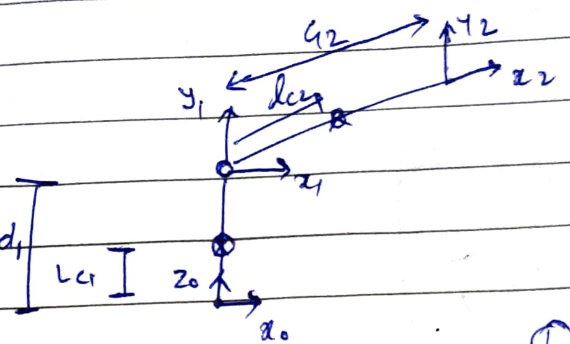
ME604

Tutorial- 6

Q1



DH parameter

	a	α	d	θ
1	0	$\pi/2$	d_1^*	0
2	a_2	0	0	θ_2^*

position of com of Link 1 in {0}

$$^0P_{c_1} = \begin{bmatrix} 0 \\ 0 \\ l_{c_1} \end{bmatrix}$$

pos. of com of Link 2 in {0}

$$^0P_{c_2} = \begin{bmatrix} l_{c_2} \cos\theta_2 \\ 0 \\ d_1 + l_{c_2} \sin\theta_2 \end{bmatrix}$$

Linear Jacobian

$$J_{v_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad J_{v_2} = \begin{bmatrix} 0 & -l_{c_2}\sin\theta_2 \\ 0 & 0 \\ 1 & l_{c_2}\cos\theta_2 \end{bmatrix}$$

$$m_1 J_{v_1}^T J_{v_1} = \begin{bmatrix} m_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (1)$$

$$m_2 J_{v_2}^T J_{v_2} = \begin{bmatrix} 1 & l_{c_2}\cos\theta_2 \\ l_{c_2}\cos\theta_2 & l_{c_2}^2 \end{bmatrix} m_2 \quad (2)$$

Angular Jacobian

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

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$J_{\omega_1}^T I_{c_1} J_{\omega_1} = 0$ ③

$J_{\omega_2}^T I_{c_2} J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & I_{c2} \end{bmatrix}$ ④

Adding ① ② ③ ④ we get D matrix

$$D = \begin{bmatrix} m_1 + m_2 & m_2 l_{c_2} \cos\theta_2 \\ m_2 l_{c_2} \cos\theta_2 & m_2 l_{c_2}^2 + I_{c_2} \end{bmatrix}$$

Now for centrifugal & coriolis components:

$$\underbrace{\begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\text{centrifugal}} + \underbrace{\begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1 \dot{q}_2]}_{\text{coriolis}}$$

$b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki})$ | $b_{iii} = 0$ | $m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$

where $m_{ij} \rightarrow$ components of 'D' matrix

$b_{122} = \frac{1}{2}(m_{122} + m_{122} - \cancel{m_{221}})$

$= -m_2 l_{c_2} \sin\theta_2$

$b_{211} = 0$

$b_{112} = 0$

$b_{212} = \frac{1}{2}[m_{212} + m_{221} - m_{122}] = 0$

$$\Rightarrow \begin{bmatrix} 0 & -m_2 l_{c_2} \sin\theta_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + [0][\dot{q}_1 \dot{q}_2]$$

$$G = -\left[J_{v_1}^T m_1 g + J_{v_2}^T m_2 g \right]$$

$$\text{where } g = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

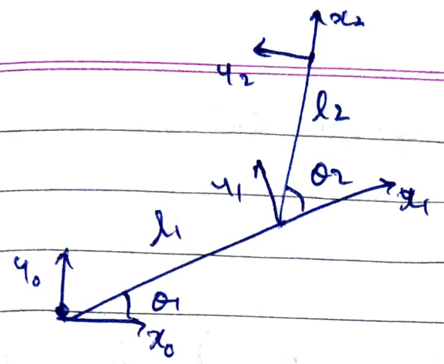
$$G = -\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -m_1 g \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -l_2 \sin 2 & 0 & l_{c_2}\cos\theta_2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -m_2 g \end{bmatrix}$$

$$\begin{bmatrix} m_1 g \\ 0 \end{bmatrix} + \begin{bmatrix} m_2 g \\ m_2 g l_{c_2} \cos\theta_2 \end{bmatrix}$$

$$\begin{bmatrix} m_1 + m_2 & m_2 l_{c_2} \cos\theta_2 \\ m_2 l_{c_2} \cos\theta_2 & m_2 l_{c_2}^2 + I_{c_2} \end{bmatrix}\begin{bmatrix} \ddot{d}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & -m_2 l_{c_2} \sin\theta_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \end{bmatrix} \ldots$$

$$\ldots + \begin{bmatrix} (m_1 + m_2) g \\ m_2 g\, l_{c_2} \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Q2



