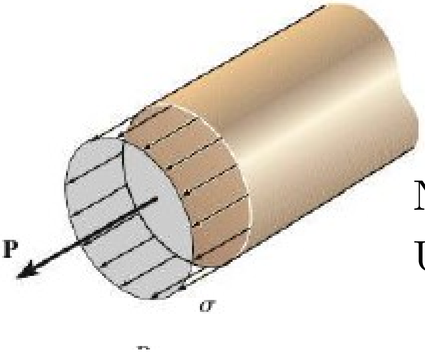


Transverse Load



Types of resisting loads and the corresponding stress distributions

Normal Load

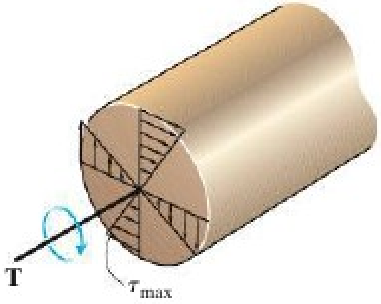


$$\sigma = \frac{P}{A}$$

Normal stress

Uniform across c/s

Torsional Load

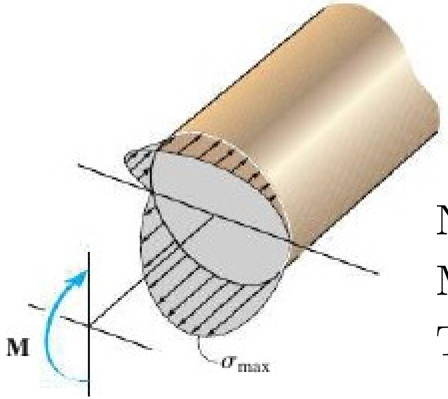


$$\tau = -\frac{Tr}{J}$$

Shear stress

Max at the surface

Bending Load



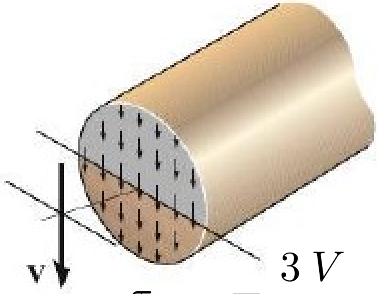
$$\sigma = -\frac{My}{I}$$

Normal stress

Max (magnitude) on the

Top and bottom surfaces

Transverse Load



$$\tau = -\frac{VQ}{bI}, \; Q = \int_{A_1} ydA$$

Shear stress

Max at the neutral axis

$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

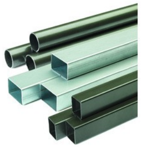
Rectangular c/s

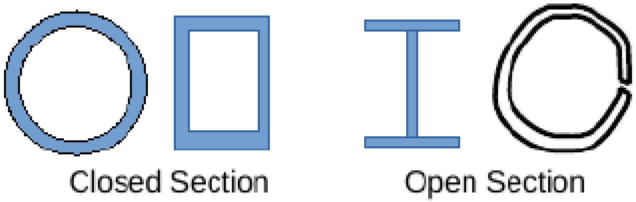
$$\tau_{max} = \frac{4}{3}\frac{V}{A}$$

Circular c/s

Torsion of Thin Walled Tubes with Closed Cross-sections

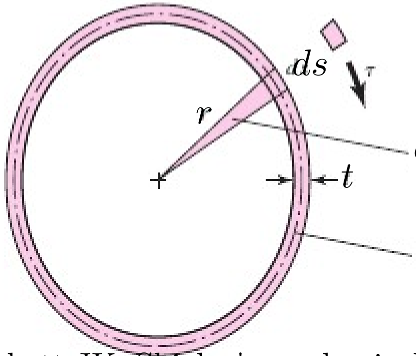
Obtain an expression for the shear stress in a thin walled cylindrical (c/s along the length of the tube are identical) tube with a closed cross section subject to a pure torsional load.





Assumptions:

- The thickness 't' of the tube may vary around the cross section but is assumed to be small as compared to the width.
- Because the thickness is small, we assume that the shear stress is distributed uniformly across the thickness (strength of material approach).
- This in-plane shear stress is the only non-zero stress component.



$$dA_m = \frac{1}{2} r\, ds$$

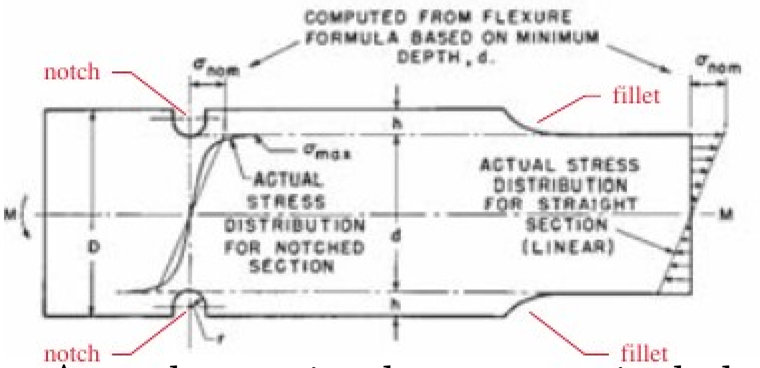
$$\tau_{average} = \frac{T}{2tA_m}$$

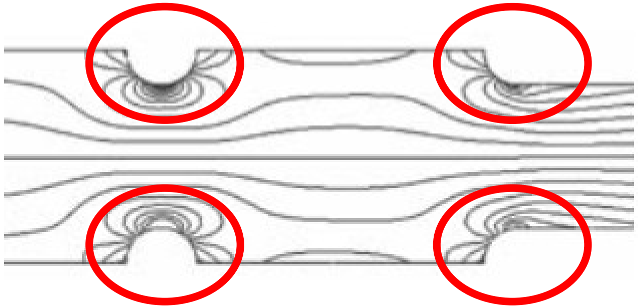
Applied torque

A_m is the mean area enclosed within the boundary of the centerline of the tube's thickness.

