Spring Semester **February 29, 2024**

IE 616: Decision Analysis and Game Theory

Mid-sem

Time: 120 minutes Max. Marks 36

Exercise 1. A two-player game is symmetric if the two players have same strategy set, i.e., $S_1 = S_2$ and the payoff functions satisfy $u_1(s_1, s_2) = u_2(s_2, s_1)$ for each $s_1, s_2 \in S_1$.

1. Prove that the set of equilibria of a two-player symmetric game is a symmetric set: if (s_1, s_2) is an equilibrium, then (s_2, s_1) is also an equilibrium. **Marks 3**

2. Provide an example of a symmetric game, where (s_1, s_2) is an NE only when $s_1 \neq s_2$ and identify all possible NE. **Marks 2**

(1) Let (s_1, s_2) is an equilibrium

$\Rightarrow$ $U_1(s_1, s_2) \geq U_1(s_j, s_2)$ $\forall s_j \in S_1$

and $U_2(s_1, s_2) \geq U_2(s_1, s_j)$ $\forall s_j \in S_2$

$\Rightarrow$ $u_2(s_2, s_1) \geq U_2(s_2, s_j)$ $\forall s_j \in S_2$ $(\because S_1 = S_2)$

and $U_1(s_2, s_1) \geq U_1(s_j, s_1)$ $\forall s_j \in S_1$

$\Rightarrow$ (s_2, s_1) is a NE.

(2) $N = \{P_1, P_2\}$; $S_1 = S_2 = \{s_1, s_2\}$

P_1 \ P_2	s_1	s_2
s_1	0,0	2,1
s_2	1,2	0,0

N.E are: (s_1, s_2), (s_2, s_1)

$\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right) \right)$

Answer:

Exercise 2. Find all the NE of the following game. Explain, how to use iterative elimination in this game. **Marks 3**

	C_1	C_2	C_3	C_4
R_1	0,7	2,5	7,0	0,1
R_2	5,2	3,3	5,2	0,1
R_3	4,0	2,5	0,7	1,1
R_4	0,0	0,-2	0,0	10,-1

Table 1: Table for Exercise 2

$b_1(C_1) = R_2$
$b_1(C_2) = R_2$
$b_1(C_3) = R_1$
$b_1(C_4) = R_4$

As a rational player, row player will never play R_3, thus, we can safely eliminate R_3.

Next,

$b_2(R_1) = C_1$
$b_2(R_2) = C_2$
$b_2(R_4) = \{C_1, C_3\}$

As a rational player, column player will never play C_4, thus, we can safely eliminate C_4.

Next,

$b_1(C_1) = R_2$
$b_1(C_2) = R_2$
$b_1(C_3) = R_1$

As a rational player, row player will never play R_3, thus, we can safely eliminate R_3.

Next,

$b_2(R_1) = C_1$
$b_2(R_2) = C_2$

As a rational player, column player will never play C_3, thus, we can safely eliminate C_3.

Next,

$b_1(C_1) = R_2$
$b_1(C_2) = R_2$

As a rational player, row player will never play R_1, thus, we can safely eliminate R_1.

Next,

$b_2(R_2) = C_2$

As a rational player, column player will never play C_1, thus, we can safely eliminate C_1.

Next,

$b_1(C_2) = R_2$.

Thus, (R_2, C_2) is the only NE of this game.

Answer:

Exercise 3. Consider the following Zero-sum game:

	C_1	C_2
R_1	1,-1	-1,1
R_2	-1,1	1,-1

Table 2: Table for Exercise 3

1. Find the mixed strategy NE of the game. **Marks 3**

2. Write an LP to find the strategy of player 1. **Marks 3**

Answer: This is a zero sum game with

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The LP to solve P1s problem derives max min mixed strategy as below:

$$\max_{x=(x_1,x_2),z} \quad z \quad \text{subject to,}$$
$$z \leq x_1 - x_2,\ z \leq x_2 - x_1,\ x_i \geq 0 \text{ and } x_1 + x_2 = 1$$

Exercise 4. In a 2×2 zero-sum matrix game

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if there is no saddle point then prove that $a + d - b - c \neq 0$. **Marks 3**

Case 1: If $a \leq b$. $\Rightarrow$ $a - b \leq 0$

then since there is no saddle point,

so, $a < c$.

again since there is no saddle point, so $c > d$

$\Rightarrow d - c < 0$

Thus, $a + d - b - c \neq 0$

Case 2: If $a > b$ $\Rightarrow$ $a - b > 0$

then since there is no saddle point,

so, $b < d$

again since there is no saddle point, so, $d > c$

$\Rightarrow d - c > 0$

$\Rightarrow a + d - b - c \neq 0$

Exercise 5. Identify the action spaces and construct the payoff function (or matrix) for the following game. Rohit and Mohit show simultaneously 1 or 2 or 3 fingers and (at the same time) guess the number of fingers shown by the opponent. If exactly one player correctly guesses the displayed fingers, the player (that guessed correctly) will receive an amount equal to the total number of fingers shown by both players. In all other cases, the player receives zero.

