1.

$$2\ddot{x} + \dot{x} = F(t)$$

Design a PD trajectory tracking controller to track a reference signal $x_d(t) = \sin t + \cos 2t$. The closed loop system should have a natural frequency less than 10 radians with a damping ratio greater than 0.707.

Set $F(t) = 2\left[K_p(x_d - x) + K_v(\dot{x}_d - \dot{x})\right] + 2\ddot{x}_d + \dot{x}$

Substituting,

$\ddot{e} + K_p e + K_v \dot{e} = 0$, $e = x_d - x$

choose K_p and K_v as needed.

2. Consider the coupled nonlinear system

$$\ddot{y}_1 + 3y_1y_2 + y_2^2 = u_1 + y_2u_2, \quad \text{— (1)}$$
$$\ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) = u_2 - (\cos y_1)^2 y_2 u_1 \quad \text{— (2)}$$

(a) Can these equations be written in the form Text

$$u_1 = f_1(\ddot{y}, \dot{y}, y); \qquad u_2 = f_2(\ddot{y}, \dot{y}, y)$$

solve (1) and (2) as linear eqns in u_1 and u_2

write your eqns in the form

$M(y)\ddot{y} + V(y, \dot{y}) + G(y) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

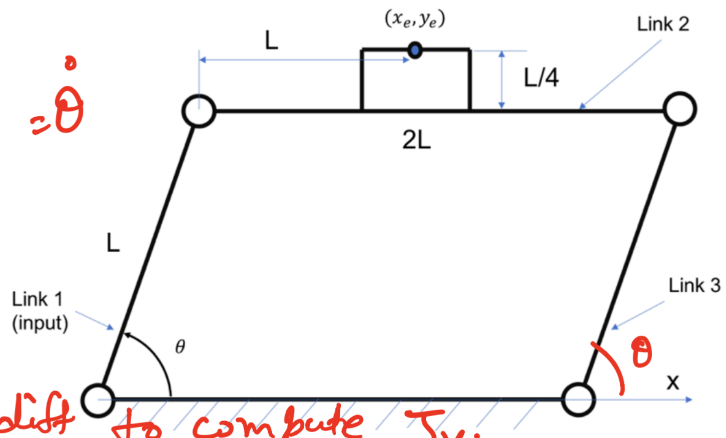
$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and design the controller.

3

Note $\omega_3 = \omega_1 = \dot{\theta}$

$\omega_2 = 0$

Write position co-ordinates of CoM of links and diff to compute J_{v_i}



The eqn. would be similar to that of a single DOF system. Design controller accordingly.

5

(a) For each link i, we have attached a frame $\{C_i\}$ to the center of mass (frame $\{2\}$ is same as frame $\{C_2\}$. Calculate matrices ${}^0_{C_1}T$ and ${}^0_{C_2}T$.

$${}^0v_{c_1} = \begin{bmatrix} -\frac{l_1}{2}s\theta_1 \\ \frac{l_1}{2}c\theta_1 \\ 0 \end{bmatrix}\dot{\theta}_1 = \underbrace{\begin{bmatrix} -\frac{l_1}{2}s\theta_1 & 0 \\ \frac{l_1}{2}c\theta_1 & 0 \\ 0 & 0 \end{bmatrix}}_{J_{c_1}}\dot{q}$$

$${}^0v_{c_2} = \begin{bmatrix} -d s\theta_1\,\dot{\theta}_1 \\ d c\theta_1\,\dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

For this two-link manipulator, the mass matrix has the form

$$M(q) = m_1 J_{v1}^T J_{v1} + m_2 J_{v2}^T J_{v2} + J_{\omega 1}^T I_{c1} J_{\omega 1} + J_{\omega 2}^T I_{c1} J_{\omega 2}$$

where, J_{vi} is the Jacobian of the center of mass of link i, $J_{\omega i}$ is the angular velocity of link i, and I_{ci} is the inertia tensor of link i expressed in frame $\{C_i\}$.

(b) Calculate ${}^0J_{v1}$ and ${}^0J_{v2}$.

(c) Calculate ${}^{c1}J_{\omega 1}$ and ${}^{c2}J_{\omega 2}$. $\rightarrow \omega_1 = \omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot{\theta}_1$

$$= \underbrace{\begin{bmatrix} -d s\theta_1 & 0 \\ d c\theta_1 & 0 \\ 0 & 1 \end{bmatrix}}_{J_{v_2}}\begin{bmatrix} \dot{\theta}_1 \\ \dot{d} \end{bmatrix}$$

(d) Calculate I_{c1} and I_{c2} in terms of the masses and dimensions of the links.

