Assignment 7

1 Solution 1:

The given system equation is:

$$2\ddot{x} + \dot{x} = F(t)$$

We want to design a PD controller:

$$F(t) = K_p e(t) + K_d \dot{e}(t)$$

where $e(t) = x_d(t) - x(t)$.

Substituting the PD controller into the system equation:

$$2\ddot{x} + \dot{x} = K_p(x_d - x) + K_d(\dot{x}_d - \dot{x})$$

$$2\ddot{x} + (1 + K_d)\dot{x} + K_p x = K_p x_d + K_d \dot{x}_d$$

The characteristic equation of the closed-loop system is:

$$2s^2 + (1 + K_d)s + K_p = 0$$

Comparing this with the standard second-order system characteristic equation:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

We have:

$$\omega_n^2 = \frac{K_p}{2} \implies K_p = 2\omega_n^2$$

$$2\zeta\omega_n = \frac{1 + K_d}{2} \implies K_d = 4\zeta\omega_n - 1$$

Given constraints: $\omega_n < 10$ rad/s and $\zeta > 0.707$

Let's choose $\omega_n = 5$ rad/s and $\zeta = 1$ (critically damped, which satisfies $\zeta > 0.707$).

Then:

$$K_p = 2(5^2) = 50$$

$$K_d = 4(1)(5) - 1 = 19$$

Therefore, the PD controller is:

$$F(t) = 50e(t) + 19\dot{e}(t)$$

2 Solution 2:

2.1 Part (a)

the equations can be written in the form $u_1 = f_1(\ddot{\mathbf{y}}, \dot{\mathbf{y}}, \mathbf{y})$ and $u_2 = f_2(\ddot{\mathbf{y}}, \dot{\mathbf{y}}, \mathbf{y})$. We can rearrange the given equations to isolate u_1 and u_2.

$$u_1 = \ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2 - y_2 u_2$$

$$u_2 = \ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) + (\cos y_1)^2 y_2 u_1$$

Substituting the expression for u_1 into the equation for u_2:

$$u_2 = \ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) + (\cos y_1)^2 y_2(\ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2 - y_2 u_2)$$

$$u_2[1+(\cos y_1)^2 y_2^2] = \ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) + (\cos y_1)^2 y_2(\ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2)$$
$$u_2 = \frac{\ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) + (\cos y_1)^2 y_2(\ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2)}{1+(\cos y_1)^2 y_2^2}$$

substitute the expression for u_2 back into the equation for u_1:

$$u_1 = \ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2 - y_2\left[\frac{\ddot{y}_2 + (\cos y_1)\dot{y}_2 + 3(y_1 - y_2) + (\cos y_1)^2 y_2(\ddot{y}_1 + 3y_1\dot{y}_2 + y_2^2)}{1+(\cos y_1)^2 y_2^2}\right]$$

Thus, $u_1 = f_1(\ddot{y}_1, \ddot{y}_2, \dot{y}_1, \dot{y}_2, y_1, y_2)$ and $u_2 = f_2(\ddot{y}_1, \ddot{y}_2, \dot{y}_1, \dot{y}_2, y_1, y_2)$.

2.2 Part (b)

To find an inverse dynamics control,

$$\ddot{y}_1 = -3y_1\dot{y}_2 - y_2^2 + u_1 + y_2 u_2 \tag{1}$$
$$\ddot{y}_2 = -(\cos y_1)\dot{y}_2 - 3(y_1 - y_2) + u_2 - (\cos y_1)^2 y_2 u_1 \tag{2}$$

$$\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix} = \begin{bmatrix}-3y_1\dot{y}_2 - y_2^2\\ -(\cos y_1)\dot{y}_2 - 3(y_1 - y_2)\end{bmatrix} + \begin{bmatrix}1 & y_2\\ -(\cos y_1)^2 y_2 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

find u_1 and u_2 such that the closed-loop system is linear and decoupled:

$$\ddot{y}_1 = \nu_1 \tag{3}$$
$$\ddot{y}_2 = \nu_2 \tag{4}$$

$$\ddot{y}_1 + 2\zeta\omega_n\dot{e}_1 + \omega_n^2 e_1 = 0 \tag{5}$$
$$\ddot{y}_2 + 2\zeta\omega_n\dot{e}_2 + \omega_n^2 e_2 = 0 \tag{6}$$

where $e_1 = y_{1d} - y_1$, $e_2 = y_{2d} - y_2$, and y_{1d} and y_{2d} are the desired trajectories.

