Engineering Control Systems - Assignment 7

1 Exercise 1: Second-Order System Control

Problem Statement: Design a proportional-derivative (PD) controller for the given second-order system:

$$2\ddot{x} + \dot{x} = F(t)$$

Required: Determine controller gains K_p and K_d to achieve critical damping, satisfying $\omega_n < 10$ rad/s and $\zeta > 0.707$.

Method and Solution

We implement a PD controller with error $e(t) = x_d(t) - x(t)$:

$$F(t) = K_p e(t) + K_d \dot{e}(t)$$

Integrating this control law into our system:

$$2\ddot{x} + \dot{x} = K_p(x_d - x) + K_d(\dot{x}_d - \dot{x})$$

After rearrangement:

$$2\ddot{x} + (1 + K_d)\dot{x} + K_p x = K_p x_d + K_d \dot{x}_d$$

This yields the characteristic equation:

$$2s^2 + (1 + K_d)s + K_p = 0$$

Comparing with the standard form $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, we establish:

$$\omega_n^2 = \frac{K_p}{2} \quad \text{and} \quad 2\zeta\omega_n = \frac{1 + K_d}{2}$$

Therefore:

$$K_p = 2\omega_n^2 \quad \text{and} \quad K_d = 4\zeta\omega_n - 1$$

To satisfy our design requirements, we select $\omega_n = 5$ rad/s and $\zeta = 1$ (critically damped), yielding:

$$K_p = 2(5^2) = 50 \quad \text{and} \quad K_d = 4(1)(5) - 1 = 19$$

Final Controller: $F(t) = 50e(t) + 19\dot{e}(t)$

2 Exercise 2: Nonlinear System Analysis

Problem Statement: Analyze and develop a control strategy for the given nonlinear system.

Part A: Control Input Expressions

The nonlinear system is described by these interconnected equations:

$$u_1 = \ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2 - y_2 u_2 \tag{1}$$

$$u_2 = \ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) + (\cos y_1)^2 y_2 u_1 \tag{2}$$

We need to express inputs u_1 and u_2 explicitly in terms of state variables and their derivatives.

First, substituting the u_1 expression into the u_2 equation:

$$u_2 = \ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) \tag{3}$$
$$+ (\cos y_1)^2 y_2(\ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2 - y_2 u_2) \tag{4}$$

Collecting u_2 terms:

$$u_2[1 + (\cos y_1)^2 y_2^2] = \ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) + (\cos y_1)^2 y_2(\ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2)$$

Solving for u_2:

$$u_2 = \frac{\ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) + (\cos y_1)^2 y_2(\ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2)}{1 + (\cos y_1)^2 y_2^2}$$

This expression for u_2 can be substituted back into the original equation for u_1.

Part B: Inverse Dynamics Control

For effective control, we transform the system equations to:

$$\ddot{y}_1 = -3y_1\dot{y}_2 - y_2^2 + u_1 + y_2 u_2 \tag{5}$$
$$\ddot{y}_2 = -(\cos y_1)\dot{y}_2 - 3(y_1 - y_2) + u_2 - (\cos y_1)^2 y_2 u_1 \tag{6}$$

In matrix notation:

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} -3y_1\dot{y}_2 - y_2^2 \\ -(\cos y_1)\dot{y}_2 - 3(y_1 - y_2) \end{bmatrix} + \begin{bmatrix} 1 & y_2 \\ -(\cos y_1)^2 y_2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

For tracking control, we define virtual control inputs:

$$\nu_1 = \ddot{y}_{1d} + 2\zeta\omega_n(\dot{y}_{1d} - \dot{y}_1) + \omega_n^2(y_{1d} - y_1) \tag{7}$$
$$\nu_2 = \ddot{y}_{2d} + 2\zeta\omega_n(\dot{y}_{2d} - \dot{y}_2) + \omega_n^2(y_{2d} - y_2) \tag{8}$$

Setting $\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix}$ and computing control inputs:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{1 + y_2^2(\cos y_1)^2} \begin{bmatrix} 1 & -y_2 \\ (\cos y_1)^2 y_2 & 1 \end{bmatrix} \left(\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} - \begin{bmatrix} -3y_1\dot{y}_2 - y_2^2 \\ -(\cos y_1)\dot{y}_2 - 3(y_1 - y_2) \end{bmatrix} \right)$$

With $\omega_n = 10$ rad/s and $\zeta = 0.5$, this controller achieves desired tracking performance.

3 Exercise 3: Robotic Manipulator Dynamics

Problem Statement: Derive the equations of motion for a single-link robotic manipulator and design a critically damped controller.