Budynas RG, Nisbett JK. Shigley's mechanical engineering design. New York: McGraw-Hill.

Stress Concentration

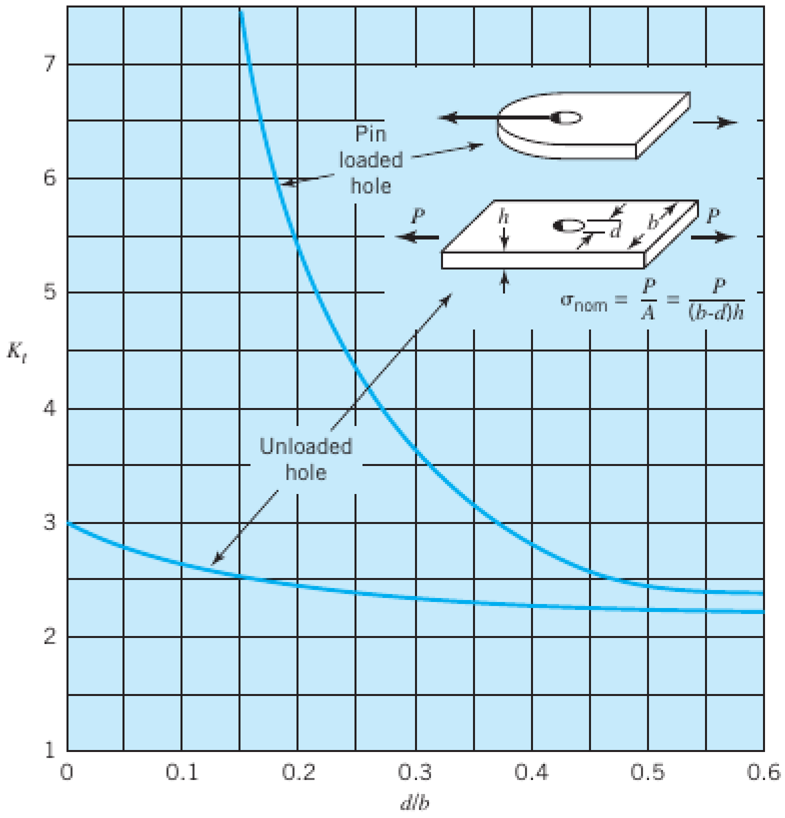




- Any change in the geometrical shape/topology of a machine part alters the stress distribution in the neighbourhood of the change.
- The elementary stress equations no longer able to describe the state of stress in the part at these locations.
- Such changes are called stress raisers, and the regions in which they occur are called areas of stress concentration.
- A *theoretical*, or *geometric*, stress-concentration factor K_t or K_{ts} is used to relate the actual maximum stress σ_{max} (τ_{max}) at the discontinuity to the nominal stress σ_{nom} (τ_{nom})

$$K_t = \frac{\sigma_{max}}{\sigma_{nom}} \quad \text{(normal stress)} \qquad K_{ts} = \frac{\tau_{max}}{\tau_{nom}} \quad \text{(shear stress)}$$

Stress Concentration – Plate with a Hole

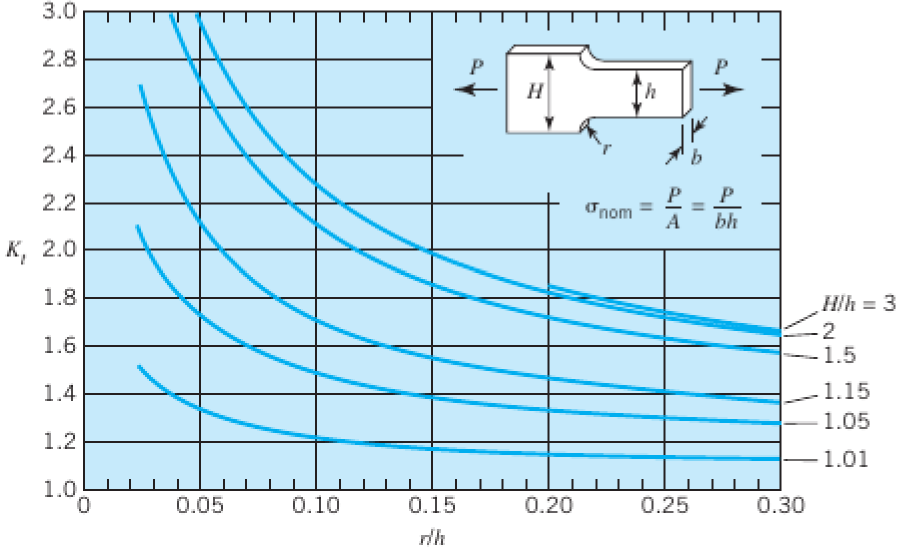


Axial Load

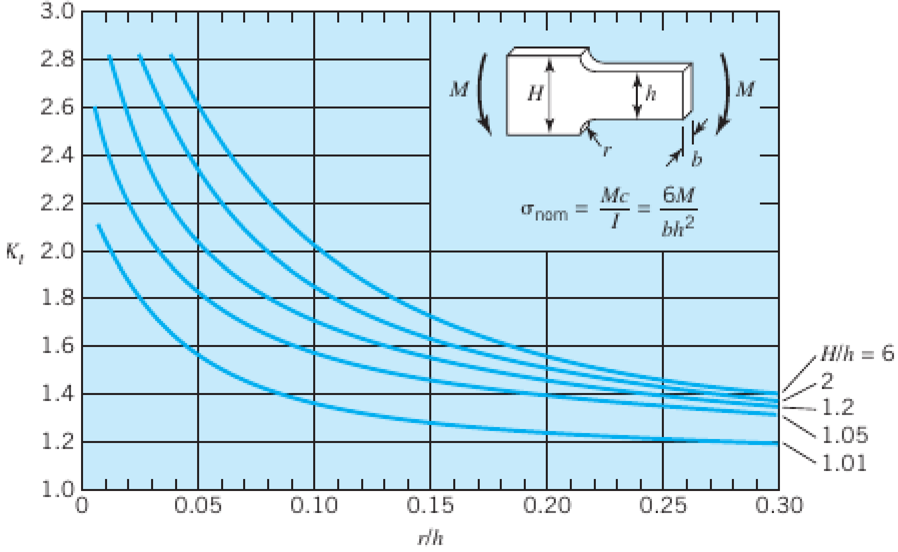
$$K_t = \frac{\sigma_{max}}{\sigma_{nom}}$$

Juvinall, R. C., and Kurt M. M. Fundamentals of machine component design. John Wiley & Sons.