$$\nu_1 = \ddot{y}_{1d} + 2\zeta\omega_n(\dot{y}_{1d} - \dot{y}_1) + \omega_n^2(y_{1d} - y_1) \tag{7}$$
$$\nu_2 = \ddot{y}_{2d} + 2\zeta\omega_n(\dot{y}_{2d} - \dot{y}_2) + \omega_n^2(y_{2d} - y_2) \tag{8}$$

$$\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix} = \begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix} = \begin{bmatrix}-3y_1\dot{y}_2 - y_2^2\\ -(\cos y_1)\dot{y}_2 - 3(y_1 - y_2)\end{bmatrix} + \begin{bmatrix}1 & y_2\\ -(\cos y_1)^2 y_2 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

Inverting the matrix

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}1 & y_2\\ -(\cos y_1)^2 y_2 & 1\end{bmatrix}^{-1}\left(\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix} - \begin{bmatrix}-3y_1\dot{y}_2 - y_2^2\\ -(\cos y_1)\dot{y}_2 - 3(y_1 - y_2)\end{bmatrix}\right)$$

The inverse of the matrix is

$$\begin{bmatrix}1 & y_2\\ -(\cos y_1)^2 y_2 & 1\end{bmatrix}^{-1} = \frac{1}{1+y_2^2(\cos y_1)^2}\begin{bmatrix}1 & -y_2\\ (\cos y_1)^2 y_2 & 1\end{bmatrix}$$

for $\omega_n = 10$ and $\zeta = 0.5$, =¿ u_1 and u_2.

3 Solution 3:

Kinetic Energy (T): $T_1 = \frac{1}{2} I_1 \dot{\theta}^2$, where I_1 is the moment of inertia of link 1 about the joint.
Potential Energy (V): $V = m_1 g h_1 + m_2 g h_2 + m_3 g h_3$, where h_i is the height of the center of mass of link i.
Lagrangian (L): The Lagrangian is defined as $L = T - V$.
Equation of Motion: Using the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau$$

Substituting the expressions for T and V into the Lagrangian and then into the Euler-Lagrange equation, we will obtain the equation of motion in the form:

$$\tau = I\ddot{\theta} + B\dot{\theta} + G(\theta)$$

2. Design a Critically Damped PD Controller
We want to design a PD controller:

$$\tau = K_p(\theta_d - \theta) + K_d(\dot{\theta}_d - \dot{\theta})$$

$\dot{\theta}_d = 0$ and $\ddot{\theta}_d = 0$.

$$\tau = K_p(\theta_d - \theta) - K_d\dot{\theta}$$

$$I\ddot{\theta} = K_p(\theta_d - \theta) - K_d\dot{\theta} + G(\theta)$$

$$I\ddot{\theta} + K_d\dot{\theta} + K_p\theta = K_p\theta_d + G(\theta)$$

The characteristic equation is:

$$Is^2 + K_d s + K_p = 0$$

Dividing by I:

$$s^2 + \frac{K_d}{I}s + \frac{K_p}{I} = 0$$

Comparing this with the standard second-order system characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = \frac{K_p}{I} \implies K_p = I\omega_n^2$$

$$2\zeta\omega_n = \frac{K_d}{I} \implies K_d = 2\zeta\omega_n I$$

Given: $\omega_n = 4$ rad/s and critically damped, so $\zeta = 1$.

