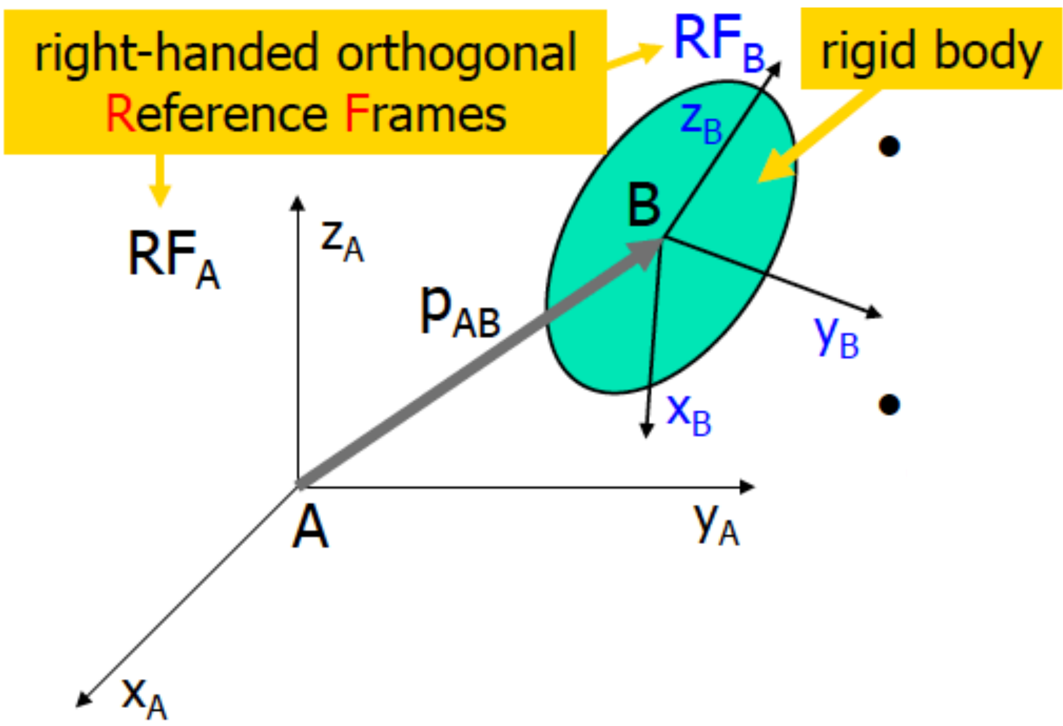
Rotation Representations : Direction cosines



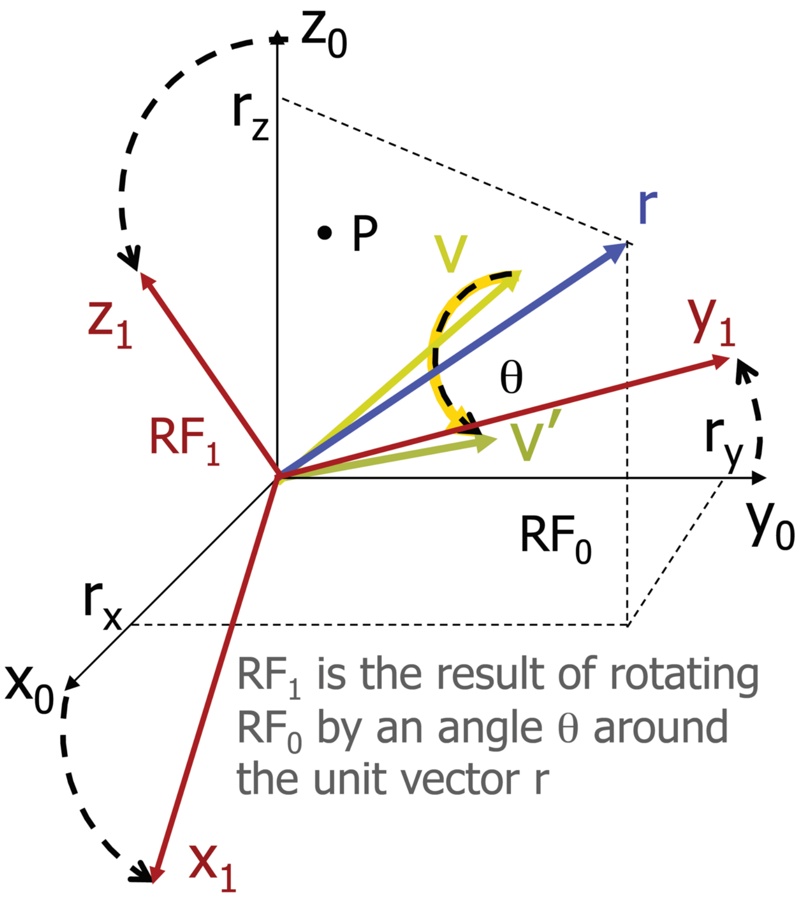
${}^{A}_{B}R = [r_1 \quad r_2 \quad r_3]$