Stress Concentration – Bar with Shoulder Fillet



Axial Load

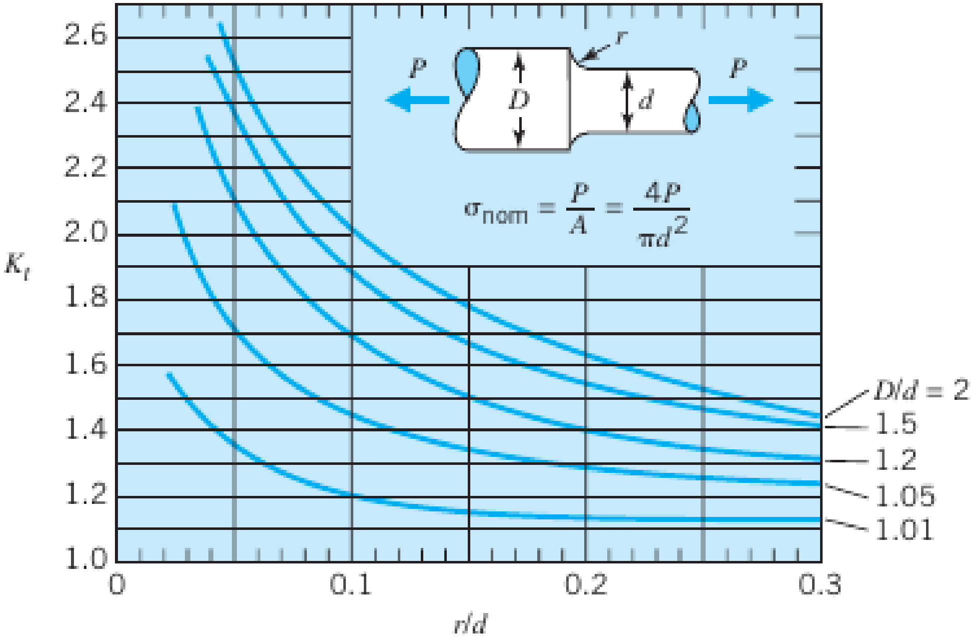


Bending Load

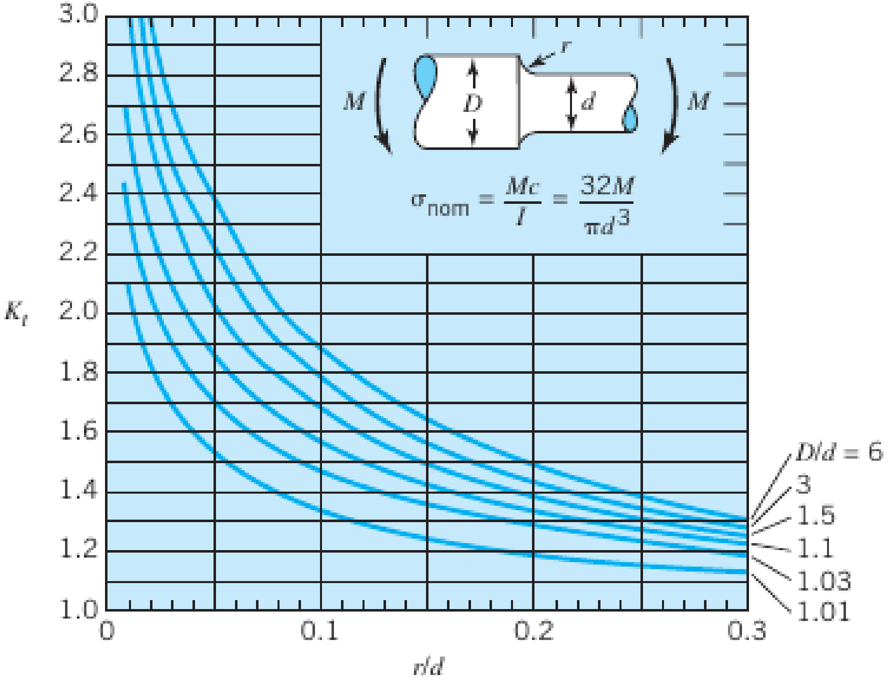
$$K_t = \frac{\sigma_{max}}{\sigma_{nom}}$$

Juvinall, R. C., and Kurt M. M. Fundamentals of machine component design. John Wiley & Sons.

