ME604
Assignment 1 Solution Key

1> We are given the direction cosines of the $\hat{i}, \hat{j}, \hat{k}$ unit vectors corresponding to 6 reference frames which are expressed in common base/reference frame RFO

Colums of these direction cosines ~~wen~~ when arranged in a row forms a matrix knowon as Rotation matrix.

$$RF1 = \underset{i}{\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}}, \underset{j}{\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}}, \underset{k}{\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} i & j & k \end{matrix} \end{array} = {}^{0}_{1}R$$

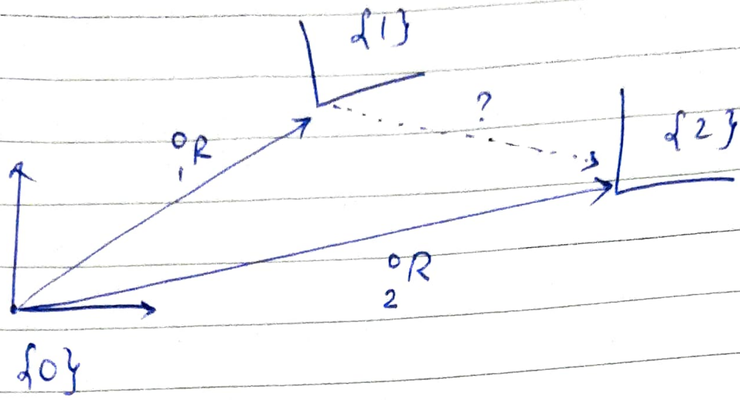
$$RF2 = \underset{i}{\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}}, \underset{j}{\begin{Bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{Bmatrix}}, \underset{k}{\begin{Bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ \begin{matrix} i & j & k \end{matrix} \end{array} = {}^{0}_{2}R$$

<u>Similarly</u>

$${}^{0}_{3}R = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \end{bmatrix}; \quad {}^{0}_{4}R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}; \quad {}^{0}_{5}R = \begin{bmatrix} 0 & -1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$${}^{0}_{6}R = \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ -1/2 & -1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

(ii) Let say we have 3 frames {0}, {1}, {2}



As shown in fig above each frame is relates to frame {0}

so,

Now to get rotation from frame {2} to frame {1} i.e. ^1_2R

we will

(i) express frame {2} in frame {0} = ^0_2R

(ii) then after frame {0} in frame {1} = ^1_0R

hence $^1_2R = \underbrace{^1_0R}_{\text{we donot have this}} \; \underbrace{^0_2R}_{\text{we have this}}$

As we know $^A_BR = {^B_AR}^T$

hence we can get $^1_0R = {^0_1R}^T$ (take Transpose)

↳ we know this!

Similarly

We know $^0_1R, \; ^0_2R, \; ^0_3R, \; ^0_4R, \; ^0_5R, \; ^0_6R$

get $^1_2R = {^1_0R}\,{^0_2R} = {^0_1R}^T\,{^0_2R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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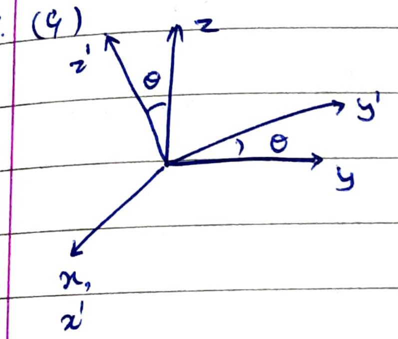
$${}^2_3R = {}^2_0R\,{}^0_3R = {}^0_2R^T\,{}^0_3R = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$$

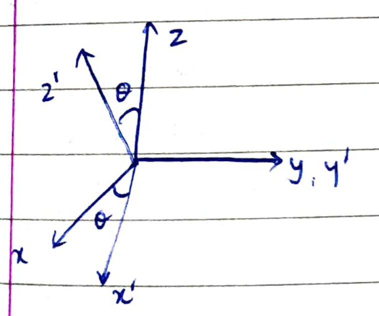
$${}^3_4R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}; \quad {}^4_5R = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}; \quad {}^5_6R = \begin{bmatrix} \frac{\sqrt{2}+1}{2\sqrt{2}} & 1/2 & \frac{\sqrt{2}-1}{2\sqrt{2}} \\ -1/2 & 1/\sqrt{2} & 1/2 \\ \frac{\sqrt{2}-1}{2\sqrt{2}} & -1/2 & \frac{\sqrt{2}+1}{2\sqrt{2}} \end{bmatrix}$$

27. (a)

