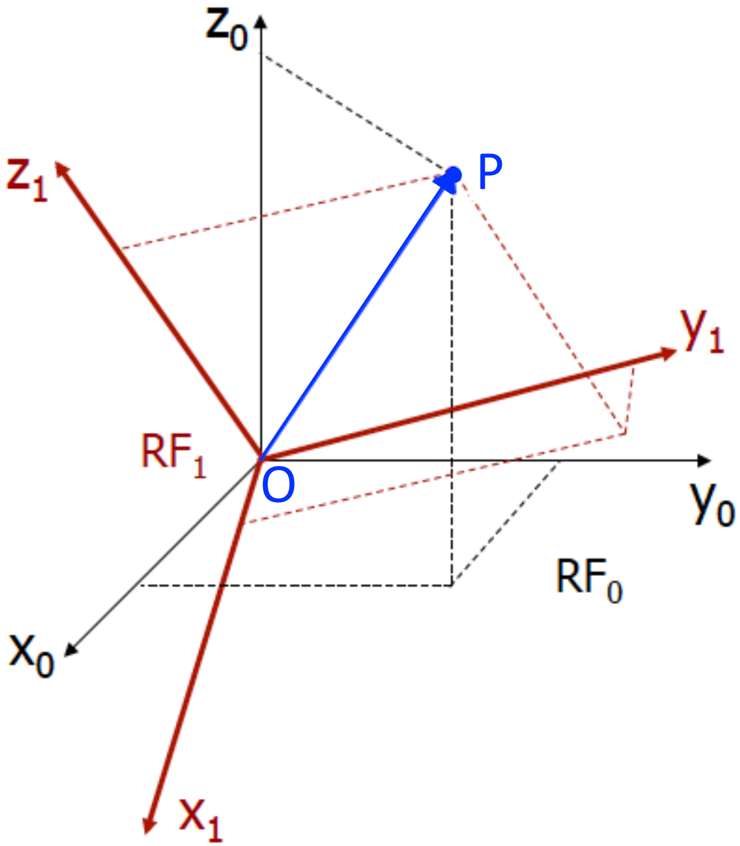
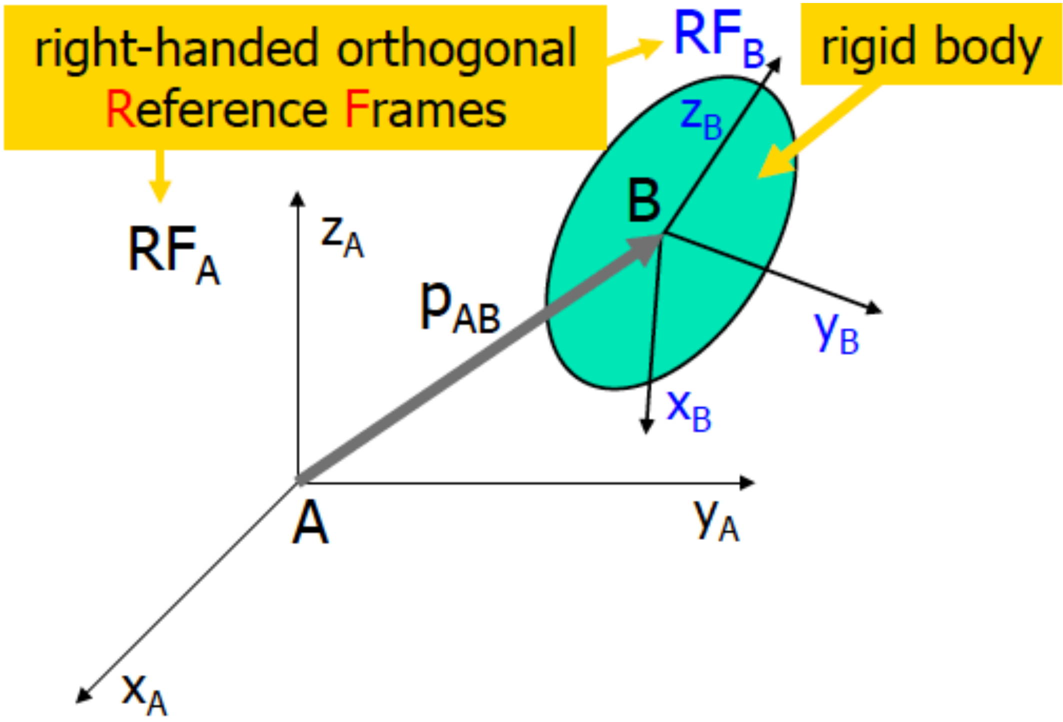
Position of a point



With respect to a fixed origin O, the position of a point P is described by the vector OP