Stress Concentration – Shaft with Fillet



Axial Load



Bending Load

$$K_t = \frac{\sigma_{max}}{\sigma_{nom}}$$

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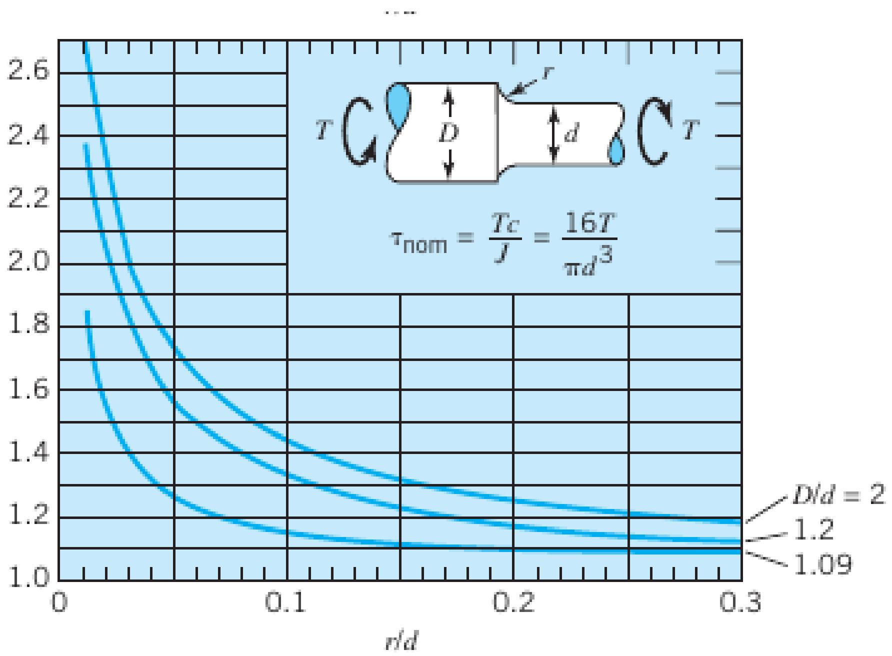
Stress Concentration – Shaft with Fillet

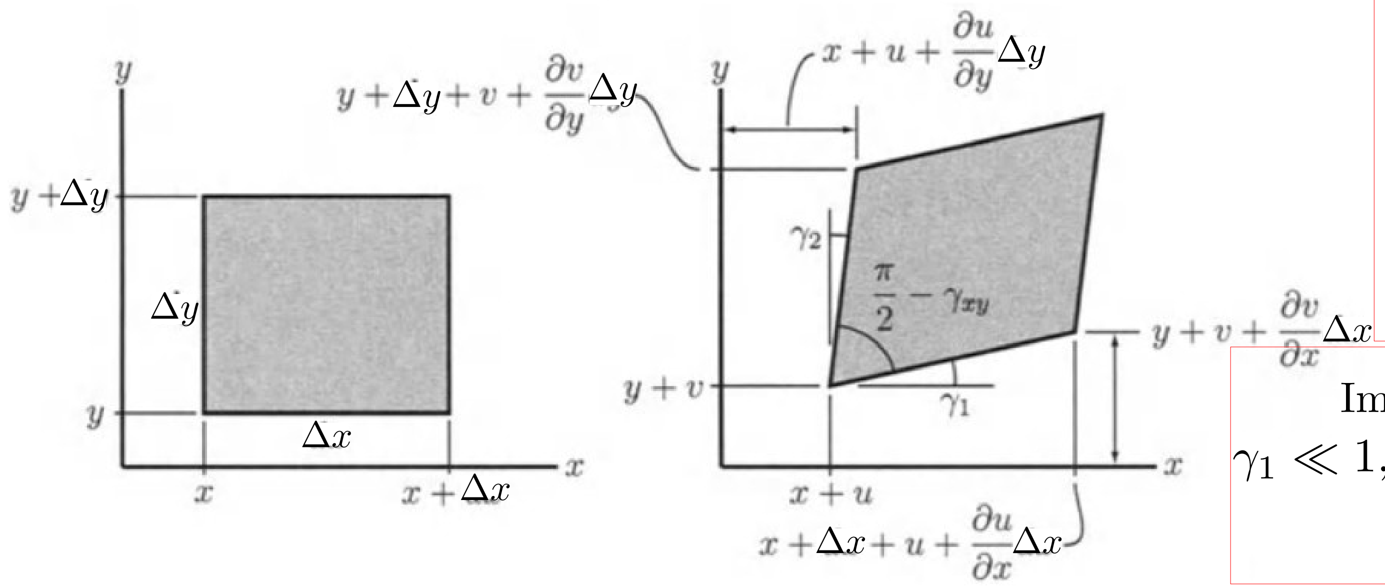


Torsional Load

$$K_{ts} = \frac{\tau_{max}}{\tau_{nom}}$$

Juvinall, R. C., and Kurt M. M. Fundamentals of machine component design. John Wiley & Sons.

Infinitesimal (Small) Strain Components



Longitudinal strain is the change in length per unit original length

Engineering shear strain is defined as the reduction in the initial right angle

Implications of small strain assumption

$$\gamma_1 \ll 1,\ \gamma_2 \ll 1, \partial u/\partial x \ll 1,\ \partial v/\partial y \ll 1$$

$$\gamma_1 \approx \frac{\partial v}{\partial x},\ \gamma_2 \approx \frac{\partial u}{\partial y}$$

Longitudinal strain$[(L - L_o)/L_o]$

$$\epsilon_{xx} = \frac{(1 + \frac{\partial u}{\partial x})\Delta x - \Delta x}{\Delta x} = \frac{\partial u}{\partial x}$$

$$\epsilon_{yy} = \frac{(1 + \frac{\partial v}{\partial y})\Delta y - \Delta y}{\Delta y} = \frac{\partial v}{\partial y}$$

Engineering Shear strain

$$\gamma_{xy} = \gamma_1 + \gamma_2$$

$$= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Tensorial Shear strain

$$\epsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Infinitesimal Strain Tensor

Infinitesimal strain tensor at point referred to Cartesian coordinate system

Two dimensions

$$\boldsymbol{\epsilon}=\begin{bmatrix}\epsilon_{xx} & \epsilon_{xy}\\ \epsilon_{yx} & \epsilon_{yy}\end{bmatrix},\ \epsilon_{xy}=\epsilon_{yx}$$

Three dimensions

$$\boldsymbol{\epsilon}=\begin{bmatrix}\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz}\\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz}\\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}\end{bmatrix},\ \epsilon_{xy}=\epsilon_{yx},\ \epsilon_{yz}=\epsilon_{zy},\ \epsilon_{xz}=\epsilon_{zx}$$

$$\epsilon_{xx}=\frac{\partial u}{\partial x},\ \epsilon_{yy}=\frac{\partial v}{\partial y},\ \epsilon_{zz}=\frac{\partial w}{\partial z}$$

$$\epsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\ \epsilon_{yz}=\frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\ \epsilon_{xz}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$\gamma_{xy}=2\epsilon_{xy},\ \gamma_{yz}=2\epsilon_{yz},\ \gamma_{xz}=2\epsilon_{xz}$$

$\epsilon_{xx},\ \epsilon_{yy},\ \epsilon_{zz}$ normal strain components
$\epsilon_{xy},\ \epsilon_{yz},\ \epsilon_{xz}$ tensorial shear strain component
$\gamma_{xy},\ \gamma_{yz},\ \gamma_{xz}$ engineering shear strain component

Infinitesimal Strain Tensor

- It is an approximate deformation measure and is valid for small shape changes and small rotations.