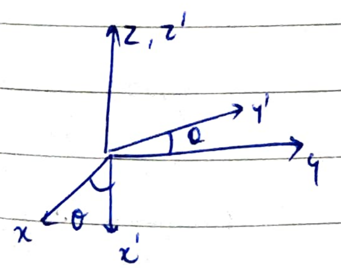


Rotation about x-axis.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

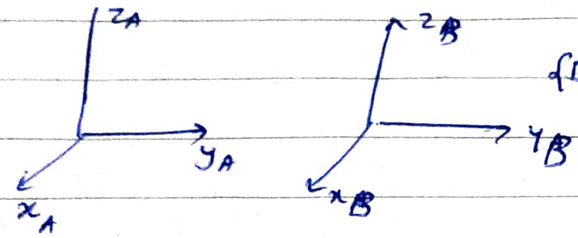


$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

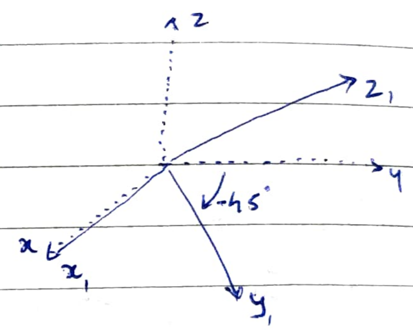
2>(b) Initially Let frame {A} & {B} be



[Note: dotted lines are previous frame & solid lines are current frame]

(i) Rotation by -45° about x-axis

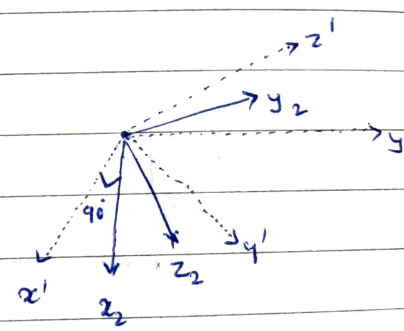
$$^{A}_{1}R = R_x(-45^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$



(ii) Rotation by 90° about world y axis

$$^{A}_{2}R = R_y(90^\circ)\cdot R_x(-45^\circ)$$

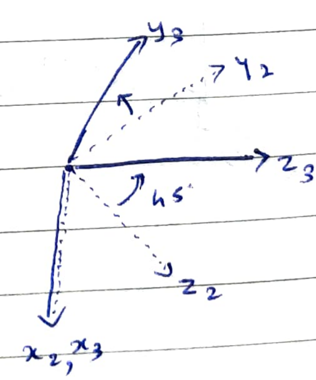
$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1 & 0 & 0 \end{bmatrix}$$



(iii) Rotation by 45° about world z axis

$$^{A}_{3}R = R_z(45^\circ)\underbrace{R_y(90^\circ)R_x(-45^\circ)}$$

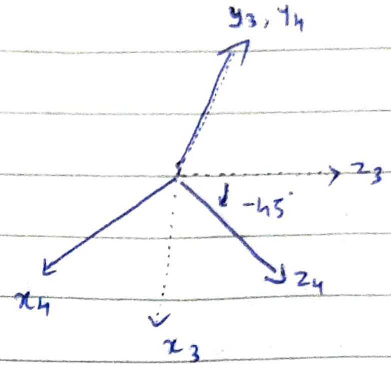
$$^{A}_{3}R = R_z(45^\circ)\,^{A}_{2}R$$



(iv) Rotation by $-45°$ about world x-axes

$$^A_4R = R_x(-45°)\underbrace{R_z(45°)R_y(90°)R_x(45°)}_{^A_3R}$$

$$= R_x(-45°)\,^A_3R$$

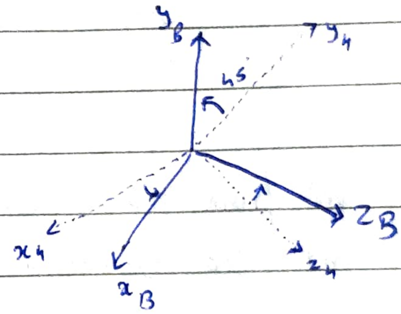


(v) Rotation by $45°$ about world z axis

$$^A_BR = R_z(45°)\underbrace{R_x(-45°)R_z(45°)R_y(90°)R_x(-45°)}_{^A_4R}$$

$$^A_BR = R_z(45°)\,^A_4R$$

$$= \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ -1/2 & -1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$



$$^A_BR = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ -1/2 & -1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

Q3. Consider unit vectors i, j, k along the edges EA, EF and EH of the cube. In the final configuration, the direction cosines of these vectors are given by

j' : {-0.707, 0.707, 0}
k' : {-0.707, -0.707, 0}
i' : k' x j' = {0, 0, 1}

Use the direction cosines to form the rotation matrix as in Prob 1.

