Spring Semester **February 3, 2024**

IE 616: Decision Analysis and Game Theory

Test 1

Time: 80 minutes Max. marks 40

Exercise 1. Find the best response function of each player and using that, find all the pure NEs of the game. Explain. **Marks 2+1**

	B	S
B	4,3	0,0
S	0,0	3,7

$BR_1(B) = \{B\}$ $BR_2(B) = \{B\}$

$BR_1(S) = \{S\}$ $BR_2(S) = \{S\}$

Pure strategy NEs of the game are: (B,B) and (S,S)

Since the best response against strategy B is B for both the players, and the best response against strategy S is S for both the players, (B, B) and (S, S) are NEs.

Exercise 2. State true or false for the following statements: **Marks 4**

(a) Every dominant strategy equilibrium is an NE. **True**

(b) A Pure Strategy NE provides insurance for a player against unilateral deviations by any other player. **False**

(c) Any game with finite players and finite action sets always has a pure strategy NE. **False**

(d) A two player game with each player having two possible strategies can have five pure strategy NEs. **False**

Exercise 3. Construct an example of a two-player game with two possible actions, where

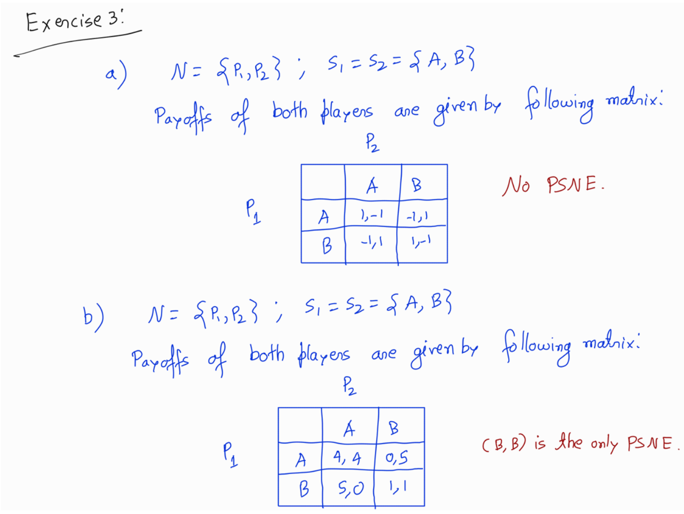
(a) no pure strategy NE exists **Marks 1**

(b) exactly one pure strategy NEs exists **Marks 1**

(c) exactly two pure strategy NEs exists **Marks 1**

(d) exactly three pure strategy NEs exists **Marks 1.5**

(e) exactly four pure strategy NEs exists **Marks 1.5**



Exercise 3:

a) $N = \{P_1, P_2\}$; $S_1 = S_2 = \{A, B\}$

Payoffs of both players are given by following matrix:

P_1 \ P_2	A	B
A	1,-1	-1,1
B	-1,1	1,-1

No PSNE.

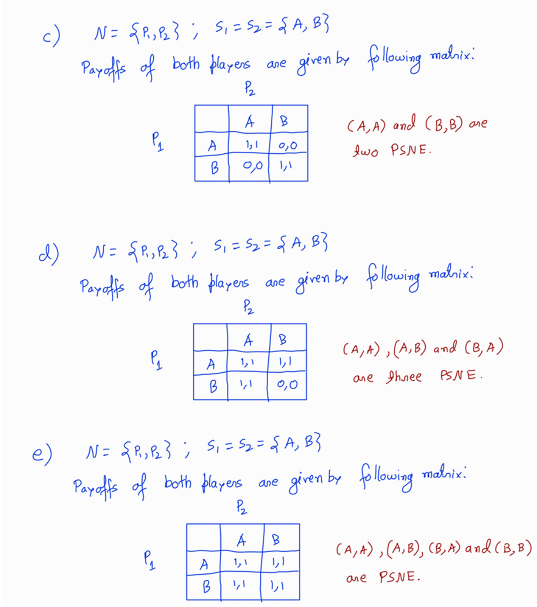
b) $N = \{P_1, P_2\}$; $S_1 = S_2 = \{A, B\}$

Payoffs of both players are given by following matrix:

P_1 \ P_2	A	B
A	4,4	0,5
B	5,0	1,1

(B, B) is the only PSNE.

