1. (a) Rotation about world 'x' axis $R_x(\phi)$

$${}^{0}_{1}R = R_x(\phi)$$

(b) Rotation about current z-axis $R_z(\theta)$

$$R_x(\phi)\, R_z(\theta)$$

(c) Rotation about world y-axis $R_Y(\psi)$

$$R_Y(\psi)\, R_x(\phi)\, R_z(\theta)$$

2. (a) Rotate by ϕ about world x-axis $R_x(\phi)$

(b) Rotate by θ about world z-axis $R_Z(\theta)$

$$R_Z(\theta)\, R_x(\phi)$$

(c) Rotate by ψ about current x-axis $R_x(\psi)$

$$R_Z(\theta)\, R_x(\phi)\, R_x(\psi)$$

(d) Rotate by θ about world z-axis $R_Z(\theta)$

$$R_z(\theta)\, R_Z(\theta)\, R_x(\phi)\, R_x(\psi)$$

3. (a) Rotation by 90° about $R_Y(90^\circ)$

(b) Rotation by 45° abou xis $R_z(45^\circ)$

$${}^{0}_{2}R = R_Y(90^\circ)\, R_z(45^\circ) = \begin{bmatrix} C & & 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$tr({}^{0}_{2}R) = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{2}\left(tr({}^{0}_{2}R) - 1\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - 1\right) = -0.1464$$

$$\theta = 98.42^\circ$$ (eq. angle of rotation)

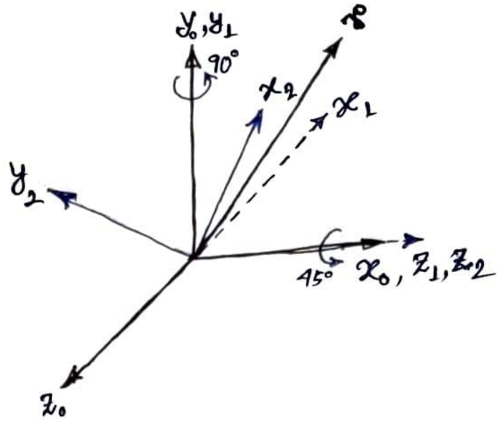
equivalant axis of rotation $\hat{k}$

$$\hat{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\tilde{k} = \frac{1}{2\sin\theta}\left({}^0_2R - {}^0_2R^T\right)$$

$$= \frac{1}{1.9784}\begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -(1+\frac{1}{\sqrt{2}}) & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} 0.3574 \\ 0.8628 \\ 0.3574 \end{bmatrix}$$



r → equivalant axis vector

3.

Consider a frame x1y1z1 affixed to the cube with origin at Oc and aligned with world frame at the begining.

Note that first 3 operations are in world frame. So we can consider the appropriate tranformation operators and apply them sequentially to get new co-ordinates for B. Hence, after the rotation about z0, translation along y0 and rotation about x0 the co-ordinates of point B in x0y0z0 frame are

T(Rx(90), 0, 0, 0)T(R(0), 0, 2, 0)T(Rz(45), 0, 0, 0) [1 1 1 1]' (' indicates transpose)

rotation translations in x, y, z

The corresponding transformation matrix between frames 1 and 2 is T(Rx(90), 0, 0, 0)T(R(0), 0, 2, 0)T(Rz(45), 0, 0, 0).

The last operation is in the frame x1y1z1 and doesn't change the location of origin of x1y1z1. Further, in x1y1z1the co-ordinates of B are still (1,1,1). The transformation matrix relating these two cube orientations is T(Rz(45), 0, 0, 0). Applying this to [1 1 1 1]' gives the new position of B wrt the origin of x1y1z1, which can then be mapped to the ground frame using the transformation matrix found earlier.

T(Rx(90), 0, 0, 0)T(R(0), 0, 2, 0)T(Rz(45), 0, 0, 0)T(Rz(45), 0, 0, 0)[1 1 1 1]'

6.

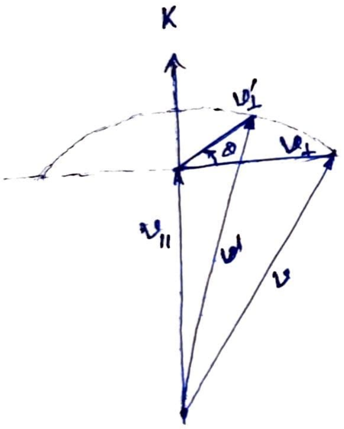
5. $\hat{K} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\theta = 90^\circ$

