Bartlett Test

Note that for calculating the confidence intervals, we assumed that the true variance σ^2 is the same for all observations and that the observations are independent.

- How can we check if this assumption is valid? **Bartlett Test**

$$H_0 : \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_m^2 = \sigma_y^2$$

$$H_1 : \text{at least one } \sigma_i^2 \neq \sigma_j^2, i \neq j$$

$$\chi^2_{cal} = \frac{M}{C}$$

where

$$M = (N-m)\ln(s_p^2) - \sum_{i=1}^{m}(n_i - 1)\ln(s_i^2)$$

$$C = 1 + \frac{1}{3(m-1)}\left[\left(\sum_{i=1}^{m}\frac{1}{n_i - 1}\right) - \frac{1}{N-m}\right]$$

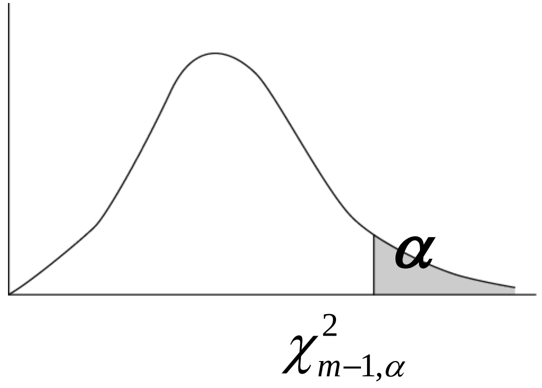
$m = 2^k$ *(Total experiments)*

Sample size = n $N = n_1 + n_2 + \ldots + n_m$

The value of M will be large if the sample variances s_i^2 differ greatly in magnitude, and will be zero if all the sample variances are exactly equal.

We will reject H_0 if χ^2_{cal} is too large, i.e.,

$$\chi^2_{\text{cal}} > \chi^2_{m-1,\alpha}$$



Bartlett Test

Example: Bartlett Test

$\chi^2_{v=m-1} = \frac{M}{C}$ where, $M = (N-m)\ln(s_p^2) - \sum_{i=1}^{m}(n_i-1)\ln(s_i^2)$, *and* $C = 1 + \frac{1}{3(m-1)}\left[\left(\sum_{i=1}^{m}\frac{1}{n_i-1}\right) - \frac{1}{N-m}\right]$

Here, N = m =

The value of M will be large if the sample variances s_i^2 differ greatly in magnitude, and will be zero if all the sample variances are exactly equal.

$S_1^2 = 24.5$, $S_2^2=21.78$, $S_3^2=134.48$, $S_4^2=242.0$,

$S_5^2=3.92$, $S_6^2=8.82$, $S_7^2=33.62$, $S_8^2=72.00$

We will reject H_0 if χ^2_{cal} is too large, i.e., $\chi^2_{cal} > \chi^2_{m-1,\alpha}$

$$S_p^{\,2} = \frac{[(y_{a1}-\bar{y}_1)^2+(y_{b1}-\bar{y}_1)^2+\text{..............}+(y_{a8}-\bar{y}_8)^2+(y_{b8}-\bar{y}_8)^2]}{(2-1)+\text{..........}+(2-1)} = 67.64$$

$$M = (16-8)\ln 67.64 - [(2-1)\ln 24.5 + (2-1)\ln 21.78 + (2-1)\ln 134.48 + (2-1)\ln 242 + (2-1)\ln 3.92 + (2-1)\ln 8.82 + (2-1)33.62 + (2-1)\ln 72.0] = 5.713$$

$$C = 1 + \frac{1}{3(8-1)}\left[\left(\sum_{i=1}^{8}\frac{1}{2-1}\right) - \frac{1}{16-8}\right] = 1 + \frac{1}{21}[8-5] = 1.357$$

$$\chi^2_{cal} = \frac{5.713}{1.357} = 4.21 \qquad \chi^2_{7,\alpha=0.05} = 14.1$$