(a)



c) $N = \{P_1, P_2\}$; $S_1 = S_2 = \{A, B\}$

Payoffs of both players are given by following matrix:

P_1 \ P_2	A	B
A	1,1	0,0
B	0,0	1,1

(A, A) and (B, B) are two PSNE.

d) $N = \{P_1, P_2\}$; $S_1 = S_2 = \{A, B\}$

Payoffs of both players are given by following matrix:

P_1 \ P_2	A	B
A	1,1	1,1
B	1,1	0,0

(A, A), (A, B) and (B, A) are three PSNE.

e) $N = \{P_1, P_2\}$; $S_1 = S_2 = \{A, B\}$

Payoffs of both players are given by following matrix:

P_1 \ P_2	A	B
A	1,1	1,1
B	1,1	1,1

(A, A), (A, B), (B, A) and (B, B) are PSNE.

(b)

Exercise 4. Consider the two-player game given by the following matrix. **Marks 1+1+1**

(a) Find all the NEs for this game.

(b) Find the only dominant strategy in the game.

Player 1 \ Player 2	L	R
T	0,0	2,1
B	3,2	1,2

(c) There is a weakly dominant strategy for this game.
We still have two NEs. Explain!

R is only weakly dominant strategy of player 2, and is not strongly dominant. Against B of player 1, both R and L yield same utility to player 2, and against L, strategy B is in BR. In all, this example shows the possibility of a dominated strategy becoming a part of the NE, as it is only weakly dominated.

Excercise 4

a) (B, L) and (T, R) are two PSNE.

b) Player 2's strategy R is the weakly dominant strategy in the game.

c) The weakly dominant strategy in the game is strategy R of Player 2. The pure strategy NEs are: (B, L) and (T, R). Having NE with Player 2's strategy L indicates that we can not eliminate a weakly dominated strategy.

Exercise 5. There are two players, each simultaneously make a binary choice whether to contribute to the public good. If one or more players contribute to the public good, then all the players receive a reward 1. Additionally, each player that contributes to the public good incurs a cost of $c > 0$. **Marks 2+1+2+2**

(a) Model the above scenario as a strategic form game. State the players, their actions, and utilities clearly.

(b) Are any strategies weakly dominant for either player? Justify the answer.

(c) Which outcome(s) have the highest total payoff across both players? Provide an intuitive explanation of the answer.

(d) Find all Pure Strategy Nash Equilibrium in this game. Justify the answer.

part(b): There is no dominant strategy when $c < 1$, strategy N is weakly dominant when $c = 1$ and strongly dominant when $c > 1$.

Exercise 6. Consider the following Prisoner's dilemma game. Suppose both the players are rational, but this information about rationality is not common knowledge. Find all the pure strategy NEs of the game. While choosing their equilibrium strategy, do players need to know

	C	D
C	4,4	0,5
D	5,0	1,1

whether the other player is rational or not? Justify your answer. **Marks 3**

Is the same true for the following game? Explain! **Marks 2**

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

Exercise 6

	C	D
C	4,4	0,5
D	5,0	1,1

The PSNE of the game is (D, D).
Here D is the strictly dominant strategy for both players. Since both players are rational, independent of other player's strategy, they will play their best response, which leads to (D,D). Thus, players do not need to know whether the other player is rational or not.

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

In the above game the only PSNE is (U,M).
Here the column player will play M only when he knows the row player is playing U. If the row player will play D the column player will instead play L. Further strategy U of row player is not his strictly best response. Thus, in the above game players need to know the other player is rational or not.

