# IE-616

# Decision Analysis and Game Theory

Ganesh Iyer

Policy: Q1 – 15%, Q2 – 15%, MS – 30%, ES – 40%

References: Narahari – Game Theory and Mechanism design
J Webb

$\max_x f(x)$ Assume $f$ is smooth

$x \in \mathcal{D}$ . $x$ can be continuous or discrete
There can also be constraints.
There may even be multiple objectives.

There can be multiple "players", each trying to optimize their respective objectives.
E.g. $f_x(x,y)$, $f_y(x,y)$.
This problem is called "Non-cooperative game".
Goal is to find some Solution/outcome.
Games may require co-operation to achieve certain objectives, such games are called "Co-operative games".

## Bandwidth Sharing Game: $X, Y$

Let $x, y$ be voice levels of two people, trying to speak.

$x, y \in [0,1]$

$f_X(x,y) = x(1-y)$ Interference from $Y$

$f_Y(x,y) = y(1-x)$ Interference from $X$

Say if Y speaks at some level $y$, $x^* = 1$ optimal. Similarly, $y^* = 1$. But if both were 1, both $f^n$s give 0. This is called "Prize of anarchy", where everyone just focuses on maximizing their own rewards.

If $\max_{x,y} \; x(1-y) + y(1-x)$ was carried out,

$$x^* = y^* = \frac{1}{2}$$

But the players may not agree, and selfishly try to maximise their own objectives. There is no "stability" in this solution.

N agent game :

↳ $\theta_1 \; \theta_2 \; --- \; \theta_n$ be values assigned by each player to an item in an auction

↳ $b_1 \; b_2 \; --- \; b_n$ be bids by playes

$f_i = \theta_i - b_i$ is utility for each player

For the auctioneer, $f_a = \max_i b_i$

Duopoly :

In a monopoly, e.g. if $p$ is price, $x(p)$ is demand, goal is to $\max_p x(p) \cdot p$

$$x_1(p_1, p_2) = \begin{cases} 0 & p_2 < p_1 \\ x(p_1) & p_1 < p_2 \\ \frac{x(p_1)}{2} & p_1 = p_2 \end{cases}$$ If two manufacturers

$\max\limits_{p_1} \left(p_1 \cdot x_1(p_1, p_2)\right)$ is goal of first manufacturer.

**Election**: Say 5 parties, won respective votes:

9, 5, 6, 7, 18 → 45

Say ≥ 30 majority seats needed to form a government. Both co-operation and competition involved. Utility can be fraction of seats you have in the government.

**Auction**: Assume auction for indivisible good. N people auctioning, $v_i$ – Value assigned by $i$
(Closed auction) $b_i$ – Bid by $i$

