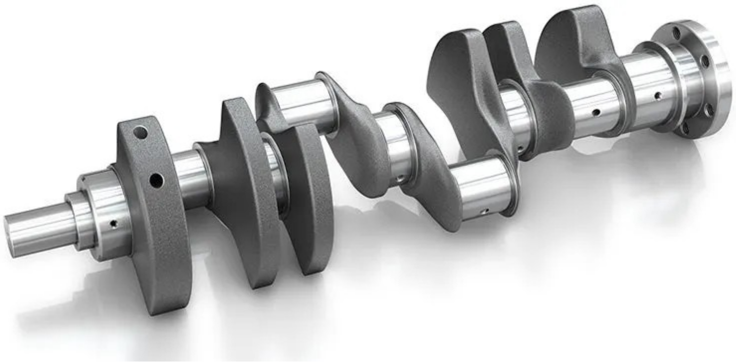
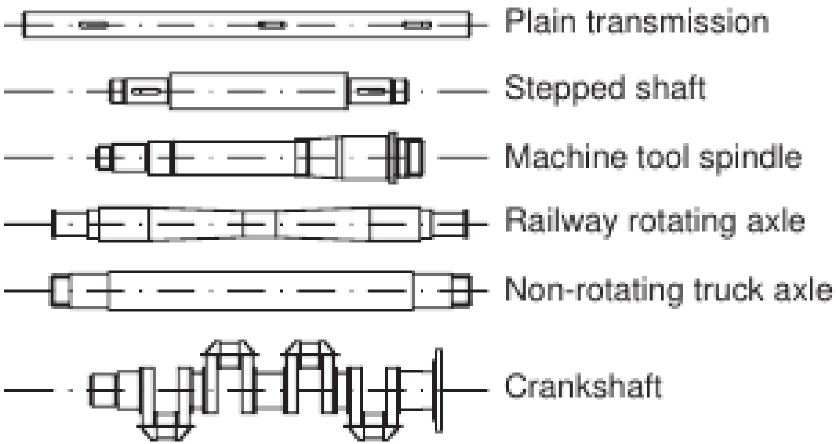
Design of Shafts



https://www.sandvik.coromant.com/en-us/industry-solutions/automotive

Shafts, Axles and Spindle

- A **shaft** is a rotating member which transmits rotary motion and torque from one location to another. The loading on the shaft is of two types – torsion and bending. Machine elements such as gears, pulleys, flywheels, cranks, sprockets are mounted on shafts
- An **axle** is a non-rotating member that carries no torque and is used to support rotating wheels, pulleys, etc. It is designed and analysed as a static beam
- A **spindle** is a short shaft which typically holds the cutting tool or workpiece of a machine tool.



Unlike many machine elements which that are manufactured in bulk and can be selected from manufacturers catalogue, shafts are usually unique and have to be designed from first principles

Childs, Mechanical Design

Shaft Design Considerations

- Material selection
- Geometric layout
- Stress and strength
    - Static strength
    - Fatigue strength
- Deflection and rigidity
    - Bending deflection
    - Torsional deflection
    - Slope at bearings and shaft-supported elements
    - Shear deflection due to transverse loading of short shafts
- Vibration due to natural frequency

- Stress analysis at a specific point on a shaft can be made using only the shaft geometry in the vicinity of the point Entire shaft geometry is not needed.
- Deflection and slope analysis cannot be carried out unless the entire geometry has been defined.

Shaft Design Considerations – Material Selection

- Most machine shafts are made from low (0.05–0.25% carbon) - to medium-carbon (0.30 – 0.60%) carbon steel (high Young's modulus), either cold rolled or hot rolled.

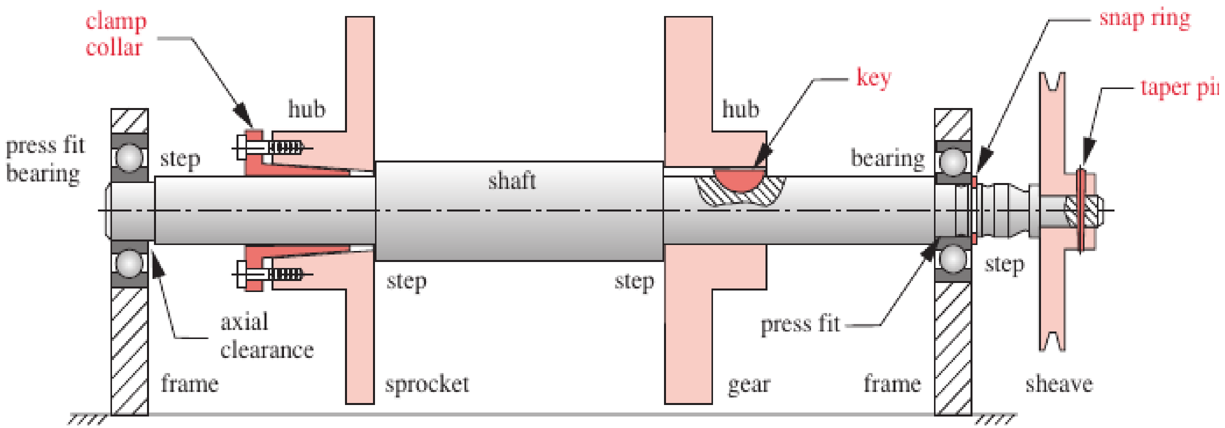
- Alloy steels are used where their higher strengths are needed.