Exercise 7. Suppose you want to purchase a bicycle. Our TAs, Prem and Tushar, each own an identical bicycle and are interested in selling them. They are free to set any price p within the range of 1000 to 5000. You decide to buy from the seller who quotes the lowest price. If both offer the same price, you decide not to buy the cycle. Prem values his cycle to be of value

2000, while Tushar values his cycle to be of value 3000. Note that after selling the cycle, the seller loses the value of the cycle in exchange to the price gained.

Marks 2+2 = 4

(a) Model the above scenario as a strategic form game. State the players, their actions, and utilities clearly.

(b) Identify all pure NEs of this game if there exists some. If pure Nash equilibria do not exist, explain why!

Answer: (a) Set of players $N = \{1, 2\}$ (with 1 representing Tushar), Action sets $A_1 = A_2 = [1000, 5000]$ and utility functions,

$$u_1(a_1, a_2) = (a_1 - 3000)1_{a_1 < a_2} \text{ and } u_2(a_1, a_2) = (a_2 - 2000)1_{a_2 < a_1}$$

(b) The Best responses

$$\mathcal{B}_1(a_2) = \arg\max_a u_1(a, a_2) = \begin{cases} [a_2, 5000] & \text{if } a_2 \leq 3000 \\ \text{does not exist} & \text{if } a_2 > 3000 \end{cases}$$

$$\mathcal{B}_2(a_1) = \arg\max_a u_2(a, a_1) = \begin{cases} [a_1, 5000] & \text{if } a_1 \leq 2000 \\ \text{does not exist} & \text{if } a_1 > 2000 \end{cases}$$

Thus all pure strategy NE are

$$\{(a, a) : a \in [1000, 2000]\}$$

and at all of these NE, both the players attain $(0, 0)$.

Exercise 8. Suppose a strategic form game $\langle N, (S_i), (u_i) \rangle$ has a mixed strategy NE $(\sigma_1^*, \cdots, \sigma_n^*)$. Then, show that

$$u_i(\sigma_1^*, \cdots, \sigma_n^*) \geq \bar{v}_i \geq \underline{v}_i \quad \forall\, i \in N,$$

where $\bar{v}_i$ is the min-max value in mixed strategies of player i and $\underline{v}_i$ is the max-min value in mixed strategies of player i (defined exactly as with pure strategies). **Marks 8**

Ex8

$\sigma_1^* \ldots \sigma_n^* \rightarrow$ Mixed Strategy NE

$$U_i(\sigma_1^* \ldots \sigma_n^*) = \max_{\sigma_i} U_i(\sigma_i \sigma_{-i}^*) \quad \forall i \quad \rightarrow (1)$$

Now observe

$$\max_{\sigma_i} U_i(\sigma_i \sigma_{-i}^*) \geq \min_{\sigma_{-i}} \max_{\sigma_i} U_i(\sigma_i \sigma_{-i}) \quad \forall i \in N \quad \rightarrow (2)$$

Thus from (1) & (2), it is clear that

$$U_i(\sigma_1^* \ldots \sigma_n^*) \geq \bar{v}_i \quad \forall i \in N$$

Now proving $\bar{v}_i \geq \underline{v}_i$

Suppose σ_{-i}^* is a minmax strategy against player i. This means

$$\bar{v}_i = \max_{\sigma_i} U_i(\sigma_i, \sigma_{-i}^*) \quad \forall i$$

Now also notice $\forall i \in N$

$$U_i(\sigma_i \sigma_{-i}^*) \geq \min_{\sigma_{-i}} U_i(\sigma_i \sigma_{-i}) \quad \forall \sigma_i$$

$$\bar{v}_i = \max_{\sigma_i} U_i(\sigma_i \sigma_{-i}^*) \geq \max_{\sigma_i} \min_{\sigma_{-i}} U_i(\sigma_i \sigma_{-i}) = \underline{v}_i \quad \forall i \in N$$

(a)

Thus $\bar{v}_i \geq \underline{v}_i \quad \forall i \in N$

Thus we get $U_i(\sigma_1^* \ldots \sigma_n^*) \geq \bar{v}_i \geq \underline{v}_i \quad \forall i \in N$

(b)