ME 346: Heat Transfer

Lecture: Conduction-Introduction

Date:

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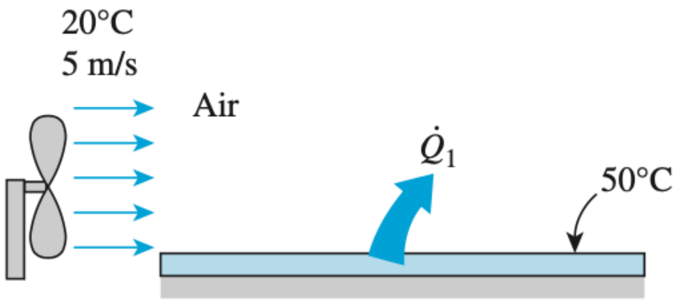
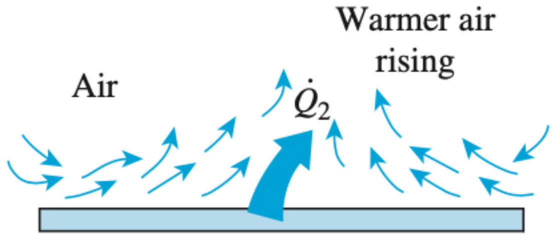
Convective Heat Transfer

Convection: involves fluid motion on top of heat conduction
Thought: is heat transfer via convection more than that by conduction?

Types of convection



(*a*) Forced convection



(*b*) Natural convection

Convective Heat Transfer Coeff.

$$q^{"} = h\,(T_s - T_\infty) \;\; [\text{Newton's law of cooling}]$$

$$@\ \text{surface: no} - \text{slip condition}$$

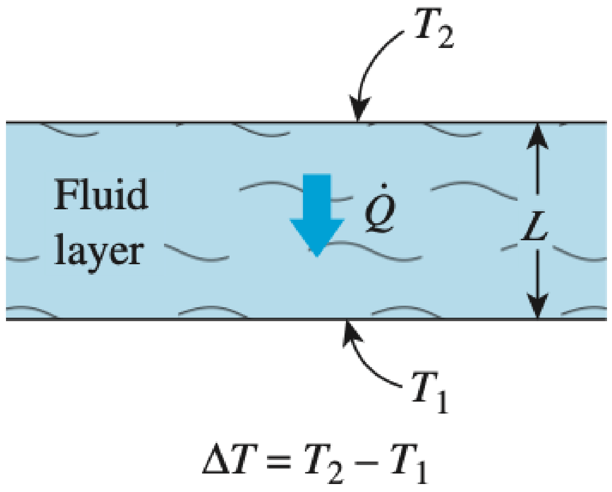
$$h(T_s - T_\infty) = -k_f \frac{\partial T}{\partial y}$$

$$\rightarrow h = \frac{-k_f \frac{\partial T}{\partial y}}{(T_s - T_\infty)} \;\; \left[\frac{W}{m^2 . K}\right]$$

Note: This is how we measure/compute *h*

Non-dimensional *h*: Nusselt Number

$$Nu = \frac{hL_c}{k_f} \quad L_c\text{: characteristic length}$$



$$q_{cond} = k\frac{\Delta T}{L}$$

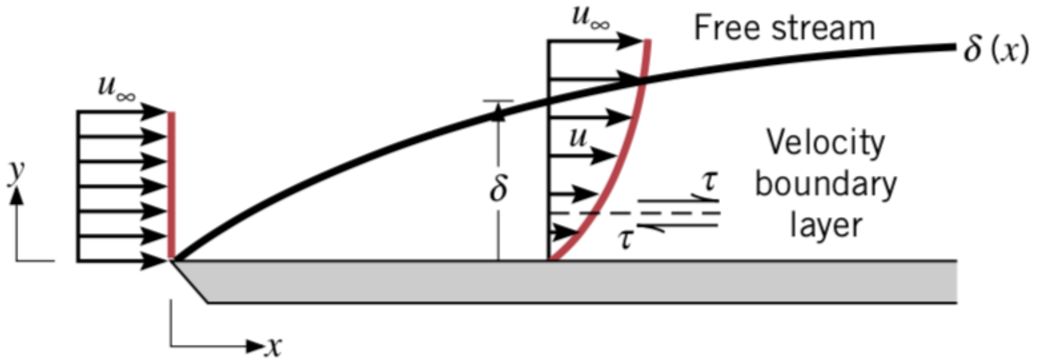
$$q_{conv} = h\,\Delta T$$

$$\frac{q_{conv}}{q_{cond}} = \frac{h\Delta T L}{k\Delta T} = \frac{hL}{k} = Nu$$