Winner – 1) Max bid as winner, $b^* = \max\limits_i b_i$
Has to pay $b^*$

2) Max bid as winner, but has to pay
$b^* = \text{second} \max\limits_i b_i$

$2^{nd}$ game such that players forced to show true value

# Non-cooperative games (Strategic Form games)

$N$ = Set of players

$S = \{ S_i, i \in N \}$

$S_i$ → Action set of player $i$

↓ Set of actions

$U = \{ U_i, i \in N \}$

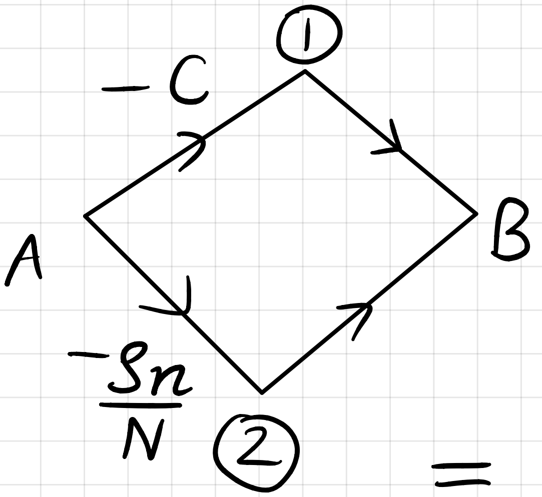
$U_i$ ← Utility of $i$

$U_i(a_i, a_{-i}) \ \forall a_i \in S_i, \forall a_{-i} \in S_{-i}$

$a_i$: Action of player $i$

$a_{-i}$: Action of all players apart from $i$

$S_{-i} = S_1 \times S_2 \times \dots S_{i-1} \times S_{i+1} \dots \times S_N$




$N = \{1, 2, \dots, N\}$

$S_1 = \{1, 2, 12\} = S_2 = \dots = S_N$

$$U_1(a_1, a_{-1}) = \begin{cases} -C & \text{if } a_1 = 1, a_{-1} \in S_{-1} \\ \dfrac{-S(n_2(a_{-1})+1)}{N} & \text{if } a_1 = 2, a_{-1} \in S_{-1} \end{cases}$$

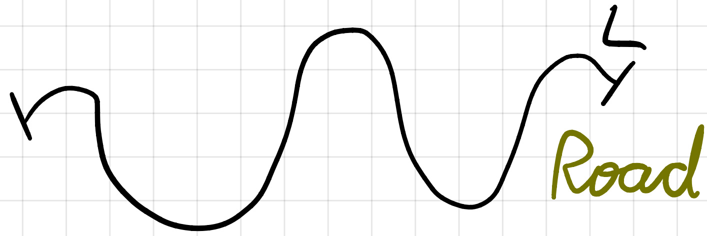
where $n_2(a_{-1})$ = # people choosing 2

For Duopoly,

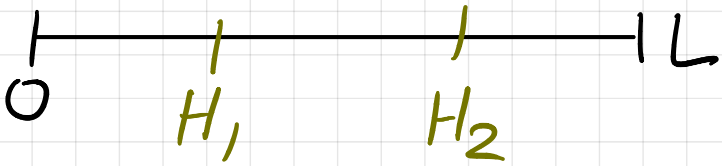
$N = \{M_1, M_2\}$, $S_1 = \mathbb{R}^+ = S_2$

$U_i(a_i, a_{-i}) = a_i x_i(a_i, a_{-i})$

## Hotelling game:



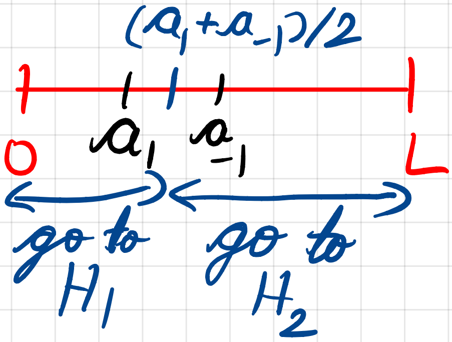

Assume people are uniformly distr. on the street.




Assume people go only to the closest hotel.

$N = \{H_1, H_2\}$ $\quad S_1 = \{0, 1, 2, \ldots, L\} = S_2$

Discrete locations




$$U_1(a_1, a_{-1}) = \begin{cases} (a_1 + a_{-1})/2 & a_1 < a_{-1} \\ 0 & a_1 = a_{-1} \\ L - (a_1 + a_{-1})/2 & a_1 > a_{-1} \end{cases}$$

Best Response (BR) against $L/2$ is $(L/2+1)$ or $(L/2-1)$.

$\Rightarrow$ order of decisions impact the game

$\rightarrow$ BR $\rightarrow$ $BR_i(a_{-i}) = \underset{a_i}{\arg\max}\, U_i(a_i, a_{-i})$

We do not consider order of decisions, all players take decisions at the same time, i.e "Simultaneous Move Games".

Since we do not know choice of other players, anticipate and choose best response.

E.g. In above game, $(a_1^*, a_2^*)$ are sol$^n$s of the game, if $a_1^* \in BR_1(a_2^*)$, $a_2^* \in BR_2(a_1^*)$

- Common knowledge
- Rational $\longleftrightarrow$ Preferences $\longleftrightarrow$ outcome $:(a_1, a_2, ---, a_N)$
- Intelligent
  - $\hookrightarrow$ knowledge about game
  - $\hookrightarrow$ computational power available

## Social Job :

$N = \{1, 2, 3, ---, N\}$

$S_i = \{P, NP\} = \{1, 0\} \;\forall i$ — clean - participate / not participate

$a_i \in \{0, 1\}$

Every player that participates : $(-v)$

Every player gets $g(n_p)$ irrespective of whether they participate or not,

here $n_p(a_{-i}) = \sum_{j \neq i} \mathbb{1}_{\{a_j = 1\}} = \sum_{j \neq i} a_j$

$$U_i(a_i, a_{-i}) = -a_i v + g(n_p(a_{-i}) + a_i)$$

$g(n_p)$ is some arbitrary function.