Coordinate frames: vector OP is independent of the coordinate frame

Rigid Body Configuration



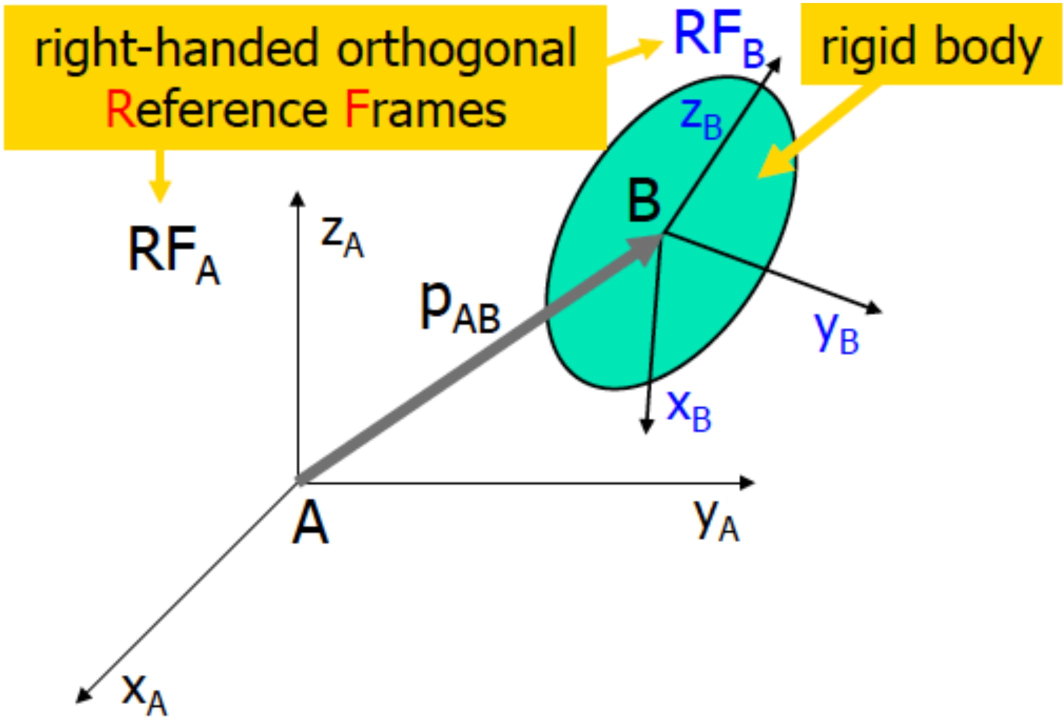
Position: ${}^{A}P_{B} = p_{AB}$

Orientation: $\{{}^{A}x_{B}, {}^{A}y_{B}, {}^{A}z_{B}\}$

Unit vectors

Describe orientation of {B} with respect to {A}

Rotation matrix



Rotation matrix: ${}^{A}_{B}R$

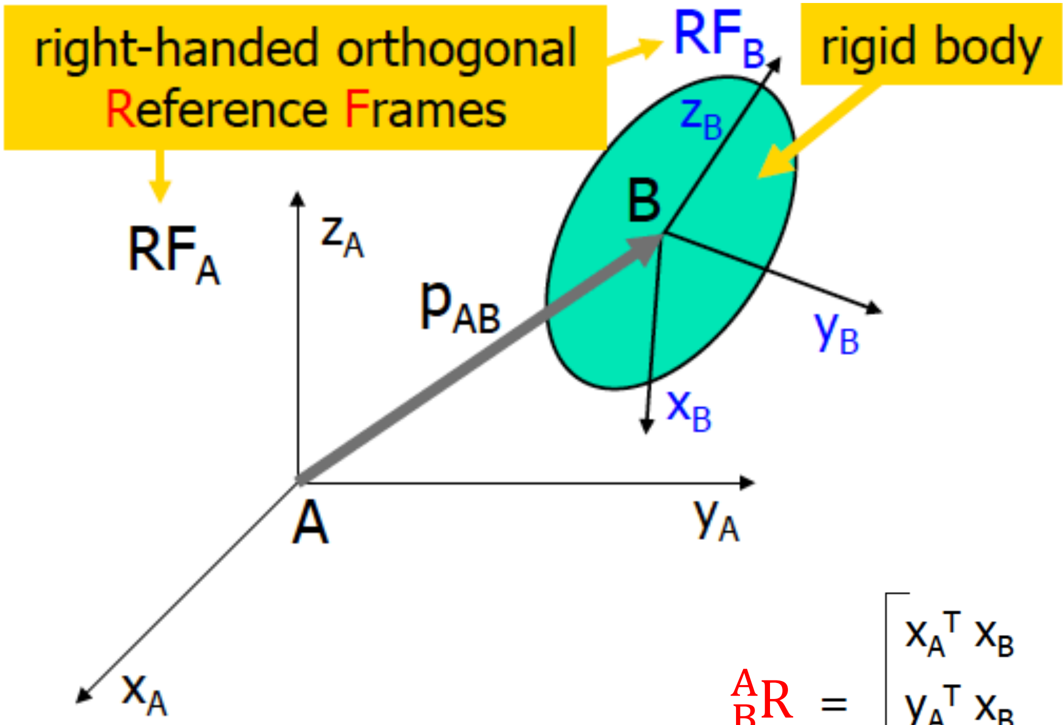
$${}^{A}x_B = {}^{A}_{B}R\,{}^{B}x_B = {}^{A}_{B}R\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$${}^{A}y_B = {}^{A}_{B}R\,{}^{B}y_B = {}^{A}_{B}R\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$${}^{A}y_B = {}^{A}_{B}R\,{}^{B}y_B = {}^{A}_{B}R\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$${}^{A}_{B}R = [{}^{A}x_B \quad {}^{A}y_B \quad {}^{A}z_B]$$