- Main advantage in using the infinitesimal strain tensor is that it is linear (does not involve products of gradients of displacements).

- Identical equations to those used for stress transformation and for finding principal stresses can be used for doing strain transformations and finding principal strains.

- Ratio of change in the volume to the original volume is given by

$$\frac{dV - dV_o}{dV_o} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \mathrm{trace}(\boldsymbol{\epsilon})$$

Constitutive Relations

Constitutive (stress - strain) relations for an **isotropic linear elastic** body in presence of temperature change

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right) + \alpha\Delta T$$

$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right) + \alpha\Delta T$$

$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\left(\sigma_{yy} + \sigma_{xx}\right)\right) + \alpha\Delta T$$

$$\gamma_{xy} = \frac{\sigma_{xy}}{G}, \ \gamma_{yz} = \frac{\sigma_{yz}}{G}, \ \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

E : Young's modulus, units GPa

G : Shear modulus, units GPa

ν : Poisson's ratio

α : Coefficient of thermal expansion

ΔT: Temperature change, units K^{-1}

Material	Mass density ρ / Mgm^{-3}	Youngs Modulus E / GNm^{-2}	Poisson Ratio ν	Expansion coeft K^{-1}
Tungsten Carbide	14 – 17	450 – 650	0.22	5×10^{-6}
Silicon Carbide	2.5 – 3.2	450	0.22	4×10^{-6}
Tungsten	13.4	410	0.30	4×10^{-6}
Alumina	3.9	390	0.25	7×10^{-6}
Titanium Carbide	4.9	380	0.19	13×10^{-6}
Silicon Nitride	3.2	320 - 270	0.22	3×10^{-6}
Nickel	8.9	215	0.31	14×10^{-6}
CFRP	1.5-1.6	70 – 200	0.20	2×10^{-6}
Iron	7.9	196	0.30	13×10^{-6}
Low alloy steels	7.8	200 - 210	0.30	15×10^{-6}
Stainless steel	7.5-7.7	190 - 200	0.30	11×10^{-6}
Mild steel	7.8	196	0.30	15×10^{-6}
Copper	8.9	124	0.34	16×10^{-6}
Titanium	4.5	116	0.30	9×10^{-6}
Silicon	2.5-3.2	107	0.22	5×10^{-6}
Silica glass	2.6	94	0.16	0.5×10^{-6}
Aluminum & alloys	2.6-2.9	69-79	0.35	22×10^{-6}
Concrete	2.4-2.5	45-50	0.3	10×10^{-6}
GFRP	1.4-2.2	7-45		10×10^{-6}
Wood, parallel grain	0.4-0.8	9-16	0.2	40×10^{-6}
Polyimides	1.4	3-5	0.1-0.45	40×10^{-6}
Nylon	1.1 – 1.2	2 – 4	0.25	81×10^{-6}
PMMA	1.2	3.4	0.35-0.4	50×10^{-6}
Polycarbonate	1.2 – 1.3	2.6	0.36	65×10^{-6}
Natural Rubbers	0.83-0.91	0.01-0.1	0.49	200×10^{-6}
PVC	1.3-1.6	0.003-0.01	0.41	70×10^{-6}

https://solidmechanics.org/Text/Chapter3_2/Chapter3_2.php

Constitutive Relations – Linear Elastic Isotropic Body

Material behaviour is characterized by **two independent material constants**

$$G = \frac{E}{2(1+\nu)}, \text{ shear modulus}(\mu)$$

$$K = \frac{E}{3(1-2\nu)}, \text{ bulk modulus}$$

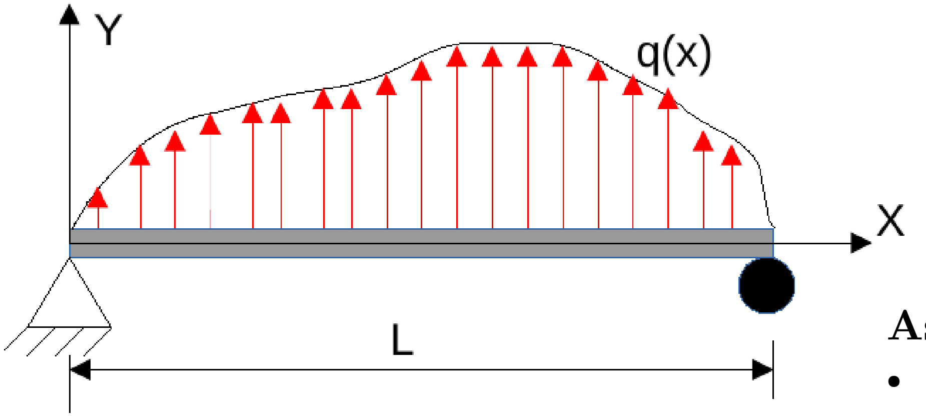
$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},$$

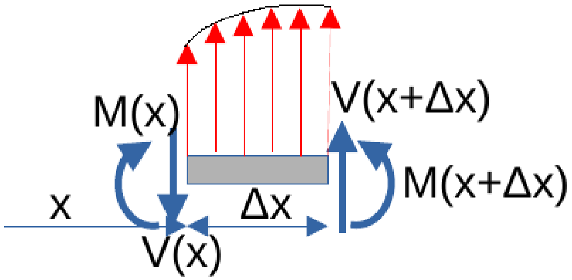
Lame's constant

- **Young's modulus** as a measure of the stiffness of the solid.
- **Poisson's ratio** is the ratio of lateral to longitudinal strain in uniaxial tensile stress. It is a measure of the compressibility of the solid.
- The **bulk modulus** quantifies the resistance of the solid to volume changes.
- The **shear modulus** quantifies its resistance to volume preserving shear deformations.
- **Coefficient of thermal expansion** quantifies the change in volume of a material if it is heated in the absence of stress.

the proper use of MIL-HDt http://www.matweb.com.