Rational + Intelligent $\Rightarrow$ Every player can find $B_i(s_{-i})\ \forall s_{-i}$

**Strong Domination:** A strategy $s' \in S_i$ is said to be strongly dominated by strategy $s \in S_i$ for player $i$ if

$$U_i(s', s_{-i}) < U_i(s, s_{-i}) \quad \forall\, s_{-i} \in S_{-i}$$

for all opponents strategy profiles

$S$ is strictly better than $S'$ irrespective of opponent's play

**Weak Domination:** A strategy $s' \in S_i$ is said to be weakly dominated by strategy $s \in S_i$ for player $i$ if

$$U_i(s', s_{-i}) \leq U_i(s, s_{-i}) \quad \forall\, s_{-i} \in S_{-i}$$

and $\exists$ at least one strategy profile $s_{-i} \in S_{-i}$ of opponent for which above inequality is strict.

**Very weak Domination:** A strategy $s' \in S_i$ is said to be weakly dominated by strategy $s \in S_i$ for player $i$ if

$$U_i(s', s_{-i}) \leq U_i(s, s_{-i}) \quad \forall\, s_{-i} \in S_{-i}$$

**Strongly Dominant Strategy:** $s_i^*$ is strongly dominant if it strongly dominates all

$$s_i \neq s_i^*,\ \forall s_i \in S_i$$

**Weakly Dominant Strategy**: $s_i^*$ is weakly dominant if it strongly dominates all

$$s_i \neq s_i^*, \forall s_i \in S_i$$

E.g. $S_i = \{1, 2, 3\}$

weakly dominant

Will never lose if we play 2. Also, There exists at least one opponent strategy $s_{-i}$ for which $U_i(2, s_{-i}) > U_i(1, s_{-i})$.
Also another $s'_{-i}$ atleast (may not be same as with 1), s.t. $U_i(2, s'_{-i}) > U_i(3, s'_{-i})$

**Very Weakly Dominant Strategy**: $s_i^*$ is very weakly dominant if it strongly dominates all

$$s_i \neq s_i^*, \forall s_i \in S_i$$

**Strongest Dominant Strategy Eqbm**:

If $(s_1^*, s_2^*, ---, s_N^*)$ exist where each strategy $s_i^*$ is strongly dominant for player $i$, then it is the eqbm $\text{sol}^n$ of the game.

There cannot be multiple strongly or weakly dominant strategies, but there can be multiple very weakly dominant strategies.

## Prisoner's Dilemma Game: $N = \{1, 2\}$

$S_1 = S_2 = \{\text{confess}(C), \text{not confess}(NC)\}$

Two prisoners commited a crime, separately interrogated.

| 1 \ 2 | NC | C |
|---|---|---|
| NC | −2, −2 | −10, −1 |
| C | −1, −10 | −5, −5 |

Penalty indicates no. of yrs of imprisonment

$U_i(s_i, s_{-i})$, Eg. $U_i(NC, C) = -10$

$i=1$, $B_i(NC) = \{C\}$, $B_i(C) = \{C\}$

$\Rightarrow$ No matter opponent's strategy, C is strictly better $\Rightarrow$ Strictly dominant.

Symmetric for 2 $\Rightarrow (C, C)$ is S.D.S.E sol$^n$.

Due to prize of anarchy. (NC, NC) could have been better, but there is no cooperation, hence (NC, NC) is not stable.

## Bandwidth game: $N = \{1, 2\}$, $s_i \in S_i = [0, 1]$

$$U_i(s_i, s_{-i}) = s_i(1 - s_{-i})$$

$$B_i(s_{-i}) = \underset{s_i}{\operatorname{argmax}}\ s_i(1 - s_{-i}) = \begin{cases} \{1\} & s_{-i} < 1 \\ [0, 1] & s_{-i} = 1 \end{cases}$$

$\Rightarrow s_i = 1$ is weakly D.S.