Energy Formulation

For a single-link manipulator, we analyze:

- Kinetic Energy: $T = \frac{1}{2}I\dot{\theta}^2$
- Potential Energy: $V = \sum_i m_i g h_i$ (sum over all masses)
- Lagrangian: $L = T - V$

Applying Euler-Lagrange formulation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau$$

This yields the standard form:

$$\tau = I\ddot{\theta} + B\dot{\theta} + G(\theta)$$

where $G(\theta)$ represents gravitational torque.

Controller Design

We implement a PD controller:

$$\tau = K_p(\theta_d - \theta) - K_d\dot{\theta}$$

assuming zero desired velocity and acceleration. The closed-loop dynamics become:

$$I\ddot{\theta} + K_d\dot{\theta} + K_p\theta = K_p\theta_d + G(\theta)$$

The characteristic equation is:

$$Is^2 + K_ds + K_p = 0$$

For critically damped response ($\zeta = 1$) with $\omega_n = 4$ rad/s:

$$K_p = I\omega_n^2 = 16I \tag{9}$$
$$K_d = 2\zeta\omega_n I = 8I \tag{10}$$

Final Controller: $\tau = 16I(\theta_d - \theta) - 8I\dot{\theta}$

4 Exercise 4: Two-Link Manipulator Analysis

Problem Statement: Analyze a two-link manipulator performing circular motion and design a controller with disturbance rejection.

Dynamic Model Derivation

The kinetic energy incorporates translational and rotational components:

$$T = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}m(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\dot{\theta}_1\dot{\theta}_2\cos\theta_2) \tag{11}$$
$$+ J_0r^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) \tag{12}$$

The potential energy accounts for gravitational effects:

$$V = mgl\cos\theta_1 + mgl\cos(\theta_1 + \theta_2)$$

Applying Lagrangian mechanics yields:

$$\tau_1 = (2ml^2 + J_0r^2)\ddot{\theta}_1 + ml^2\cos\theta_2\ddot{\theta}_2 - ml^2\sin\theta_2\dot{\theta}_2^2 \tag{13}$$
$$- 2ml^2\sin\theta_2\dot{\theta}_1\dot{\theta}_2 + mgl\sin\theta_1 + mgl\sin(\theta_1 + \theta_2) + B_0r\dot{\theta}_1 \tag{14}$$

$$\tau_2 = ml^2\cos\theta_2\ddot{\theta}_1 + (ml^2 + J_0r^2)\ddot{\theta}_2 + ml^2\sin\theta_2\dot{\theta}_1^2 \tag{15}$$
$$+ mgl\sin(\theta_1 + \theta_2) + B_0r\dot{\theta}_2 \tag{16}$$

Circular Trajectory Generation

For end-effector positions:

$$x = l\cos\theta_1 + l\cos(\theta_1 + \theta_2) \tag{17}$$
$$y = l\sin\theta_1 + l\sin(\theta_1 + \theta_2) \tag{18}$$

The desired circular trajectory with radius $R = \frac{3l}{2}$ and angular velocity $v = 10\pi$ rad/s:

$$x = \frac{3l}{2}\cos(10\pi t) \tag{19}$$
$$y = \frac{3l}{2}\sin(10\pi t) \tag{20}$$

PD Controller Design

For each joint, we implement:

$$\tau_i = K_{p_i}(\theta_{id} - \theta_i) + K_{d_i}(\dot{\theta}_{id} - \dot{\theta}_i)$$

With $\omega_n = 36$ rad/s and critical damping:

$$K_{p1} = 2592ml^2 \tag{21}$$
$$K_{d1} = 144ml^2 \tag{22}$$
$$K_{p2} = 1296ml^2 \tag{23}$$
$$K_{d2} = 72ml^2 \tag{24}$$

Disturbance Analysis

With constant disturbance τ_d acting on joint 1, the steady-state error:

$$e_{ss} = -\frac{\tau_d}{K_{p1}} = -\frac{\tau_d}{2592ml^2}$$

This demonstrates the controller's disturbance rejection capability.

5 Exercise 5: Forward Kinematics and Dynamics

Problem Statement: Develop transformation matrices, Jacobians, and inertia models for a two-link robotic manipulator.

Transformation Matrices

For centers of mass locations with $q_1 = \theta_1$ and $q_2 = \theta_2$:

$$^{0}_{C_1}T = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & \frac{L_1}{2}\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & \frac{L_1}{2}\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{0}_{C_2}T = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & L_1\cos\theta_1 + \frac{L_2}{2}\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & L_1\sin\theta_1 + \frac{L_2}{2}\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Velocity Jacobians