Test #	X1	X2	X3	Y_{ai} (kpsi)	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	84	91	87.5
2	1	-1	-1	90.6	84	87.3
3	-1	1	-1	69.6	86	77.8
4	1	1	-1	76	98	87
5	-1	-1	1	77.7	80.5	79.1
6	1	-1	1	99.7	95.5	97.6
7	-1	1	1	82.7	74.5	78.6
8	1	1	1	93.7	81.7	87.7

Trick: Effects Calculation Matrix

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Calculation Matrix

	Main Effects			Interactions				
Test	X_1	X_2	X_3	X_1X_2	X_1X_3	X_2X_3	$X_1X_2X_3$	$\bar{y}$
1	-1	-1	-1	1	1	1	-1	87.5
2	1	-1	-1	-1	-1	1	1	87.3
3	-1	1	-1	-1	1	-1	1	77.8
4	1	1	-1	1	-1	-1	-1	87
5	-1	-1	1	1	-1	-1	1	79.1
6	1	-1	1	-1	1	-1	-1	97.6
7	-1	1	1	-1	-1	1	-1	78.6
8	1	1	1	1	1	1	1	87.7

Statistical Significance

In general, for 2-factor design, we could have 'a' levels of factor A, and 'b' levels of factor B.

Each combination is replicated 'n' times

General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	. . .	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

What is the effects model and hypothesis test?

		Factor B			
		1	2	. . .	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	$\vdots$				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

Effects Model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

In the two-factor factorial, both row and column factors (or treatments), A and B, are of al interest. Specifically, we are interested in **testing hypotheses** about the equality of row treatment effects, say

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$$
$$H_1: \text{at least one } \tau_i \neq 0 \tag{5.2a}$$

and the equality of column treatment effects, say

$$H_0: \beta_1 = \beta_2 = \cdots = \beta_b = 0$$
$$H_1: \text{at least one } \beta_j \neq 0 \tag{5.2b}$$

We are also interested in determining whether row and column treatments *interact*. Thus, we also wish to test

$$H_0: (\tau\beta)_{ij} = 0 \qquad \text{for all } i, j$$
$$H_1: \text{at least one } (\tau\beta)_{ij} \neq 0 \tag{5.2c}$$

We now discuss how these hypotheses are tested using a **two-factor analysis of variance**.

ANOVA for Two-Factor Factorial Design

The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

α

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	
A treatments	SS_A	$a-1$	$MS_A = \dfrac{SS_A}{a-1}$	$F_0 = \dfrac{MS_A}{MS_E}$	$F_{1-\alpha,\, a-1,\, ab(n-1)}$
B treatments	SS_B	$b-1$	$MS_B = \dfrac{SS_B}{b-1}$	$F_0 = \dfrac{MS_B}{MS_E}$	$F_{1-\alpha,\, b-1,\, ab(n-1)}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \dfrac{MS_{AB}}{MS_E}$	$F_{1-\alpha,\, (a-1)(b-1),\, ab(n-1)}$
Error	SS_E	$ab(n-1)$	$MS_E = \dfrac{SS_E}{ab(n-1)}$		
C Total	SS_T	$abn-1$			

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
	130	155		34	40		20	70		
1	74	180	539	80	75	229	82	58	230	998
	150	188		136	122		25	70		
2	159	126	623	106	115	479	58	45	198	1300
	138	110		174	120		96	104		
3	168	160	576	150	139	583	82	60	342	1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130	155	134.75	34	40	57.25	20	70	57.5	998
	74	180		80	75		82	58		
2	150	188	155.75	136	122	119.75	25	70	49.5	1300
	159	126		106	115		58	45		
3	138	110	144	174	120	145.75	96	104	85.5	1501
	168	160		150	139		82	60		
$y_{.j.}$	1738			1291			770			3799 = $y_{...}$