$$K_p = I(4^2) = 16I$$

$$K_d = 2(1)(4)I = 8I$$

The PD controller:

$$\tau = 16I(\theta_d - \theta) - 8I\dot{\theta}$$

Solution 4

Part (a): Dynamic Equations

$$T = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}m(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\dot{\theta}_1\dot{\theta}_2\cos\theta_2) + J_0(\dot{\theta}_1 r)^2 + J_0(\dot{\theta}_2 r)^2$$

$$V = mgl\cos\theta_1 + mgl\cos(\theta_1 + \theta_2)$$

Applying the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) - \frac{\partial T}{\partial \theta_i} + \frac{\partial V}{\partial \theta_i} = \tau_i$$

$$\tau_1 = (2ml^2 + J_0 r^2)\ddot{\theta}_1 + ml^2 \cos\theta_2 \ddot{\theta}_2 - ml^2 \sin\theta_2 \dot{\theta}_2^2 - 2ml^2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 + mgl \sin\theta_1 + mgl \sin(\theta_1 + \theta_2) + B_0 r \dot{\theta}_1$$

$$\tau_2 = ml^2 \cos\theta_2 \ddot{\theta}_1 + (ml^2 + J_0 r^2)\ddot{\theta}_2 + ml^2 \sin\theta_2 \dot{\theta}_1^2 + mgl \sin(\theta_1 + \theta_2) + B_0 r \dot{\theta}_2$$

Part (b): Joint Trajectories for Circular Motion

The end-effector position is given by:

$$\begin{aligned} x &= l\cos\theta_1 + l\cos(\theta_1 + \theta_2) \\ y &= l\sin\theta_1 + l\sin(\theta_1 + \theta_2) \end{aligned}$$

$$\begin{aligned} x &= R\cos(vt) = \frac{3l}{2}\cos(10\pi t) \\ y &= R\sin(vt) = \frac{3l}{2}\sin(10\pi t) \end{aligned}$$

Part (c): PD Trajectory Tracking Controller

The PD controller for each joint is:

$$\tau_i = K_{p_i}(\theta_{id} - \theta_i) + K_{d_i}(\dot{\theta}_{id} - \dot{\theta}_i)$$

$$2ml^2\ddot{\theta}_1 + K_{d_1}\dot{\theta}_1 + K_{p_1}\theta_1 = K_{p_1}\theta_{1d} + ml^2\cos\theta_2\ddot{\theta}_2 - ml^2\sin\theta_2\dot{\theta}_2^2 - 2ml^2\sin\theta_2\dot{\theta}_1\dot{\theta}_2 + mgl\sin\theta_1 + mgl\sin(\theta_1 + \theta_2)$$

$$ml^2\ddot{\theta}_2 + K_{d_2}\dot{\theta}_2 + K_{p_2}\theta_2 = K_{p_2}\theta_{2d} + ml^2\cos\theta_2\ddot{\theta}_1 + ml^2\sin\theta_1\dot{\theta}_1^2 + mgl\sin(\theta_1 + \theta_2)$$

For critical damping

$$\begin{aligned} K_{p_i} &= I_i\omega_n^2 \\ K_{d_i} &= 2\zeta\omega_n I_i = 2\omega_n I_i \end{aligned}$$

$$\begin{aligned} K_{p_1} &= 2ml^2(36^2) = 2592ml^2 \\ K_{d_1} &= 2(36)(2ml^2) = 144ml^2 \\ K_{p_2} &= ml^2(36^2) = 1296ml^2 \\ K_{d_2} &= 2(36)(ml^2) = 72ml^2 \end{aligned}$$

Part (d): Steady State Error with Disturbance Torque

With a disturbance torque ¡span class=”math-inline”¿
tau_d¡/span¿ on joint 1, the closed-loop equation for joint 1 becomes:

$$2ml^2\ddot{\theta}_1 + K_{d_1}\dot{\theta}_1 + K_{p_1}\theta_1 = K_{p_1}\theta_{1d} + \tau_d$$

At steady state, ¡span class=”math-inline”¿
ddot{
theta}_1 ¯
dot{
theta}_1 0̄¡/span¿, so:

$$K_{p_1}\theta_1 = K_{p_1}\theta_{1d} + \tau_d$$

The steady-state error is:

$$e_{ss} = \theta_{1d} - \theta_1 = -\frac{\tau_d}{K_{p_1}} = -\frac{\tau_d}{2592ml^2}$$