Linear velocity Jacobians relate joint rates to linear velocities of centers of mass:

$$^{0}J_{v1} = \begin{bmatrix} -\frac{L_1}{2}\sin\theta_1 & 0 \\ \frac{L_1}{2}\cos\theta_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$^{0}J_{v2} = \begin{bmatrix} -L_1\sin\theta_1 - \frac{L_2}{2}\sin(\theta_1+\theta_2) & -\frac{L_2}{2}\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + \frac{L_2}{2}\cos(\theta_1+\theta_2) & \frac{L_2}{2}\cos(\theta_1+\theta_2) \\ 0 & 0 \end{bmatrix}$$

Angular velocity Jacobians map joint rates to angular velocities:

$$^{c1}J_{\omega 1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$^{c2}J_{\omega 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Inertia Properties

For links with uniform density and square cross-section:

$$I_{c1} = \frac{1}{12}m_1(L_1^2 + h^2)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$I_{c2} = \frac{1}{12}m_2(L_2^2 + h^2)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Manipulator Inertia Matrix

The joint-space inertia matrix incorporates all mass effects:

$$\begin{aligned}D(q) &= m_1 J_{v1}^T J_{v1} + m_2 J_{v2}^T J_{v2} + J_{\omega1}^T I_{c1} J_{\omega1} + J_{\omega2}^T I_{c2} J_{\omega2} && (25)\\ &= \begin{bmatrix}D_{11} & D_{12}\\D_{21} & D_{22}\end{bmatrix} && (26)\end{aligned}$$

Where:

$$\begin{aligned}D_{11} &= \frac{1}{4}m_1L_1^2 + m_2\left(L_1^2 + L_1L_2\cos\theta_2 + \frac{1}{4}L_2^2\right) + \frac{1}{12}m_1(L_1^2 + h^2) && (27)\\ D_{12} = D_{21} &= m_2\left(\frac{1}{2}L_1L_2\cos\theta_2 + \frac{1}{4}L_2^2\right) && (28)\\ D_{22} &= \frac{1}{4}m_2L_2^2 + \frac{1}{12}m_2(L_2^2 + h^2) && (29)\end{aligned}$$

Complete Dynamic Equations

The complete motion equations include Coriolis/centrifugal effects:

$$C(q,\dot{q}) = \begin{bmatrix}-m_2L_1L_2\sin\theta_2\dot{\theta}_2 & -m_2L_1L_2\sin\theta_2(\dot{\theta}_1 + \dot{\theta}_2)\\ m_2L_1L_2\sin\theta_2\dot{\theta}_1 & 0\end{bmatrix}$$

And gravitational effects:

$$G(q) = \begin{bmatrix}\frac{1}{2}m_1gL_1\cos\theta_1 + m_2g\left(L_1\cos\theta_1 + \frac{1}{2}L_2\cos(\theta_1+\theta_2)\right)\\ \frac{1}{2}m_2gL_2\cos(\theta_1+\theta_2)\end{bmatrix}$$

6 Exercise 6: Task-Space Control Design

Problem Statement: Develop task-space dynamics and controller for a two-link manipulator.

Task-Space Dynamics Derivation

For end-effector kinematics:

$$\begin{aligned}x &= l_1\cos q_1 + l_2\cos(q_1+q_2) && (30)\\ y &= l_1\sin q_1 + l_2\sin(q_1+q_2) && (31)\end{aligned}$$

The differential relationship is:

$$\dot{x} = J(q)\dot{q}$$

with Jacobian:

$$J(q) = \begin{bmatrix}-l_1\sin q_1 - l_2\sin(q_1+q_2) & -l_2\sin(q_1+q_2)\\ l_1\cos q_1 + l_2\cos(q_1+q_2) & l_2\cos(q_1+q_2)\end{bmatrix}$$

Transforming joint-space dynamics to task-space:

$$F = \Lambda(q)\ddot{x} + \mu(q,\dot{q}) + p(q)$$

where:

$$\Lambda(q) = (J(q)^{-T} D(q) J(q)^{-1})^{-1} \tag{32}$$

$$\mu(q, \dot{q}) = \Lambda(q) J(q)^{-T} \left(C(q, \dot{q}) - D(q) J(q)^{-1} \dot{J}(q) \right) J(q)^{-1} \dot{x} \tag{33}$$

$$p(q) = \Lambda(q) J(q)^{-T} G(q) \tag{34}$$

Nonlinear Decoupling PD Controller

The task-space control law implements:

$$F = \Lambda(q)(\ddot{x}_d + K_v \dot{e} + K_p e) + \mu(q, \dot{q}) + p(q)$$

Converting to joint torques:

$$\tau = J^T(q) F$$

With critically damped response at $\omega_n = 36$ rad/s:

$$K_p = \begin{bmatrix} 1296 & 0 \\ 0 & 1296 \end{bmatrix}, \quad K_v = \begin{bmatrix} 72 & 0 \\ 0 & 72 \end{bmatrix}$$

The final controller compensates for all nonlinear effects while ensuring precise trajectory tracking in the task space.