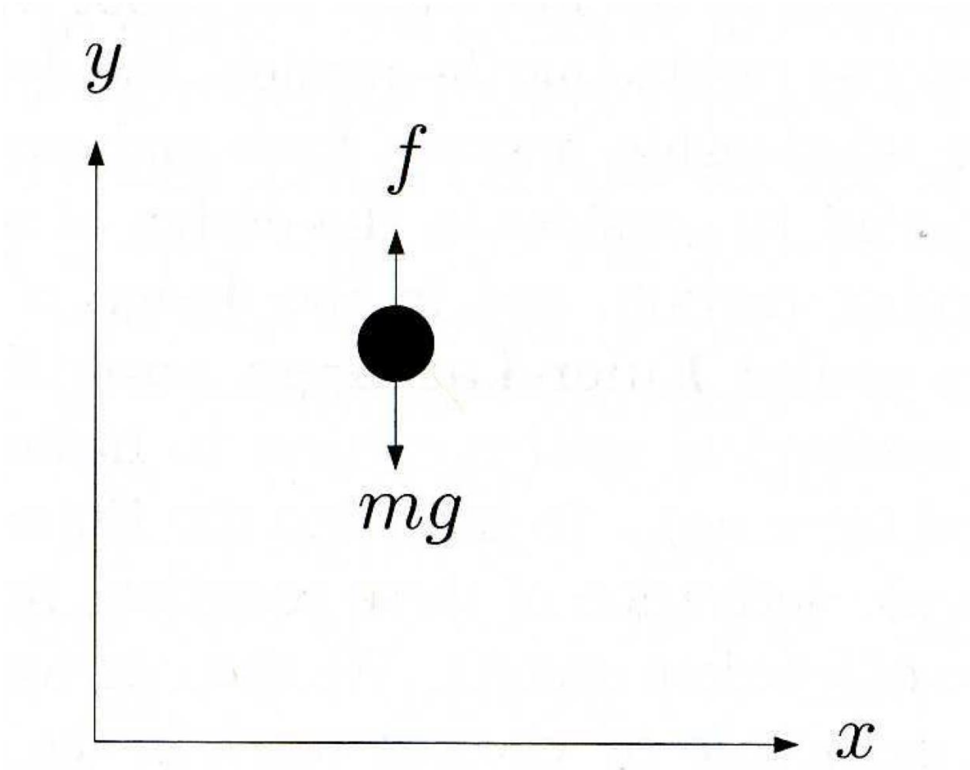
Example: One-DOF system



- f is the external force
- mg is the force acting on the particle due to gravity

Equation of motion as per Newton's second law

$$m\,\ddot{y} = \Sigma F_i = f - mg$$

Example: One-DOF system

The equation of motion of the particle

$$m\,\ddot{y} = \Sigma F_i = f\ - mg$$

can be rewritten in a different way!

$$m\ddot{y} = \frac{d}{dt}\left(m\frac{dy}{dt}\right) = \frac{d}{dt}(m\,\frac{\partial}{\partial \dot{y}}[\frac{1}{2}\dot{y}^2] = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{y}}\right)$$

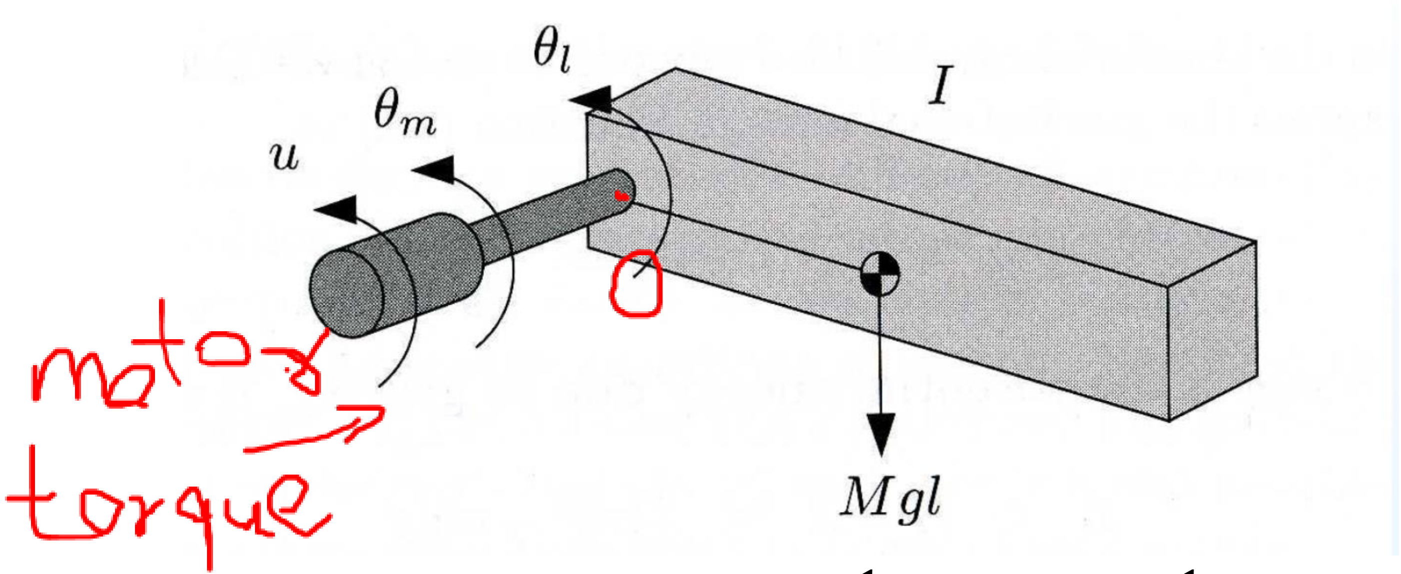
$$mg = \frac{\partial}{\partial y}[mgy] = \frac{\partial}{\partial y}P$$

with $K = \frac{1}{2}m\dot{y}^2$ and $P = mgy$ as the kinetic and potential energy.

Newton's second law can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\mathcal{L}\right) - \frac{\partial}{\partial y}\mathcal{L} = f$$ with the Lagrangian, $\mathcal{L}(y, \dot{y}) = K\ - P$.