$r_1 = {}^{A}x_B$
$r_2 = {}^{A}y_B$
$r_3 = {}^{A}z_B$

Constraints:

$|r1| = |r2| = |r3| = 1$
$r1.r2 = r2.r3 = r3.r1 = 0$

Axis-angle representation



RF$_1$ is the result of rotating RF$_0$ by an angle θ around the unit vector r

DATA

- unit vector r ($\|r\| = 1$)
- θ (positive if counterclockwise, as seen from an "observer" oriented like r with the head placed on the arrow)

DIRECT PROBLEM

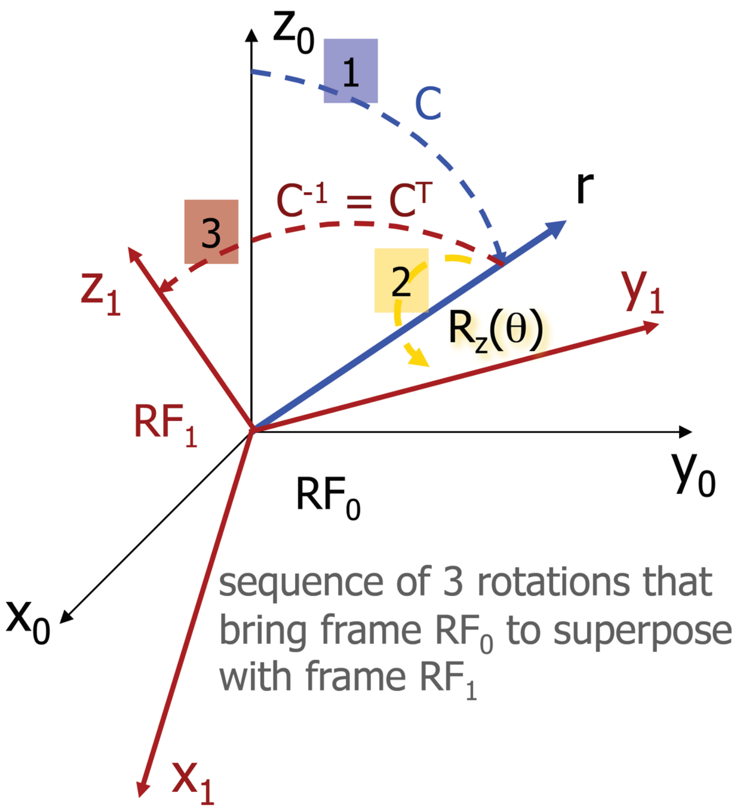
find

$$R(\theta, r) = [{}^0x_1 \; {}^0y_1 \; {}^0z_1]$$

such that

$$ {}^0P = R(\theta,r)\,{}^1P \qquad {}^0v' = R(\theta,r)\,{}^0v $$

Axis-angle representation



sequence of 3 rotations that bring frame RF_0 to superpose with frame RF_1

$$\boxed{R(\theta, r) = C\, R_z(\theta)\, C^T}$$

sequence of three rotations

$$C = \begin{bmatrix} n & s & r \end{bmatrix}$$

after the first rotation the z-axis coincides with r

n and s are orthogonal unit vectors such that

$n \times s = r$, or

$$n_y s_z - s_y n_z = r_x$$
$$n_z s_x - s_z n_x = r_y$$
$$n_x s_y - s_x n_y = r_z$$

Axis-angle representation

$$R(\theta, r) =$$

$$\begin{bmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_x r_y(1-\cos\theta)-r_z\sin\theta & r_x r_z(1-\cos\theta)+r_y\sin\theta \\ r_x r_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_y r_z(1-\cos\theta)-r_x\sin\theta \\ r_x r_z(1-\cos\theta)-r_y\sin\theta & r_y r_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