$N = abn$
$= 3 \times 3 \times 4$
$= 36$

36 terms

$$SS_{Total} = \sum_i \sum_j \sum_k y_{ijk}^2 = (130^2 + 155^2 + 74^2 + 180^2 + 34^2 + 40^2 + \cdots + 96^2 + 104^2 + 82^2 + 60^2)$$

$$= 478647$$

$$\text{Grand Mean} = \frac{\sum\sum\sum y_{ijk}}{36} = \frac{3799}{36} = \bar{\bar{y}}$$

$$= 105.53$$

$$SS_{mean} = N\bar{\bar{y}}^2 = 36\left(\frac{3799}{36}\right)^2 = 400900$$

3 terms

$$SS_{material} = \underset{b}{3} \times \underset{n}{4} \times \left[\left(\frac{998}{12} - 105.53\right)^2 + \left(\frac{1300}{12} - 105.53\right)^2 + \left(\frac{1501}{12} - 105.53\right)^2\right]$$

$$= 10683.12$$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130	155	134.75	34	40	57.25	20	70	57.5	998
	74	180		80	75		82	58		
2	150	188	155.75	136	122	119.75	25	70	49.5	1300
	159	126		106	115		58	45		
3	138	110	144	174	120	145.75	96	104	85.5	1501
	168	160		150	139		82	60		
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$

3 terms

$$SS_{temp} = \underset{a}{3} \times \underset{w}{4} \times \left[\left(\frac{1738}{12} - 105.53\right)^2 + \left(\frac{1291}{12} - 105.53\right)^2 + \left(\frac{770}{12} - 105.53\right)^2 \right]$$

$$= 39118.72$$

36 terms

For replicates,

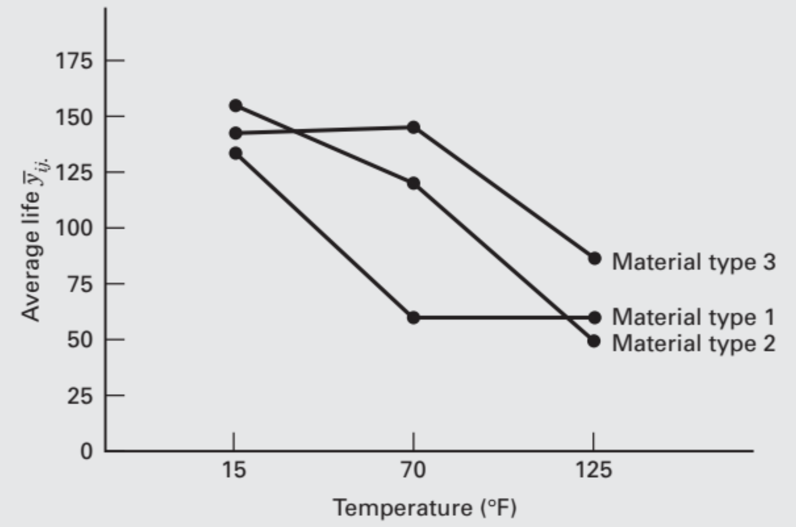
$$SS_E = \sum\sum (y_{ijk} - \bar{y})^2 = \left((130 - 134.75)^2 + (155 - 134.75)^2 + \ldots + (34 - 57.25)^2 + (40 - 57.25)^2 \right.$$

$$SS_{interaction} = [SS_T - SS_{mean}] - SS_{material} - SS_{temp} - SS_E$$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
	130	155	134.75	34	40	57.25	20	70	57.5	
1	74	180		80	75		82	58		998
	150	188	155.75	136	122	119.75	25	70	49.5	
2	159	126		106	115		58	45		1300
	138	110	144	174	120	145.75	96	104	85.5	
3	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$



Analysis of Variance for Battery Life Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Material types	10,683.72	2	5,341.86	7.91	0.0020
Temperature	39,118.72	2	19,559.36	28.97	< 0.0001
Interaction	9,613.78	4	2,403.44	3.56	0.0186
Error	18,230.75	27	675.21		
Total	77,646.97	35			

ccessed by others.

How would you check Model Adequacy?