- Cold-rolled steel is more often used for smaller-diameter shafts (< about 75 mm dia) and hot-rolled used for larger sizes.

Shaft Design Considerations – Shaft Layout

Points to consider while deciding the shaft layout

- Axial layout/positioning of the various components (minimize shaft deflection, max. moment)
    - Shaft shoulders, ring and groove, sleeve, setscrews, pins, tapered hub, nut and washer
- Provision for supporting axial loads – use of taper bearings
- Providing for torque transfer
    - Keys, splines, setscrews, pins, press or shrink fits, tapered fits
- Ease of assembly and disassembly



Mounted parts

- Bearings
- Sprocket
- Gear
- Sheave/pulley

Axial Location

- Shoulder/Step
- Snap ring
- Taper pin

Torque transfer methods

- Press fit
- Clamp collar
- Key
- Taper pin

Shaft Design Considerations – Stress and Strength

- Identify critical locations - will usually be on the outer surface, at axial locations where the bending moment is large, where the torque is present, and where stress concentrations exist.

- **Torsion**: Most shafts will transmit torque through a portion of the shaft. The torque is often relatively constant at steady state operation. The shear stress due to the torsion will be greatest on outer surfaces.

- **Bending**: Since most shaft problems incorporate gears or pulleys that introduce forces in two planes, the shear and bending moment diagrams will generally be needed in two planes. Resultant moments are obtained by summing moments as vectors at points of interest along the shaft. A steady bending moment will produce a completely reversed moment on a rotating shaft. The normal stress due to bending moments will be greatest on the outer surfaces.

- **Axial Force**: Axial stresses on shafts due to the axial components transmitted through helical gears or tapered roller bearings will almost always be negligibly small compared to the bending moment stress. They are often also constant, so they contribute little to fatigue. Consequently, it is usually acceptable to neglect the axial stresses induced by the gears and bearings when bending is present in a shaft. **But verify**.

Shaft Design Considerations – Static Strength

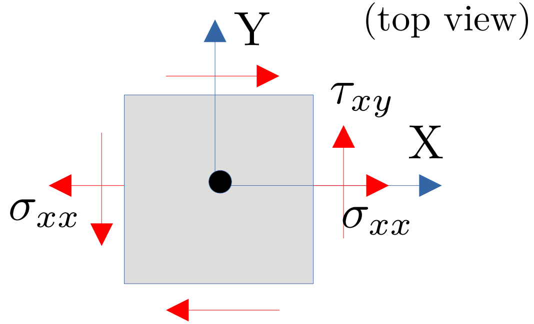
- Though a shaft is under dynamic loading (fatigue), a static analysis gives a preliminary estimate of the dimensions of the required dimensions.
- For a point on the surface of a circular solid shaft of diameter d located at a critical location with constant bending moment M, constant torque T and constant axial force F (x-axis along the axis, y - horizontal)

$$\sigma_{xx} = \frac{32M}{\pi d^3} + \frac{4F}{\pi d^2}$$

(the second component can be either positive or negative (tension/compression))

$$\tau_{xy} = \frac{16T}{\pi d^3}$$

(stress concentration effects are neglected for ductile material)



- Since the point is assumed to be on an unloaded surface, it is in the state of plane stress.