Nu: enhancement in heat transfer across a fluid layer due to convection compared to conduction

$$Nu = 1?$$

Boundary Layers

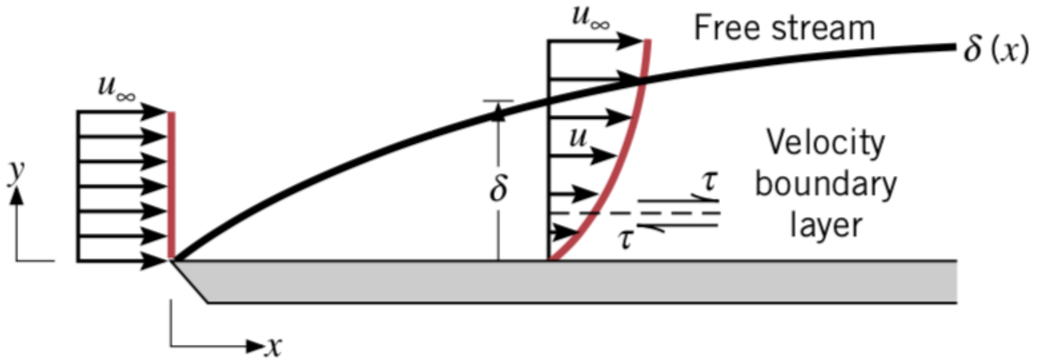


Velocity Boundary Layer:

Consider flow over a flat-plate:

- Fluid particles assume zero-velocity at the surface: no-slip boundary condition
- These particles retard the motion of particles in the adjoining fluid layer, which act to retard the motion of particles in the next layer, and so on until, at a distance $y = \delta$ from the surface, the effect becomes negligible.
- δ increases with x, i.e., boundary layer grows

Boundary Layers



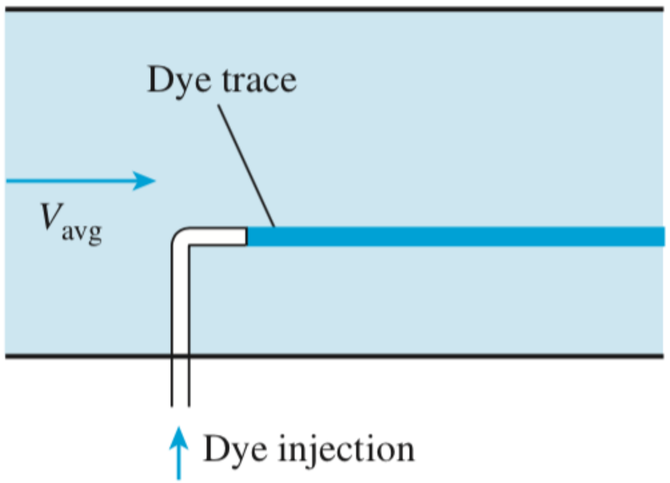
Velocity Boundary Layer:

Consider flow over a flat-plate:

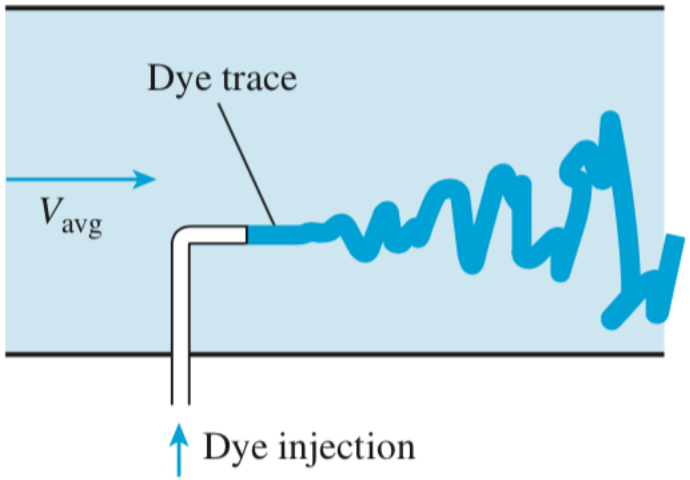
- u varies from $[u = 0 \text{ @ } y = 0]$ to $[u = 0.99\, u_\infty \text{ @ } y = \delta]$ $\rightarrow \tau_s$ (shear stress)
- Friction coefficient: $C_f \equiv \dfrac{\tau_s}{\rho u_\infty^2/2}$
- Newtonian fluid: $\tau_s \propto \dfrac{\partial u}{\partial y}\Big|_{y=0} \rightarrow \tau_s = \mu \dfrac{\partial u}{\partial y}\Big|_{y=0}$; μ: dynamic viscosity
- C_f varies with x