Symmetric for $i=2 \Rightarrow (1,1)$ is W.D.S.E.
Again, Prize of Anarchy.

Tragedy of Commons: N people

$S = \{1,0\}$, to sit or not.

$$U_i(s_i, s_{-i}) = s_i - \frac{5}{N}\sum_{j=1}^{N} s_j$$

$$= s_i\left(1-\frac{5}{N}\right) - \frac{5}{N}\underbrace{\sum_{j\neq i}^{N} s_j}_{s_{-i}}$$

$$B_i(s_{-i}) = \begin{cases} \{1\} & N>5 \quad \forall s_{-i} \\ \{0\} & N<5 \quad \forall s_{-i} \\ \{0,1\} & N=5 \quad \forall s_{-i} \end{cases}$$

S.D.S (for the first two cases); Very weakly D.S (for $N=5$)

Secondary Bid Auction: $S_i = [0,\infty)$

$v_1, v_2, ---, v_N \rightarrow$ value of object

$$U_i(b_i, b_{-i}) = (v_i - \max_{j\neq i} b_j)\,\mathbb{1}(b_i = \max_j b_j)$$

Assuming no ties

$\max_{j\neq i} b_j$ ↓ If $i$ is highest, this represents second highest

Say $b_{-i}$ is s.p of opponent.

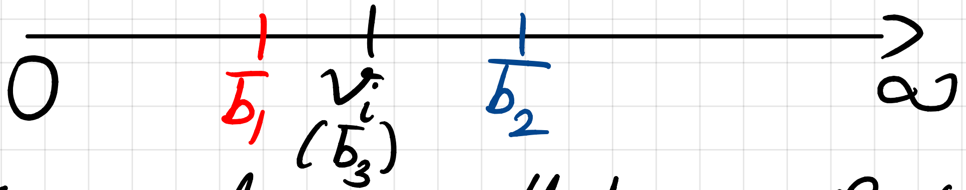
$$\begin{array}{ll} & B_i(b_{-i}) = \begin{cases} (\bar{b}_1, \infty) & v_i > \max\limits_{j \neq i} b_j = \bar{b}_1 \\ [0, \bar{b}_2] & v_i < \max\limits_{j \neq i} b_j = \bar{b}_2 \\ [0, \infty) & v_i = \max\limits_{j \neq i} b_j = \bar{b}_3 \end{cases} \end{array}$$

$u_i = v_i - \bar{b}_1$

$u_i = 0$

can be ] if we say 0 to all if tie

$u_i = 0$




We can clearly see that $v_i \in B_i(b_{-i})$ no matter what $b_{-i}$ is. Clearly, $v_i$ will not lose to $s_i$ irrespec. of $b_{-i}$. If we choose any action $[0, \bar{b}_1)$, it is strictly worser than if we choose $v_i$ $\exists\, \bar{b}_1 < v_i$. Taking limit of $\bar{b}_1 \to v_i$, all actions $[0, v_i)$ strictly worser than $v_i$ $\exists$ one s.p. Similarly, all actions $(\bar{b}_2, \infty)$ worser than $v_i$ $\exists\, \bar{b}_2 > v_i$, taking limit $\bar{b}_2 \to v_i$, All actions $(v_i, \infty)$ worser than $v_i$. In case 3, all actions equivalent. $\Rightarrow$ Weakly dominant strategy is $\{v_i\}$.

W.D.S.E is $\{v_1, v_2, \ldots, v_N\}$. Every player forced to bid their values.

Best response may not exist if utility $f^n$ is unbounded, or the strategy set not compact.

Show that : BR sets of all players have a common element, which is the Nash Equilibrium $(s_1^*, s_2^*, s_3^*, \ldots, s_N^*) \ni \forall i \; s_i^* \in B_i(s_{-i}^*)$ (s.t.)

Equilibrium : Stable against unilateral deviations.