- The three principal stresses are given by:

$$\sigma_A, \sigma_B = \frac{\sigma_{xx}}{2} \pm \left[\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}, \quad \sigma_C = 0$$

Shaft Design Considerations – Static Strength

- The maximum shear stress is then given by

$$\tau_{max} = \frac{\sigma_A - \sigma_B}{2} = \left[\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

- The equivalent or the von Mises stress is then given by

$$\sigma_{eq} = (\sigma_A^2 + \sigma_B^2 - \sigma_A\sigma_B)^{1/2} = (\sigma_{xx}^2 + 3\tau_{xy}^2)^{1/2}$$

- Substituting the expression for the stress components:

$$\tau_{max} = \frac{2}{\pi d^3}\sqrt{(Fd + 8M)^2 + (8T)^2} \qquad \sigma_{eq} = \frac{4}{\pi d^3}\sqrt{(Fd + 8M)^2 + 48T^2}$$

- If N is the factor of safety and σ_y is the yield strength of the material in tension, then the diameter d is chosen such that

$$\frac{\sigma_y}{2N} = \frac{2}{\pi d^3}\sqrt{(Fd + 8M)^2 + (8T)^2}$$

(Tresca criterion)

$$\frac{\sigma_y}{N} = \frac{4}{\pi d^3}\sqrt{(Fd + 8M)^2 + 48T^2}$$

(Distortion Energy criterion)

If $F = 0$, it is easier to solve these equations for d

Shaft Design Considerations – Fatigue Strength

- For a point on the surface of a circular solid shaft of diameter d located at a critical location with fluctuating bending moment (mean component M_m, alternating component M_a) and fluctuating torque (mean component T_m, alternating component T_a) [axial loading is ignored]

$$\sigma_m = K_f \frac{32M_m}{\pi d^3}, \quad \sigma_a = K_f \frac{32M_a}{\pi d^3}$$

$$\tau_m = K_{fs} \frac{16T_m}{\pi d^3}, \quad \tau_a = K_{fs} \frac{16T_a}{\pi d^3}$$

K_f and K_{fs} are the fatigue stress concentration factors at the critical location

- The mean and alternating von Mises (equivalent) stresses are given by

$$\sigma_{eqm} = (\sigma_m^2 + 3\tau_m^2)^{1/2} \qquad \sigma_{eqa} = (\sigma_a^2 + 3\tau_a^2)^{1/2}$$

or

$$\sigma_{eqm} = \left[\left(K_f \frac{32M_m}{\pi d^3}\right)^2 + 3\left(K_{fs} \frac{16T_m}{\pi d^3}\right)^2\right]^{1/2} \qquad \sigma_{eqa} = \left[\left(K_f \frac{32M_a}{\pi d^3}\right)^2 + 3\left(K_{fs} \frac{16T_a}{\pi d^3}\right)^2\right]^{1/2}$$

Shaft Design Considerations – Fatigue Strength

or

$$\sigma_{eqm} = \frac{1}{\pi d^3}\left[K_f^2(32M_m)^2 + 3K_{fs}^2(16T_m)^2\right]^{1/2} \qquad \sigma_{eqa} = \frac{1}{\pi d^3}\left[K_f^2(32M_a)^2 + 3K_{fs}^2(16T_a)^2\right]^{1/2}$$

- If N is the factor of safety (given), the Goodman criterion can be written as

$$\frac{\sigma_{eqa}}{\sigma_e} + \frac{\sigma_{eqm}}{\sigma_{ult}} = \frac{1}{N}$$

- Substituting the expressions for σ_{eqa} and σ_{eqm} obtained earlier one gets an equation for d.

- On solving the equation, we get the estimate of the minimum diameter which needs to be used.

- One can also use the the other fatigue criterion – Gerber, etc

Shaft Design Considerations – Fatigue Strength

- In addition one also needs to check for static failure in the first cycle itself – for the estimated d, we calculate the maximum equivalent (von Mises) stress

$$\sigma_{eqmax} = \left((\sigma_m + \sigma_a)^2 + 3(\tau_m + \tau_a)^2\right)^{1/2}$$

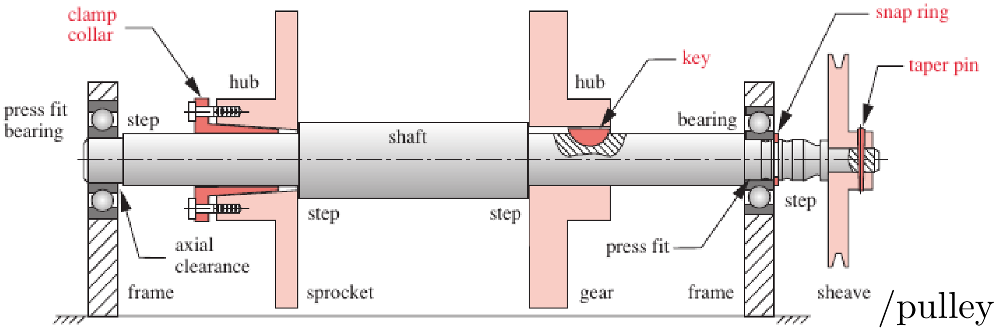
  and compare it with $\dfrac{\sigma_y}{N}$

- Another class of problems consists of given d, estimate the factor safety, N. This is typically easier to solve.