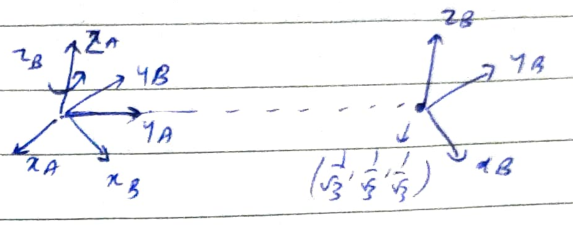
Q4

$$R = \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ -1/2 & -1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \qquad t = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Rotation about some axis of A

translate 1 unit along (1,1,1) direction.

(a) (i) Rotation
(ii) translation



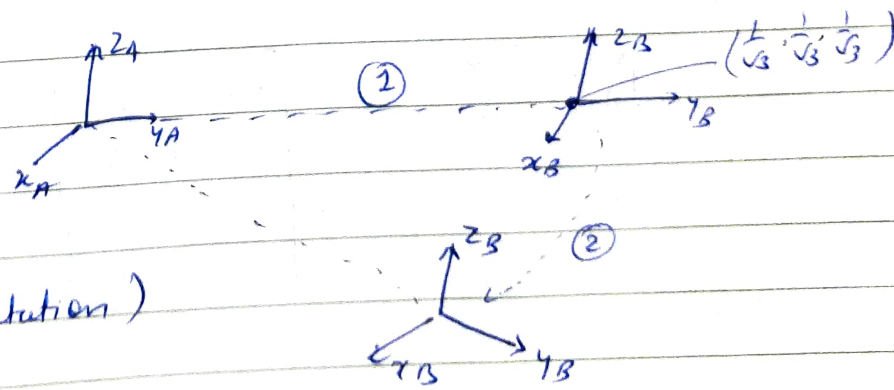
(Just a Representation for understanding)

$${}^{A}_{B}T = \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 & 1/\sqrt{3} \\ -1/2 & -1/\sqrt{2} & 1/2 & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{B}p = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \; ; \quad {}^{A}p = {}^{A}_{B}T \, {}^{B}p = {}^{A}_{B}T \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.87 \\ -1.129 \\ -0.836 \\ 1 \end{bmatrix}$$

$${}^{A}p = \begin{bmatrix} 0.87 \\ -1.129 \\ -0.836 \end{bmatrix}$$

(b) (i) translation
(ii) Rotation



(Just a Representation)

Steps of transformation

(i) translation of frame{B} at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(ii) Rotation of frame{B} about {A} by R

Let a point in {A} be O_B i.e origin of {B} after translation

$$O_B = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

O_B Rotates about {A} by R

$$O_{B\,new} = \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ -1/2 & -1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -0.4082 \\ -0.4082 \\ -0.8162 \end{bmatrix}$$

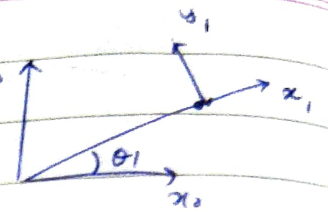
Hence transformation matrix from {B} to {A}

$${}^A_BT = \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 & -0.4082 \\ -1/2 & -1/\sqrt{2} & 1/2 & -0.4082 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} & -0.8162 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

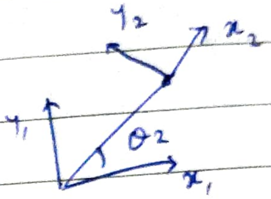
$$P_A = {}^A_BT\,{}^BP = \begin{bmatrix} -0.1153 \\ -2.1153 \\ -2.2304 \\ 1 \end{bmatrix}, \quad {}^AP = \begin{bmatrix} -0.1153 \\ -2.1153 \\ -2.2304 \end{bmatrix}$$

Q5 (a)

(i)
$$
{}^0_1T = \begin{array}{c} {}^0_1R \\ \left[\begin{array}{ccc|c} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}
$$



$$
{}^1_2T = \begin{array}{c} {}^1_2R \\ \left[\begin{array}{ccc|c} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}
$$