$m_1 = m_2 = m$

$l_1 = l_2 = l$

$I_{C_1} = I_{C_2} = 0$

Pos. of com of Link 1 in {0}

$${}^0P_{C_1} = \begin{bmatrix} l_1 C_1 \\ l_2 S_1 \\ 0 \end{bmatrix}$$

Pos. of com of Link 2 in {0}

$${}^0P_{C_2} = \begin{bmatrix} l_1 C_1 + l_2 C_{12} \\ l_1 S_1 + l_2 S_{12} \\ 0 \end{bmatrix}$$

$$J_{v_1} = \begin{bmatrix} -l_1 S_1 & 0 \\ l_1 C_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} \\ +l_1 C_1 + l_2 C_{12} & l_2 C_{12} \\ 0 & 0 \end{bmatrix}$$

$$J_{\omega_1} = [\bar{\varepsilon} z_1 \quad 0]$$

$$J_{\omega_2} = [\bar{\varepsilon} z_1 \quad \bar{\varepsilon} z_2]$$

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$D(q) = m J_{v_1}^T J_{v_1} + m J_{v_2}^T J_{v_2} + J_{\omega_1}^T I_{C_1} J_{\omega_1} + J_{\omega_2}^T I_{C_2} J_{\omega_2}$$

$$= m\begin{bmatrix} l^2 & 0 \\ 0 & 0 \end{bmatrix} + m\begin{bmatrix} 2l^2 + 2l^2 C_2 & l^2 + l^2 C_2 \\ l^2 + l^2 C_2 & l^2 \end{bmatrix}$$

$$D(q) = \begin{bmatrix} 3ml^2 + 2ml^2 C_2 & ml^2 + ml^2 C_2 \\ ml^2 + ml^2 C_2 & ml^2 \end{bmatrix}$$

now Centrifugal & Coriolis ter-

~~$B(q)\dot{q}$~~

$$b_{ijk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right) ; \; b_{iii} = 0$$

where, $m_{ijk} = \dfrac{\partial m_{ij}}{\partial q_k}$

$m_{ij} \to$ elements of matrix $[D]$

$$\begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

$b_{111} = 0 \; ; \; b_{222} = 0$

$b_{122} = -ml^2 S_2$

$b_{211} = ml^2 S_2$

$b_{112} = -ml^2 S_2$

$b_{212} = 0$

$$\begin{bmatrix} 0 & -ml^2 S_2 \\ ml^2 S_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} -2ml^2 S_2 \\ 0 \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

Gravity

$$G = -\left[J_{v_1}^T mg + J_{v_2}^T mg \right]$$

here $g = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$

$$G = \begin{bmatrix} mglc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} mglc_1 + mglc_{12} \\ mglc_{12} \end{bmatrix}$$

But we need to derive equation in term of motor angle & we need to consider K.E of motors also.

hence we know that

$$r\theta_{l_1} = \theta_{m_1} \quad \Big| \quad r\theta_{l_2} = \theta_{m_2}$$

$$\theta_{l_1} = \frac{\theta_{m_1}}{r} \qquad \theta_{l_2} = \frac{\theta_{m_2}}{r}$$

$$D(\theta_m) = \begin{bmatrix} 3ml^2 + 2ml^2\cos\left(\frac{\theta_{m_2}}{r}\right) + J_m & ml^2 + ml^2\cos\left(\frac{\theta_{m_2}}{r}\right) \\ ml^2 + ml^2\cos\left(\frac{\theta_{m_2}}{r}\right) & ml^2 + J_m \end{bmatrix}$$

$$C(\theta_m) = \begin{bmatrix} 0 & -\frac{ml^2\sin\left(\frac{\theta_{m_2}}{r}\right)}{r} \\ \frac{ml^2\sin\left(\frac{\theta_{m_2}}{r}\right)}{r} & 0 \end{bmatrix}$$