Prove all dominant E are N.E.

| $P_1 \downarrow$ / $P_2 \rightarrow$ | A | B |
|---|---|---|
| A | 1,−1 | −1,1 |
| B | −1,1 | 1,−1 |

$B_1(A) = \{A\}$, $B_1(B) = \{B\}$

$B_2(A) = \{B\}$, $B_2(B) = \{A\}$

No N.E exists (Pure strategy)

## Co-ordination game :

2 players, can study or play

| $P_1 \downarrow$ / $P_2 \rightarrow$ | St | Pl |
|---|---|---|
| St | (2,1) | (0,0) |
| Pl | (0,0) | (1,2) |

$B_1(St) = \{St\}$, $B_1(Pl) = \{Pl\}$

$B_2(St) = \{St\}$, $B_2(Pl) = \{Pl\}$

$\Rightarrow (St, St), (Pl, Pl)$ are N.E.

## Companies :

| | A | B |
|---|---|---|
| A | (2,1) | (0,0) |
| B | (0,0) | (1,2) |

A, B two products. If diff, confusion, 0 for both. But if they promote same product, get reward. But, one company better, higher reward for one product, similar to prev. example.

**Duopoly**: $x(p) \to$ Demand, $p \to$ Price

$x(p)$ decreases with $p$. In monopoly, $\sup_p (p-\underset{\downarrow \text{ prod. cost}}{c}) \cdot x(p)$

In duopoly, competition exists.

Ex: $x(p) = 1/p$

$\max_p \; 1 - \frac{c}{p} \Rightarrow p^* \to \infty$

$$x_1(p_1, p_2) = \begin{cases} 0 & p_1 > p_2 \\ x(p_1) & p_1 < p_2 \\ \frac{x(p_1)}{2} & p_1 = p_2 \end{cases} \qquad x(\cdot) > 0$$

$$U_i(p_i, p_{-i}) = (p_i - c) \cdot x_i(p_i, p_{-i})$$

$$B_1(p_2) = \operatorname{argmax}_{p_1} (p_1 - c) \cdot x_1(p_1, p_2)$$

Case 1: $p_2 < c$

$$\operatorname{argmax}_{p_1} \begin{cases} (\underbrace{p_1 - c}_{-ve}) x(p_1) & p_1 < p_2 \\ 0 & p_1 > p_2 \\ \underbrace{(p_2 - c)}_{-ve} \frac{x(p_2)}{2} & p_1 = p_2 \end{cases}$$

$\Rightarrow p_1 = (p_2, \infty)$

Case 2: $p_2 = c$

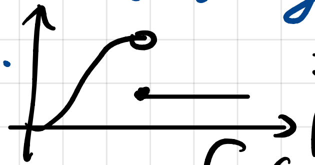
$$\operatorname{argmax}_{p_1} \begin{cases} (\underbrace{p_1 - c}_{-ve}) x(p_1) & p_1 < p_2 = c \\ 0 & p_1 > p_2 \\ 0 & p_1 = p_2 = c \end{cases}$$

$\Rightarrow p_1 = [\underset{\overset{\shortparallel}{c}}{p_2}, \infty)$

Case 3: $p_2 > c$

$$\underset{p_1}{\operatorname{argmax}} \begin{cases} (p_1 - c)\,x(p_1) & p_1 < p_2 \\ 0 & p_1 > p_2 \\ (p_2 - c)\,\dfrac{x(p_2)}{2} & p_1 = p_2 \end{cases}$$

If this $f^n$ increases, no max, only sup exists.



For a given $p_2$ No BR in this case

$$\Rightarrow B_1(p_2) = \begin{cases} (p_2, \infty) & \text{if } p_2 < c \\ [c, \infty) & \text{if } p_2 = c \\ \text{BR may not exist} & \text{if } p_2 > c \end{cases}$$

$$B_2(p_1) = \begin{cases} (p_1, \infty) & \text{if } p_1 < c \\ [c, \infty) & \text{if } p_1 = c \\ \text{BR may not exist} & \text{if } p_1 > c \end{cases}$$

Clearly, at least one N.E exists, i.e $(c, c)$.