Example: Single-link Arm



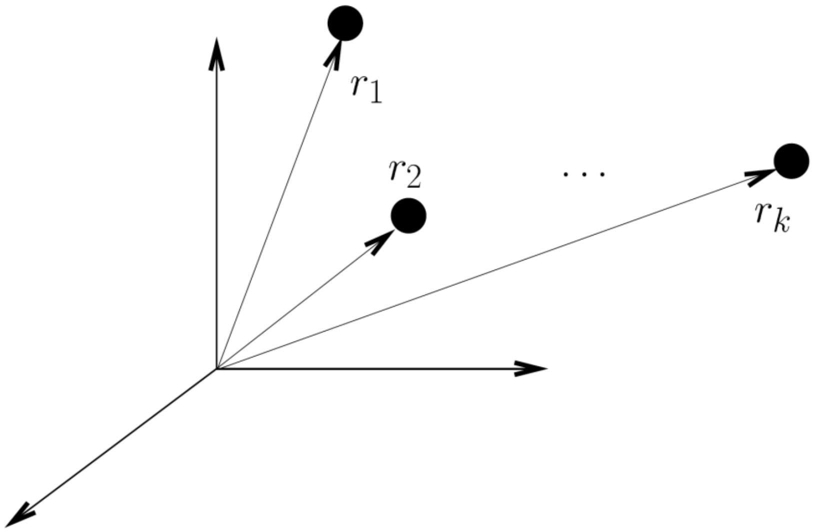
A rigid link (θ_l) coupled to a DC motor (θ_m), through a gear box.

$$\theta_m = r\theta_l$$

- Kinetic energy: $K = \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_l\dot{\theta}_l^2 = \frac{1}{2}(r^2J_m + J_l)\dot{\theta}_l^2$
- Potential energy: $P = Mgl(1 - \cos\theta_l)$
- The Lagrangian is $\mathcal{L} = K - P$, and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}\mathcal{L}\right) - \frac{\partial}{\partial\theta}\mathcal{L} = (r^2J_m + J_l)\ddot{\theta} + Mgl\sin\theta_l = ru$$

Holonomic constraints



- Unconstrained system of k particles
- Degrees of freedom are $3k$
- The number of DoFs is less if the system is constrained.

A constrained imposed on k particles (with coordinates $r_1, r_2, \ldots, r_k \in R^3$) is called holonomic, if it is an equality constraint of the form

$$g_i(r_1, r_2, \ldots, r_k) = 0 \qquad i = 0, 1, 2, \ldots, l$$

and non-holonomic otherwise.

Presence of constraint implies presence of a constraint force, that forces this constraint to hold.

Holonomic constraints

Example: Two particles joined by a massless rigid wire of length l.

$$r_1, r_2 \in R^3 : \left|\left|r_1 - r_2\right|\right|^2 = (r_1 - r_2)^T (r_1 - r_2) = l^2$$

In general,

$$g_i(r_1, r_2, \ldots, r_k) = 0 \qquad i = 0, 1, 2, \ldots, l$$

Differentiating,

$$\frac{d}{dt} g_i(r_1, r_2, \ldots, r_k) = \frac{\partial g_i}{\partial r_1}\frac{dr_1}{dt} + \frac{\partial g_i}{\partial r_2}\frac{dr_2}{dt} + \cdots + \frac{\partial g_i}{\partial r_k}\frac{dr_k}{dt} = 0$$

or,

$$\frac{\partial g_i}{\partial r_1} dr_1 + \frac{\partial g_i}{\partial r_2} dr_2 + \cdots + \frac{\partial g_i}{\partial r_k} dr_k = 0$$

Generalized coordinates

If the system is subject to holonomic constraints then

- If a system consists of k particles, it may be possible to express their coordinates as a functions of fewer than $3k$ variables

$$r_1 = r_1(q_1, \dots, q_n), r_2 = r_2(q_1, \dots, q_n), \dots, r_k = r_k(q_1, \dots, q_n)$$

- The smallest set of variables is called generalized coordinates

- The smallest number n is called the number of degrees of freedom

- If the system consists of an infinite number of particles, it might have finite number of degrees of freedom

Virtual displacements

Given a set of k particles and a holonomic constraints

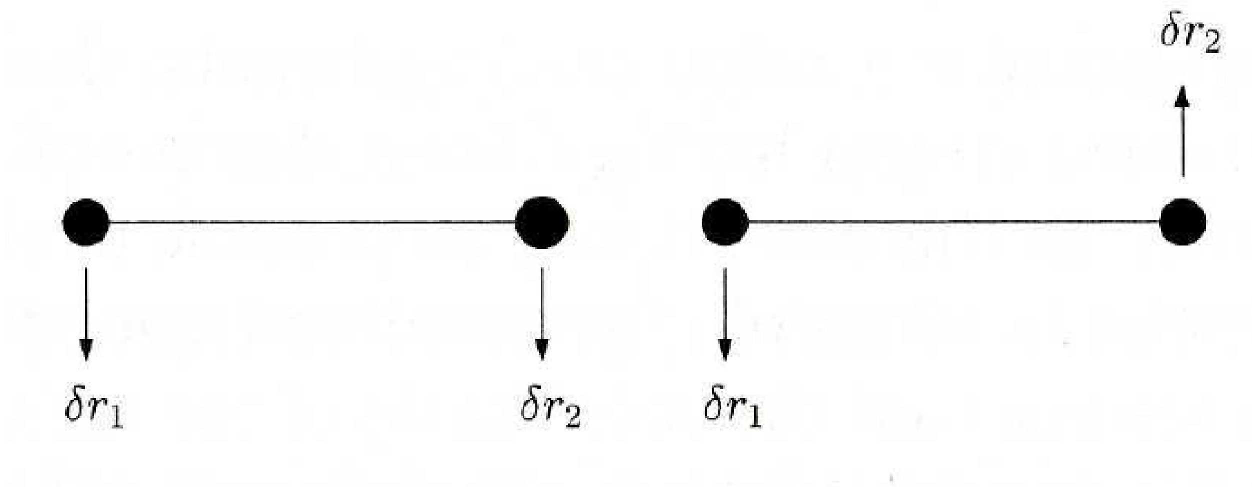
$$g_i(r_1, \dots, r_k) = 0, \qquad i = 1, 2, \dots, l$$

a set of infinitesimal displacements $\delta r_1, \delta r_2, \dots, \delta r_k$ that are consistent with the constraint, i.e.

$$\frac{\partial g_i}{\partial r_1}\delta r_1 + \frac{\partial g_i}{\partial r_2}\delta r_2 + \dots + \frac{\partial g_i}{\partial r_k}\delta r_k = 0, \qquad i = 1, 2, \dots, l$$

are called virtual displacements.