$$B(\theta_m) = \begin{bmatrix} -\frac{2ml^2\sin\left(\frac{\theta_{m_2}}{r}\right)}{r} \\ 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 2mgl\cos\left(\frac{\theta_{m_1}}{r}\right) + mgl\cos\left(\frac{\theta_{m_1}+\theta_{m_2}}{r}\right) \\ mgl\cos\left(\frac{\theta_{m_1}+\theta_{m_2}}{r}\right) \end{bmatrix}$$

$$\Rightarrow D(\theta_m)\begin{bmatrix} \ddot{\theta}_{m_1} \\ \ddot{\theta}_{m_2} \end{bmatrix} + C(\theta_m)\begin{bmatrix} \dot{\theta}_{m_1}^2 \\ \dot{\theta}_{m_2}^2 \end{bmatrix} + B(\theta_m)\left[\dot{\theta}_{m_1}\dot{\theta}_{m_2}\right] + \ldots$$

$$\ldots + G(\theta_m) = \frac{KV}{R}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - B_m\begin{bmatrix} \dot{\theta}_{m_1} \\ \dot{\theta}_{m_2} \end{bmatrix}$$

Q3 (a) Position of the end-effector in terms of Input (θ)

$$X_e = L\cos(\theta) + L$$
$$Y_e = L\sin(\theta) + L/4$$

Velocity of end effector in terms of Input velo. ($\dot{\theta}$)

$$\begin{bmatrix} \dot{X}_e \\ \dot{Y}_e \end{bmatrix} = \begin{bmatrix} -L\sin\theta \\ L\cos\theta \end{bmatrix} \dot{\theta}$$

(b) 2X1 Jacobian matrix relating velocity of COM of links 1, 2, & 3. to the Input joint velocity.

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} L/2\cos(\theta) \\ L/2\sin(\theta) \end{bmatrix} \quad \Big| \quad \begin{bmatrix} \dot{X}_1 \\ \dot{Y}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -L/2\sin(\theta) \\ L/2\cos(\theta) \end{bmatrix}}_{J_1} \dot{\theta}$$

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} L\cos(\theta) + L \\ L\sin(\theta) \end{bmatrix} \quad \Big| \quad \begin{bmatrix} \dot{X}_2 \\ \dot{Y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -L\sin(\theta) \\ -L\cos(\theta) \end{bmatrix}}_{J_2} \dot{\theta}$$

$$\begin{bmatrix} X_3 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 2L + L/2\cos\theta \\ L/2\sin\theta \end{bmatrix} \quad \Big| \quad \begin{bmatrix} \dot{X}_3 \\ \dot{Y}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -L/2\sin\theta \\ L/2\cos\theta \end{bmatrix}}_{J_3} \dot{\theta}$$

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(c) K.E expression for manipulator

We know that $\omega_1 = \omega_2 = \dot{\theta}\hat{k}$

While $\omega_2 = 0$

Hence

$$KE = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} m_2 v_2^T v_2 + \frac{1}{2} m_3 v_3^T v_3 + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_3 \dot{\theta}^2$$

$$= \frac{1}{2} m_1 \left(\frac{L^2}{4}\right)\dot{\theta}^2 + \frac{1}{2} m_2 (L^2)\dot{\theta}^2 + \frac{1}{2}\left(\frac{L^2}{4}\right)\dot{\theta}^2 + \frac{1}{2}(I_1 + I_3)\dot{\theta}^2$$

$$m_2 = 2m$$

Let $m_1 = m_3 = m$; $I_1 = I_3 = I$

$$KE = \frac{1}{2} m \left[\frac{L^2}{2} + 2L^2\right]\dot{\theta}^2 + I\dot{\theta}^2$$

now $I = \frac{mL^2}{12}$

$$\frac{1}{2} m \left[\frac{L^2}{2} + 2L^2\right] + = \left[\frac{1}{2}\cdot\frac{5}{2} mL^2 + \frac{mL^2}{12}\right]\dot{\theta}^2$$

$$KE = \frac{16}{12} mL^2\dot{\theta}^2 = \frac{4}{3} mL^2\dot{\theta}^2$$

(d) Equation of motion

Potential energy $= m_1 g \frac{L}{2}\sin\theta + m_2 g \frac{L}{2}\sin\theta + m_2 g L\sin\theta$