Shaft Design Considerations – Fatigue Strength

- **Estimates of the Stress Concentration Factors**

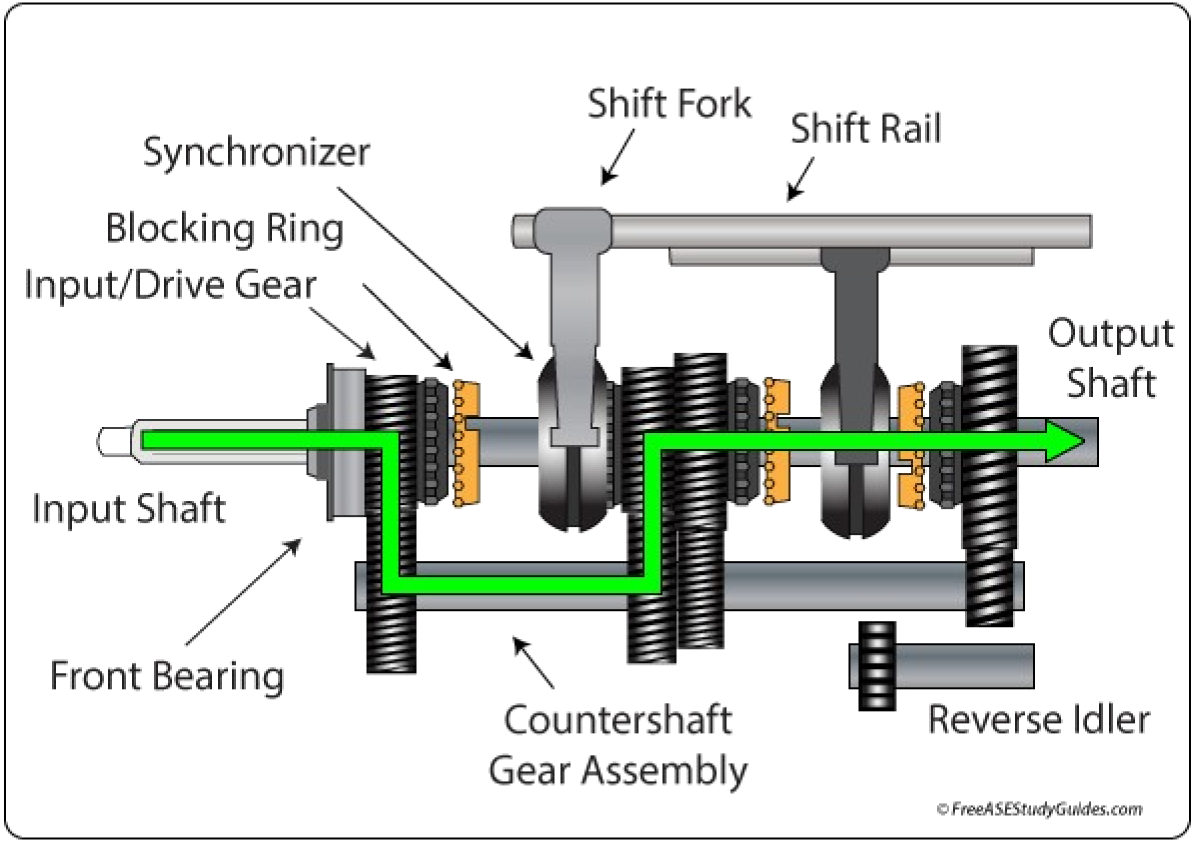
  The design of the shaft is greatly influenced by the presence of geometric features such as shoulders, holes, keyways which lead to stress concentration.



Stress concentration factors depend on the dimensions of the shaft which is not available. Hence estimates of the these factors are needed and are given by

	Bending	Torsional	Axial
Shoulder fillet—sharp ($r/d = 0.02$)	2.7	2.2	3.0
Shoulder fillet—well rounded ($r/d = 0.1$)	1.7	1.5	1.9
End-mill keyseat ($r/d = 0.02$)	2.14	3.0	—
Sled runner keyseat	1.7	—	—
Retaining ring groove	5.0	3.0	5.0

What is a Countershaft



Countershaft: It is a secondary shaft, which is driven by the main shaft and from which The power is supplied to a machine component. Often, the countershaft is driven from the main shaft by means of a pair of spur or helical gears and thus rotates 'counter' to the direction of the main shaft.