Also, $v' = v\cos\theta + (r \times v)\sin\theta + (1-\cos\theta)(r^T v)\, r$

Axis-angle representation : Inverse Problem

Given a rotation matrix R,

find a corresponding unit vector r and angle θ such that

$R(\theta, r) = R$

$$tr(R) = R_{11} + R_{22} + R_{33} = 1 + 2\cos\theta$$

$$\cos\theta = \frac{R_{11} + R_{22} + R_{33} - 1}{2}$$

$$\|r\| = 1 \Rightarrow \pm\frac{1}{2}\sqrt{(R_{12} - R_{21})^2 + (R_{23} - R_{32})^2 + (R_{31} - R_{13})^2}$$

$$\theta = \mathrm{ATAN2}\,\{\sqrt{(R_{12} - R_{21})^2 + (R_{23} - R_{32})^2 + (R_{31} - R_{13})^2}, R_{11} + R_{22} + R_{33} - 1\}$$

$$r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

Axis-angle representation : Singular cases

- If $\theta = 0$, there is no solution for r (rotation axis is undefined)

- If $\theta = \pm\pi$, then $\sin\theta = 0$, $\cos\theta = -1$, $R = 2rr^T - I$

$$
\boxed{r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \pm\sqrt{(R_{11}+1)/2} \\ \pm\sqrt{(R_{22}+1)/2} \\ \pm\sqrt{(R_{33}+1)/2} \end{bmatrix}}
\quad \text{with} \quad
\boxed{\begin{array}{l} r_x r_y = R_{12}/2 \\ r_x r_z = R_{13}/2 \\ r_y r_z = R_{23}/2 \end{array}}
\leftarrow
$$

Resolve ambiguities (always two solutions of opposite signs)

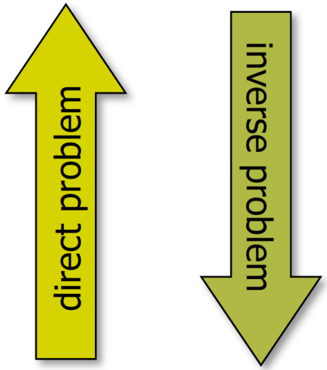
Example:

$$
R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
$$

Three angle ('Minimal') representations

- rotation matrices: 9 elements
  - 3 orthogonality relationships
  - 3 unitary relationships

  = 3 independent variables

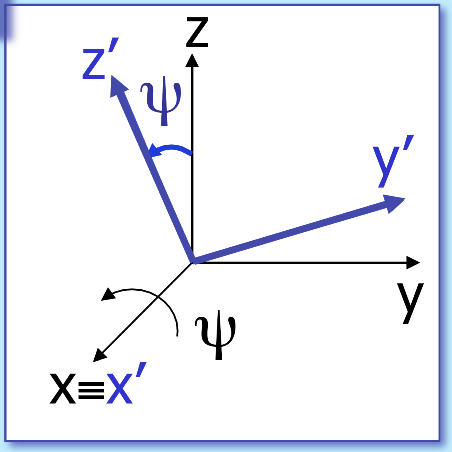


- sequence of 3 rotations around independent axes
  - fixed (a_i) or moving/current (a'_i) axes
    - generically called Roll-Pitch-Yaw (fixed axes) or Euler (moving axes) angles
  - 12 + 12 possible different sequences (e.g., XYX)
  - actually, only 12 since

$$\{(a_1\ \alpha_1), (a_2\ \alpha_2), (a_3\ \alpha_3)\} \equiv \{(a'_3\ \alpha_3), (a'_2\ \alpha_2), (a'_1\ \alpha_1)\}$$