Virtual displacements of a rigid bar



These infinitesimal motions do not destroy the constraint

$$(r_1 - r_2)^T(r_1 - r_2) = l^2$$

If r_1 and r_2 are perturbed

$$r_1 \rightarrow r_1 + \delta r_1 \qquad r_2 \rightarrow r_2 + \delta r_2$$

that is

$$(r_1 + \delta r_1 - r_2 - \delta r_2)^T(r_1 + \delta r_1 - r_2 - \delta r_2) = l^2$$

or,

$$(r_1 - r_2)^T(\delta r_1 - \delta r_2) = 0$$

D'Alembert's principle

Consider a system of k particles, suppose that

- The system has holonomic constraints, that is some of the particles are exposed to constraint forces f_i^c.
- There are externally applied forces f_i^e on the particles.
- The system is moving

Then the work done by all forces applied to the i^{th} particle along each set of virtual displacements is zero, if we add the inertia forces

$$\sum_i \left(f_i^e - \frac{d}{dt}[m\dot{r}_i] \right)^T \delta r_i = 0$$

D'Alembert's principle

Virtual displacements are computed as

$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j, \qquad i = 1, 2, \dots, k$$

Then

$$\sum_{i=1}^{k} f_i^{eT} \delta r_i = \sum_{i=1}^{k} f_i^{eT} \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} f_i^{eT} \frac{\partial r_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_{j=1}^{n} \psi_j \delta q_j$$

Functions ψ_j are called generalized forces.

D'Alembert's principle

The second term can be rewritten as

$$\sum_{i=1}^{k} \frac{d}{dt} m_i \dot{r}_i{}^T \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i^T \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i^T \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j$$

Now,

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i}^{k} \left\{ \frac{d}{dt} \left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i^T \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_j} \right] \right\}$$

D'Alembert's principle

Hence,

$$\sum_{i=1}^{k}\sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\left\{\frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j}\right] - m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]\right\}\right]\delta q_j$$

Now,

$$v_i = \dot{r}_i = \sum_{j=1}^{n}\frac{\partial r_i}{\partial q_j}\delta\dot{q}_j \;\Rightarrow\; \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

$$\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] = \sum_{l=1}^{n}\frac{\partial^2 r_i}{\partial q_j \partial q_l}\delta\dot{\mathrm{q}}_l = \frac{\partial}{\partial q_j}\left[\underbrace{\sum_{l=1}^{n}\frac{\partial r_i}{\partial q_l}\delta\dot{q}_l}_{v_i}\right] = \frac{\partial v_i}{\partial q_j}$$

D'Alembert's principle

Hence,

$$\sum_{i=1}^{k}\sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\left\{\frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j}\right] - m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]\right\}\right]\delta q_j$$

$$= \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\left\{\frac{d}{dt}\left[m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j}\right] - m_i v_i^T \frac{\partial v_i}{\partial q_j}\right\}\right]\delta q_j$$

$$= \sum_{j=1}^{n}\left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}\right]\delta q_j$$

where,

$$K = \sum_{i=1}^{n}\frac{1}{2} m_i v_i^T v_i$$

D'Alembert's principle

To summarize,

$$\sum_i \left(f_i^e - \frac{d}{dt}[m\dot{r}_i] \right)^T \delta r_i = 0$$

with

$$\sum_{i=1}^{k} \frac{d}{dt} m_i \dot{r}_i^{\,T} \delta r_i = \sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \right] \delta q_j , \qquad \sum_{i=1}^{k} f_i^{eT} \delta r_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

is

$$\sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j \right] \delta q_j = 0$$

If δq_j are independent

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j = 0, \qquad i = 1, \dots, n$$

D'Alembert's principle

If ψ_j are the sum of an externally applied generalized force and another one due to a potential field, then

$$\psi_j = -\frac{\partial P}{\partial q_j} + \tau_j,$$

And the Euler-Lagrange equations of motion are,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j,$$

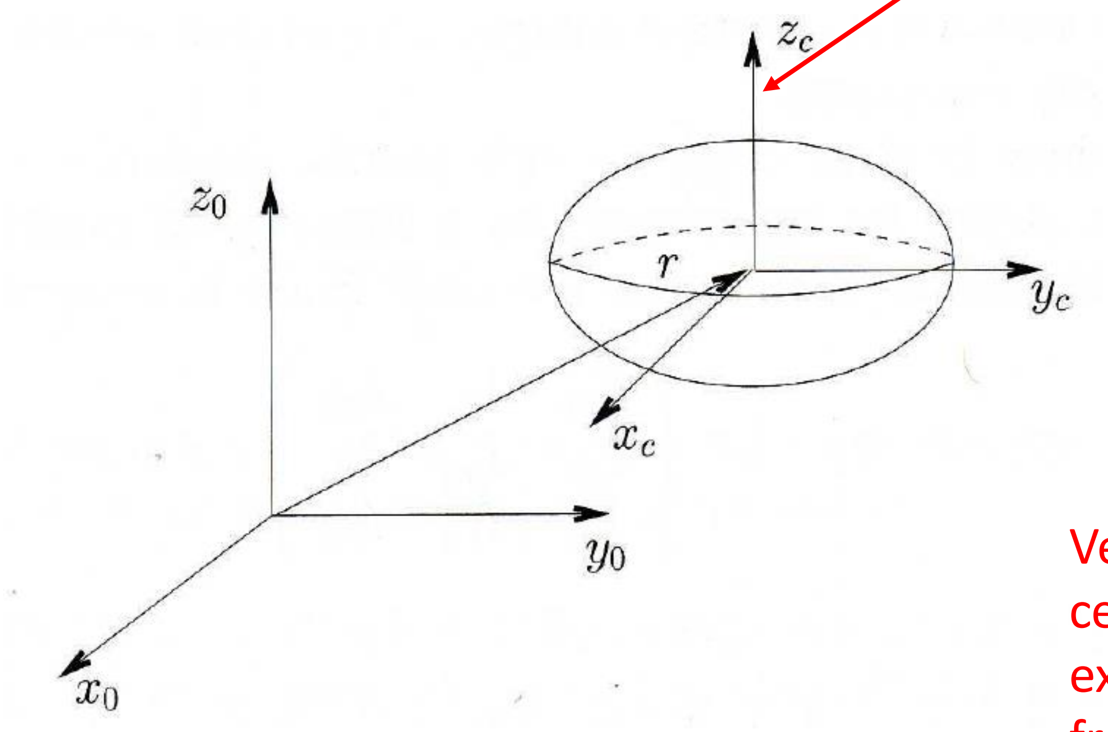
Lagrangian

$$\mathcal{L} = K - P,$$

$$i = 1, \dots, n$$

Computing kinetic energy of a rigid body

Frame attached to the center of mass



Kinetic energy of a rigid body comprises of kinetic energy of translation and kinetic energy of rotation

$$K = \frac{1}{2} m|v_c|^2 + \frac{1}{2}\,\omega^T I_i \omega$$

Inertia tensor in the inertial (fixed) frame

Vector of velocity of center of mass expressed in inertial frame

Vector of angular velocity of the body expressed in inertial frame

Computing kinetic energy : Obtaining I_i

Computing angular velocity

$$S(\omega) = \dot{R}(t)R^T(t) \quad \rightarrow \omega$$

Matrix I_i is the inertia tensor.