Rotation matrix



$${}^A_BR = [{}^Ax_B \quad {}^Ay_B \quad {}^Az_B]$$

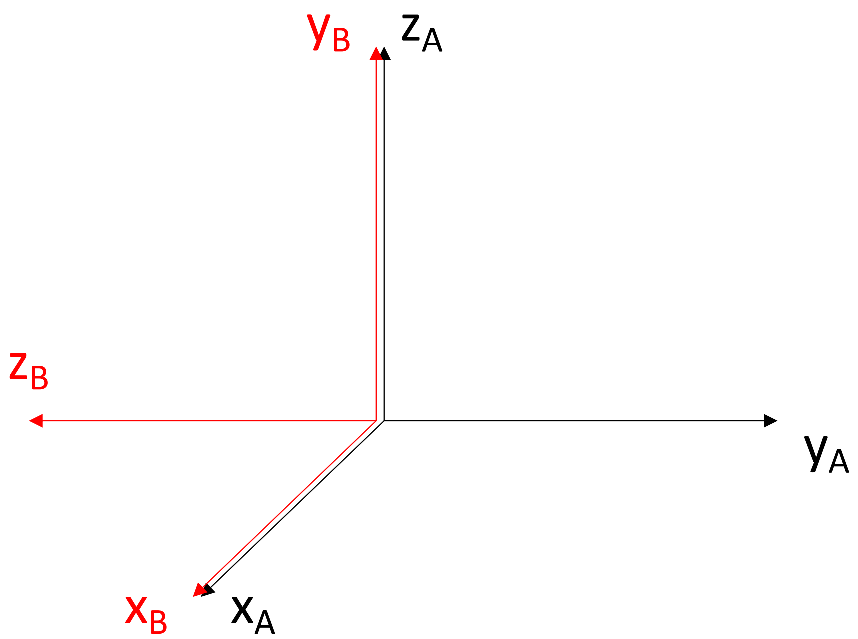
$${}^Ax_B = \begin{bmatrix} x_B.xA \\ x_B\,.yA \\ x_B.zA \end{bmatrix}$$

$${}^A_BR = \begin{bmatrix} x_A^T\, x_B & x_A^T\, y_B & x_A^T\, z_B \\ y_A^T\, x_B & y_A^T\, y_B & y_A^T\, z_B \\ z_A^T\, x_B & z_A^T\, y_B & z_A^T\, z_B \end{bmatrix}$$

direction cosine of z_B w.r.t. x_A

${}^Bz_A^T$

Example



$$
{}^{A}_{B}R = \begin{array}{cl}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{array}{l} {}^{B}x_A{}^{T} \\ {}^{B}y_A{}^{T} \\ {}^{B}z_A{}^{T} \end{array} \\
\begin{array}{ccc} {}^{A}x_B & {}^{A}y_B & {}^{A}z_B \end{array} &
\end{array}
$$

Rotation matrix

$$ {}^{A}_{B}R = [{}^{A}x_B \quad {}^{A}y_B \quad {}^{A}z_B] = \begin{bmatrix} {}^{B}x_A{}^{T} \\ {}^{B}y_A{}^{T} \\ {}^{B}z_A{}^{T} \end{bmatrix} = [{}^{B}x_A \quad {}^{B}y_A \quad {}^{B}z_A]^T = {}^{B}_{A}R^T $$

Inverse of Rotation matrices ${}^{A}_{B}R^{-1} = {}^{B}_{A}R = {}^{A}_{B}R^{T}$ (Orthonormal matrices)

chain rule property

$$ {}^{k}_{i}R.\ {}^{i}_{j}R = {}^{k}_{j}R $$

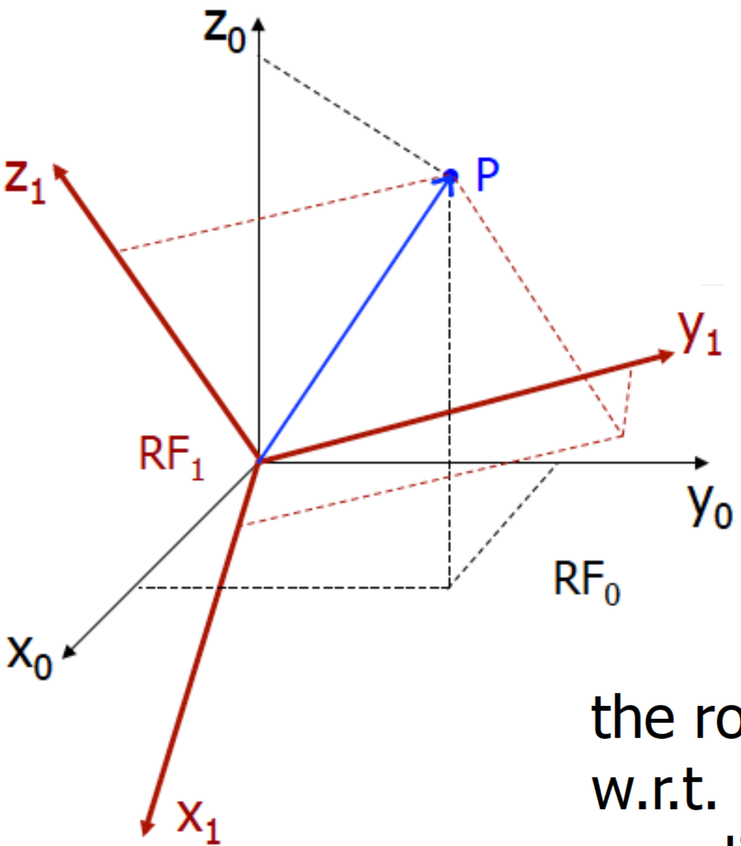
Orientation of RF_i w.r.t RF_k

Orientation of RF_j w.r.t RF_i

Orientation of RF_j w.r.t RF_k

In general, the product of rotation matrices does not commute!