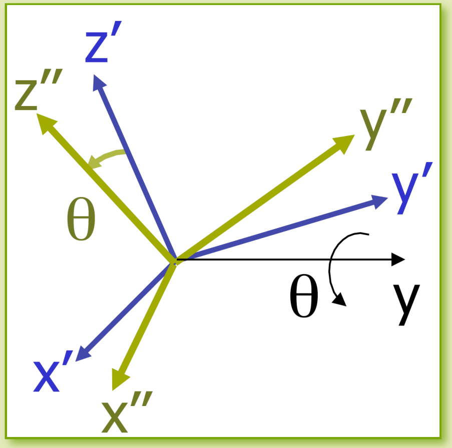
Fixed XYZ (Roll-Pitch-Yaw) angles

1 **ROLL**



$$R_X(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

PITCH **2**

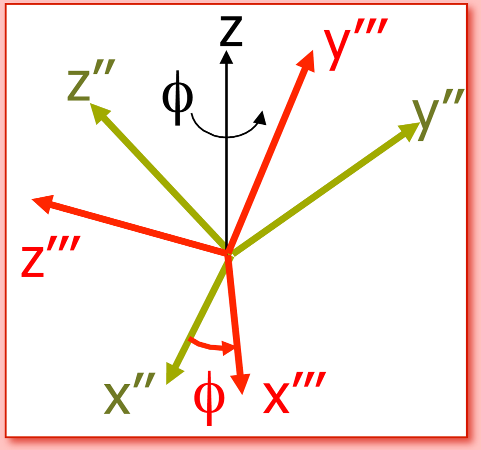


$C_1 R_Y(\theta) C_1^T$

with $R_Y(\theta) =$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

3 **YAW**



$C_2 R_Z(\phi) C_2^T$

with $R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Fixed angles

- direct problem: given ψ , θ , ϕ ; find R

  $R_{RPY}(\psi, \theta, \phi) = R_Z(\phi)\, R_Y(\theta)\, R_X(\psi)$

  order of definition

$$= \begin{bmatrix} c\phi\, c\theta & c\phi\, s\theta\, s\psi - s\phi\, c\psi & c\phi\, s\theta\, c\psi + s\phi\, s\psi \\ s\phi\, c\theta & s\phi\, s\theta\, s\psi + c\phi\, c\psi & s\phi\, s\theta\, c\psi - c\phi\, s\psi \\ - s\theta & c\theta\, s\psi & c\theta\, c\psi \end{bmatrix}$$

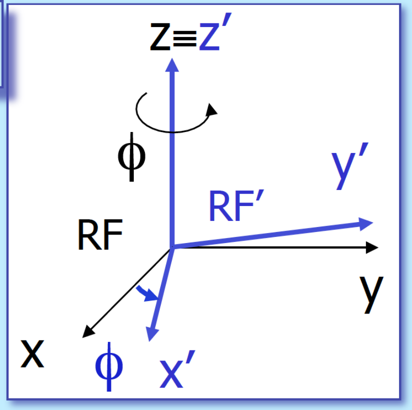
- inverse problem: given $R = \{r_{ij}\}$; find ψ , θ , ϕ
- $r_{32}^2 + r_{33}^2 = c^2\theta$, $r_{31} = -s\theta$ $\Rightarrow$ $\boxed{\theta = \text{ATAN2}\{-r_{31}, \pm\sqrt{r_{32}^2 + r_{33}^2}\}}$

  for $r_{31}<0$, two symmetric values w.r.t. $\pi/2$
- if $r_{32}^2 + r_{33}^2 \neq 0$ (i.e., $c\theta \neq 0$)

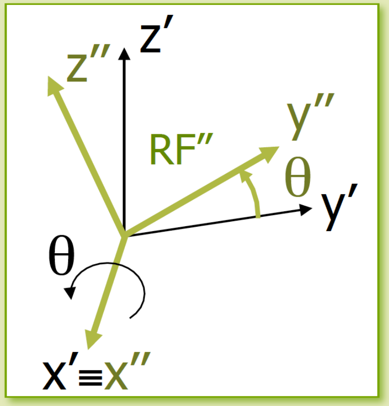
  $r_{32}/c\theta = s\psi$, $r_{33}/c\theta = c\psi$ $\Rightarrow$ $\boxed{\psi = \text{ATAN2}\{r_{32}/c\theta, r_{33}/c\theta\}}$
- similarly ... $\boxed{\phi = \text{ATAN2}\{ r_{21}/c\theta, r_{11}/c\theta \}}$
- singularities for $\theta = \pm\,\pi/2$

