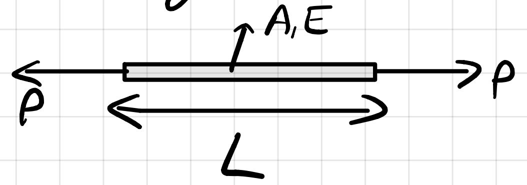
Ganesh Iyer, 210100059 ME-423 : Tutorial-3

1. Derive the material indices for the following cases:
   a. A light truss with stiffness greater than S^*
   b. A light shaft with stiffness greater than S^*

a) Objective : min m Constraint : $S \geq S^*$



$$\delta = \frac{PL}{EA}$$ Truss → Axial

$$S = \frac{P}{\delta} = \frac{EA}{L} \geq S^*$$

$$m = \rho AL \Rightarrow A = \frac{m}{\rho L}$$

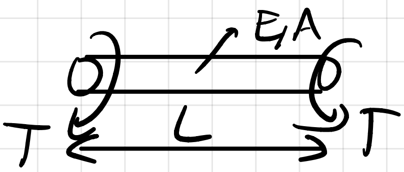
$$\Rightarrow \frac{E}{L} \times \frac{m}{\rho L} \geq S^* \Rightarrow m \geq S^* \times \left(\frac{\rho}{E}\right) \times L^2$$

Material

$$\min(m) \Rightarrow \max\left(\frac{E}{\rho}\right)$$

$\frac{E}{\rho}$: material index

b) Objective : min m Constraint : $S \geq S^*$



$$\theta = \frac{TL}{GJ}$$ Shaft → Torsion

$$S = \frac{T}{\theta} = \frac{GJ}{L} \geq S^*$$

$$m = \rho AL \Rightarrow A = \frac{m}{\rho L}$$ Let us take circular c/s

$$\Rightarrow A = \pi d^2, \quad J = \frac{\pi d^4}{32} \Rightarrow J = \frac{A^2}{32\pi}$$

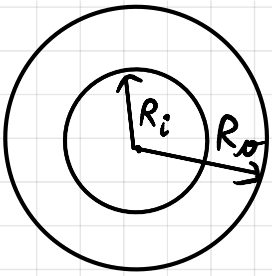
$$\Rightarrow J = \frac{m^2}{32\pi \rho^2 L^2}$$

Now, $$\frac{G}{L}\left(\frac{m^2}{32\pi\rho^2 L^2}\right) \geqslant S^*$$

$$\Rightarrow m \geqslant S^{*\,1/2} \times \left(\frac{\rho}{\sqrt{G}}\right) \times (32\pi L^3)^{1/2}$$

$$\min(m) \Rightarrow \max\left(\frac{G^{1/2}}{\rho}\right) \text{ material index}$$

2. Derive the shape factor for annular cross-section with inner radius R_i and outer radius R_o for torsional stiffness.



$$J = \frac{\pi(R_o^4 - R_i^4)}{2}$$

Reference c/s : $\cancel{\pi} R^2 = \cancel{\pi}(R_o^2 - R_i^2)$

$$J_{ref} = \frac{\pi R^4}{2} = \frac{\pi(R_o^2 - R_i^2)^2}{2}$$

$$S_{tors.} = \frac{GJ}{L} \propto J$$

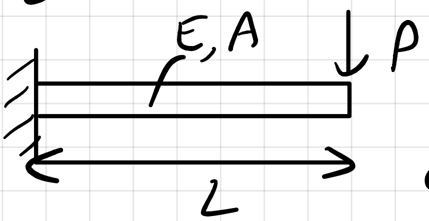
$$\Rightarrow \varphi = \frac{J}{J_{ref}} = \frac{R_o^4 - R_i^4}{(R_o^2 - R_i^2)^2} = \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2}$$

3. Derive the performance indices for the following cases:

   a. A light beam with maximum stress less than or equal to σ_f (material property)

   b. A light hollow shaft with stiffness greater than S^*

a) Objective : min m , Constraint: $\sigma_{max} \leq \sigma_f$



$$M_{max} = PL$$

$$\sigma_{max} = \frac{M_{max}\, y}{I} = \frac{M_{max}}{Z}$$

For bending strength, shape factor

$$\varphi_B^f = 6Z/A^{3/2} \Rightarrow Z = \varphi_B^f \cdot A^{3/2}/6$$

$$\Rightarrow \sigma_{max} = \frac{6\, M_{max}}{\varphi_B^f A^{3/2}}, \quad A = \frac{m}{\rho L} \Rightarrow \sigma_{max} = \frac{6(PL)(\rho L)^{3/2}}{\varphi_B^f m^{3/2}}$$