Mapping: Change of Coordinates



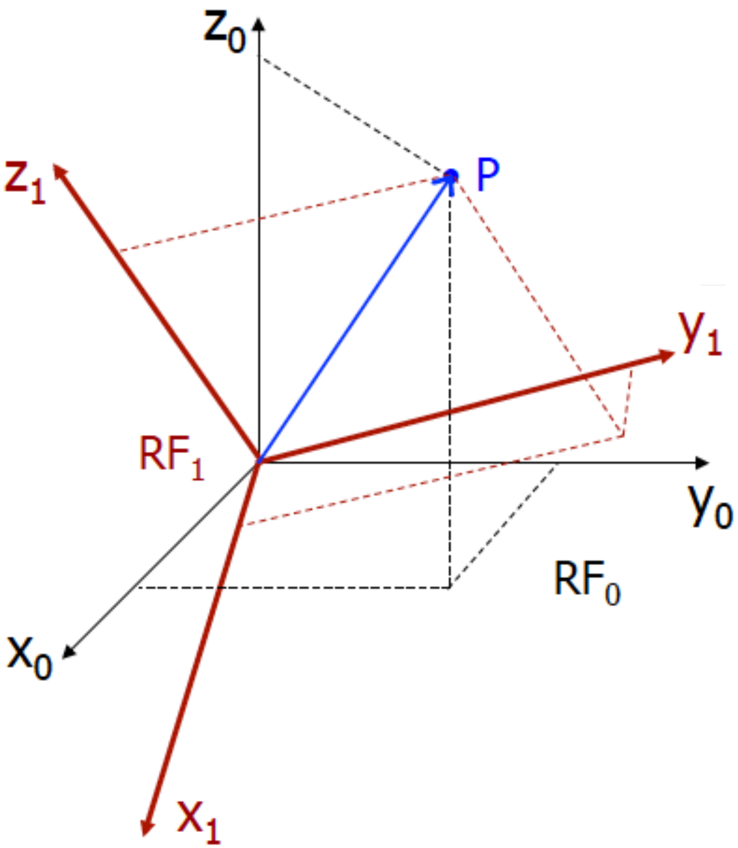
$$^0p = \begin{bmatrix} x_0.P \\ y_0\,.P \\ z_0.P \end{bmatrix} = \begin{bmatrix} x_0{}^T P \\ y_0{}^T P \\ z_0{}^T P \end{bmatrix} = \begin{bmatrix} x_0{}^T \\ y_0{}^T \\ z_0{}^T \end{bmatrix} P$$

If P is given in RF_1

$$^0p = \begin{bmatrix} {}^1x_0.{}^1P \\ {}^1y_0\,.{}^1P \\ {}^1z_0.{}^1P \end{bmatrix} = \begin{bmatrix} {}^1x_0{}^T \\ {}^1y_0{}^T \\ {}^1z_0{}^T \end{bmatrix} {}^1P = {}^0_1R\,{}^1P$$

the rotation matrix ${}^0_1R$ (i.e., the orientation of RF_1 w.r.t. RF_0) represents also the change of coordinates of a vector from RF_1 to RF_0

Operator

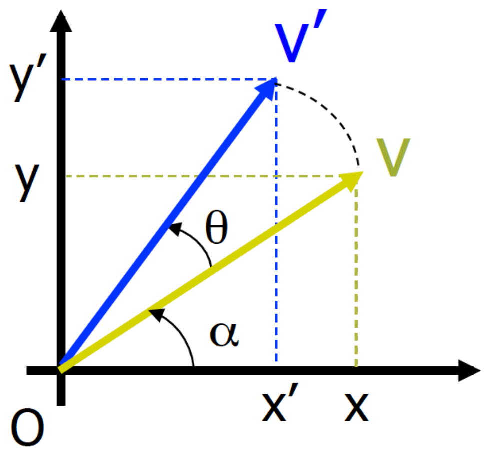


Moving points (within the same frame)

Mapping: ${}^{0}p = {}^{0}_{1}R\,{}^{1}P$

Operator: ${}^{0}P_2 = \quad R\,{}^{0}P_1$

Rotation about z-axis



$$x = |v| \cos \alpha$$
$$y = |v| \sin \alpha$$

$$x' = |v| \cos(\alpha + \theta) = |v| (\cos \alpha \cos \theta - \sin \alpha \sin \theta)$$
$$= x \cos \theta - y \sin \theta$$

$$y' = |v| \sin(\alpha + \theta) = |v| (\sin \alpha \cos \theta + \cos \alpha \sin \theta)$$
$$= x \sin \theta + y \cos \theta$$

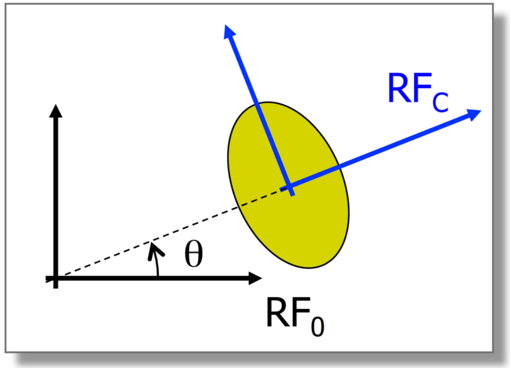
$$z' = z$$

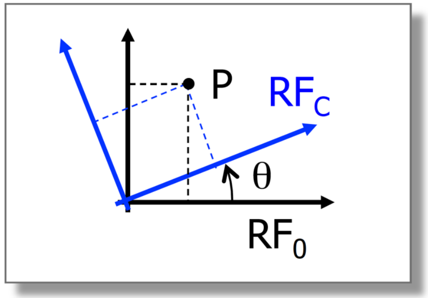
or...