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130	155	134.75	34	40	57.25	20	70	57.5	
	74	180		80	75		82	58		998
2	150	188	155.75	136	122	119.75	25	70	49.5	
	159	126		106	115		58	45		1300
3	138	110	144	174	120	145.75	96	104	85.5	
	168	160		150	139		82	60		1501
$y_{.j.}$		1738			1291			770		$3799 = y_{...}$

Regression Model

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Main Effects

Ambient temperature (E_1) 9150 psi

Wind Velocity (E_2) - 5100 psi

Bar Size (E_3) 850 psi

Two-Variable Interactions

Ambient temperature-Wind Velocity (E_{12}) 0 psi

Ambient temperature-Bar Size (E_{13}) 4650 psi

Wind Velocity-Bar Size (E_{23}) -100 psi

Three-Variable Interaction

Ambient temperature-Wind Velocity-Bar Size (E_{123}) -4700 psi

$$y_1 = \beta_0 + \beta_1(-1) + \beta_2(-1) + \beta_3(-1) + \beta_{12}(+1) + \beta_{13}(+1) + \beta_{23}(+1) + \beta_{123}(-1)$$

What is the regression model?

$$y = f(x_1, x_2, x_3)$$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3$$

we want to find & β_i ✓

```
[ 8.53250000e+01 4.57500000e+00 -2.55000000e+00 4.25000000e-01 -3.55271368e-15 2.32500000e+00 -5.00000000e-02 -2.35000000e+00]
```

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

8×4

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

For each experimental data, we can find the error (ϵ_i) between the model predicted value ($\hat{y}_i$) and observed experimental value (y_i)

$$\epsilon_i = y_i - \hat{y}_i$$

With 'least square fit', the aim is to find such coefficients, that minimizes the total sum of squares of error.

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each β_i

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each $\boldsymbol{\beta_i}$

$\hat{y}_1 = \beta_0 + \beta_1 x_{11}$

$\hat{y}_2$ -

$\hat{y}_3$ -

$\hat{y}_4$

$\hat{y}_8$

We can write the model in a matrix format

$$[Y_{exp}] = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_8 \end{bmatrix}, \quad [\hat{Y}] = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_8 \end{bmatrix}_{8\times1} = [X]_{8\times4}[\beta]_{4\times1}$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

where,

$$[x] = \begin{bmatrix} 1 & x_{11} & x_{2i} & x_{31} \\ 1 & x_{12} & x_{22} & x_{33} \\ 1 & \vdots & & \\ \vdots & & & \\ 1 & x_{18} & x_{28} & x_{38} \end{bmatrix}$$

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each $\boldsymbol{\beta_i}$

Thus, $L = \sum \varepsilon_i^2 = [\varepsilon]^T [\varepsilon]$

$= [Y_{exp} - \hat{Y}]^T [Y_{exp} - \hat{Y}]$

$= [Y_{exp} - X\beta]^T [Y_{exp} - X\beta]$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_8 \end{bmatrix} = [Y_{exp}] - [\hat{Y}]$$

$$\Rightarrow \boxed{L = [Y_{exp}]^T [Y_{exp}] - 2[\beta]^T [X]^T [Y_{exp}] + [\beta]^T [X]^T [X][\beta]}$$

How to Find Regression Coefficients?

Test #	*X1*	*X2*	*X3*	*Average (kpsi)*
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each $\boldsymbol{\beta_i}$

$$\Rightarrow \boxed{L = [Y_{exp}]^T [Y_{exp}] - 2[\beta]^T [X]^T [Y_{exp}] + [\beta]^T [X]^T [X][\beta]}$$

minimize L wrt β $\Rightarrow \frac{\partial L}{\partial [\beta]} = 0$

$$\Rightarrow -2[X]^T [Y_{exp}] + 2[X]^T [X][\beta] = 0$$

$$\Rightarrow \boxed{[\beta] = \left(\underset{4\times 8}{[X]^T}\underset{8\times 4}{[X]}\right)^{-1} \underset{4\times 8}{[X]^T} \underset{8\times 1}{[Y_{exp}]}}$$

How to Find Regression Coefficients?