In the body frame, it is constant $I_c = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

To compute inertia tensor in the inertial frame, we can use the formula

$$I_i = R(t)I_cR^T(t)$$

Kinetic energy for n-link manipulator

Kinetic energy of link i

$$K = \frac{1}{2}m\left|v_{c_i}\right|^2 + \frac{1}{2}\,\omega_i^T I_i \omega_i$$

The generalized coordinates q are usually the the joint angles (for revolute joints) and positions (for prismatic joints)

We need to express

- $v_{c_i} = \dot{r}_{c_i}$ as a function of generalized coordinates q and velocities $\dot{q}$

  where, r_{c_i} is the position of the center of mass of link i

- ω_i as a function of generalized coordinates q and velocities $\dot{q}$

$$v_{c_i} = J_{v_{c_i}}(q)\dot{q}, \qquad \omega_i = J_{\omega_i}(q)\dot{q}$$

Total energy for n-link manipulator

Total kinetic energy

$$K = \frac{1}{2}\dot{q}^T \left[\boxed{\sum_{i=1}^{n} m_i J_{v\,c_i}^T(q) J_{v\,c_i}(q) \; + J_{\omega\,i}^T(q) R_i(q) I_{ci} R_i^T(q) J_{\omega\,i}(q)} \right] \dot{q}$$

$$= \frac{1}{2}\dot{q}^T D(q)\dot{q} \quad = \sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j$$

nxn symmetric matrix
D(q)

Total potential energy

$$P = \sum_{i}^{n} P_i = \sum_{i}^{n} m_i g r_{c\,i}$$

Computing Jacobians $J_{v_{c_i}}$ and J_{ω_i}

Follow the same approach that was used to determine end-effector velocities

Using D-H frames,

$$J_{v_{c_i}}^{(k)} = \begin{cases} {}^{0}z_{k-1} & \text{for prismatic joint, k < i} \\ {}^{0}z_{k-1} \times \left[r_{c_i} - o_{k-1}\right], & \text{for revolute joint, } k < i \\ 0 & k > i \end{cases}$$

$$J_{\omega_i}^{(k)} = \begin{cases} 0 & \text{for prismatic joint, k < i} \\ {}^{0}z_{k-1} & \text{for revolute joint, } k < i \\ 0 & k > i \end{cases}$$

Equations of motion

$$\psi_k = -\frac{\partial P}{\partial q_k} + \tau_k$$

where τ_i is the joint torque applied at the k^{th} joint.

Hence, if we define $\mathcal{L} = K - P$

Equations of motion are given by

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_k} - \frac{\partial (K-P)}{\partial q_k}, \qquad \mathrm{k} = 1, \dots, \mathrm{n}$$

Equations of motion

$$\frac{\partial K}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q_k}}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q}\right] = \sum_{j=1}^{n} d_{kj}\dot{q}_j$$

$$\Rightarrow \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_k} = \frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot{q}_j\right] = \sum_{j=1}^{n} d_{kj}\ddot{q}_j \ + \sum_{j=1}^{n}\frac{d}{dt}[d_{kj}(q)]\dot{q}_j$$

$$= \sum_{j=1}^{n} d_{kj}\ddot{q}_j \ + \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial d_{kj}(q)}{\partial q_i}\dot{q}_i\right)\dot{q}_j$$

$$= \sum_{j=1}^{n} d_{ij}\ddot{q}_j \ + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}(\frac{\partial d_{kj}(q)}{\partial q_i} + \frac{\partial d_{ki}(q)}{\partial q_j})\dot{q}_i\dot{q}_j$$

Equations of motion

$$\frac{\partial(K-P)}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} - P\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q} - \frac{\partial}{\partial q_k}P$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial}{\partial q_k}P$$

Combining, the equations of motion are

$$\sum_{j=1}^{n} d_{kj}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\underbrace{\boxed{\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)}}_{c_{ijk}}\dot{q}_i\dot{q}_j + \underbrace{\boxed{\frac{\partial}{\partial q_k}P}}_{g(q)} = \tau_k$$

$$k = 1, 2, \ldots, n$$

Matrix form of Equations of Motion

$$\sum_{j=1}^{n} d_{kj}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \frac{\partial}{\partial q_k}P = \tau_k, \qquad k = 1, 2, \dots, n$$

$$\sum_{j=1}^{n} d_{kj}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\,\dot{q}_i\dot{q}_j + \frac{\partial}{\partial q_k}P = \tau_k, \qquad k = 1, 2, \dots, n$$