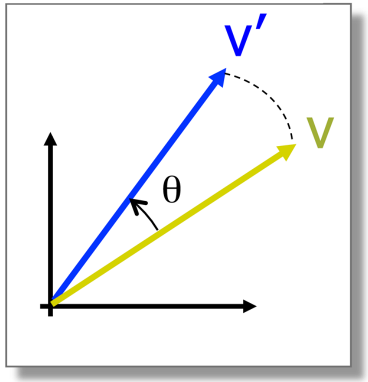
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Interpretations of rotation matrices

the same rotation matrix, e.g., $R_z(\theta)$, may represent:







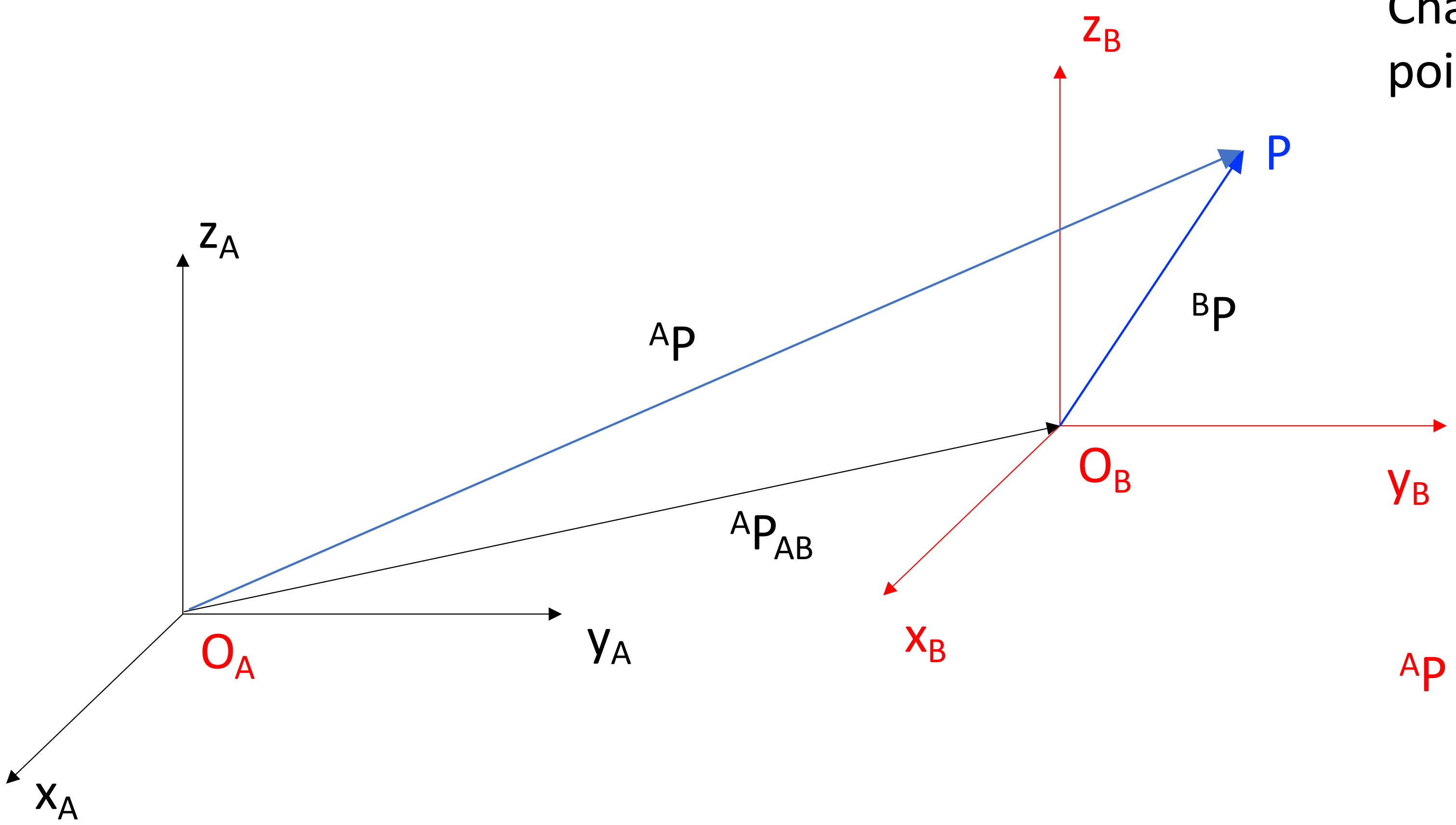
the orientation of a rigid body with respect to a reference frame RF_0
ex: $[{}^0x_c \; {}^0y_c \; {}^0z_c] = R_z(\theta)$

the change of coordinates from RF_C to RF_0
ex: ${}^0P = R_z(\theta)\, {}^CP$

the vector rotation operator
ex: $v' = R_z(\theta)\, v$

the rotation matrix ${}^0R_C$ is an operator superposing frame RF_0 to frame RF_C