## Bandwidth (BW) sharing game:

$N$ players $\quad S_i = [0, 1/N]$

$$U_i(s_i, s_{-i}) = s_i\Big(1 - \sum_{j=1}^{N} s_j\Big) \quad \text{Power transmitted}$$

$j = 1$, $s_{-1} = (s_2, s_3, \dots, s_N)$

$B_1(s_{-1})$: $U_1(s_1, s_{-1}) = s_1(1 - s_1 - t_1)$

where $t_i = \sum_{j \neq i}^{N} s_j$

For sup, diff: $1 - 2s_1 - t_1 = 0$

$\Rightarrow s_1^* = (1 - t_1)/2$

diff: $-2 < 0 \Rightarrow$ max

$\Rightarrow B_1(s_{-1}) = \left\{ \frac{1 - t_1}{2} \right\} \Rightarrow B_i(s_{-i}) = \left\{ \frac{1 - \sum_{j \neq i} s_j}{2} \right\}$

$s_1^*, s_2^*, \ldots, s_N^*$ is N·E if

$s_i^* = (1 - \sum_{j \neq i} s_j^*)/2 \quad \forall i$ as BR set has unique element. (!)

$2s_i^* = 1 - \sum_{j \neq i} s_j^*$

$\Rightarrow s_i^* = 1 - \sum_{j=1}^{N} s_j^* \quad \forall i \Rightarrow$ Symmetric N·E if it exists

Assume $s^*$.

$\Rightarrow s^* = 1 - Ns^* \Rightarrow s^* = \frac{1}{(N+1)}$

$! \text{NE} \Rightarrow \left( \frac{1}{N+1}, \frac{1}{N+1}, \ldots, \frac{1}{N+1} \right) \Rightarrow \sum U_i(s_i^*, s_{-i}^*) = \frac{1}{(N+1)^2}$

Purposes of N·E:

1) Prescription ✓ iff N·E

2) Prediction → elimination → outcome

$P_1 \downarrow$ $P_2 \rightarrow$

| | L | M | R |
|---|---|---|---|
| U | 1,3 | 2,2 | 0,1 |
| D | 0,1 | 1,3 | 1,2 |

M strongly dominates R

$\Rightarrow$ R can be eliminated

| | L | M | ~~R~~ |
|---|---|---|---|
| U | 1,3 | 2,2 | ~~0,1~~ |
| D | 0,1 | 1,3 | ~~1,2~~ |

U dominates D

$\Rightarrow$ D can be eliminated

| | L | M | ~~R~~ |
|---|---|---|---|
| U | 1,3 | 2,2 | ~~0,1~~ |
| ~~D~~ | ~~0,1~~ | ~~1,3~~ | ~~1,2~~ |

M is dominated by L

$\Rightarrow$ M can be eliminated

| | L | ~~M~~ | ~~R~~ |
|---|---|---|---|
| U | 1,3 | ~~2,2~~ | ~~0,1~~ |
| ~~D~~ | ~~0,1~~ | ~~1,3~~ | ~~1,2~~ |

The choice left is the N.E. $U \in B_1(L)$, $L \in B_2(U)$

If prediction possible, gives unique N.E.

3) Self-Enforcement (No external force needed to come to an agreement)

All of this is under the assumption that all players are intelligent and rational. But, if some players not rational, N.E may not be achieved. Need robustness in analysis.

Say player $i$ plays $s_i$.

$\Rightarrow$ Worst that player $i$ gets $= \min_{s_{-i}} U_i(s_i, s_{-i})$

$\Rightarrow$ Player $i$ tries to maximise the worst he can get, i.e $\underline{v}_i = \max_{s_i} \left( \min_{s_{-i}} U_i(s_i, s_{-i}) \right)$

$\underline{v}_i$ is called the security level, and $\underline{s}_i^*$ is the safety strategy (maxmin strategy) that gets him $\underline{v}_i$.

$\underline{v}_i$ is the value that player $i$ is guaranteed to get irrespective of what the opponent does.

## Min Max Value and Strategy:

Opponent wants to hurt player $i$ the most. They know that $i$ is rational

$\Rightarrow$ $i$ tries to do $\max_{s_i} U_i(s_i, s_{-i})$

Opponents try to minimize the max. value $i$ hopes to get.

$$\overline{v}_i = \min_{s_{-i}} \left( \max_{s_i} U_i(s_i, s_{-i}) \right)$$

and $\overline{s}_{-i}^*$ is the opp. strategy for this.

If every player wants to protect themselves, $(\underline{s}_1^*, \underline{s}_2^*, ---, \underline{s}_N^*)$ is the outcome, robust.

Say $(s_1^*, s_2^*, ---, s_N^*)$ is the N.E.