Matrix form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

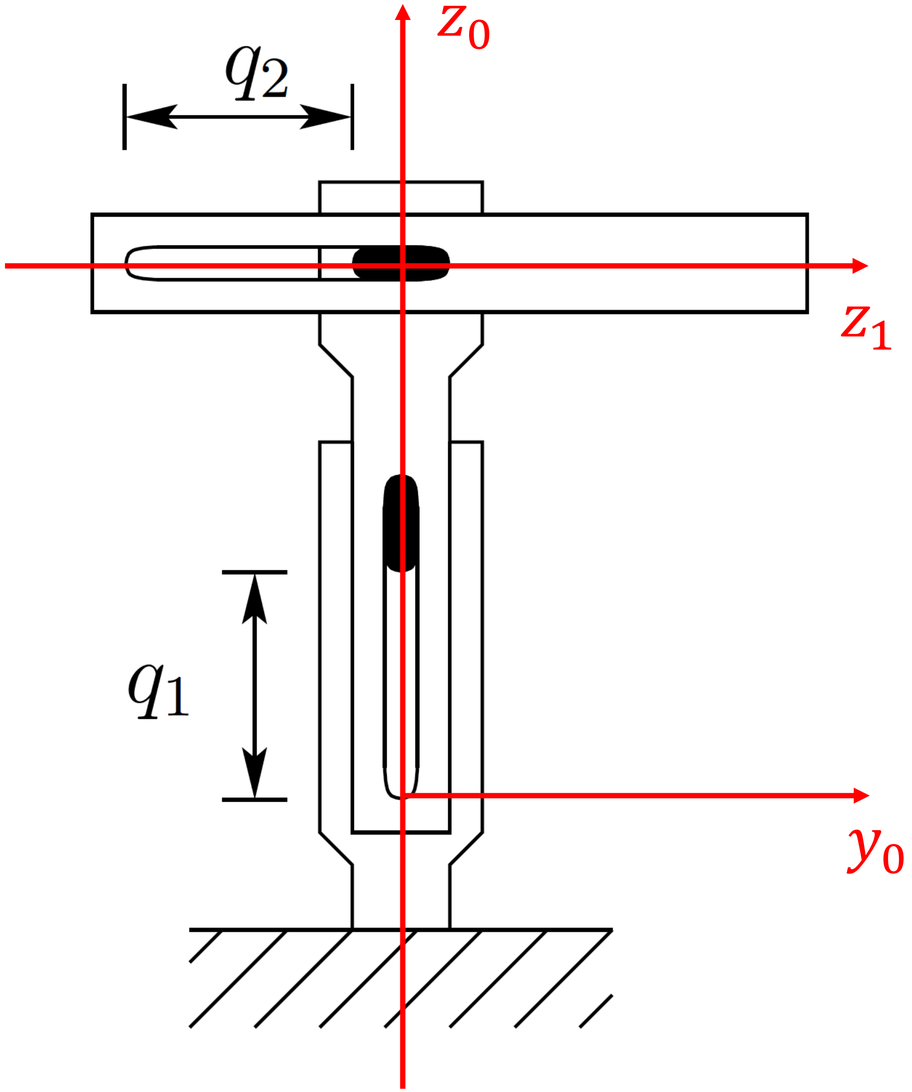
D(q) is the inertia matrix

Centrifugal and Coriolis terms

Vector of gravity terms

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$$

Example: Cartesian manipulator



	$\boldsymbol{\theta}$	d	$\boldsymbol{\alpha}$	$\boldsymbol{a}$
1	0	q_1	$-\frac{\pi}{2}$	0
2	0	q_2	0	0

DH parameters

Only prismatic joints: $J_\omega = 0$

$$J_{v_{c_1}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad J_{v_{c_2}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Cartesian manipulator: equations of motion

$$v_{c_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{q}, \qquad v_{c_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \dot{q}$$

Hence, the kinetic and potential energy are

$$K = \frac{1}{2} \dot{q}^T \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix} \dot{q}, \qquad P = g(m_1 + m_2) q_1 + Const$$

The equations of motion are:

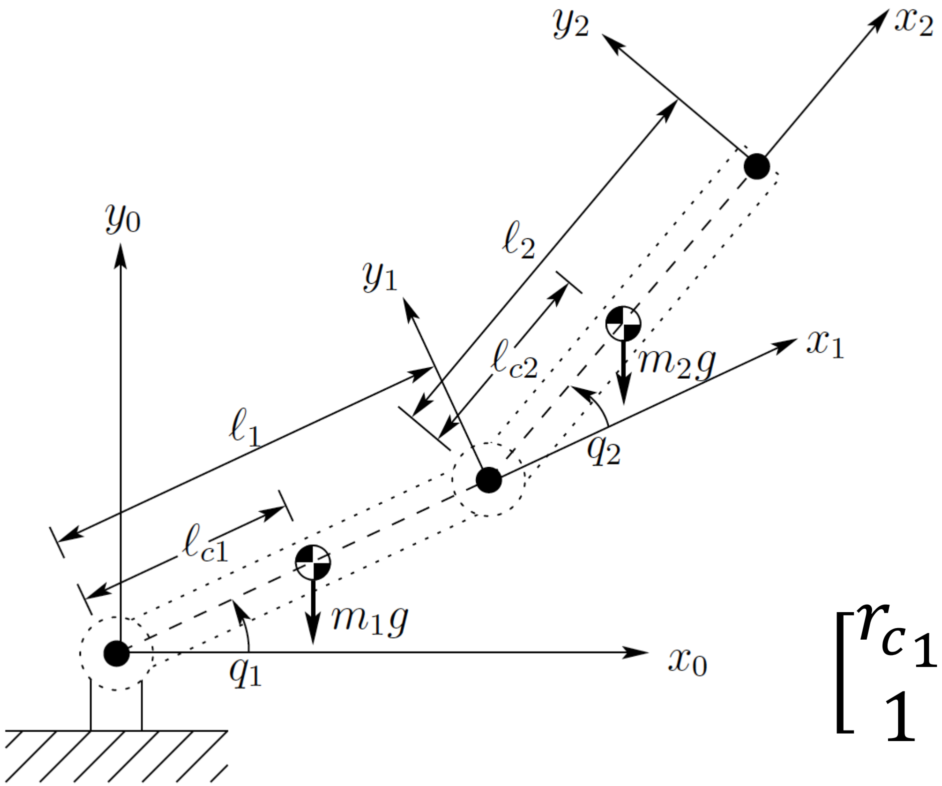
$$\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_i} - \frac{\partial (K - P)}{\partial q_i} = \tau_i$$

or,

$$(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) = \tau_1$$

$$m_2 \ddot{q}_2 = \tau_2$$

Example: Two-link manipulator



	$\boldsymbol{\theta}$	d	$\boldsymbol{\alpha}$	$\boldsymbol{a}$
1	q_1	0	0	l_1
2	q_2	0	0	l_2

DH parameters

$$\begin{bmatrix} r_{c_1} \\ 1 \end{bmatrix} = {}^0_1T \begin{bmatrix} l_1 - l_{c_1} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} r_{c_2} \\ 1 \end{bmatrix} = {}^0_2T \begin{bmatrix} l_2 - l_{c_2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Jacobians: Two-link manipulator

$$J_{v\,c_1}^{1} = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}l_{c_1}c_1\\l_{c_1}s_1\\0\end{bmatrix} = \begin{bmatrix}-l_{c_1}s_1\\l_{c_1}c_1\\0\end{bmatrix}, \qquad J_{v\,c_1}^{2} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \qquad J_{v\,c_1} = [J_{v\,c_1}^{1} \quad J_{v\,c_1}^{2}]$$

$$J_{v\,c_2}^{1} = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}l_1c_1 + l_{c_2}c_{12}\\l_1s_1 + l_{c_2}s_{12}\\0\end{bmatrix} = \begin{bmatrix}-l_1s_1 - l_{c_2}s_{12}\\l_1c_1 + l_{c_2}c_{12}\\0\end{bmatrix},$$

$$J_{v\,c_2}^{2} = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}l_{c_2}c_{12}\\l_{c_2}s_{12}\\0\end{bmatrix} = \begin{bmatrix}-l_{c_2}s_{12}\\l_{c_2}c_{12}\\0\end{bmatrix}, \qquad J_{v\,c_2} = [J_{v\,c_2}^{1} \quad J_{v\,c_2}^{2}]$$

$$J_{\omega_1} = [z_0 \quad 0] = \begin{bmatrix}0 & 0\\0 & 0\\1 & 0\end{bmatrix}, \qquad J_{\omega_2} = [z_0 \quad z_1] = \begin{bmatrix}0 & 0\\0 & 0\\1 & 1\end{bmatrix}$$