(e) Calculate the mass matrix $M(q)$.

(f) Calculate the other terms (gravity vector, Coriolis and centrifugal terms) and write out the equations of motion as:

$$\tau_1 = f_1(\ddot{q}, \dot{q}, q); \qquad \tau_2 = f_2(\ddot{q}, \dot{q}, q)$$

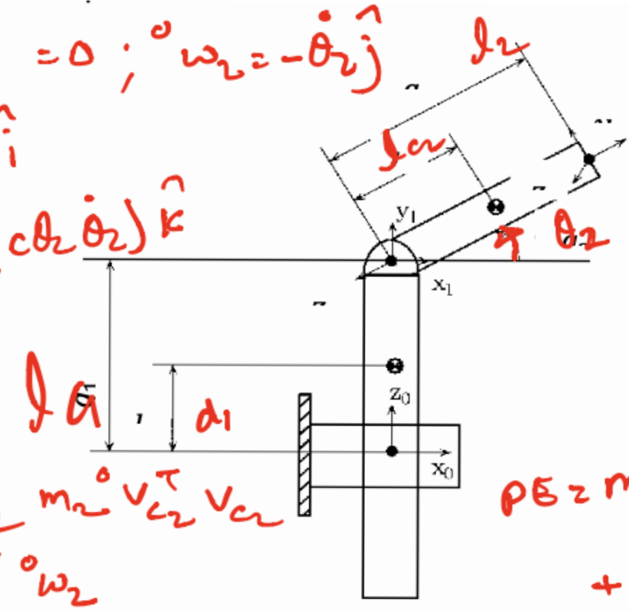
g) $q_{1d} = c_1 t$, $q_{2d} = c_2 t$

$\tau = M(q)\,[\overset{\text{unit mass}}{\text{PD control}}] + C\dot{q} + g(q)$

6

$^0v_{c1} = \dot{d}_1 \hat{k}$; $^0\omega_1 = 0$; $^0\omega_2 = -\dot{\theta}_2 \hat{j}$

$^0v_{c2} = -l_{c2}\, s\theta_2\, \dot{\theta}_2\, \hat{i} + (\dot{d}_1 + l_{c2}\, c\theta_2\, \dot{\theta}_2)\hat{k}$



$K.E. = \frac{1}{2} m_1\, {}^0v_{c1}^T\, v_{c1} + \frac{1}{2} m_2\, {}^0v_{c2}^T\, v_{c2} + \frac{1}{2} I_{c2}\, {}^0\omega_2^T\, {}^0\omega_2$

$PE = m_1 g d_1 + m_2 g l_{c2}\, s\theta_2$

use Lagrangian & Add friction term

a) Derive the dynamic equations governing the motion of this manipulator. Consider the coefficient of viscous friction at the prismatic joint to be c_1, and revolute joint to be frictionless. Note that the distance l_{C1} as link 1 moves!
b) Transform the equations in part (a) to obtain the equations of motion in task space
c) Design a task-space nonlinear decoupling PD trajectory controller with ω = 36 rad/s to follow some desired trajectory $[x_{\cdot}(t),y_{\cdot}(t)]^{T}$.

follow formula derived in class

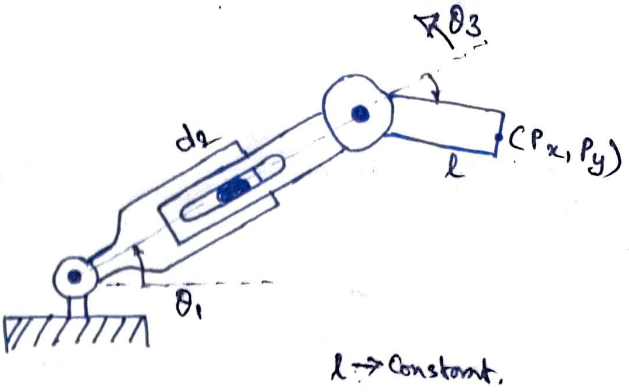
Follow lecture notes to cancel non-linearities and get task-space eqns. in the form $\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}a_x\\ a_y\end{bmatrix}$

end-effector position

design a_x, a_y as PD controllers for unit mass case.