Thm: $U_i(s_i^*, s_{-i}^*) \geq \bar{v}_i^* \geq \underline{v}_i^* \; \forall i$

Proof: 1) $\bar{v}_i^* \geq \underline{v}_i^*$

By def$^n$, 
$$\bar{v}_i^* = \max_{s_i} U_i(s_i, \bar{s}_{-i}^*)$$
$$\geq U_i(s_i, \bar{s}_{-i}^*) \; \forall s_i \in S_i$$

$$\Rightarrow \bar{v}_i^* \geq \min_{s_{-i}} U_i(s_i, s_{-i}) \; \forall s_i \in S_i$$

$$\Rightarrow \bar{v}_i^* \geq \max_{s_i}\left(\min_{s_{-i}} U_i(s_i, s_{-i})\right) = \underline{v}_i^*$$

$$\Rightarrow \bar{v}_i^* \geq \underline{v}_i^*$$

2) $U_i(s_i^*, s_{-i}^*) \geq \bar{v}_i^*$

By def$^n$ of N.E.,
$$U_i(s_i^*, s_{-i}^*) = \max_{s_i} U_i(s_i, s_{-i}^*)$$
$$\geq \min_{s_{-i}}\left(\max_{s_i} U_i(s_i, s_{-i})\right)$$
$$= \bar{v}_i^*$$

$$\Rightarrow U_i(s_i^*, s_{-i}^*) \geq \bar{v}_i^*$$

In BW sharing game, if they co-ordinate

**Social Objective:**

$$U_{so}(\underline{s}) \triangleq \sum_{i=1}^{N} U_i(s_i, s_{-i}) \;,\; \underline{s} = (s_1, \ldots, s_N)$$

$$\max_{\underline{s} \in [0, \frac{1}{N}]^N} U_{so}(\underline{s})$$ optimize $\bar{s}$

At optim, $s_1^* = s_2^* = \ldots = s_N^* = \frac{1}{2N}$

$$\Rightarrow U_{so}(\underline{s}^*) = \frac{1}{4N}$$

Ratio of social optimal utility to N.E. utility:

$$\frac{1/4N}{\frac{1}{(N+1)^2}} = \frac{(N+1)^2}{4N} = \left(1 + \frac{1}{N}\right)\left(\frac{N+1}{4}\right)$$

Costly loss if large N.

Difference between the two utilities is Price of Anarchy:

$$PoA = \frac{\sum_i U_{i,so} - \sum U_{N.E}^{\text{worst}}}{\sum_i U_{i,so}}$$

**Ex. Find max-min and min-max for this game:**

| $P_1 \downarrow$ $P_2 \rightarrow$ | $C_1$ (W.U = 1) | $C_2$ (W.U = 2) | $C_3$ (W.U = 3) for $P_2$ | for $P_1$ |
|---|---|---|---|---|
| $R_1$ | 3, 2 | 1, 4 | 5, 3 | W.U = 1 |
| $R_2$ | 4, 3 | 2, 2 | 3, 5 | W.U = 2 |
| $R_3$ | 2, 1 | 3, 5 | 4, 4 | W.U = 2 |

Max-Min $P_2$ = $\{C_3\}$

$P_1$ Max-min $= \{R_2, R_3\}$

$\Rightarrow$ Max - Min : 1) $(R_2, C_3)$ 2) $(R_3, C_3)$

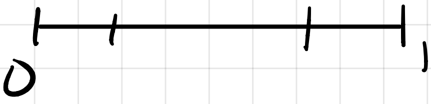
Value: $\underline{v}_1 = 2$, $\underline{v}_2 = 3$