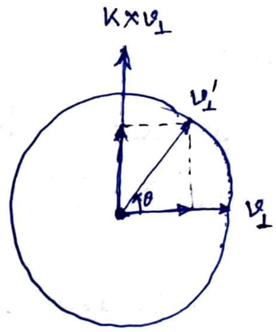
$$R_{\hat{K},\theta} = \begin{bmatrix} K_1^2(1-C\theta)+C\theta & K_1K_2(1-C\theta)-K_3S\theta & K_1K_3(1-C\theta)+K_2S\theta \\ K_1K_2(1-C\theta)+K_3S\theta & K_2^2(1-C\theta)+C\theta & K_2K_3(1-C\theta)-K_1S\theta \\ K_1K_3(1-C\theta)\pm K_2S\theta & K_2K_3(1-C\theta)+K_1S\theta & K_3^2(1-C\theta)+C\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 - 1/\sqrt{3} & 1/3 + 1/\sqrt{3} \\ 1/3 + 1/\sqrt{3} & 1/3 & 1/3 - 1/\sqrt{3} \\ 1/3 - 1/\sqrt{3} & 1/3 + 1/\sqrt{3} & 1/3 \end{bmatrix}$$

$\cos\theta = 0$
$\sin\theta = 1$

(b) $v' = R_{\hat{k},\theta}\, v$





≡ Top view ≡

$$\begin{Bmatrix} v = v_{\parallel} + v_\perp \\ v' = v_{\parallel} + v_\perp' \end{Bmatrix}$$

$v_{\parallel} = (\hat{k}.v)\hat{k} = (K^T v)\hat{k}$

$v_\perp = v - (K^T v)\hat{k}$

$$v_\perp' = |v_\perp'| \cos\theta \frac{v_\perp}{|v_\perp|} + \frac{(\hat{k} \times v_\perp)}{|\hat{k} \times v_\perp|} |v_\perp'| \sin\theta$$

$$= v_\perp \cos\theta + (\hat{k} \times v_\perp)\sin\theta$$

* $v_\perp$ and $v_\perp'$ has equal magnitude.
* $\hat{k}$ & $v_\perp$ perpendicular to eachother $|\hat{k} \times v_\perp| = |v_\perp|$

$$v' = (K^T v)\hat{k} + v_\perp \cos\theta + (\hat{k} \times v_\perp)\sin\theta$$

substituting $v_\perp$,

$$v' = (K^T v)\hat{k} + (v - (K^T v)\hat{k})\cos\theta + \left(\hat{k} \times (v - (K^T v)\hat{k})\right)\sin\theta$$

or, $v' = v\cos\theta + (1-\cos\theta)(K^T v)\hat{k} + (\hat{k} \times v)\sin\theta + 0$

(proved)

6. All possible sets of Eular angles

XYX	YXY	ZXY
XYZ	YXZ	ZXZ
XZX	YZX	ZYX
XZY	YZY	ZYZ

12 possible sets of Eular angles

(b) ZZX not possible !

Two consecutive rotation of same axis do not produce two independant rotation, It is equivalant to a single rotation.

So, ZZX can not produce 3 independant rotation.

(c) ZYZ Eular angles, rotation $\{\pi/2, 0, \pi/2\}$

$$R_z(\pi/2)\, R_y(0)\, R_z(\pi/2)$$

$$= R_z(\pi/2)\, I\, R_z(\pi/2)$$

$$= R_z(\pi)$$

The direction of the x_1 axis will be opposite to x_0.

7. Quaternion : $Q = q_0 + iq_1 + jq_2 + kq_3$

unit vector, $r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$

(a) a rotation of θ about r given by $Q = \left(\cos\frac{\theta}{2} \;\; r_x \sin\theta/2 \;\; r_y \sin\theta/2 \;\; r_z \sin\theta/2\right)$

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = \cos^2\theta/2 + r_x^2 \sin^2\theta/2 + r_y^2 \sin^2\theta/2 + r_z^2 \sin^2\theta/2$$

$$= \cos^2\theta/2 + (r_x^2 + r_y^2 + r_z^2)\sin^2\theta/2$$

r is a unit vector, norm of r is 1.

$$r_x^2 + r_y^2 + r_z^2 = 1$$

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = \cos^2\theta/2 + \sin^2\theta/2 = 1 .$$

(b)

$$R_{\hat{r},\theta} = \begin{bmatrix} r_x^2(1-C\theta)+C\theta & r_x r_y(1-C\theta)-r_z S\theta & r_x r_z(1-C\theta)+r_y S\theta \\ r_x r_y(1-C\theta)+r_z S\theta & r_y^2(1-C\theta)+C\theta & r_y r_z(1-C\theta)-r_x S\theta \\ r_x r_z(1-C\theta)-r_y S\theta & r_y r_z(1-C\theta)+r_x S\theta & r_z^2(1-C\theta)+C\theta \end{bmatrix}$$

given,

- $q_0 = \cos\theta/2$
- $q_1 = r_x \sin\theta/2$
- $q_2 = r_y \sin\theta/2$
- $q_3 = r_z \sin\theta/2$