Marks 1+2

Answer: The action spaces are $S_i = \{1,2,3\} \times \{1,2,3\}$. We represent the strategy of player i by $\mathbf{s}_i = (s_{i,1}, s_{i,2})$, where $s_{i,1}$ is the finger shown by player i and $s_{i,2}$ is its guess of the other players choice. The pay-off function of player i is given by:

$$u_i(\mathbf{s}_i, \mathbf{s}_{-i}) = 1_{s_{i,2}=s_{-i,1}} 1_{s_{-i,2} \neq s_{i,1}} (s_{i,1} + s_{-i,1}) \tag{1}$$

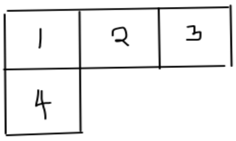


Figure 1: Modified Tic-Toc

Exercise 6. Consider a modified tic-tac-toe game with 2×3 squares (2 rows and 3 columns), but with some deleted entries as in figure 1 and two players. The first player starts the game and plays in odd rounds, and it always places circles 'o' in one of the empty entries. The second player plays in even rounds and places crosses '$\times$' in empty entries. The game is stopped either when a player manages to 'majorly complete' a horizontal line (at least 2 out of 3) with declaring him as the winner or when all the options are exhausted. At the start of the game, the first player has only two options $\{4,3\}$, and at all other rounds the players have all the left over squares as options. Model this as an extensive form game (mention all the ingredients describing the game), where the winner gets 5 while the others get 0. **Marks 4**

Answer:

$N = \{1,2\}$

$A_i = \{1,2,3,4\}$

$H = \{ \{3,1,2\}, \{3,1,4,2\}, \ldots \}$ set of all histories.

$S_H = \{ e, \{3\}, \{4\}, \{3,1\}, \{3,1,4\}, \ldots, \{4,3,2\} \}$

... set of all subhistories

$I_1 = \{ \epsilon, \{3,1\}, \{3,2\}, \{3,4\}, \{4,1\}, \ldots \{4,3\} \}$

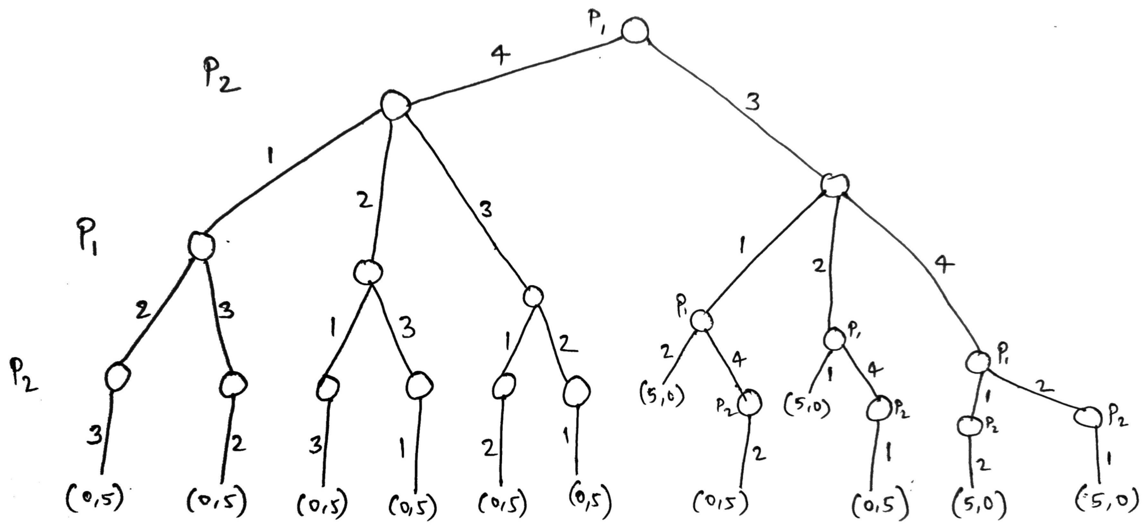
$I_2 = \{ \{3\}, \{4\}, \{3,1,4\}, \ldots \{4,1,3\}, \{4,2,1\}, \ldots, \{4,3,2\} \}$

$p(\epsilon) = p(\{3,1\}) = \ldots = p(\{4,3\}) = 1$

$p(\{3\}) = p(\{4\}) = p(\{3,1,4\}) = \ldots = p(\{4,3,2\}) = 2$

These are player assignment functions.