N.E: $BR_1(C_3) = \{R_1\}$, $BR_2(R_1) = \{C_2\}$

$BR_1(C_2) = \{R_3\}$, $BR_2(R_3) = \{C_3\}$

$\Rightarrow (R_3, C_2)$ is N.E

## Hotelling game revisited :



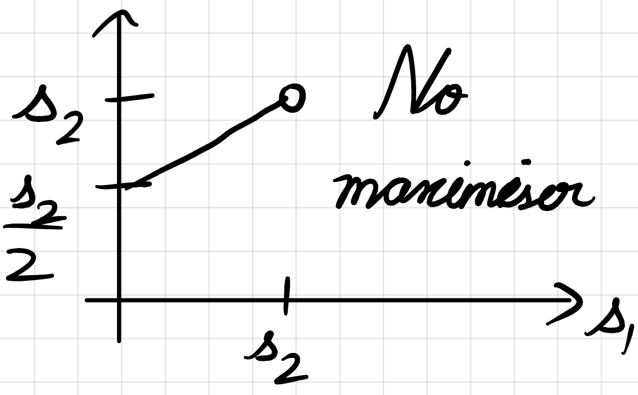

$S_1 = S_2 = [0, 1]$

$$U_1(s_1, s_2) = \begin{cases} (s_1 + s_2)/2 & s_1 < s_2 \\ 1/2 & s_1 = s_2 \\ 1 - (s_1 + s_2)/2 & s_1 > s_2 \end{cases}$$

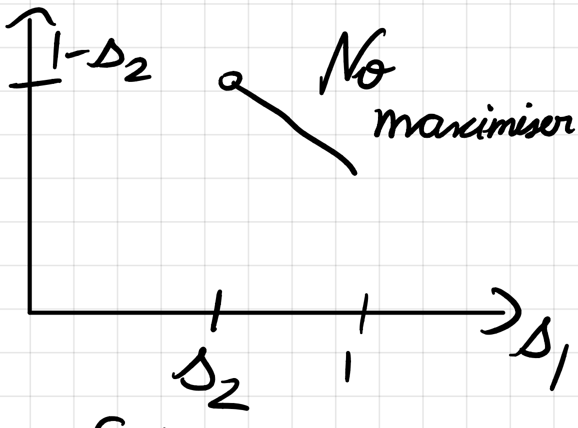
$$B_1(s_2) = \arg\max_{s_1} U_1(s_1, s_2)$$

$$= \max \Big\{ \underbrace{\tfrac{1}{2}}_{s_1 = s_2}, \underbrace{\sup_{s_1 < s_2} (s_1 + s_2)/2}_{\text{II}}, \underbrace{\sup_{s_1 > s_2} 1 - (s_1 + s_2)/2}_{\text{III}} \Big\}$$

II :




III :




$$B_1(s_2) = \max\{1/2, s_2, 1 - s_2\} = \begin{cases} 1 - s_2 & s_2 < 1/2 \\ 1/2 & s_2 = 1/2 \\ s_2 & s_2 > 1/2 \end{cases}$$

No B.R possible other than $s_2 = 1/2$ because sup.

$B_1(s_2 = 1/2) = \{1/2\}$, $B_2(s_1 = 1/2) = \{1/2\}$

$\Rightarrow (1/2, 1/2)$ is N.E

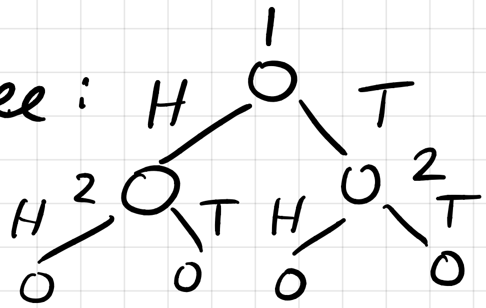
## Extended Form Game :

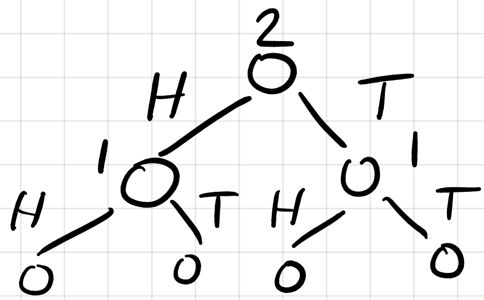
→ Players choose actions one after the other

→ can observe some information and store history

→ Termination

Game Tree :




$\langle N, S, u \rangle$ + Set of Histories $\mathcal{H} = \{\{HH\}, \{HT\}, \{TH\}, \{TT\}\}$

Sub-Histories : $\{\{H\}, \{T\}, \{\varepsilon\}\}$

→ one comp → (I) , — second comp → (O)

↳ enter → in

↳ or not → out

$(I) \begin{cases} \to b \\ \to B \end{cases}$