Beam Bending and Deflection





Assumptions:

- straight, prismatic beam, made with homogeneous material
- cross-section of the beam is symmetric about the plane of loading.
- Euler-Bernoulli assumptions

To calculate the beam bending and deflection

$$EI\frac{d^2v}{dx^2} = M \qquad \text{(A)}$$

$$EI\frac{d^3v}{dx^3} = \frac{dM}{dx} = -V \qquad \text{(B)}$$

$$EI\frac{d^4v}{dx^4} = -\frac{dV}{dx} = q \qquad \text{(C)}$$

Moment curvature relation

$$\frac{1}{\rho} = \frac{M}{EI_{zz}}$$

Curvature

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left(1 + \left(\frac{dv}{dx}\right)^2\right)^{3/2}} \approx \frac{d^2v}{dx^2}$$

(small slope)

Method of Superposition

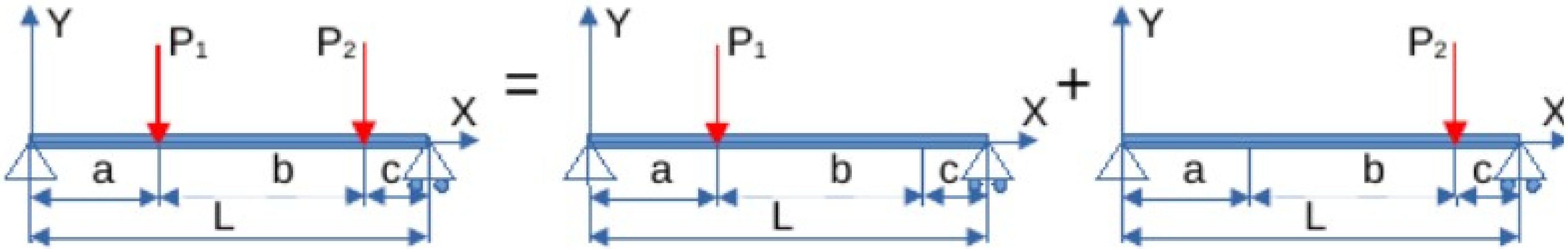
- Consider the equations governing the deflection of the beam (equation of the elastic curve)

$$EI\frac{d^2v}{dx^2} = M(x), \text{ or } EI\frac{d^4v}{dx^4} = q(x)$$

  subject to appropriate boundary conditions. Note that the second equation is derived assuming that the flexural rigidity EI does not change along the length of the beam.

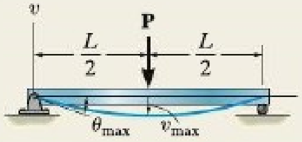
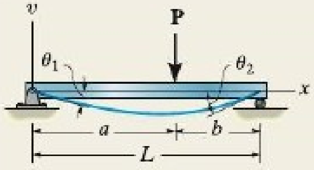
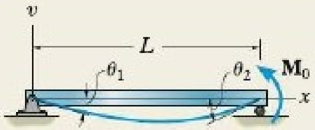
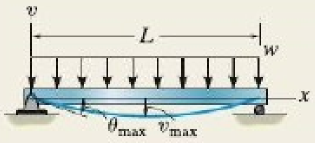
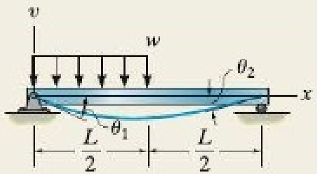
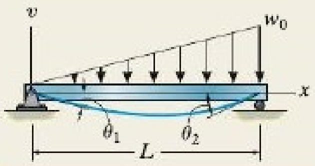
- These are **linear** differential equations. Consequences:
  - If $v(x)$ is the deflection due to load P, then the deflection due to load αP (α is a constant) is $\alpha v(x)$. It is assumed that the location of point application does not change.
  - If $v_1(x)$ is the deflection due to the load P_1 acting at $x = a$ alone and $v_2(x)$ is the deflection due to the load P_2 acting at $x = b$ alone, then $v_1(x) + v_2(x)$ is the deflection due P_1 acting at $x = a$ and P_2 acting at $x = b$ simultaneously.

Method of Superposition



Beam Bending Formulae

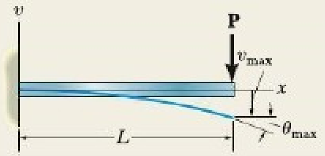
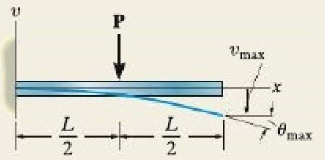
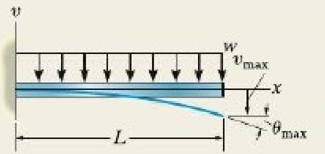
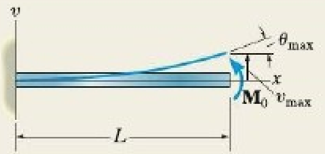
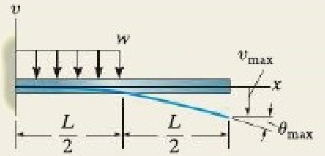
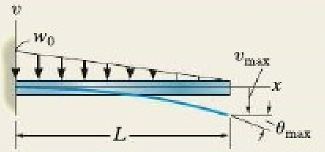
Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{16EI}$	$v_{max} = \dfrac{-PL^3}{48EI}$	$v = \dfrac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \dfrac{-Pab(L+b)}{6EIL}$ $\theta_2 = \dfrac{Pab(L+a)}{6EIL}$	$v\Big\vert_{x=a} = \dfrac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \dfrac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \dfrac{-M_0L}{6EI}$ $\theta_2 = \dfrac{M_0L}{3EI}$	$v_{max} = \dfrac{-M_0L^2}{9\sqrt{3}EI}$ at $x = 0.5774L$	$v = \dfrac{-M_0x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \dfrac{-wL^3}{24EI}$	$v_{max} = \dfrac{-5wL^4}{384EI}$	$v = \dfrac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \dfrac{-3wL^3}{128EI}$ $\theta_2 = \dfrac{7wL^3}{384EI}$	$v\Big\vert_{x=L/2} = \dfrac{-5wL^4}{768EI}$ $v_{max} = -0.006563\dfrac{wL^4}{EI}$ at $x = 0.4598L$	$v = \dfrac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \dfrac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \dfrac{-7w_0L^3}{360EI}$ $\theta_2 = \dfrac{w_0L^3}{45EI}$	$v_{max} = -0.00652\dfrac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \dfrac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