Test #	*X1*	*X2*	*X3*	*Average (kpsi)*
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

$$\Rightarrow \boxed{[\beta] = \left([X]^T [X]\right)^{-1} [X]^T [Y_{exp}]}$$

What if we want to fit a model like

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{123} x_1 x_2 x_3 + \beta_{11} x_1^2$$

x_4 x_5 x_6

then, rename $x_1 x_2 = x_4$, $x_1 x_2 x_3 = x_5$, $x_1^2 = x_6$

$\beta_{12} = \beta_4$, $\beta_{123} = \beta_5$, $\beta_{11} = \beta_6$

Then do the same procedure as before

Example

The yield from a certain chemical depends on either the chemical formulation of the input materials or the mixer speed, or both.

A 2-level factorial design was run with three replicates and the yield data are shown below.

Find main effects and interaction effects.

x_1	x_2	y_a	y_b	y_c	$\bar{y}$
-1	-1	10	20	30	20
1	-1	40	30	50	40
-1	1	60	30	60	50
1	1	40	45	50	45

$$\underline{y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2}$$

Example

Consider following factorial design with 2 variables (k = 2), and 3 levels each

Each combination replicated 4 times (n = 4)

Life (in hours) Data for the Battery Design Example

Material	Temperature (°F)					
Type	15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

What is the effects model and hypothesis test?

Example

The yield form, a certain chemical depends on

either the **chemical formulation of the input materials** or **the mixer speed**, **or both**.

A 2-level factorial design was run with three replicates and the yield data are shown below.

Find Main and Interaction Effects and their Confidence Intervals, and Significance using ANOVA

x_1	x_2	y_a	y_b	y_c	$\bar{y}$
-1	-1	10	20	30	20
1	-1	40	30	50	40
-1	1	60	30	60	50
1	1	40	45	50	45

x_1	x_2	y_a	y_b	y_c	$\bar{y}$
-1	-1	10	20	30	20
1	-1	40	30	50	40
-1	1	60	30	60	50
1	1	40	45	50	45

$x_1 x_2$

$E_1 = \frac{1}{2}\left(-20+40+(-50)+45\right)$

$E_1 = 7.5$

$E_2 = \frac{1}{2}\left(-20-40+50+45\right) = 17.5$

$E_{12} = \frac{1}{2}\left(20-40-50+45\right) = -12.5$

What are sample variances?

How to find confidence intervals for E_1, E_2, E_{12}?

$S_1^2 = \sum \frac{(y_i-\bar{y})^2}{n-1} = \frac{(10-20)^2+(20-20)^2+(30-20)^2}{2} = 100$

$S_2^2 = \frac{(40-40)^2+(30-40)^2+(50-40)^2}{2} = 100$

$S_3^2 = 300$

$S_4^2 = 25$

Pooled variance

$S_p^2 = \frac{\sum DOF \times S_i^2}{T\,DOF} = \frac{2(100)+2(100)+2(300)+2(25)}{2+2+2+2}$

$= 131.25$

x_1	x_2	y_a	y_b	y_c	$\bar{y}$
-1	-1	10	20	30	20
1	-1	40	30	50	40
-1	1	60	30	60	50
1	1	40	45	50	45

Confidence interval for E_1 $\quad V(cY) = c^2 V(\bar{y})$

$$V(E_1) = V\left(\frac{\bar{y}_2 - \bar{y}_1 + \bar{y}_4 - \bar{y}_3}{2}\right)$$

$$= \frac{1}{4} V\left(\frac{y_{2a}+y_{2b}+y_{2c}}{3} - \frac{y_{1a}+y_{1b}+y_{1c}}{3} + \frac{y_{4a}+y_{4b}+y_{4c}}{3} - \frac{y_{3a}+y_{3b}+y_{3c}}{3}\right)$$

$$= \frac{1}{36}\, 12\, V(y_{1a}) = \frac{12}{36}\sigma^2 = \frac{1}{3}\sigma^2$$

$V(E_1) = \sigma^2/3$

$V(E_2) = \sigma^2/3 = V(E_{12})$

Confidence interval $\quad E_i \pm t_{\nu,\alpha}\sqrt{\frac{S_p^2}{3}}$

at 95% confidence $\quad t_{\nu,\alpha} = t_{8,0.025} = 2.306$ from table

$E_i \pm 2.306\sqrt{131.25/3} = E_i \pm 15.25$

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