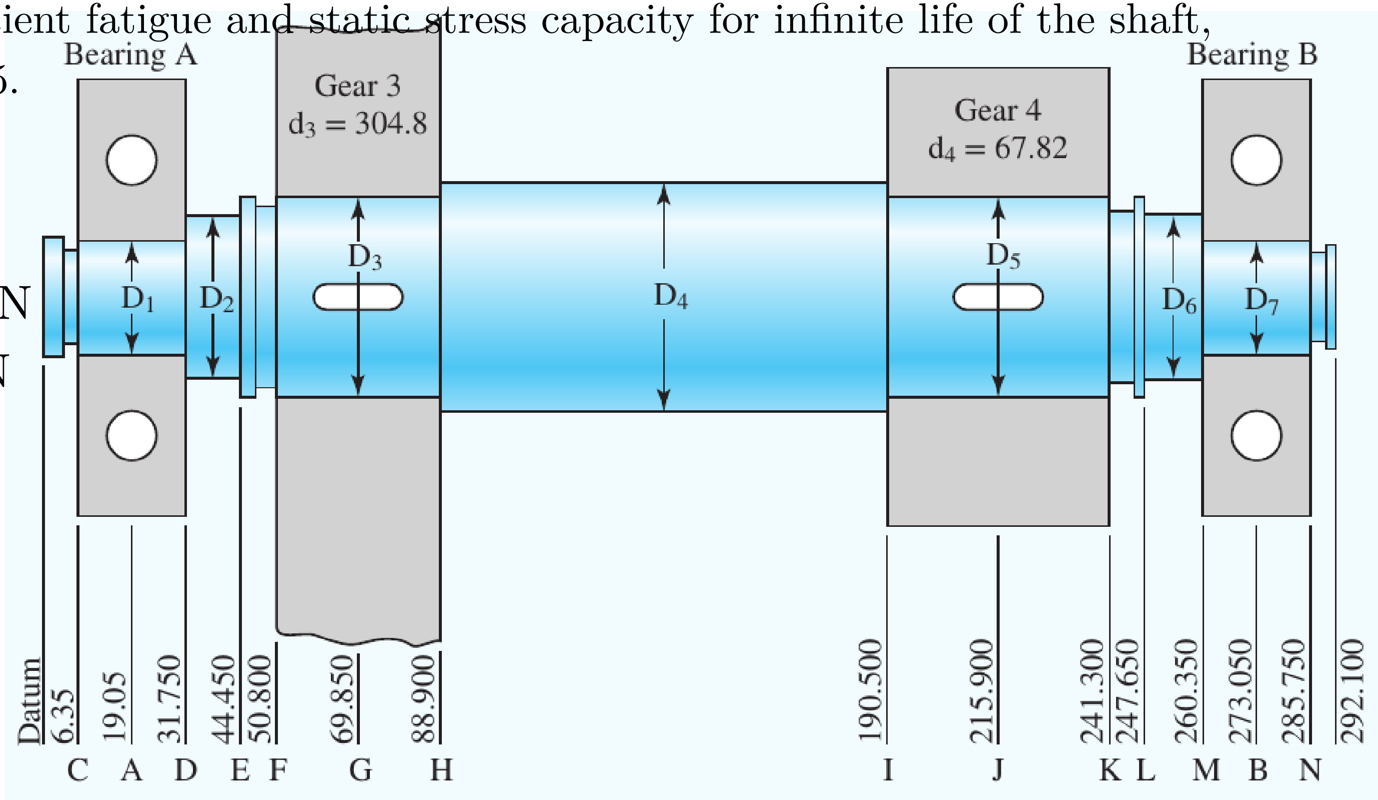
https://www.freeasestudyguides.com/manual-transmission-countershafts.html

Problem – Fatigue Strength

The general layout and axial dimensions of the countershaft carrying two spur gears has been is shown in Figure. The gears and bearings are located and supported by shoulders, and held in place by retaining rings. The gears transmit torque through keys. Gears have been specified as shown, allowing the tangential and radial forces transmitted through the gears to the shaft are given. Choose appropriate material. Determine $D_1,\ldots,D_7$, based on providing sufficient fatigue and static stress capacity for infinite life of the shaft, with minimum safety factors of 1.5.