1.(a)

End effector position (P_x, P_y)



$$\left.\begin{cases} P_x = d_2 \cos\theta_1 + l\cos(\theta_1+\theta_3) \\ P_y = d_2 \sin\theta_1 + l\sin(\theta_1+\theta_3) \end{cases}\right\} \text{—(A)}$$

Squaring and adding the both eq^n

$$P_x^2 + P_y^2 = d_2^2 + l^2 + 2d_2 l\left(\cos\theta_1\cos(\theta_1+\theta_3) + \sin\theta_1\sin(\theta_1+\theta_3)\right)$$

$$= d_2^2 + l^2 + 2d_2 l\cos\theta_3$$

$$\cos\theta_3 = \frac{P_x^2 + P_y^2 - (d_2^2 + l^2)}{2d_2 l} \; ; \; \theta_3 = \cos^{-1}\left[\frac{P_x^2 + P_y^2 - (d_2^2 + l^2)}{2d_2 l}\right]$$

again,
$$P_x = d_2\cos\theta_1 + l\cos\theta_1\cos\theta_3 - l\sin\theta_1\sin\theta_3 = (d_2 + l\cos\theta_3)\cos\theta_1 - l\sin\theta_3\sin\theta_1$$
$$P_y = d_2\sin\theta_1 + l\sin\theta_1\cos\theta_3 + l\cos\theta_1\sin\theta_3 = (d_2 + l\cos\theta_3)\sin\theta_1 + l\sin\theta_3\cos\theta_1$$

$$\Rightarrow \left.\begin{cases} P_x = a C\theta_1 - b S\theta_1 \\ P_y = a S\theta_1 + b C\theta_1 \end{cases}\right\} \Longrightarrow \sin\theta_1 = \frac{P_y - \frac{b}{a}P_x}{(a - b^2/a)}$$

Where d_2 is assumed arbitrary as we have 2 eq^n with 3 variables. Infinite solution possible.

(b). Position (P_x, P_y) as well as orientation of endeffector is known, $\psi = (\theta_1 + \theta_3)$ w.r.t ground.

eq^n (A) simplifies into $\Longrightarrow$
$$\left.\begin{aligned} P_x &= d_2 C\theta_1 + l C\psi \\ P_y &= d_2 S\theta_1 + l S\psi \end{aligned}\right\} \text{— (B)}$$

$$\left.\begin{aligned} d_2 C\theta_1 &= P_x - l C\psi \quad \text{—①} \\ d_2 S\theta_1 &= P_y - l S\psi \quad \text{—②} \end{aligned}\right\} \Rightarrow \theta_1 = \tan^{-1}\left[\frac{P_y - lS\psi}{P_x - lC\psi}\right] \text{—(A1)}$$

Sq & add ①

$$d_2 = \sqrt{P_x^2 + P_y^2 + l^2 - 2P_x l C\psi - 2P_y l S\psi} \quad \text{—— (A2)}$$

$$\theta_3 = \psi - \theta_1 \quad \text{—— (A3)}$$

2.