Salil S. Kulkarni

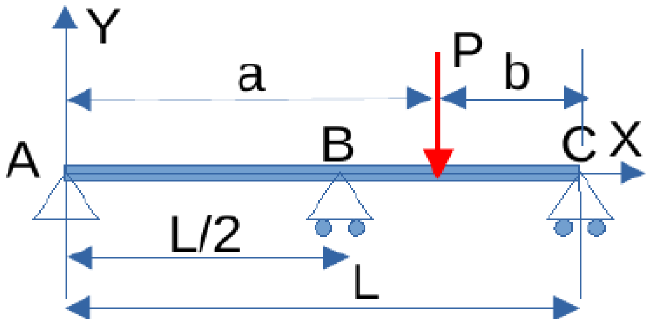
Beam Bending Formulae

Cantilevered Beam Slopes and Deflections

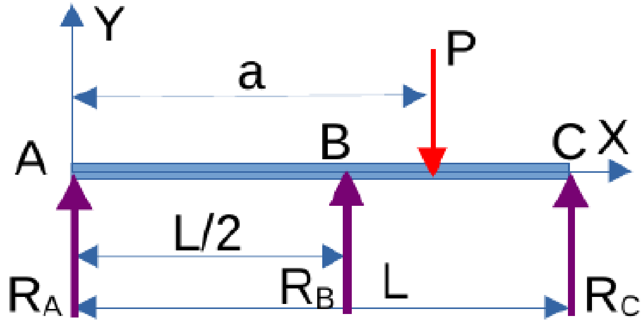
Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{2EI}$	$v_{max} = \dfrac{-PL^3}{3EI}$	$v = \dfrac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \dfrac{-PL^2}{8EI}$	$v_{max} = \dfrac{-5PL^3}{48EI}$	$v = \dfrac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \dfrac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \dfrac{-wL^3}{6EI}$	$v_{max} = \dfrac{-wL^4}{8EI}$	$v = \dfrac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \dfrac{M_0L}{EI}$	$v_{max} = \dfrac{M_0L^2}{2EI}$	$v = \dfrac{M_0x^2}{2EI}$
	$\theta_{max} = \dfrac{-wL^3}{48EI}$	$v_{max} = \dfrac{-7wL^4}{384EI}$	$v = \dfrac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $0 \le x \le L/2$ $v = \dfrac{-wL^3}{384EI}(8x - L)$ $L/2 \le x \le L$
	$\theta_{max} = \dfrac{-w_0L^3}{24EI}$	$v_{max} = \dfrac{-w_0L^4}{30EI}$	$v = \dfrac{-w_0x^2}{120EIL}(10L^3 - 10L^2x + 5Lx^2 - x^3)$

Beam Bending and Deflection

Find the reaction at point B



The FBD of the beam is given by

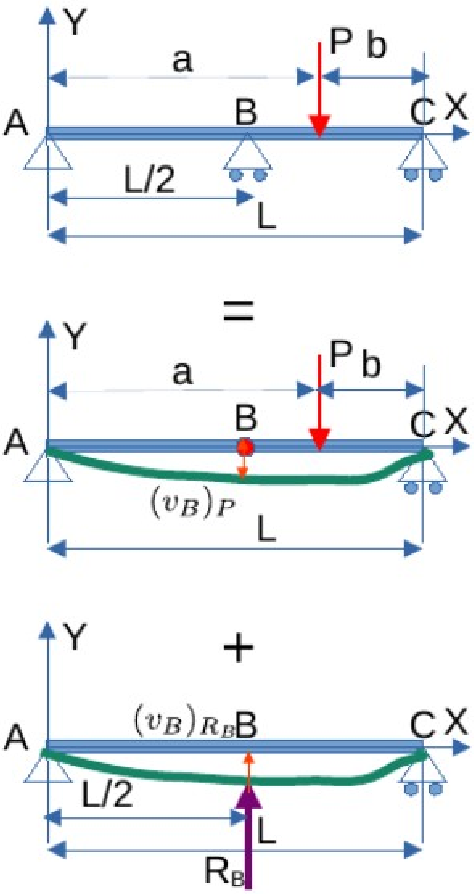


$\boxed{\text{statically indeterminate problem}}$

There are three unknowns and two equilibrium equations and hence it is a statically indeterminate problem.

Beam Bending and Deflection

Statically Indeterminate Problem



- Remove R_B to make the problem statically determinate.

- The deflection at point B due to P is

$$(v_B)_P = \frac{Pb}{12EI}\left(\left(\left(L^2 - b^2\right) - \frac{L}{2}\right)^2\right) \downarrow$$

- Now consider the beam with only R_B acting.

- The deflection at B due to R_B is

$$(v_B)_{R_B} = \frac{R_B L^3}{48EI} \uparrow$$

- The net deflection at B is zero due to the support. Hence

$$\frac{Pb}{12EI}\left(\left(\left(L^2 - b^2\right) - \frac{L}{2}\right)^2\right) = \frac{R_B L^3}{48EI}$$

- Solve for R_B

Column Buckling

Buckling is defined as a large sudden deflection of a structure due to a small increase in the existing load.