$$W^t_{23} = 2401 \text{ N} \qquad W^t_{54} = 10814 \text{ N}$$

$$W^r_{23} = 876 \text{ N} \qquad W^r_{54} = 3937 \text{ N}$$



Budinas, Shigley'S Mechanical Engineering Design (SIE), 9th Ed (pg 374-378)

Salil S. Kulkarni

Solution

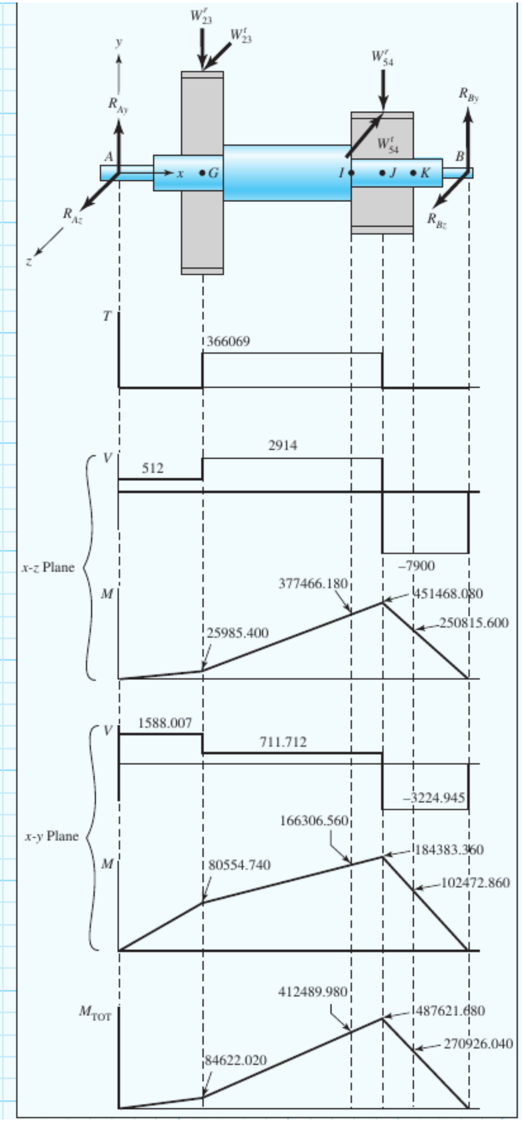
Perform free body diagram analysis to get reaction forces at the bearings.

$$R_{Az} = 511.54\ \text{N}$$
$$R_{Ay} = 1586.67\ \text{N}$$
$$R_{Bz} = 7900.00\ \text{N}$$
$$R_{By} = 3226.28\ \text{N}$$

From ΣM_x, find the torque in the shaft between the gears,
$T = W_{23}^t(d_3/2) =$
$2402(304.8/2) =$
$366069.07\ \text{N} \cdot \text{mm}$.

Generate shear-moment diagrams for two planes.

Combine orthogonal planes as vectors to get total moments, e.g., at J, $\sqrt{451468^2 + 184383^2} =$ $487622\ \text{N} \cdot \text{mm}$.



Start with point I, where the bending moment is high, there is a stress concentration at the shoulder, and the torque is present.

$$\text{At } I,\ M_a = 412489.98\ \text{N} \cdot \text{mm},\ T_m = 366069.07\ \text{N} \cdot \text{mm},\ M_m = T_a = 0$$

Assume generous fillet radius for gear at I.

From Table 7-1, estimate $K_t = 1.7$, $K_{ts} = 1.5$. For quick, conservative first pass, assume $K_f = K_t$, $K_{fs} = K_{ts}$.

Choose inexpensive steel, 1020 CD, with $S_{ut} = 470$ MPa. For S_e,

Equation (6–18) $\quad k_a = aS_{ut}^b = 3.04(470)^{-0.217} = 0.8$

Guess $k_b = 0.9$. Check later when d is known.

$$k_c = k_d = k_e = 1$$

Equation (6–17) $\quad S_e = (0.8)(0.9)(0.5)(470) = 169.18$ MPa

For first estimate of the small diameter at the shoulder at point I, use the DE-Goodman criterion of Equation (7-8). This criterion is good for the initial design, since it is simple and conservative. With $M_m = T_a = 0$, Equations (7-6) and (7-8) reduce to

$$d = \left\{\frac{16n}{\pi}\left(\frac{2(K_f M_a)}{S_e} + \frac{[3(K_{fs}T_m)^2]^{1/2}}{S_{ut}}\right)\right\}^{1/3}$$

$$d = \left\{\frac{16(1.5)}{\pi}\left(\frac{2(1.7)(412489.98)}{169.18} + \frac{\{3[(1.5)(366069.07)]^2\}^{1/2}}{470}\right)\right\}^{1/3}$$

$$d = 42.35\ \text{mm}$$

All estimates have probably been conservative, so select the next standard size below 1.69 in and check, $d = 41.275$ mm.