$$= \frac{mgL\sin\theta}{2} + mg\frac{L}{2}\sin\theta + 2mgL\sin\theta$$

$$= 3mgL\sin\theta$$

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Lagrange $L = K - P$

$$\frac{4}{3} m l^2 \dot{\theta}^2 - 3mg\, l \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right) - \frac{\partial}{\partial \theta}(K - P) = \tau$$

$$\frac{\partial K}{\partial \dot{\theta}} = \frac{8}{3} m l^2 \dot{\theta} \quad \Big| \quad \frac{\partial}{\partial \dot{\theta}}(K - P) = -3mg\, l \cos\theta$$

$$\frac{\partial}{\partial t}\left(\frac{\partial K}{\partial \dot{\theta}}\right) = \frac{8}{3} m l^2 \ddot{\theta}$$

$$\Rightarrow \frac{8}{3} m l^2 \ddot{\theta} + 3mg\, l \cos\theta = \tau$$

(e) Max & min value of Inertia

(Q4) $K = \frac{1}{2}\left(I_{xx} w_x^2 + I_{yy} w_y^2 + I_{zz} w_z^2\right)$

General Euler angle of ϕ, θ, ψ

$$w_x = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$
$$w_y = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$
$$w_z = \dot{\phi} \cos\theta + \dot{\psi}$$

Eom wrt ψ.

$$\frac{\partial}{\partial t}\left(\frac{\partial k}{\partial \dot{\psi}}\right) - \frac{\partial k}{\partial \psi} = 0$$

$$\frac{\partial k}{\partial \dot{\psi}} = \frac{\partial}{\partial \dot{\psi}}\left[\frac{1}{2} I_{xx} w_x^2 + \frac{1}{2} I_{yy} w_y^2 + \frac{1}{2} I_{zz} w_z^2\right]$$

$$= \frac{1}{2} I_{xx} \cancel{\frac{\partial}{\partial \dot{\psi}}(w_x^2)}^{\,0} + \frac{1}{2} I_{yy} \cancel{\frac{\partial}{\partial \dot{\psi}}}(w_y^2)^{\,0} + \frac{1}{2} I_{zz} \frac{\partial}{\partial \dot{\psi}}(w_z)^2$$

$$\frac{\partial}{\partial t}\left(\frac{\partial k}{\partial \dot{\psi}}\right) = \frac{\partial}{\partial t}\left(I_{zz} w_z\right) = I_{zz} \dot{w}_z$$

$$\frac{\partial k}{\partial \psi} = \frac{2}{2} I_{xx} w_x \left(\dot{\phi} \sin\theta \cos\psi \mp \dot{\theta} \sin\psi\right)$$
$$+ \frac{2}{2} I_{yy} w_y \left(-\dot{\phi} \sin\theta \sin\psi - \dot{\theta} \cos\psi\right)$$
$$+ \frac{2}{2} I_{zz} w_z (0)$$

$$I_{zz} \dot{w}_z - I_{xx} w_x w_y + I_{yy} w_y w_x = 0$$

$$\boxed{I_{zz} \dot{w}_z + (I_{yy} - I_{xx}) w_x w_y = 0}$$

Similarly

$$\omega_z = \dot{\theta} \sin\psi \sin\phi + \dot{\psi} \cos\phi$$
$$\omega_x = \dot{\theta} \sin\psi \cos\phi - \dot{\psi} \sin\phi$$
$$\omega_y = \dot{\theta} \cos\psi + \dot{\phi}$$

Now $\underbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\phi}}\right)}_{I_{yy}\dot{\omega}_y} - \underbrace{\frac{\partial K}{\partial \phi}}_{I_{xx}\omega_x\omega_z - I_{zz}\omega_z\omega_x} = 0$

following the same way as done on previous page

$$I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z = 0$$

Similarly we can also write

$$\omega_x = \dot{\psi} \cos\phi + \dot{\theta}$$
$$\omega_y = \dot{\psi} \sin\phi \sin\theta + \dot{\phi} \cos\theta$$
$$\omega_z = \dot{\psi} \sin\phi \cos\theta - \dot{\phi} \sin\theta$$

& $\underbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right)}_{I_{xx}\dot{\omega}_x} - \underbrace{\frac{\partial K}{\partial \theta}}_{I_{zz}\omega_z\omega_y - I_{yy}\omega_z\omega_y} = 0$

following same way as shown in previous page

$$\cancel{I_{zz}}\, I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z = 0$$