Kinetic Energy: Two-link manipulator

Linear and angular velocities

$$v_{c_1} = J_{v_{c_1}}\dot{q}, \quad \omega_1 = J_{\omega_1}\dot{q}, \qquad\qquad v_{c_2} = J_{v_{c_2}}\dot{q}, \quad \omega_2 = J_{\omega_2}\dot{q}$$

Translational kinetic energy

$$K_{trans} = \frac{1}{2}m_1 v_{c_1}^T v_{c_1} + \frac{1}{2}m_2 v_{c_2}^T v_{c_2} = \frac{1}{2}\dot{q}^T\left[m_1 J_{v_{c_1}}{}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}{}^T J_{v_{c_2}}\right]\dot{q}$$

Rotational kinetic energy

$$K_{rot} = \frac{1}{2}\dot{q}^T\left[J_{\omega_1}^T(q)R_1(q)I_1R_1^T(q)J_{\omega_1}(q) + J_{\omega_2}^T(q)R_2(q)I_2R_2^T(q)J_{\omega_1}(q)\right]\dot{q}$$

$$= \frac{1}{2}\dot{q}^T\left\{(I_{33})_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + (I_{33})_2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\}\dot{q}$$

Hence,

$$D(q) = m_1 J_{v_{c_1}}{}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}{}^T J_{v_{c_2}} + \begin{bmatrix}(I_{33})_2 + (I_{33})_1 & (I_{33})_2\\ (I_{33})_2 & (I_{33})_2\end{bmatrix}$$

Equations of motion: Two-link manipulator

$$d_{11} = m_1 l_{c\,1}^2 + m_2\left(l_1^2 + l_{c\,2}^2 + 2 l_1 l_{c_2} c_2\right) + (I_{33})_1 + (I_{33})_2$$

$$d_{12} = d_{21} = m_2\left(l_{c_2}^{\,2} + l_1 l_{c_2} c_2\right) + (I_{33})_2, \qquad d_{22} = m_2 l_{c\,2}^2 + (I_{33})_2$$

$$c_{ijk} = \left(\frac{\partial d_{ik}}{\partial q_j} + \frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{jk}}{\partial q_i}\right)$$

$$c_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0,$$

$$c_{112} = c_{121} = \frac{1}{2}\left(\frac{\partial d_{12}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_1}\right) = -m_2 l_1 l_{c_2} s_2 =: h,$$

$$c_{122} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h, \qquad c_{211} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h,$$

$$c_{212} = c_{221} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0, \qquad c_{222} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = -h$$

Equations of motion: Two-link manipulator

Potential energy

$$\mathrm{P}_1 = m_1 g l_{c_1} s_1, \qquad \mathrm{P}_2 = m_2 g\left(l_1 s_1 + l_{c_2} s_{12}\right), \qquad P = P_1 + P_2$$

Hence,

$$\phi_1 = \frac{\partial P}{\partial q_1} = \left(m_1 l_{c_1} + m_2 l_1\right) g c_1 + m_2 l_{c_2} g c_{12}, \qquad \phi_2 = \frac{\partial P}{\partial q_2} = m_2 g l_{c_2} c_{12}$$

Equations of motion

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{111}\dot{q}_1^2 + c_{112}\dot{q}_1\dot{q}_2 + c_{121}\dot{q}_2\dot{q}_1 + c_{122}\dot{q}_2^2 + \phi_1 = \tau_1,$$
$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{211}\dot{q}_1^2 + c_{212}\dot{q}_1\dot{q}_2 + c_{221}\dot{q}_2\dot{q}_1 + c_{222}\dot{q}_2^2 + \phi_1 = \tau_1$$