$\rightarrow 2\cos^2\theta/2 = 1+\cos\theta$

$\cos\theta = (2q_0^2-1)$

$\rightarrow 2q_0q_1 = 2\cos\theta/2\, r_x \sin\theta/2 = r_x\sin\theta$

$\rightarrow 2q_0q_2 = r_y\sin\theta$

$\rightarrow 2q_0q_3 = r_z\sin\theta$

$\rightarrow \sin\theta = \cancel{2q_0\sqrt{q_0^2-1}} = 2q_0\sqrt{1-q_0^2}$

$\rightarrow (1-\cos\theta) = 2(1-q_0^2)$

$\rightarrow r_x = \dfrac{2q_0q_1}{\sin\theta} = \dfrac{2q_0q_1}{2q_0\sqrt{1-q_0^2}}$

$\rightarrow r_y = \dfrac{q_2}{\sqrt{1-q_0^2}}$

$\rightarrow r_z = \dfrac{q_3}{\sqrt{1-q_0^2}}$

- $r_x^2(1-\cos\theta)+\cos\theta = \dfrac{q_1^2}{\cancel{(1-q_0^2)}}\cdot 2\cancel{(1-q_0^2)} + (2q_0^2-1) = 2q_1^2+2q_0^2-1$
- $r_y^2(1-\cos\theta)+\cos\theta = 2q_2^2+2q_0^2-1$
- $r_z^2(1-\cos\theta)+\cos\theta = 2(q_3^2+q_0^2)-1$
- $r_x r_y(1-\cos\theta) \pm r_z\sin\theta = 2q_1q_2 \pm 2q_0q_3 = 2(q_1q_2 \pm q_0q_3)$
- $r_x r_z(1-\cos\theta) \pm r_y\sin\theta = 2(q_1q_3 \pm q_0q_2)$
- $r_y r_z(1-\cos\theta) \pm r_x\sin\theta = 2(q_2q_3 \pm q_0q_1)$

$$R_{\hat{r},\theta} = \begin{bmatrix} 2(q_1^2+q_0^2)-1 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & 2(q_2^2+q_0^2)-1 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & 2(q_3^2+q_0^2)-1 \end{bmatrix}$$

(c) Let,

$$X = x_0 + ix_1 + jx_2 + kx_3 = x_0 + \mathbf{x}$$
$$Y = y_0 + iy_1 + jy_2 + ky_3 = y_0 + \mathbf{y}$$
$$Z = z_0 + \mathbf{z}$$

$i \to j \to k$

$$XY = x_0y_0 - (x_1y_1 + x_2y_2 + x_3y_3) + x_0(iy_1 + jy_2 + ky_3)$$
$$+ y_0(ix_1 + jx_2 + kx_3) + i(x_2y_3 - x_3y_2)$$
$$+ j(x_3y_1 - x_1y_3) + k(x_1y_2 - x_2y_1)$$

$$= \underbrace{x_0y_0 - \mathbf{x}.\mathbf{y}}_{z_0} + \underbrace{x_0\mathbf{y} + y_0\mathbf{x} + \mathbf{x}\times\mathbf{y}}_{\mathbf{z}}$$

$$x_0y_0 - \mathbf{x}.\mathbf{y} = x_0y_0 - \mathbf{x}^T\mathbf{y} = z_0$$

$$x_0\mathbf{y} + y_0\mathbf{x} + \mathbf{x}\times\mathbf{y} = \mathbf{z}$$

$$XY = Z$$

(d)

$$\mathbf{v} = [v_x \; v_y \; v_z]^T \qquad \mathbf{q} = [q_1\, q_2\, q_3]^T$$

$$Q = (q_0 + \mathbf{q}) \qquad Q^* = (q_0 - \mathbf{q}) \qquad V = (0 + \mathbf{v})$$

$$QV = 0 - \mathbf{q}.\mathbf{v} + q_0\mathbf{v} + \mathbf{q}\times\mathbf{v}$$

$$QVQ^* = -q_0(\mathbf{q}.\mathbf{v}) + q_0(\mathbf{v}.\mathbf{q}) + (\mathbf{q}\times\mathbf{v}).\mathbf{q} + (\mathbf{q}.\mathbf{v})\mathbf{q} + q_0^2\mathbf{v} + q_0(\mathbf{q}\times\mathbf{v})$$
$$- q_0(\mathbf{v}\times\mathbf{q}) - \mathbf{q}\times\mathbf{v}\times\mathbf{q}$$

$$= (\mathbf{q}.\mathbf{v})\mathbf{q} + q_0^2\mathbf{v} + 2q_0(\mathbf{q}\times\mathbf{v}) - \mathbf{q}\times\mathbf{v}\times\mathbf{q}$$

expanding and simplifying all these elements, can be arranged as $(A_{11}i + A_{21}j + A_{31}k)v_x + (A_{12}i + A_{22}j + A_{32}k)v_y + (\cdots)v_z$

$$QVQ^* = R(\theta)\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \mathbf{v}'$$

[$R(\theta)$ is similar matrix as 2.7(b)]

$\mathbf{v}' \to$ rotated co-ordinates of vector $\mathbf{v}$