Translations

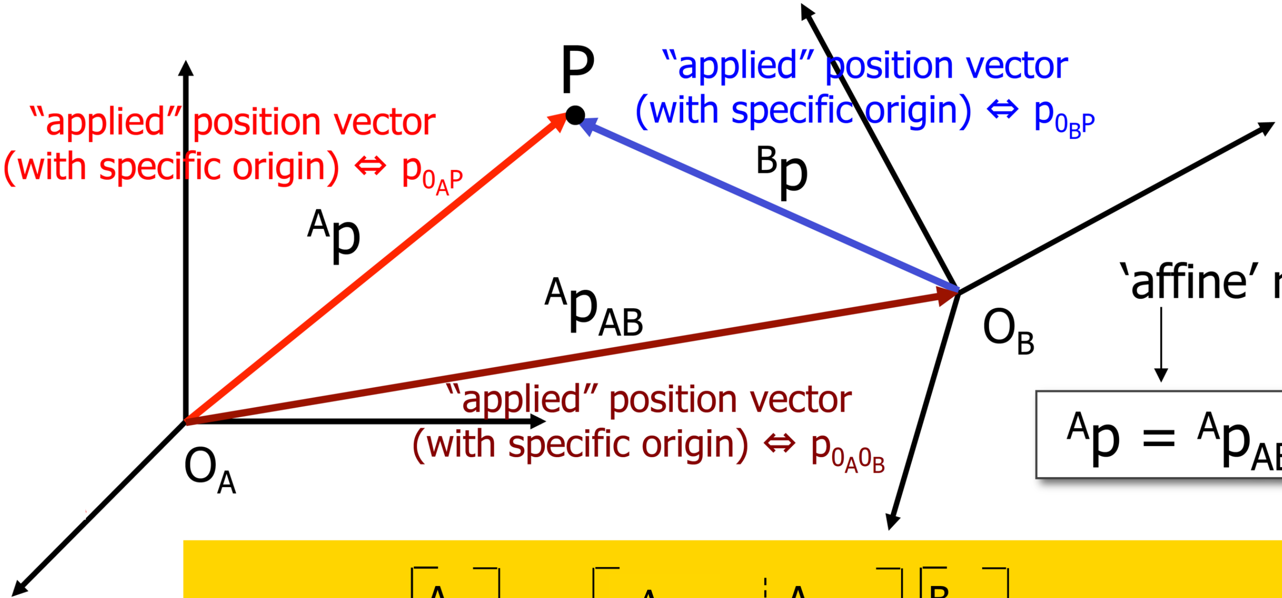


Changing the description of point P

$O_BP \Rightarrow O_AP$

$^AP = O_AP = {^AP_{AB}} + {^BP}$

Homogeneous transformations



'affine' relationship

$$\boxed{{}^A p = {}^A p_{AB} + {}^A_B R \; {}^B p}$$

$$\boxed{{}^A p_{hom} = \left[\begin{array}{c} {}^A p \\ \hline 1 \end{array}\right] = \left[\begin{array}{ccc|c} & {}^A_B R & & {}^A p_{AB} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} {}^B p \\ \hline 1 \end{array}\right] = {}^A_B T \; {}^B p_{hom}}$$

linear relationship

vector in homogeneous coordinates

4x4 matrix of homogeneous transformation

Properties of T matrix

describes the relation between reference frames (relative pose = position & orientation)

transforms the representation of a position vector (applied vector starting from the origin of the frame) from a given frame to another frame