- $\theta_1 = \tan^{-1}(P_y/P_x)$ —— ①

- $P_z = d_2 + 1m$

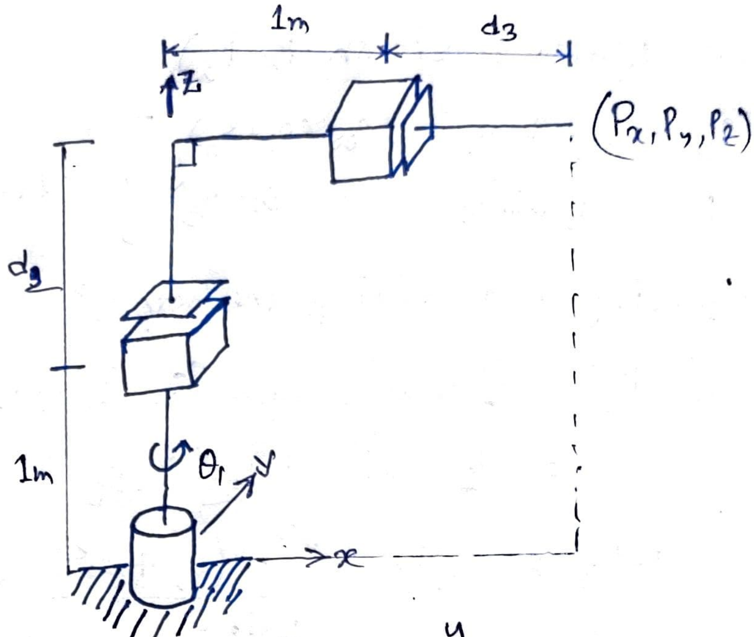
or, $d_2 = (P_z - 1)$ —— ②

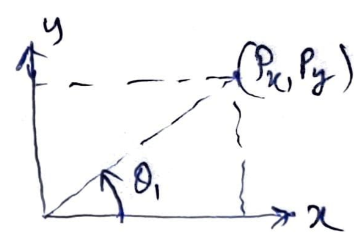
- ~~$P_z^2 = P_x^2 + P_y^2$~~

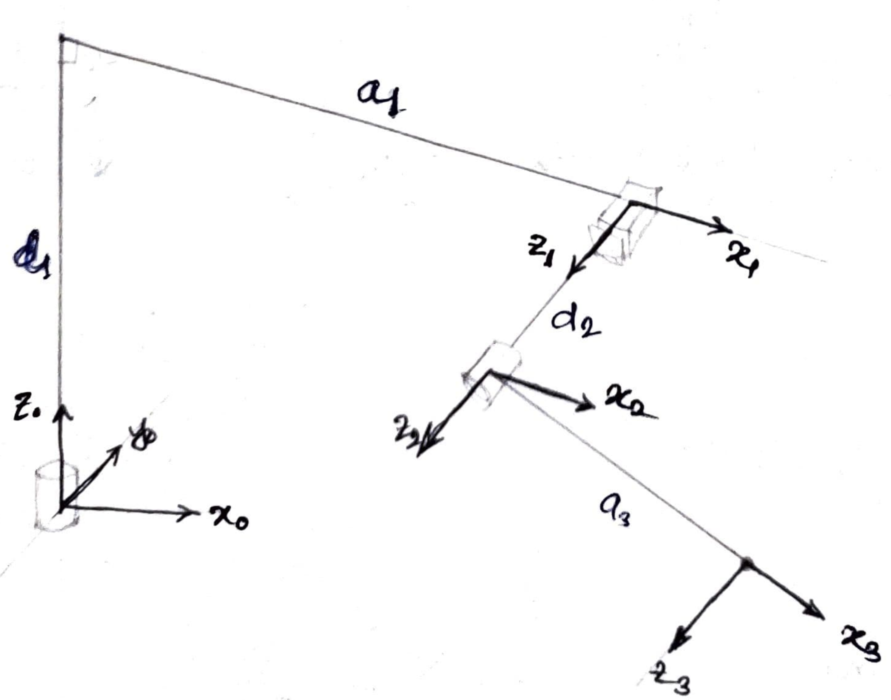
$P_z^2 + (1+d_3)^2 = (P_z^2 + P_y^2 + P_x^2)$

or $(1+d_3)^2 = P_x^2 + P_y^2$

$d_3 = \sqrt{(P_x^2 + P_y^2)} - 1$

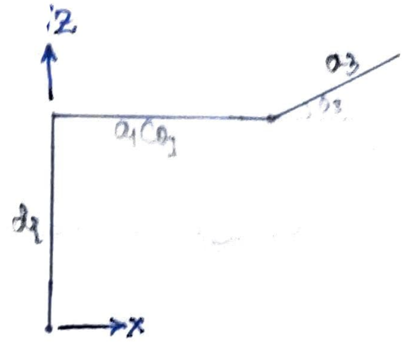






3 (a) end effector position known (P_x, P_y, P_z)

[From Tutorial 3, we get]

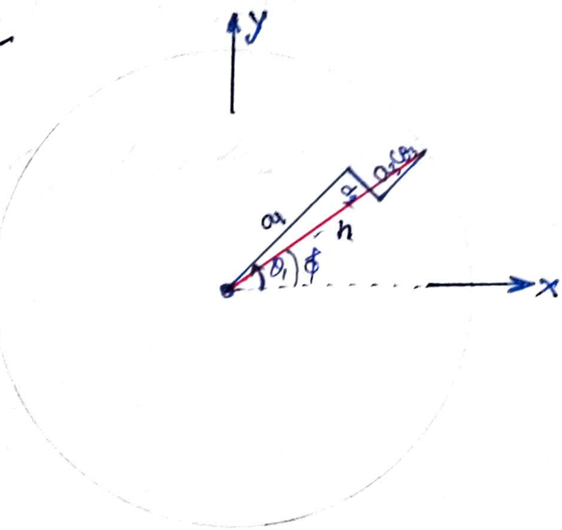


$$\begin{cases} P_x = a_1 C_{\theta_1} + d_2 S_{\theta_1} + a_3 C_{\theta_1} C_{\theta_3} & \text{—— (1)} \\ P_y = a_1 S_{\theta_1} - d_2 C_{\theta_1} + a_3 S_{\theta_1} C_{\theta_3} & \text{—— (2)} \\ P_z = d_1 + a_3 S_{\theta_3} & \text{—— (3)} \end{cases}$$