Solution 5

(a) Homogeneous Transformation Matrices ${}^0_{C_1}T$ and ${}^0_{C_2}T$
Let $q_1 = \theta_1$ and $q_2 = \theta_2$.

$${}^0_{C_1}T = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & \frac{L_1}{2}\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & \frac{L_1}{2}\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_{C_2}T = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & L_1\cos\theta_1 + \frac{L_2}{2}\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & L_1\sin\theta_1 + \frac{L_2}{2}\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Jacobians ${}^0J_{v1}$ and ${}^0J_{v2}$

$${}^0J_{v1} = \begin{bmatrix} -\frac{L_1}{2}\sin\theta_1 & 0 \\ \frac{L_1}{2}\cos\theta_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$${}^0J_{v2} = \begin{bmatrix} -L_1\sin\theta_1 - \frac{L_2}{2}\sin(\theta_1+\theta_2) & -\frac{L_2}{2}\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + \frac{L_2}{2}\cos(\theta_1+\theta_2) & \frac{L_2}{2}\cos(\theta_1+\theta_2) \\ 0 & 0 \end{bmatrix}$$

(c) Angular Velocity Jacobians ${}^{c1}J_{\omega 1}$ and ${}^{c2}J_{\omega 2}$

$${}^{c1}J_{\omega 1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$${}^{c2}J_{\omega 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Inertia Tensors I_{c1} and I_{c2}
For a bar of uniform density with square cross-section:

$$I_{c1} = \begin{bmatrix} \frac{1}{12}m_1(L_1^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m_1(L_1^2+h^2) & 0 \\ 0 & 0 & \frac{1}{12}m_1(L_1^2+h^2) \end{bmatrix}$$

$$I_{c2} = \begin{bmatrix} \frac{1}{12}m_2(L_2^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m_2(L_2^2+h^2) & 0 \\ 0 & 0 & \frac{1}{12}m_2(L_2^2+h^2) \end{bmatrix}$$

(e) Inertia Matrix $D(q)$

$$D(q) = m_1 J_{v1}^T J_{v1} + m_2 J_{v2}^T J_{v2} + J_{\omega 1}^T I_{c1} J_{\omega 1} + J_{\omega 2}^T I_{c2} J_{\omega 2}$$

$$D(q) = \begin{bmatrix} \frac{1}{4}m_1L_1^2 + m_2(L_1^2 + L_1L_2\cos\theta_2 + \frac{1}{4}L_2^2) + \frac{1}{12}m_1(L_1^2+h^2) & m_2(\frac{1}{2}L_1L_2\cos\theta_2 + \frac{1}{4}L_2^2) \\ m_2(\frac{1}{2}L_1L_2\cos\theta_2 + \frac{1}{4}L_2^2) & \frac{1}{4}m_2L_2^2 + \frac{1}{12}m_2(L_2^2+h^2) \end{bmatrix}$$

(f) Equations of Motion

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q)$$

Calculating $C(q,\dot{q})$ and $G(q)$:

$$C(q,\dot{q}) = \begin{bmatrix} -m_2L_1L_2\sin\theta_2\dot{\theta}_2 & -m_2L_1L_2\sin\theta_2(\dot{\theta}_1+\dot{\theta}_2) \\ m_2L_1L_2\sin\theta_2\dot{\theta}_1 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} \frac{1}{2}m_1gL_1\cos\theta_1 + m_2g(L_1\cos\theta_1 + \frac{1}{2}L_2\cos(\theta_1+\theta_2)) \\ \frac{1}{2}m_2gL_2\cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\tau_1 = D_{11}\ddot{\theta}_1 + D_{12}\ddot{\theta}_2 - m_2L_1L_2\sin\theta_2\dot{\theta}_2\dot{\theta}_1 - m_2L_1L_2\sin\theta_2\dot{\theta}_2^2 + \frac{1}{2}m_1gL_1\cos\theta_1 + m_2g(L_1\cos\theta_1 + \frac{1}{2}L_2\cos(\theta_1+\theta_2))$$

$$\tau_2 = D_{21}\ddot{\theta}_1 + D_{22}\ddot{\theta}_2 + m_2L_1L_2\sin\theta_2\dot{\theta}_1^2 + \frac{1}{2}m_2gL_2\cos(\theta_1+\theta_2)$$

$$x_d(t) = R\cos(\omega t)$$
$$y_d(t) = R\sin(\omega t)$$
$$z_d(t) = pt$$

where R is the radius of the helix and $\omega = v/R$ is the angular velocity.