it is a roto-translation operator on vectors in the three-dimensional space

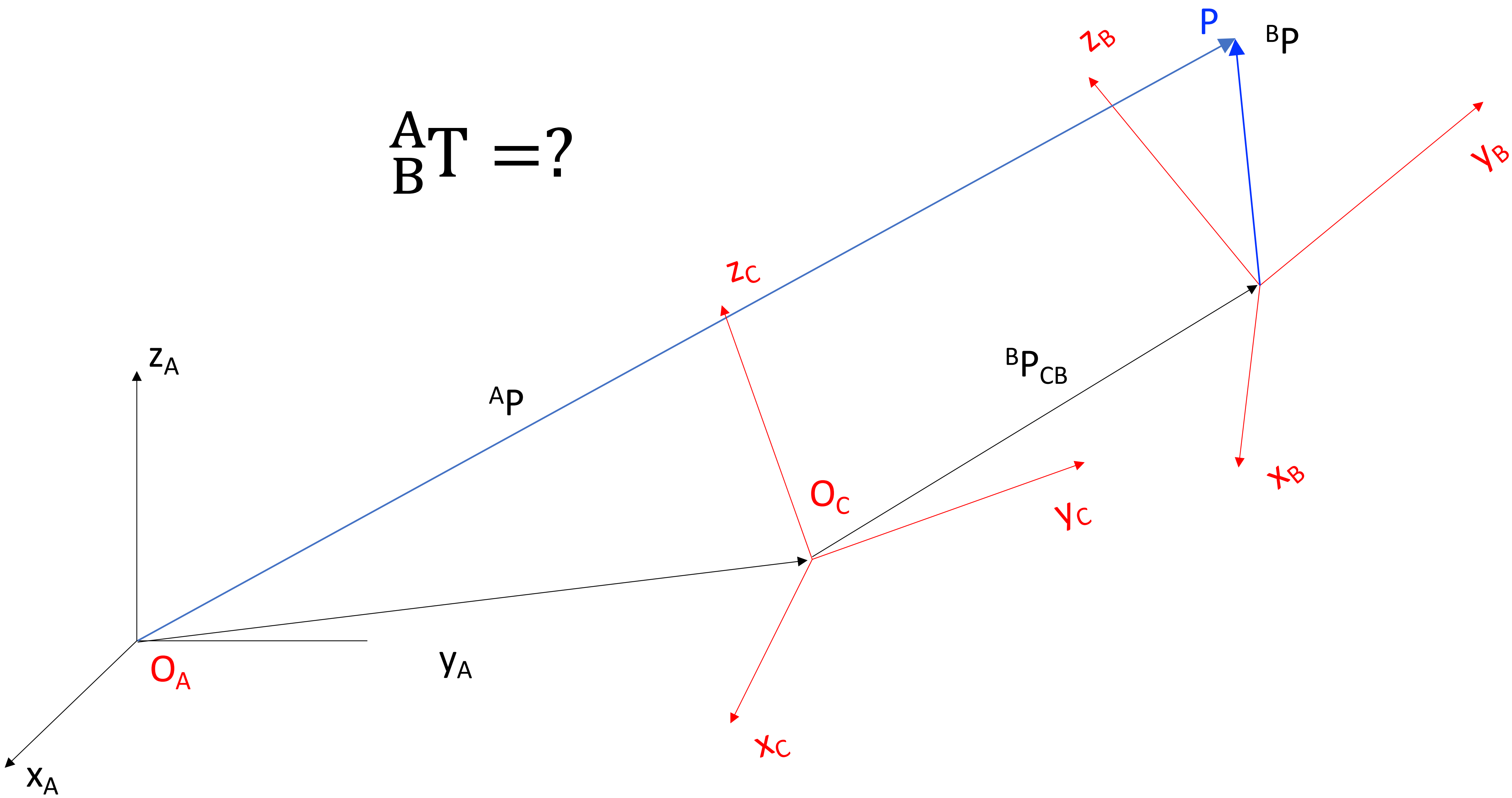
it is always invertible $\left({}^{A}_{B}T \right)^{-1} = {}^{B}_{A}T$

can be composed, i.e., ${}^{A}_{C}T = {}^{A}_{B}T \, {}^{B}_{C}T$ ← note: it does not commute!

Inverse of a homogeneous transformation

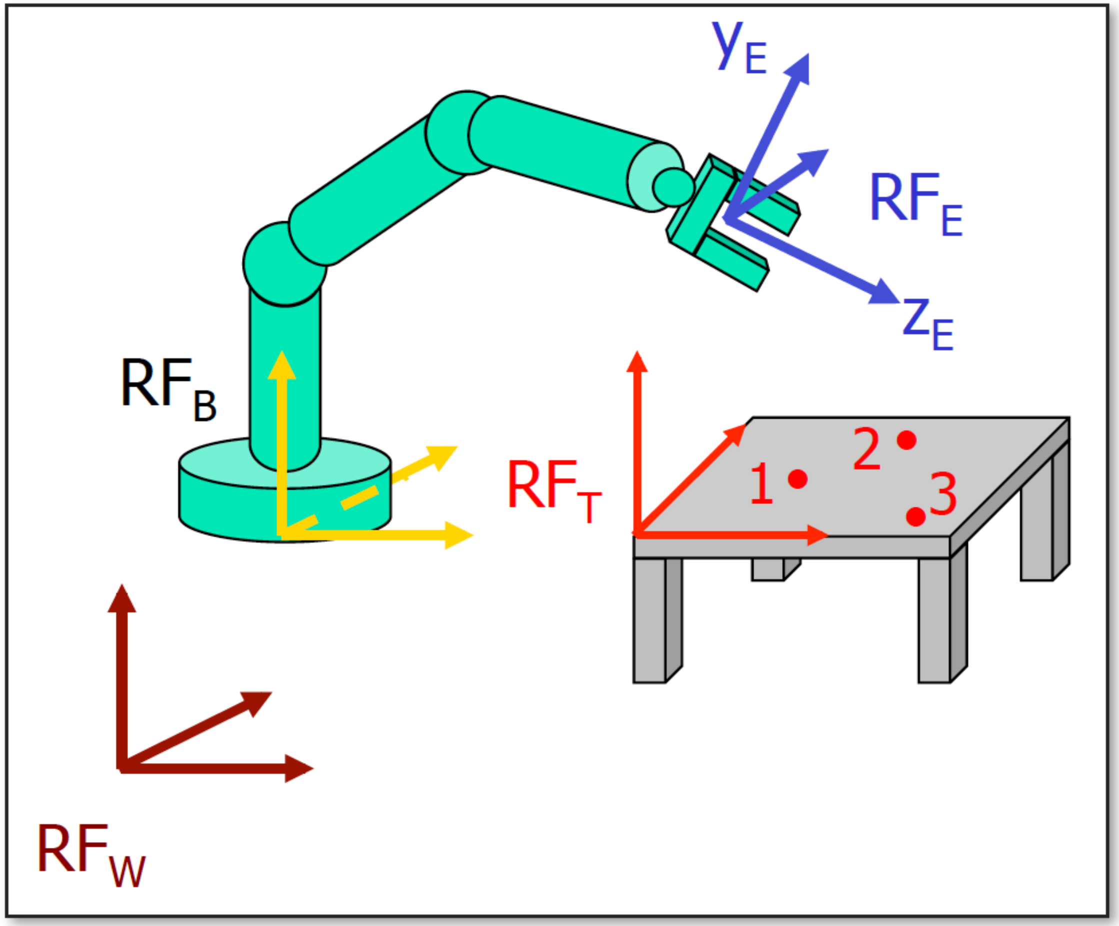
$$^{A}p = {}^{A}p_{AB} + {}^{A}_{B}R\,{}^{B}p \qquad {}^{B}p = {}^{B}p_{BA} + {}^{B}_{A}R\,{}^{A}p = -\,{}^{A}R_{B}^{T}\,{}^{A}p_{AB} + {}^{B}_{A}R^{T}\,{}^{A}p$$