From (3): $\theta_3 = \sin^{-1}\left[\dfrac{P_z - d_1}{a_3}\right] \Rightarrow \cos\theta_3 = \sqrt{\dfrac{a_3^2 - (P_z - d_1)^2}{a_3^2}}$

$$\theta_3 = \tan^{-1}\left[\frac{P_z - d_1}{\sqrt{a_3^2 - (P_z - d_1)^2}}\right] ✓$$

recollecting (1) & (2)



$$P_x = (a_1 + a_3 C_{\theta_3}) C_{\theta_1} + d_2 S_{\theta_1} \quad \text{—— (4)}$$

$$P_y = (a_1 + a_3 C_{\theta_3}) S_{\theta_1} - d_2 C_{\theta_1} \quad \text{—— (5)}$$

$4 \times C_{\theta_1}$ & $5 \times S_{\theta_1}$ & addition:

$$P_x C_{\theta_1} + P_y S_{\theta_1} = (a_1 + a_3 C_{\theta_3})$$

or $h \cos(\theta_1 - \phi) = a_1 + a_3 C_{\theta_3}$

$$\begin{cases} h = \sqrt{P_x^2 + P_y^2} \\ \phi = \tan^{-1}\left(\frac{P_y}{P_x}\right) \end{cases}$$

or, $\cos(\theta_1 - \phi) = \dfrac{a_1 + a_3 C_{\theta_3}}{h}$

$$\begin{cases} P_x = h C_\phi \\ P_y = h S_\phi \end{cases}$$

$$\theta_1 = \phi + \cos^{-1}\left[\frac{a_1 + a_3 C_{\theta_3}}{\sqrt{P_x^2 + P_y^2}}\right]$$

From eq^n (5): $d_2 = (a_1 + a_3 C_{\theta_3}) \tan\theta_1 - P_y / C_{\theta_1}$ ✓

(b) orientation of the last link of the manipulator (3-link) w.r.t. Base frame:

$$ {}^0_3R = \begin{bmatrix} -0.8 & 0.4242 & 0.4242 \\ 0.6 & 0.5656 & 0.5656 \\ 0 & 0.707 & -0.707 \end{bmatrix} $$

desired orientation of the end effector: ${}^0_6R = R_z(45°)\, R_y(30°)\, R_z(45°)$

$$ = \begin{bmatrix} -0.0670 & -0.9330 & 0.3536 \\ 0.9330 & 0.0670 & 0.3536 \\ -0.3536 & 0.3536 & 0.8660 \end{bmatrix} $$

orientation of wrist w.r.t. end of 3rd link of manipulator

$$ {}^3_6R = ({}^0_3R)^T\, {}^0_6R = \begin{bmatrix} 0.6134 & 0.7866 & -0.0707 \\ 0.2493 & -0.1079 & 0.9622 \\ 0.7493 & -0.6079 & -0.2623 \end{bmatrix} $$

$$ = \begin{bmatrix} C\theta_4 C\theta_5 C\theta_6 - S\theta_4 S\theta_6 & -C\theta_4 C\theta_5 S\theta_6 - S\theta_4 C\theta_6 & C\theta_4 S\theta_5 \\ S\theta_4 C\theta_5 C\theta_6 + C\theta_4 S\theta_6 & -S\theta_4 C\theta_5 S\theta_6 + C\theta_4 C\theta_6 & S\theta_4 S\theta_5 \\ -S\theta_5 C\theta_6 & S\theta_5 S\theta_6 & C\theta_5 \end{bmatrix} $$

$\theta_4 = \text{atan2}(r_{23}, r_{13}) = 94.2$

$\theta_5 = \text{atan2}\left(\sqrt{r_{13}^2 + r_{23}^2},\ r_{33}\right) = 109.21$

$\theta_6 = \text{atan2}(r_{32}, -r_{31}) = -140.9$