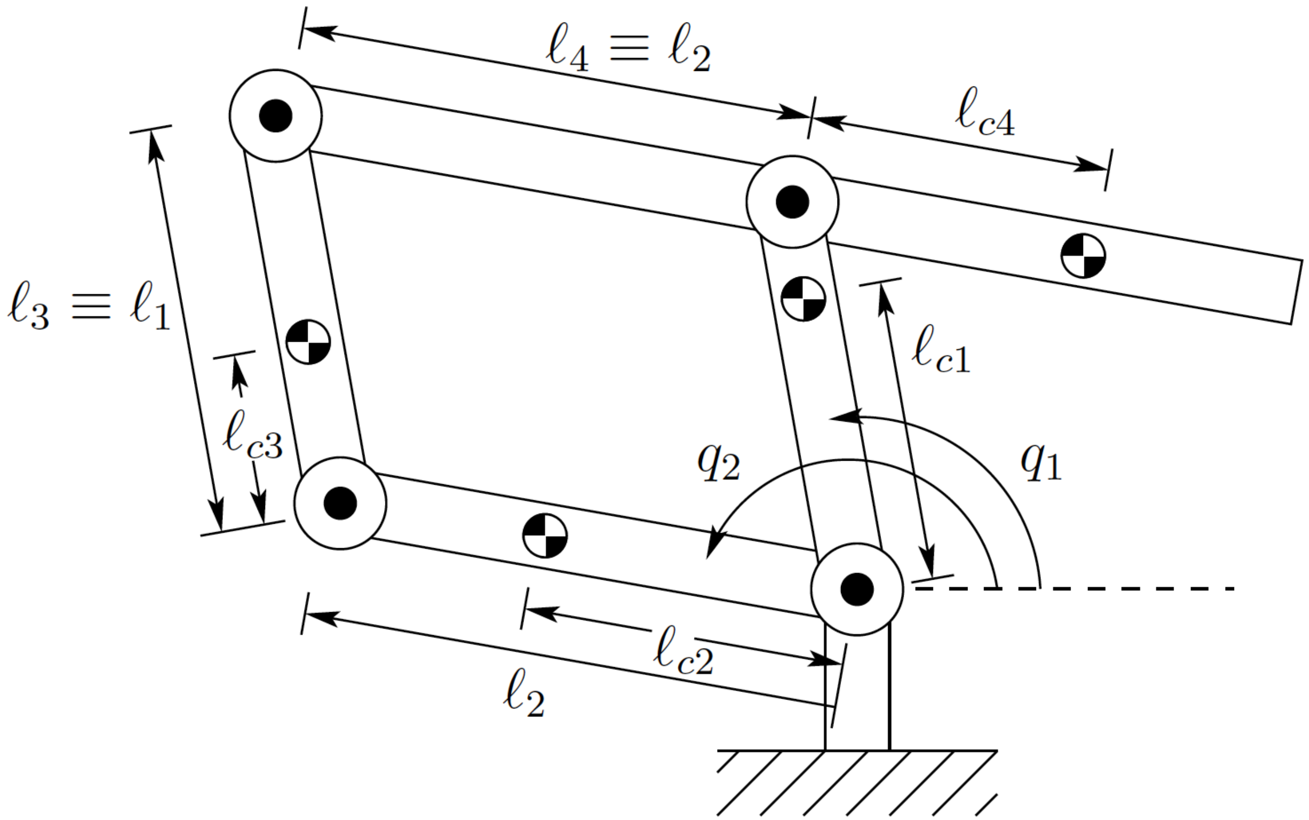
In matrix form,

$$D(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

where,

$$C(q,\dot{q}) = \begin{bmatrix} h\dot{q}_2 & h\dot{q}_2 + h\dot{q}_1 \\ -h\dot{q}_1 & 0 \end{bmatrix}; \qquad G(q) = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Example: Five bar linkage



$$\begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = \begin{bmatrix} l_{c1} c_1 \\ l_{c1} s_1 \end{bmatrix},$$

$$\begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = \begin{bmatrix} l_{c2} c_2 \\ l_{c2} s_2 \end{bmatrix},$$

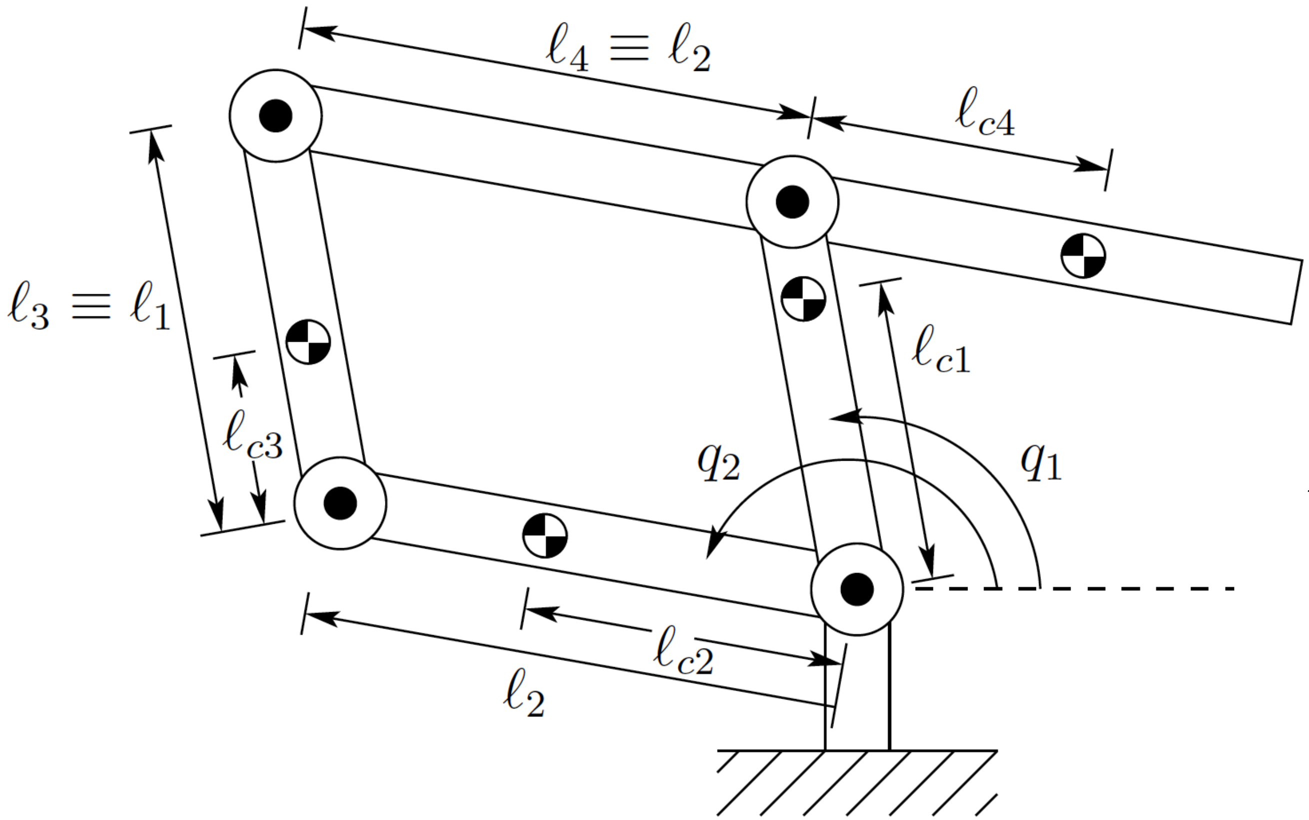
$$\begin{bmatrix} x_{c3} \\ y_{c3} \end{bmatrix} = \begin{bmatrix} l_2 c_2 \\ l_2 s_2 \end{bmatrix} + \begin{bmatrix} l_{c3} c_1 \\ l_{c3} s_1 \end{bmatrix},$$

$$\begin{bmatrix} x_{c4} \\ y_{c4} \end{bmatrix} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \end{bmatrix} - \begin{bmatrix} l_{c4} c_2 \\ l_{c4} s_2 \end{bmatrix}$$

$$v_{c1} = \begin{bmatrix} -l_{c1} s_1 & 0 \\ l_{c1} c_1 & 0 \end{bmatrix} \dot{q}\,, \qquad v_{c2} = \begin{bmatrix} 0 & -l_{c2} s_2 \\ 0 & l_{c2} c_2 \end{bmatrix} \dot{q},$$

$$v_{c3} = \begin{bmatrix} -l_{c3} s_1 & -l_2 s_2 \\ l_{c3} c_1 & l_2 c_2 \end{bmatrix} \dot{q}, \qquad v_{c4} = \begin{bmatrix} -l_1 s_1 & l_{c4} s_2 \\ l_1 c_1 & -l_{c4} c_2 \end{bmatrix} \dot{q}$$