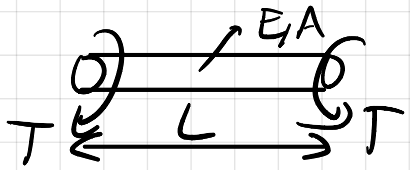
$$\sigma_{max} \leq \sigma_f$$

$$\Rightarrow \frac{6P\rho^{3/2}L^{5/2}}{\varphi_B^f m^{3/2}} \leq \sigma_f$$

$$\Rightarrow m \geq (6P)^{2/3} \cdot (L)^{5/3} \cdot \left(\frac{\rho}{\varphi_B^{2/3}\, \sigma_f^{2/3}} \right)$$

$$\min(m) \Rightarrow \max\left(\frac{\sigma_f^{2/3} \cdot \varphi_B^{2/3}}{\rho} \right) : \text{Material Index}$$

b) Objective: min m Constraint: $S \geq S^*$



$$\theta = \frac{TL}{GJ}$$ Shaft → Torsion

$$S = \frac{T}{\theta} = \frac{GJ}{L} \geq S^* \qquad m = \rho A L$$

For shaft, $\varphi_T^e = \frac{2\pi J}{A^2}$ from eq.(2).

$$J = \frac{\varphi_T^e}{2\pi} A^2 = \frac{\varphi_T^e (m/\rho L)^2}{2\pi} = \frac{\varphi_e^T m^2}{2\pi \rho^2 L^2}$$

$$\Rightarrow \frac{G}{L} \times \left(\frac{\varphi_e^T m^2}{2\pi \rho^2 L^2}\right) \geq S^*$$

$$\Rightarrow m \geq (2\pi S^*)^{1/2} \times (L)^{3/2} \times \left(\frac{\rho}{\sqrt{G \varphi_e^T}}\right)$$

Performance Index: $\dfrac{\sqrt{G \varphi_e^T}}{\rho}$

4. Derive the material index to maximize the slenderness ratio (L/r) of a column with circular c/s subject to the constraint that it must not buckle under a given load F.

$$P_{cr} = C\pi^2 \frac{EI}{L^2}$$ C depends on B.Cs, some constant

Objective: $\max\left(\frac{L}{r}\right)$ Constraint: $P_{cr} \geq F$

$$C\pi^2 \frac{EI}{L^2} \geq F \Rightarrow \frac{C\pi^3 E r^4}{4L^2} \geq F$$

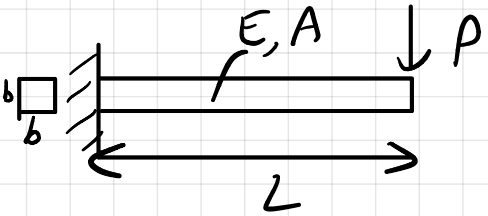
$$\Rightarrow \frac{C\pi^3 E L^2}{4}\left(\frac{r}{L}\right)^4 \geq F$$

$$\Rightarrow \frac{L}{r} \leq \left(\frac{C\pi^3}{4}\right)^{1/4} E^{1/4} \times \frac{1}{F^{1/4}} \times L^{1/2}$$

$\max\left(\frac{L}{r}\right) \Rightarrow \max\left(E^{1/4}\right)$, material index

5. Derive the material index to minimize the cost of a beam with stiffness greater than S^*. Note that the cost of the beam, C, can be assumed to be directly proportional to the mass of the beam, i.e. $C = C_m \boldsymbol{m}$ where C_m is the cost per unit mass and is a material property. Your material index will now include E, ρ and C_m.

Objective: min C, Constraint: $S \geq S^*$



$$\delta = \frac{PL^3}{3EI}, \quad S = \frac{P}{\delta} = \frac{3EI}{L^3}$$

$$S = \frac{3Eb^4}{12L^3} = \frac{EA^2}{4L^3} \quad (A = b^2)$$

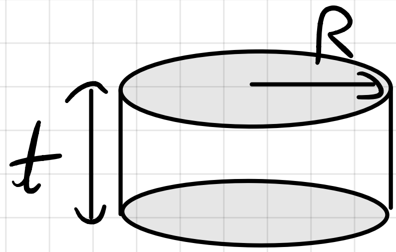
$$C = C_m \times m = C_m \rho A L \Rightarrow A = C/(C_m \rho L)$$

$$S = \frac{E C^2}{4 L^5 C_m^2 \rho^2} \geq S^*$$

$$\Rightarrow C \geq (4L^5)^{1/2} \times \left(\frac{C_m \rho}{E^{1/2}} \right)$$

$$\min(C) \Rightarrow \max \left(\frac{E^{1/2}}{C_m \rho} \right) \leftarrow \text{index}$$

6. Derive the material index to maximize the energy stored per unit mass in a flywheel of fixed outer radius R, radius t and rotating with angular speed ω. Note that the maximum stress induced in the flywheel should be less than or equal to the failure stress σ_f, a material property. Note that at this stress the flywheel bursts. The maximum principal stress in a spinning disk of radius R with uniform thickness is $\sigma_{max} = \frac{3+\nu}{8}\rho R^2 \omega^2$.



$$E = \frac{1}{2} I \omega^2 = \frac{1}{4} m R^2 \omega^2$$

$$E/m = \frac{1}{4} R^2 \omega^2$$

$$\sigma_{max} = \frac{3+\nu}{8} \rho R^2 \omega^2 \leq \sigma_f$$

$$\Rightarrow \frac{3+\nu}{2} \rho \left(\frac{E}{m} \right) \leq \sigma_f$$

$$\Rightarrow \frac{E}{m} \leq \frac{2\sigma_f}{\rho(3+\nu)} \leftarrow \text{material index}$$