(b)
$$
{}^0_2T = {}^0T_1 \, {}^1T_2 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & l_2\cos(\theta_1+\theta_2) + l_1\cos\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & l_2\sin(\theta_1+\theta_2) + l_2\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(ii) (a) Transformation matrix from 1 to 0

$$
{}^0_1T = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right];
$$

Transformation matrix from 2 to 1

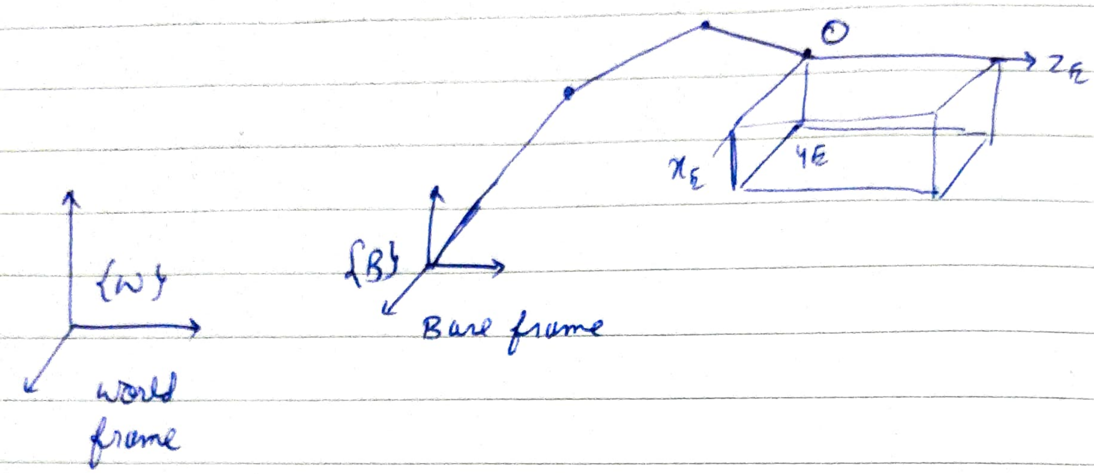
$$
{}^1_2T = \left[\begin{array}{ccc|c} \sin\theta & \cos\theta & 0 & a \\ 0 & 0 & -1 & 0 \\ -\cos\theta & \sin\theta & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

transformation matrix from 3 to 2

$$
{}^2_3T = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & L \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

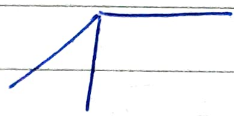
(b)
$$
{}^0_3T = {}^0_1T \, {}^1_2T \, {}^2_3T = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & a + L\cos\theta \\ \sin\theta & \cos\theta & 0 & d + L\sin\theta \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Q6 Given

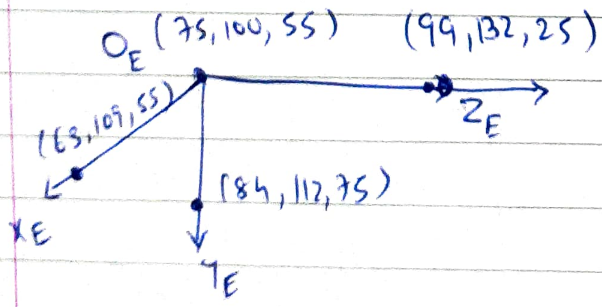


At 'O' endeffector frame & the vertex of the box coincide

frame {O}



And from the coordinates of the vertex of box we can find the direction cosines of frame {O}



$X_E - O_E = (63, 109, 55) - (75, 100, 55)$

$= (-12, 9, 0)$

~~Unit Vector = $\frac{X_E - O_E}{\sqrt{\ }}$~~

$$\text{Unit Vector} = \frac{X_E - O_E}{|X_E - O_E|}$$

$$= \frac{(-12, 9, 0)}{\sqrt{12^2 + 9^2}}$$

$$= (-0.8, 0.6, 0)$$

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Similarly

$Y_E - O_E = (9, 12, 20)$

Unit Vector $= \dfrac{Y_E - O_E}{|Y_E - O_E|}$

$= \dfrac{(9, 12, 20)}{\sqrt{9^2 + 12^2 + 20^2}}$

$= (0.36, 0.48, 0.8)$

$Z_E - O_E = (24, 32, -30)$

Unit vector $= \dfrac{Z_E - O_E}{|Z_E - O_E|}$

$= \dfrac{(24, 32, -30)}{\sqrt{24^2 + 32^2 + 30^2}}$

$= (0.48, 0.64, -0.6)$

frame

$$RF'O_E' = \begin{array}{ccc} \left\{\begin{matrix} -0.8 \\ 0.6 \\ 0 \end{matrix}\right\}, & \left\{\begin{matrix} 0.36 \\ 0.48 \\ 0.8 \end{matrix}\right\}, & \left\{\begin{matrix} 0.48 \\ 0.64 \\ -0.6 \end{matrix}\right\} \\ i & \hat{j} & K \end{array}$$