Example: Five bar linkage



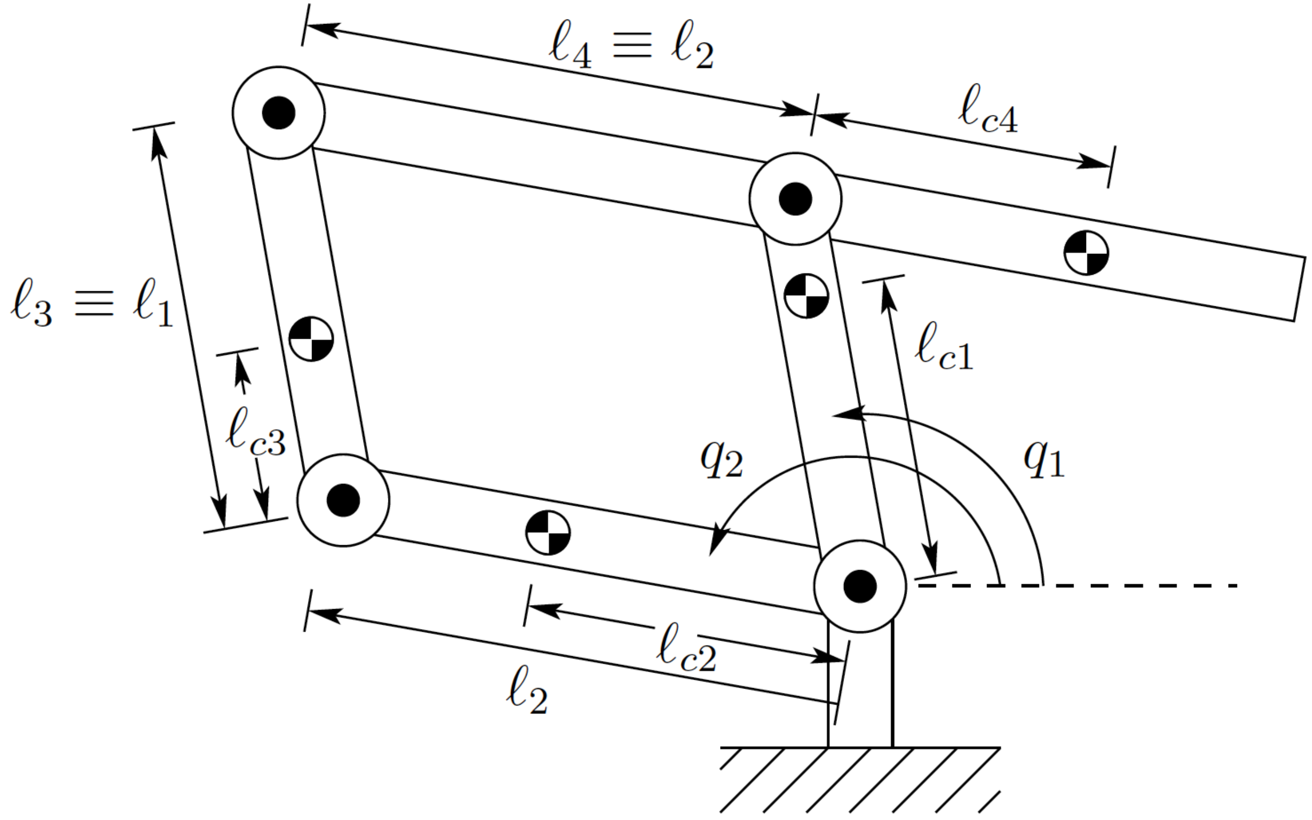
$$\omega_1 = \omega_3 = \dot{q}_1 \hat{k}, \qquad \omega_2 = \omega_4 = \dot{q}_2$$

$$D(q) = \sum_{i=1}^{4} m_i J_{vci}^T J_{vci} + \begin{bmatrix} I_1 + I_3 & 0 \\ 0 & I_2 + I_4 \end{bmatrix}$$

0 if $m_2 l_2 l_{c3} = m_4 l_1 l_{c4}$

$$\begin{aligned} d_{11}(q) &= m_1 l_{c1}^2 + m_3 l_{c3}^2 + m_4 l_4^2 + I_1 + I_3, \\ d_{21}(q) &= d_{12}(q) = (m_2 l_2 l_{c3} - m_4 l_1 l_{c4}) \cos(q_2 - q_1), \\ d_{22}(q) &= m_2 l_{c2}^2 + m_3 l_2^2 + m_4 l_{c4}^2 + I_2 + I_4 \end{aligned}$$

Example: Five bar linkage



Constant diagonal inertia matrix

$\downarrow$

no Coriolis/Centrifugal terms!

Potential energy

$$P = \sum_{i=1}^{4} y_{ci} = g(m_1 l_{c1} + m_3 l_{c3} + m_4 l_1)s_1$$
$$+ g(m_2 l_{c2} - m_4 l_{c4} + m_3 l_2)s_2$$

Hence,

$$\phi_1 = g(m_1 l_{c1} + m_3 l_{c3} + m_4 l_1)c_1$$
$$\phi_2 = g(m_2 l_{c2} - m_4 l_{c4} + m_3 l_2)c_2$$

Equations of motion:

$d_{11}\ddot{q}_1 + \phi_1 = \tau_1,$ $\qquad$ $d_{22}\ddot{q}_2 + \phi_2 = \tau_2$

Properties: Skew-symmetry and passivity

- Matrix $N(q, \dot{q}) = \dot{D}(q) - 2C(q, \dot{q})$ is skew-symmetric

$$\dot{d}_{kj} = \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \,,$$

$$n_{kj} = \dot{d}_{kj} - 2c_{kj} = \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} - \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \right] \dot{q}_i = \sum_{i=1}^{n} \left[\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j} \right] \dot{q}_i$$

$$= -n_{jk}$$

- Passivity property: There exists a constant $\beta \geq 0$, such that

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta \geq -\beta, \qquad \forall T \geq 0$$

Amount of energy produced by the system has a lower bound given by $-\beta$.

Properties: Passivity and total energy

- Total energy in the system $H = \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$
- Along the system trajectory,

$$\begin{aligned}\dot{H} &= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T \frac{\partial P}{\partial q} \\ &= \dot{q}^T\big(\tau - C(q,\dot{q}) - G(q)\big) + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T \frac{\partial P}{\partial q} \\ &= \dot{q}^T\tau + \frac{1}{2}\dot{q}^T\left(\dot{D}(q) - 2C(q,\dot{q})\right)\dot{q} = \dot{q}^T\tau\end{aligned}$$

- Integrating,

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta = H(T) - H(0) \geq -H(0)$$