- Buckling is an example of **structural instability**. Machine elements where structural stability is of prime importance include long slender beams, thin sheets, narrow beams, vacuum tanks, casings of rockets, etc.
- For a machine elements to **buckle** it should be subjected to a **compressive loads**

Column Buckling

The critical column buckling load, P_{cr}, is estimated using Euler's formula

$$P_{cr} = c\pi^2 \frac{EI}{L^2} \quad \text{or} \quad P_{cr} = \pi^2 \frac{EI}{(kL)^2}$$

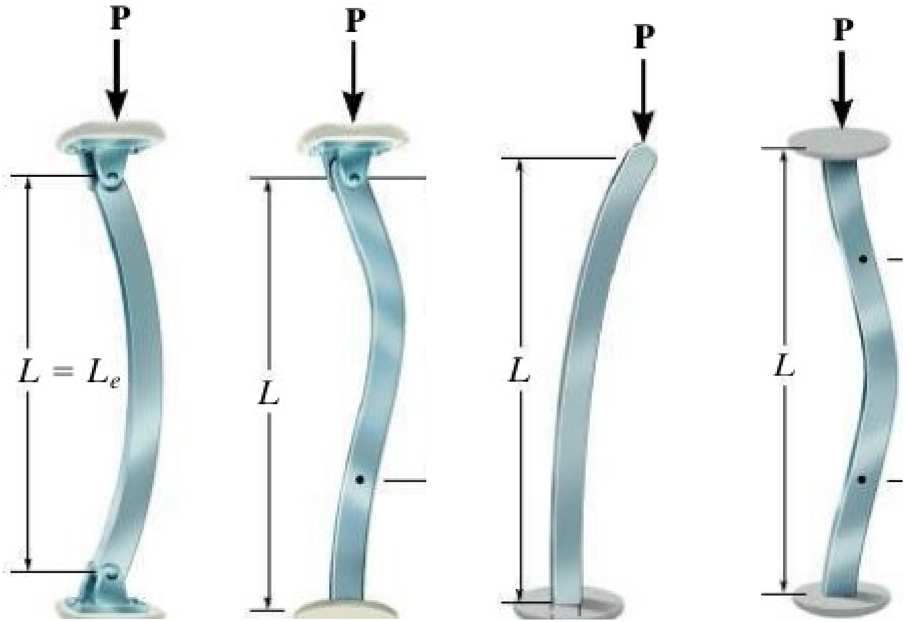
where the constants c (and k) depend on the boundary conditions ($k = 1/\sqrt{c}$).

	Boundary conditions	c	k
1.	P-P	1.00	1.00
2.	P-C	2.05	0.70
3.	C-F	0.25	2.00
4.	C-C	4.00	0.50

P-Pinned, C-Clamped, F-Free

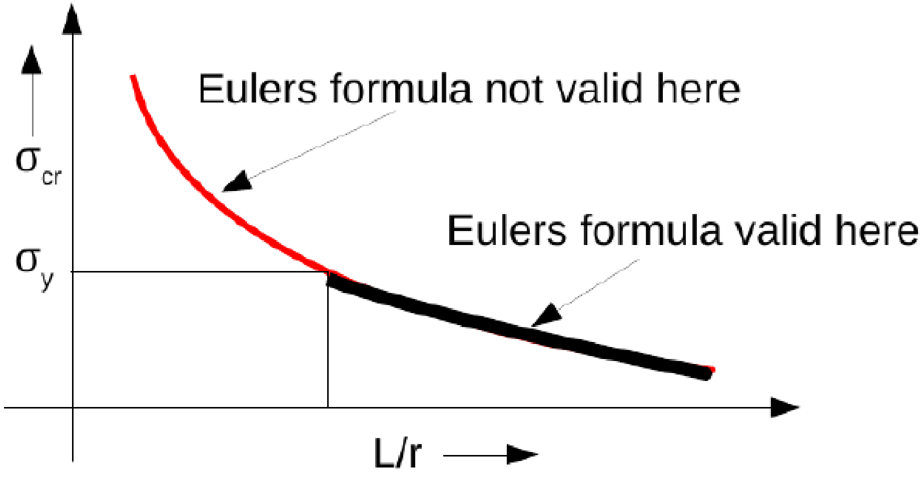
Assumptions:

- Column is perfectly straight before loading.
- It is made up homogeneous material.
- The material behaves as a linear elastic material.
- The load is applied through the centroid of the cross section and the column bends in a single plane.



P-P C-F C-F C-C

Column Buckling



L/r slenderness ratio

$\sigma_{cr} = P_{cr}/A$

r minimum radius of gyration

Validity of Euler's Formula

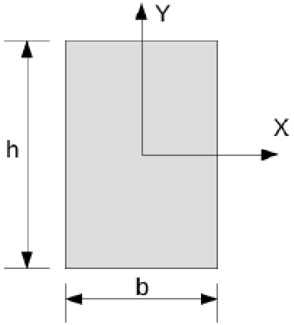
Material	E (GPa)	σ_y (MPa)	$L/r >$
Steel	200	250	89
Pure Aluminum	70	90	87
Al 6061-T6	70	270	51

Radius of Gyration

The radius of gyration, r, relates the moment of inertia of the cross-section, I, to its area, A. The radius of gyration is obtained from

$$I = Ar^2$$

It is the distance from the axis where the the entire area can be assumed to be concentrated so as to give the same moment of inertia. Consider a rectangular cross-section with breadth b and height h and $b < h$.



- We have

$$A = bh, \ I_{xx} = \frac{bh^3}{12}, \ I_{yy} = \frac{b^3h}{12}$$

- Now $I = Ar^2$. Therefore

$$r_x = \frac{h}{2\sqrt{3}}, \ r_y = \frac{b}{2\sqrt{3}}$$

- Since $b < h$, $r_y < r_x$ and $r_{min} = r_y$.

The radius of gyration can be used to compare the susceptibility of buckling of a column about different axes. In this case the column is more likely to buckle about the $Y-$ axis.