for world frame $= \begin{array}{ccc} \left\{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right\}, & \left\{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right\}, & \left\{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right\} \\ i & j & k \end{array}$

$$^{W}_{OE}R \text{ or } ^{W}_{E}R = \begin{bmatrix} i_E \cdot i_W & j_E \cdot i_W & k_E \cdot i_W \\ i_E \cdot j_W & j_E \cdot j_W & k_E \cdot j_W \\ i_E \cdot k_W & j_E \cdot k_W & k_E \cdot k_W \end{bmatrix}$$

$$= \begin{bmatrix} -0.8 & 0.36 & 0.48 \\ 0.6 & 0.48 & 0.64 \\ 0 & 0.8 & -0.6 \end{bmatrix}$$

position of O_E in world frame is $(75, 100, 55)$

$$^{W}_{E}T = \left[\begin{array}{c|c} ^{W}_{E}R & ^{W}P_E \\ \hline 0 & 1 \end{array}\right]$$

$$ {}^{W}_{E}T = \begin{bmatrix} -0.8 & 0.36 & 0.48 & 75 \\ 0.6 & 0.48 & 0.64 & 100 \\ 0 & 0.8 & -0.6 & 55 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(b) We know

${}^{W}_{E}T$ & ${}^{B}_{E}T$ we need to find ${}^{W}_{B}T$

$$ {}^{W}_{B}T = {}^{W}_{E}T \; {}^{E}_{B}T $$

$$ \hookrightarrow {}^{E}_{B}T = {}^{B}_{E}T^{T} \quad (\text{transpose}) $$

Hence $ {}^{W}_{B}T = {}^{W}_{E}T \left({}^{B}_{E}T\right)^{T} $

Also we observe that
$$ {}^{B}_{E}T = \left[\begin{array}{ccc|c} -0.8 & 0.36 & 0.48 & 68 \\ 0.6 & 0.48 & 0.64 & 174 \\ 0 & 0.8 & -0.6 & -60 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{B}_{E}R = {}^{W}_{E}R $$

this implies that world frame $\{W\}$ & base frame $\{B\}$ are ~~align~~ aligned same to frame $\{E\}$

Or in other words

Orientation of $\{W\}$ is same as $\{B\}$

difference lies in just translation.

hence $ {}^{W}_{B}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $; (oriented same)

translation $= {}^{W}P_{E} - {}^{B}P_{E} = \begin{bmatrix} 75 \\ 100 \\ 55 \end{bmatrix} - \begin{bmatrix} 68 \\ 174 \\ -60 \end{bmatrix} = \begin{bmatrix} 7 \\ -74 \\ 115 \end{bmatrix}$

$$ {}^{B}_{B}T = \begin{bmatrix} 1 & 0 & 0 & +7 \\ 0 & 1 & 0 & -74 \\ 0 & 0 & 1 & +115 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Q8 (a) $$ {}^{0}_{1}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad {}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} 0 & 1 & 0 & -0.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(b) $ {}^{0}_{2}T = {}^{0}_{1}T \, {}^{1}_{2}T $

$\Rightarrow$ Initial origin of frame $\{2\}$ be $O_2 = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix}$ in frame $\{1\}$

$\Rightarrow$ Rotation about z_1 of $\{1\}$ by 45°

$$ O_{2new} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} $$

Also translated in x_0 direction which is same as x_1 direction by 0.5

$$ = \begin{bmatrix} -1/\sqrt{2} + 0.5 \\ 0 \\ 0 \end{bmatrix} $$

Q7.
i: Write the 4x4 homogenous transfromation matrix using the given
rotation matrix and vector OP.
ii: Note that co-ordinates of S in F2 are {-1, 0, 0}.
iii: Use the homogenous transformation matrix to find F1 co-ordinates of S
iv: To get the equation of plane in F2 co-ordinates, you can use the homogenous
transformation matrix to get co-ordinates of points A, B and C in F2 co-ordinates,
and then use those to get the equation of the plane.

Note that F2 co-ordinates of A can be obtained from the sum of the first column of R
with vector OP (why?)
Similarly, F2 co-ordinates of B can be obtained from the sum of the second column of R with vector OP
(why?)
and F2 co-ordinates of C can be obtained from the sum of the third column of R with vector OP (why?)