A typical D/d ratio for support at a shoulder is $D/d = 1.2$, thus, $D = 1.2(41.275) = 49.5300$ mm. Increase to $D = 50.8$ mm. A nominal 2-in. cold-drawn shaft diameter can be used. Check if estimates were acceptable.

$$D/d = 50.8/41.275 = 1.23$$

Assume fillet radius $r = d/10 = 4.1275$ mm, $\quad r/d = 0.1$

$$K_t = 1.6 \text{ (Fig. A–15–9)},\ q = 0.82 \text{ (Fig. 6–26)}$$

Equation (6–32) $\quad K_f = 1 + 0.82(1.6 - 1) = 1.49$

$$K_{ts} = 1.35 \text{ (Fig. A–15–8)},\ q_s = 0.85 \text{ (Fig. 6–27)}$$

$$K_{fs} = 1 + 0.85(1.35 - 1) = 1.30$$

$$k_a = 0.801 \text{ (no change)}$$

Equation (6–19) $\quad k_b = \left(\frac{41.275}{7.62}\right)^{-0.107} = 0.835$

$$S_e = (0.801)(0.835)(0.5)(470.000) = 157.12\ \text{MPa}$$

Equation (7–4) $\quad \sigma_a' = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.49)412489.98(1000)}{\pi(41.275)^3} = 89149.95\ \text{kPa}$

Equation (7–5) $\quad \sigma_m' = \left[3\left(\frac{16K_{fs}T_m}{\pi d^3}\right)^2\right]^{1/2} = \frac{\sqrt{3}(16)(1.30)366069.07(1000)}{\pi(41.275)^3}$

$= 59585.40$ kPa

Draw the FBD and calculate the reaction forces.
Draw the torque, shear force and bending moment diagrams (BMD).
Draw the resultant BMD.
Identify the critically stressed locations (guess).
Start with the one you think is the most critical (I) in the present solution.
Estimate the fatigue stress concentration factors at that location.
Choose the material.
Find the corrected endurance limit.
Find the diameter using the Goodman criterion and the given FOS.

Using Goodman criterion

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{89149.95/157.12}{1000} + \frac{59585.40/470}{1000} = 0.696$$

$$n_f = 1.44$$

Note that we could have used Equation (7-7) directly.

Check yielding.

$$n_y = \frac{S_y}{\sigma'_{\max}} > \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{392997.900}{89149.9464 + 59585.4043} = 2.64$$

Also check this diameter at the end of the keyway, just to the right of point I, and at the groove at point K. From moment diagram, estimate M at end of keyway to be $M = 423675\ \mathrm{N \cdot mm}$.

Assume the radius at the bottom of the keyway will be the standard $r/d = 0.02$, $r = 0.02d = 0.02(41.275) = 0.8255$ mm.

$$K_t = 2.14 \text{ (Table 7–1)}, \ q = 0.65 \text{ (Figure 6–26)}$$

$$K_f = 1 + 0.65(2.14 - 1) = 1.74$$

$$K_{ts} = 3.0 \text{ (Table 7–1)}, \ q_s = 0.71 \text{ (Figure 6–27)}$$

$$K_{fs} = 1 + 0.71(3 - 1) = 2.42$$

$$\sigma'_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.74)(423675\ \mathrm{N \cdot mm})}{\pi(41.275)^3} = 106849.00\ \mathrm{kPa}$$

$$\sigma'_m = \sqrt{3}(16)\frac{K_{fs}T_m}{\pi d^3} = \frac{\sqrt{3}(16)(2.42)(366069.07)}{\pi(41.275)^3} = 111134.24\ \mathrm{kPa}$$

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{106849.00}{157(1000)} + \frac{111134.2416}{470(1000)} = 0.919$$

$$n_f = 1.09$$

The keyway turns out to be more critical than the shoulder. We can either increase the diameter or use a higher strength material. Unless the deflection analysis shows a need for larger diameters, let us choose to increase the strength. We started with a very low strength and can afford to increase it some to avoid larger sizes. Try 1050 CD with $S_{ut} = 690$ MPa.