$$\underbrace{\left[\begin{array}{ccc:c} & {}^{A}_{B}R & & {}^{A}p_{AB} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]}_{{}^{A}_{B}T} \qquad \underbrace{\left[\begin{array}{ccc:c} & {}^{B}_{A}R & & {}^{B}p_{BA} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]}_{{}^{B}_{A}T} = \underbrace{\left[\begin{array}{ccc:c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}p_{AB} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]}_{({}^{A}_{B}T)^{-1}}$$



$^{A}_{B}T = ?$
P
^{B}P
z_B
y_B
x_B
z_C
y_C
x_C
O_C
$^{B}P_{CB}$
^{A}P
z_A
y_A
x_A
O_A

Transform Equation



absolute definition of task

task definition relative to the robot end-effector

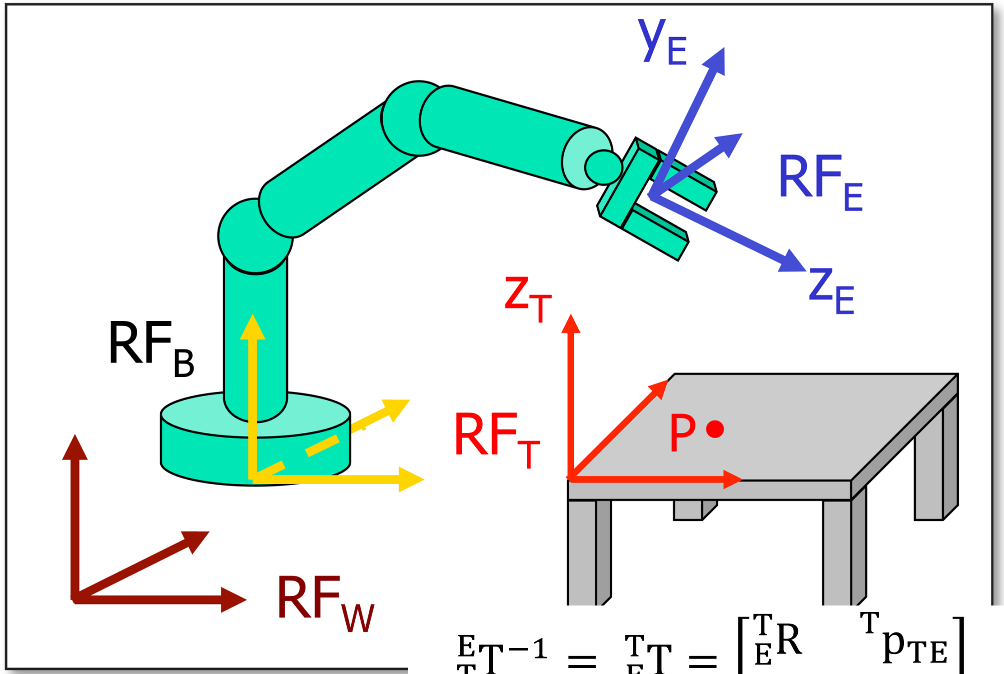
$$ {}^{W}_{T}T = {}^{W}_{B}T \quad {}^{B}_{E}T \quad {}^{E}_{T}T $$

known, once the robot is placed

direct kinematics of the robot arm (function of q)

$$ {}^{B}_{E}T\,(q) = {}^{W}_{B}T^{\,-1} \quad {}^{W}_{T}T \quad {}^{E}_{T}T^{\,1} = \text{constant} $$

Example



$$ {}^{E}_{T}T^{-1} = {}^{T}_{E}T = \begin{bmatrix} {}^{T}_{E}R & {}^{T}p_{TE} \\ 0 & 1 \end{bmatrix} $$

$$ {}^{T}_{E}R = {}^{E}_{T}R^{T} $$

$$ {}^{T}p_{TE} = {}^{T}p - {}^{T}_{E}R\,{}^{E}p = \begin{bmatrix} p_x \\ p_y \\ h \end{bmatrix} $$

- the robot carries a depth camera (e.g., a Kinect) on the end-effector
- the end-effector should go to a pose above the point P on the table, pointing its approach axis downward and being aligned with the table sides

$$ {}^{E}_{T}R = ? $$

- point P is known in the table frame RF_T

$$ {}^{T}p = \begin{pmatrix} p_x \\ p_y \\ 0 \end{pmatrix} $$

- the depth camera proceeds centering point P in its image until it senses a distance h from the table (in RF_E)

$$ {}^{E}p = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} $$