ME604: Introduction to Robotics

Spring 2025

Assignment 2

1. Consider the following sequence of rotations
   a. Rotate by ϕ about the world x-axis
   b. Rotate by θ about the current (rotated) z-axis
   c. Rotate by ψ about the world y-axis

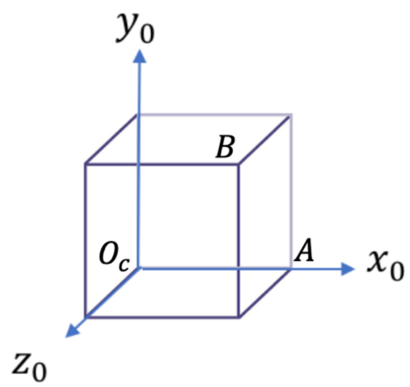
Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2. Consider the following sequence of rotations
   a. Rotate by ϕ about the world x-axis
   b. Rotate by θ about the world z-axis
   c. Rotate by ψ about the current x-axis
   d. Rotate by θ about the world z-axis

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

3. A unit cube is initially placed at the origin of frame $x_0y_0z_0$ as shown. The cube is rotated about the z_0 axis by 45 degrees, translated by 2 units in y_0 direction, rotated about x_0 by 90 degrees and finally rotated about its edge O_cA by 45 degrees. Note that point O_c represents one of the corners of the cube.

Determine the new coordinates of point B of the cube, which was initially located at (1, 1, 1).



4. Suppose R represents a rotation of 90 degrees about y_0(world frame), followed by a rotation of 45 degrees about z_1 (current frame). Find the equivalent axis/angle to represent R. Sketch the initial and final frames, and the equivalent axis vector r.

5. Let $k = \frac{1}{\sqrt{3}}[1\ 1\ 1]^{\mathrm{T}}$, θ = 90°. Find $R_{k,\theta}$.

   If vector $v' = R_{k,\theta}v$, verify that

   $v' = v\cos\theta + (k \times v)\sin\theta + (1 - \cos\theta)(k^{\mathrm{T}}v)v$

6. List all possible sets of Euler angles (we discussed ZXZ in class). Is it possible to have ZZX Euler angles? Why or why not?

Find the rotation matrix corresponding to the set of ZYZ Euler angles $\{\frac{\pi}{2}, 0, \frac{\pi}{2}\}$. What is the direction of x_1 axis relative to the base frame.

7. **Quaternions (Bonus)**
Complex numbers can be generalized by defining three independent square roots for -1 that obey the multiplication rules

$$-1 = i^2 = j^2 = k^2$$

$$i = jk = -kj\ ; j = ki = -ik; k = ij = -ji$$

Using these, we define a **quaternion** by $Q = q_0 + iq_1 + jq_2 + kq_3$, which is typically represented by a 4-tuple (q_1, q_2, q_3, q_4).

A rotation by θ about the unit vector $r = [r_x\ r_y\ r_z]^{\mathrm{T}}$ can be represented by the unit quaternion $Q = (\cos\frac{\theta}{2}, r_x\sin\frac{\theta}{2}, r_y\sin\frac{\theta}{2}, r_z\sin\frac{\theta}{2})$.

1. Show that such a quaternion has unit norm, i.e., that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.
2. Using the results for axis-angle representation, determine the rotation matrix R that corresponds to the rotation represented by unit quaternion Q = (q_0, q_1, q_2, q_3).
3. The quaternion $Q = (q_0, q_1, q_2, q_3)$ can be thought of as having a scalar component q_0 and a vector component $q = [q_1, q_2, q_3]^{\mathrm{T}}$. Show that the product of two quaternions, $Z = XY$ is given by

   $z_0 = x_0y_0 - x^{\mathrm{T}}y$
   $z = x_0y + y_0x + x \times y$

   Hint: perform the multiplication $(x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3)$ and simplify the result.

4. Let v be a vector whose coordinates are given by $v = [v_x\ v_y\ v_z]^{\mathrm{T}}$. If the quaternion represents a rotation, show that the new, rotated coordinates of v are given by $Q\,(0, v_x, v_y, v_z)Q^*$, in which $(0, v_x, v_y, v_z)$ is a quaternion with zero as its real component and $Q^* = (q_0, -q_1, -q_2, -q_3)$ is the inverse of Q.