Recalculate factors affected by S_{ut}, i.e., $k_a \rightarrow S_e$; $q \rightarrow K_f \rightarrow \sigma'_a$

$$k_a = 3.04(690)^{-0.217} = 0.736, \quad S_e = 0.736(0.835)(0.5)(690) = 211.91\ \mathrm{MPa}$$

$$q = 0.72, \ K_f = 1 + 0.72(2.14 - 1) = 1.82$$

$$\sigma'_a = \frac{32(1.82)(423675)}{\pi(41.275)^3} = 111746\ \mathrm{kPa}$$

$$\frac{1}{n_f} = \frac{111746}{211.91(1000)} + \frac{111134.2416}{690(1000)} = 0.689$$

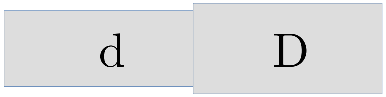
$$n_f = 1.45$$

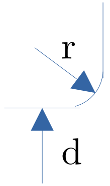
This is slightly under the goal for the design factor to be 1.5. If we round to 2 significant figures, which is actually more realistic for fatigue, then we get 1.5. If the situation calls for a more conservative decision, a higher strength material can be selected.

Check at the groove at K, since K_t for flat-bottomed grooves are often very high. From the torque diagram, note that no torque is present at the groove. From the moment diagram, $M_a = 270926\ \mathrm{N \cdot mm}$,

Check for static failure at the chosen location. If there is adequate FOS against static failure, find the exact local dimensions at the location

Typical D/d = 1.2, d/r = 10.





Repeat the strength calculations at the location using the exact dimensions.

If the FOS is acceptable, proceed to the next critically stressed location

Keyway to the right of location I:.

Radius at the bottom of the keyway is typically r/d' = 0.02.

Calculate r and use it to find the fatigue stress concentration factors.

Then calculate the FOS. In the present case, the FOS is not sufficient at the base of the keyway – can change either change D or the material.

For strength calculations, one can go with a better material if one has started with the most basis materials in the first place. Once this location is made safe, proceed to to the next critical locations.

Continue till all the selected critical locations have an adequate FOS.

$M_m = T_a = T_m = 0$. To quickly check if this location is potentially critical, just use $K_f = K_t = 5.0$ as an estimate, from Table 7-1.

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(5)(270926(1000))}{\pi(41.275)^3} = 196227 \text{ kPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{211.91(1000)}{196227} = 1.08$$

This is low. We will look up data for a specific retaining ring to obtain K_f more accurately. With a quick online search of a retaining ring specification using the website www.globalspec.com, appropriate groove specifications for a retaining ring for a shaft diameter of 41.275 mm are obtained as follows: width, $a = 1.727$ mm; depth, $t = 1.219$ mm; and corner radius at bottom of groove, $r = 0.254$ mm. From Figure A-15-16, with $r/t = 0.254/1.219 = 0.208$, and $a/t = 1.727/1.219 = 1.42$

$$K_t = 4.3, \; q = 0.65 \text{ (Figure 6–26)}$$

$$K_f = 1 + 0.65(4.3 - 1) = 3.15$$

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(3.15)(270926)}{\pi(41.275)^3} = 123427 \text{ kPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{211(1000)}{123427} = 1.71$$

Quickly check if point M might be critical. Only bending is present, and the moment is small, but the diameter is small and the stress concentration is high for a sharp fillet required for a bearing. From the moment diagram, $M_a = 108347.82$ N · mm, and $M_m = T_m = T_a = 0$.

Estimate $K_t = 2.7$ from Table 7-1, $d = 25.4$ mm, and fillet radius r to fit a typical bearing.

$$r/d = 0.02, \; r = 0.02(25.4) = 0.5080$$

$$q = 0.7 \text{ (Figure 6–26)}$$

$$K_f = 1 + (0.7)(2.7 - 1) = 2.19$$

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(2.19)(108347.82)}{\pi(25.4)^3} = 147490.23 \text{ kPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{211.91(1000)}{147490.23} = 1.44$$

This location is more critical than perhaps anticipated. The estimate for stress concentration is likely on the high side, so we will choose to continue and recheck after a specific bearing is selected.

With the diameters specified for the critical locations, fill in trial values for the rest of the diameters, taking into account typical shoulder heights for bearing and gear support.

$$D_1 = D_7 = 25.4 \text{ mm}$$

$$D_2 = D_6 = 35.56 \text{ mm}$$

$$D_3 = D_5 = 41.275 \text{ mm}$$

$$D_4 = 50.8 \text{ mm}$$

The bending moments are much less on the left end of the shaft, so D_1, D_2, and D_3 could be smaller. However, unless weight is an issue, there is little advantage to requiring more material removal. Also, the extra rigidity may be needed to keep deflections small.