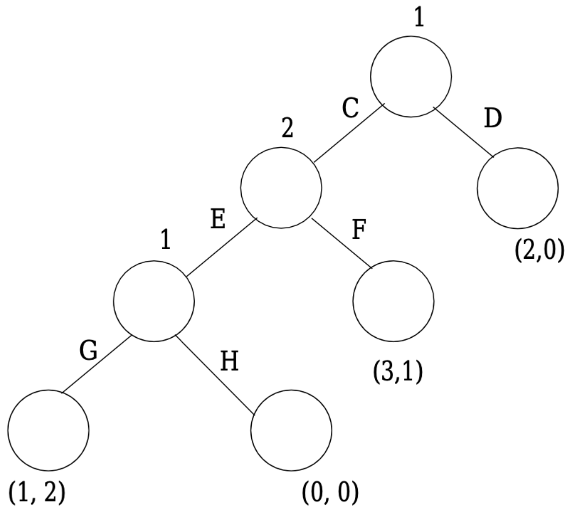
Extensive Form Games – Subgame Perfect Equilibrium (SGPE)

We consider the game as discussed in class. In this game, there are two players, Player 1 and player 2. Below figure shows the game tree. Observe that



$$N = \{1, 2\}; \quad A_1 = \{C, D, G, H\}; \quad A_2 = \{E, F\}.$$

The terminal histories are given by

$$\mathbb{H} = \{(C, E, G), (C, E, H), (C, F), (D)\}.$$

The proper subhistories of terminal histories are given by

$$S_{\mathbb{H}} = \{\epsilon, (C), (C, E)\}.$$

The player function is given by

$$P(\epsilon) = 1; \quad P(C) = 2; \quad P(C, E) = 1.$$

The information sets are given by

$$\mathbb{I}_1 = \{\{\varepsilon\}, \{(C, E)\}\}; \mathbb{I}_2 = \{\{(C)\}\}.$$

Player 1 has four strategies given by

$$\begin{aligned} s_{11} &: \{\varepsilon\} \to C; \quad \{(C, E)\} \to G \\ s_{12} &: \{\varepsilon\} \to C; \quad \{(C, E)\} \to H \\ s_{13} &: \{\varepsilon\} \to D; \quad \{(C, E)\} \to G \\ s_{14} &: \{\varepsilon\} \to D; \quad \{(C, E)\} \to H. \end{aligned}$$

For the sake of convenience, let us denote the above strategies by CG, CH, DG, and DH, respectively. Player 2 has two strategies given by

$$\begin{aligned} s_{21} &: \{C\} \to E \\ s_{21} &: \{C\} \to F \end{aligned}$$

For the sake of convenience, let us denote the above strategies by E and F, respectively. If S_1 and S_2 are the sets of strategies of players 1 and 2 respectively, it can be seen that

$$S_1 = \{CG, CH, DG, DH\}$$
$$S_2 = \{E, F\}$$

The set of strategy profiles, $S_1 \times S_2$, is given by

$$S_1 \times S_2 = \{(CG, E), (CG, F), (CH, E), (CH, F), (DG, E), (DG, F), (DH, E), (DH, F)\}$$

Note that a strategy profile uniquely determines a terminal history. For example, the profile (CG, E) corresponds to the terminal history (C, E, G); the profile (CG, F) corresponds to the terminal history (C, F); the profiles (DH, E) as well as (DH, F) correspond to the terminal history (D), etc.

Corresponding payoffs are given by the table below.

Player 1	**Player 2**	
	$s_{21} = E$	$s_{22} = F$
$s_{11} = DG$	2,0	2,0
$s_{12} = DH$	2,0	2,0
$s_{13} = CG$	1,2	3,1
$s_{14} = CH$	0,0	3,1

Table 1: Payoffs obtained in this game

One can verify that (DG, E) is NE as none of the players finding beneficial to deviate from this strategy profile. With the same arguments, (DH, E) and (CH, F) are also NE.

Subgame Perfect Equilibrium (SGPE):

Consider an extensive form game $\Gamma = \left\langle N, (A_i)_{i \in N}, \mathbb{H}, P, (\mathbb{I}_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$. Now, for any $s \in S_{\mathbb{H}}$, $\mathcal{S}_G(s)$ is a subgame. For example, when $s = \varepsilon$, then $\mathcal{S}_G(s) = \mathcal{S}_G(\varepsilon)$ is complete game. Now, a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is an SGPE if

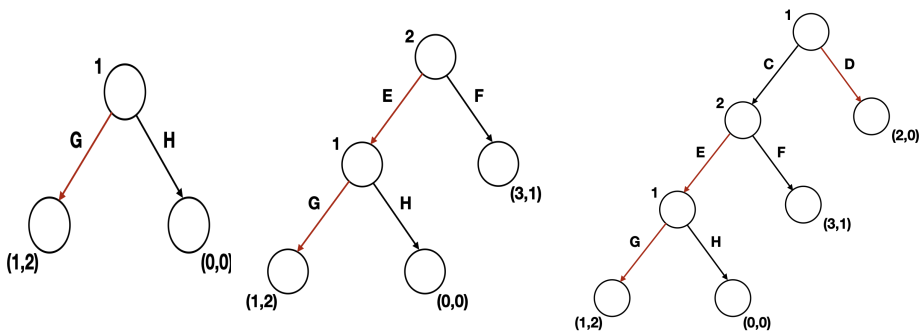
(i) s^* is NE for complete game $\mathcal{S}_G(\varepsilon)$.

(ii) for any history $s_h \in S_{\mathbb{H}}$, let $s^*(s_h) = (s_1^*(s_h), \ldots, s_n^*(s_h))$ be the corresponding strategy profile for the subgame $\mathcal{S}_G(s_h)$. Then $s^*(s_h)$ is NE for $\mathcal{S}_G(s_h)$.

For example, if $s = C$, then $s_1^* = DG$ and $s_2^* = E$. Also, observe $s_1^*(C) = G$ and $s_2^*(C) = E$.

Backward Induction Method:

Now, we determine the Subgame Perfect Equilibrium (SGPE) for this game using the backward induction method. We begin at the last decision node (excluding terminal nodes). At this



stage, Player 1 always chooses action G, as it provides a higher utility compared to action H. Knowing this, we now analyze Player 2's strategy by moving one step up in the game tree, as illustrated in the figure.

Player 2, aware that Player 1 will choose G, evaluates the possible outcomes. If Player 2 selects action E, they receive a utility of 2, whereas choosing F results in a utility of 1, which is lower. Consequently, Player 2 will prefer E, anticipating Player 1's choice of G.

Following the same reasoning, we now move one more step up in the game tree to consider the entire game. At this point, Player 1 recognizes the sequence of actions if they choose C: Player 2 will opt for E, followed by Player 1 selecting G. This sequence results in a utility of 1 for Player 1.

However, if Player 1 chooses D, they receive a guaranteed utility of 2, which is higher than 1. Therefore, Player 1 will prefer action D (as indicated by the red directions in the figure). This is subgame perfect equilibrium.

In the case of extensive game explained above, we have seen that the profiles (DH, E), (DG, E), and (CH, F) are all Nash equilibria. However, only the profile (DG, E) is a subgame perfect equilibrium.