The utilities are given in game tree.



Exercise 7. Location Game: Two competing coffee house chains, Pete's Coffee and Caribou Coffee, are seeking locations for new branch stores in Cambridge. The town is comprised of only one street, along which all the residents live. Each of the two chains therefore needs to choose a single point within the interval [0, 1], which represents the exact location of the branch store along the road. It is assumed that each resident will go to the coffee house that is nearest to his place of residence. If the two chains choose the exact same location, they will each attract an equal number of customers. Each chain, of course, seeks to maximize its number of customers. To simplify the analysis required here, suppose that each point along the interval [0, 1] represents a town resident, and that the fraction of residents who frequent each coffee house is the fraction of points closer to one store than to the other.

(i) Describe this situation as a two-player strategic-form game. **Marks 2**

(ii) Prove that the only equilibrium in this game is that given by both chains selecting the location $x = 1/2$. **Marks 3**

Answer: We have $N = \{P, C\}$ and action sets are $\mathcal{A}_i = [0, 1]$ for each i with s_i representing the location of its store. Now the utility of player i is the fraction of players that it attracts, which equals

$$u_i(s_i, s_{-i}) = \begin{cases} \frac{s_i+s_{-i}}{2} & \text{if } s_i < s_{-i} \\ 1 - \frac{s_i+s_{-i}}{2} & \text{if } s_i > s_{-i} \\ \frac{1}{2} & \text{else.} \end{cases}$$

Clearly the best response for any player i is:

$$\mathcal{BR}_i(1/2) = \{1/2\}$$

implying $(1/2, 1/2)$ is a NE.

If possible say (x, y) is a NE with say $y \neq 1/2$. But the BR of player 1 against such a y is obtained by solving the following:

$$\sup_{s\in[0,y)} \frac{s+y}{2}, \quad \inf_{s\in(y,1]} \frac{s+y}{2}$$

and comparing the respective maximum values with $u_1(y, y) = 1/2$. Clearly when $y > 1/2$ we have

$$\sup_{s\in[0,1]} u_1(s, y) = \sup_{s\in[0,y)} \frac{s+y}{2} = y,$$

but does not have a maximizer and hence does not have a best response. Similarly if $y < 1/2$

$$\sup_{s\in[0,1]} u_1(s,y) = 1 - \inf_{s\in(y,1]} \frac{s+y}{2} = 1 - y$$

and again there is no best response. This is a contradiction.

Exercise 8. In a **War of Attrition**, two players compete for a resource of value v. This could be two animals competing for ownership of a breeding territory or two supermarkets engaged in a price war. The strategy for player i (for any i) is a choice of a persistence time (the time for which it continues to compete), t_i. The model makes three assumptions:
1. The cost of the contest is related only to its duration. There are no other costs (e.g., risk of injury).
2. The player that persists the longest gets all of the resource and hence the corresponding reward. If both the players quit at the same time, then neither gets the resource.
3. The cost paid by each player is proportional to the shortest persistence time chosen. (That is, no costs are incurred after one player quits and the contest ends.) Also let c be the cost per unit of persistence time.

(i) Model it as a strategic form game. **Marks 2**

(ii) Find the best response sets and then the NE. **Marks 3**

(iii) Say the two players decide to optimize their joint utility (or sum utility) and share the profit, then find the optimizers. **Marks 2**

Answer: The set of players $N = \{1, 2\}$. The strategy set of each player is $S_i = [0, \infty)$ for $i = 1, 2$. The utilities are given by:

$$u_i(t_1, t_2) = v1_{t_i > t_{-i}} - c\min\{t_1, t_2\}$$

Thus the BR set for any player i is given by:

$$\mathcal{BR}_i(t_{-i}) = \begin{cases} \{0\} & \text{if } t_{-i} > \frac{v}{c} \\ \{0\} \cup (\frac{v}{c}, \infty) & \text{if } t_{-i} = \frac{v}{c} \\ (t_{-i}, \infty) & \text{if } t_{-i} < \frac{v}{c} \end{cases} \tag{2}$$

Thus the NE of the game are $(0, t)$ and $(t, 0)$ with any $t \geq v/c$.

part (iii) Now they optimize jointly the following objective function:

$$\max_{t_1,t_2} \left(v \left(1_{t_1>t_2} + 1_{t_2>t_1} \right) - 2c\min\{t_1, t_2\} \right) = \max_{t_1,t_2} \left(v\left(1_{t_1 \neq t_2}\right) - 2c\min\{t_1, t_2\} \right)$$

Thus clearly the set of optimizers coincide with the set of NE, i.e.,

$$\arg\max \text{ set } = \{(t, 0) : t > 0\} \cup \{(0, t) : t > 0\}.$$