Z-X-Z Euler angles

1



$$R_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

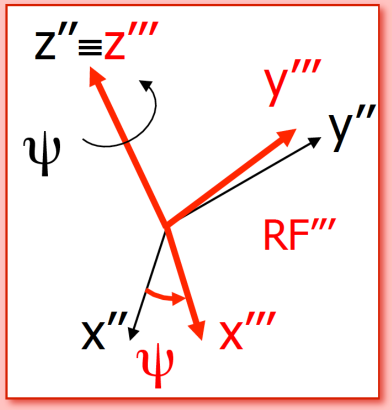


2

$$R_{x'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3

$$R_{z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Z-X-Z Euler angles

- direct problem: given ϕ , θ , ψ ; find R

  $$R_{ZX'Z''}(\underrightarrow{\phi,\ \theta,\ \psi}) = R_Z(\phi)\, R_{X'}(\theta)\, R_{Z''}(\psi)$$

  order of definition in concatenation

  $$= \begin{bmatrix} c\phi\, c\psi - s\phi\, c\theta\, s\psi & -c\phi\, s\psi - s\phi\, c\theta\, c\psi & s\phi\, s\theta \\ s\phi\, c\psi + c\phi\, c\theta\, s\psi & -s\phi\, s\psi + c\phi\, c\theta\, c\psi & -c\phi\, s\theta \\ s\theta\, s\psi & s\theta\, c\psi & c\theta \end{bmatrix}$$

- given a vector $v''' = (x''',y''',z''')$ expressed in RF''', its expression in the coordinates of RF is

  $$v = R_{ZX'Z''}(\phi,\ \theta,\ \psi)\ v'''$$

Z-X-Z Euler angles

- inverse problem: given $R = \{r_{ij}\}$; find ϕ , θ , ψ

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\phi\, c\psi - s\phi\, c\theta\, s\psi & -c\phi\, s\psi - s\phi\, c\theta\, c\psi & s\phi\, s\theta \\ s\phi\, c\psi + c\phi\, c\theta\, s\psi & -s\phi\, s\psi + c\phi\, c\theta\, c\psi & -c\phi\, s\theta \\ s\theta\, s\psi & s\theta\, c\psi & c\theta \end{bmatrix}$$

- $r_{13}^2 + r_{23}^2 = s^2\theta, \;\; r_{33} = c\theta \;\Rightarrow$ $\boxed{\theta = \text{ATAN2}\{\pm\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\}}$

  two values differing just for the sign

- if $r_{13}^2 + r_{23}^2 \neq 0$ (i.e., $s\theta \neq 0$)

  $r_{31}/s\theta = s\psi \,, \quad r_{32}/s\theta = c\psi \;\Rightarrow$ $\boxed{\psi = \text{ATAN2}\{r_{31}/s\theta, r_{32}/s\theta\}}$

- similarly... $\boxed{\phi = \text{ATAN2}\{r_{13}/s\theta, -r_{23}/s\theta\}}$
- there is always a pair of solutions
- there are always singularities (here $\theta = 0, \pm\pi$)

Why this order?

$$R_{RPY}(\psi, \theta, \phi) = R_Z(\phi)\, R_Y(\theta)\, R_X(\psi)$$

order of definition

"reverse" order in the product (pre-multiplication…)

- need to refer each rotation in the sequence to one of the original fixed axes
  - use of the angle/axis technique for each rotation in the sequence: $C\, R(\alpha)\, C^T$, with C being the rotation matrix reverting the previously made rotations (= go back to the original axes)

$$\begin{aligned} R_{RPY}(\psi, \theta, \phi) &= [R_X(\psi)]\, [R_X^T(\psi)\, R_Y(\theta)\, R_X(\psi)] \\ &\quad [R_X^T(\psi)\, R_Y^T(\theta)\, R_Z(\phi)\, R_Y(\theta)\, R_X(\psi)] \\ &= R_Z(\phi)\, R_Y(\theta)\, R_X(\psi) \end{aligned}$$

Unit quaternion

- to eliminate undetermined and singular cases arising in the axis/angle representation, one can use the *unit quaternion* representation

$$\mathcal{Q} = \{\underbrace{\eta}_{\text{a scalar}}, \underbrace{\boldsymbol{\varepsilon}}_{\text{3-dim vector}}\} = \{\cos(\theta/2), \sin(\theta/2)\,\mathbf{r}\}$$

- $\eta^2 + \|\boldsymbol{\varepsilon}\|^2 = 1$ (thus, "unit ...")
- $(\theta, \mathbf{r})$ and $(-\theta, -\mathbf{r})$ gives the same quaternion $\mathcal{Q}$
- the absence of rotation is associated to $\mathcal{Q} = \{1, \mathbf{0}\}$