$$\tau = D(q)\ddot{q}_{des} + C(q,\dot{q})\dot{q}_{des} + G(q)$$

Solution to Question 6

Part a) Task Space Dynamic Equations

$$x = l_1\cos q_1 + l_2\cos(q_1+q_2)$$
$$y = l_1\sin q_1 + l_2\sin(q_1+q_2)$$

$$\dot{x} = J(q)\dot{q}$$

$$J(q) = \begin{bmatrix} -l_1\sin q_1 - l_2\sin(q_1+q_2) & -l_2\sin(q_1+q_2) \\ l_1\cos q_1 + l_2\cos(q_1+q_2) & l_2\cos(q_1+q_2) \end{bmatrix}$$

The dynamic equations in joint space are generally of the form:

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q)$$

$$\ddot{x} = \dot{J}(q)\dot{q} + J(q)\ddot{q}$$
$$\ddot{q} = J^{-1}(q)(\ddot{x} - \dot{J}(q)\dot{q})$$

$$\tau = D(q)J^{-1}(q)(\ddot{x} - \dot{J}(q)\dot{q}) + C(q,\dot{q})\dot{q} + G(q)$$

$$J^{-T}(q)\tau = J^{-T}(q)D(q)J^{-1}(q)\ddot{x} - J^{-T}(q)D(q)J^{-1}(q)\dot{J}(q)\dot{q} + J^{-T}(q)C(q,\dot{q})\dot{q} + J^{-T}(q)G(q)$$

$$J^{-T}(q)\tau = J^{-T}(q)D(q)J^{-1}(q)\ddot{x} - J^{-T}(q)D(q)J^{-1}(q)\dot{J}(q)J^{-1}(q)\dot{x} + J^{-T}(q)C(q,\dot{q})J^{-1}(q)\dot{x} + J^{-T}(q)G(q)$$

5. Simplified Task Space Dynamics: The dynamic equations in task space can be written as:

$$F = \Lambda(q)\ddot{x} + \mu(q,\dot{q}) + p(q)$$

Part b Task-Space Nonlinear Decoupling PD Controller

$$F = \Lambda(q)(\ddot{x}_d + K_v\dot{e} + K_p e) + \mu(q,\dot{q}) + p(q)$$

$$\tau = J^T(q)\left[\Lambda(q)(\ddot{x}_d + K_v\dot{e} + K_p e) + \mu(q,\dot{q}) + p(q)\right]$$

$$K_p = \begin{bmatrix} k_{p1} & 0 \\ 0 & k_{p2} \end{bmatrix}, \quad K_v = \begin{bmatrix} k_{v1} & 0 \\ 0 & k_{v2} \end{bmatrix}$$

For critical damping

$$k_{p_i} = \omega_n^2 = 36^2 = 1296$$

$$k_{v_i} = 2\zeta\omega_n = 2 \times 1 \times 36 = 72$$

So,

$$K_p = \begin{bmatrix} 1296 & 0 \\ 0 & 1296 \end{bmatrix}, \quad K_v = \begin{bmatrix} 72 & 0 \\ 0 & 72 \end{bmatrix}$$

The final control law is:

$$\tau = J^T(q)\left[\Lambda(q)\left(\ddot{x}_d + \begin{bmatrix} 72 & 0 \\ 0 & 72 \end{bmatrix}\dot{e} + \begin{bmatrix} 1296 & 0 \\ 0 & 1296 \end{bmatrix} e\right) + \mu(q,\dot{q}) + p(q)\right]$$