Laminar and Turbulent Flow

Laminar Flow: highly-ordered flow with smooth streamlines

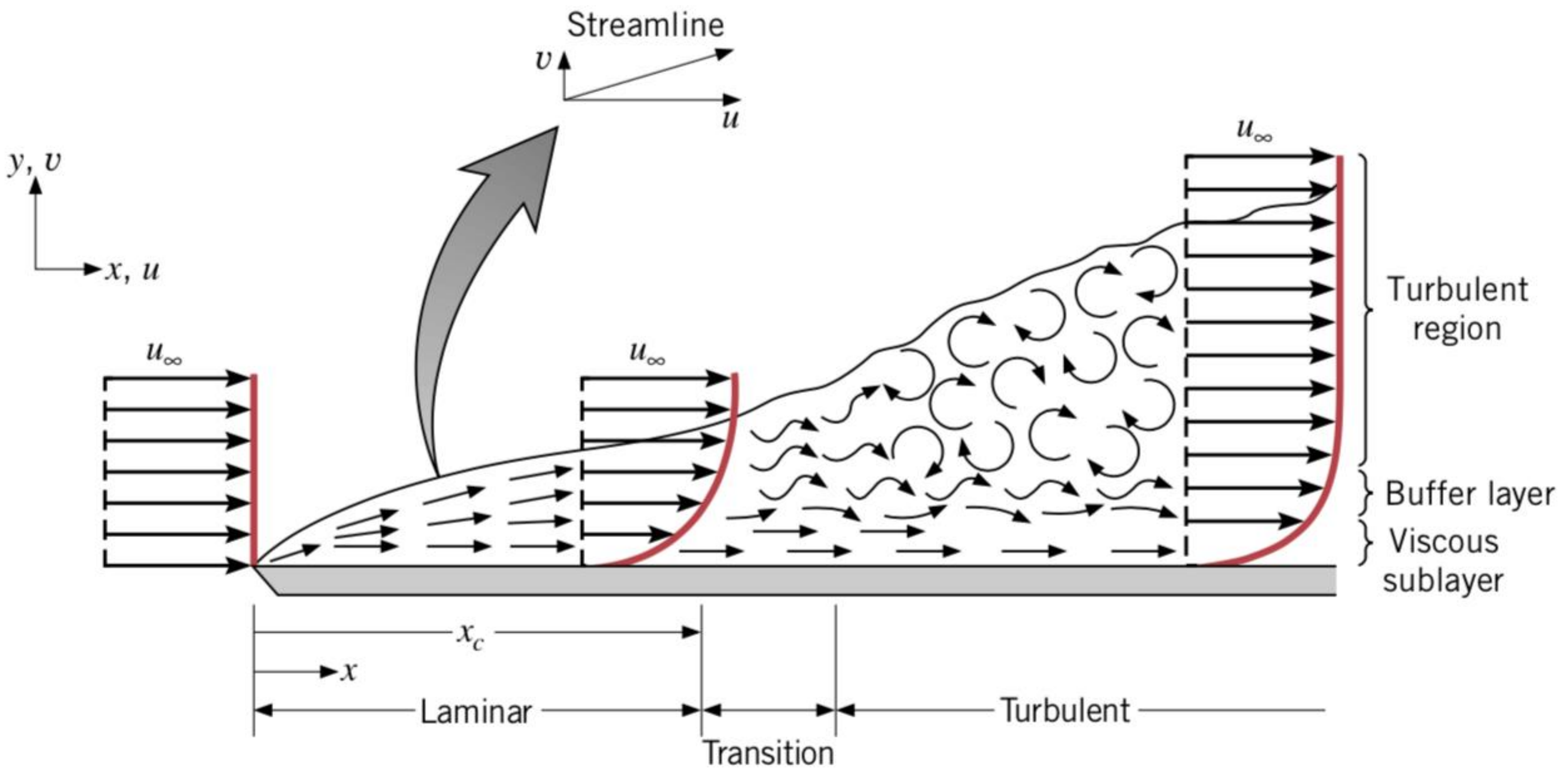


Turbulent Flow: Chaotic, highly irregular flow



intense mixing of the fluid in turbulent flow
→ large friction force and convection heat transfer rate

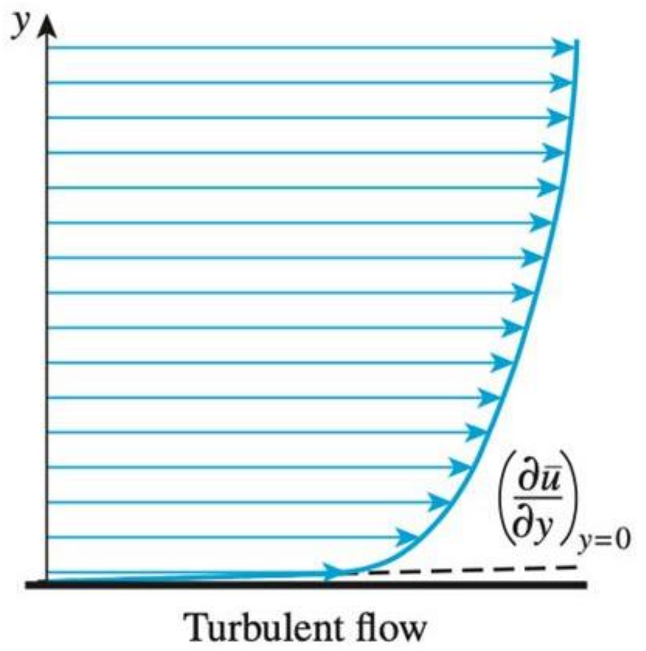
Laminar and Turbulent Flow



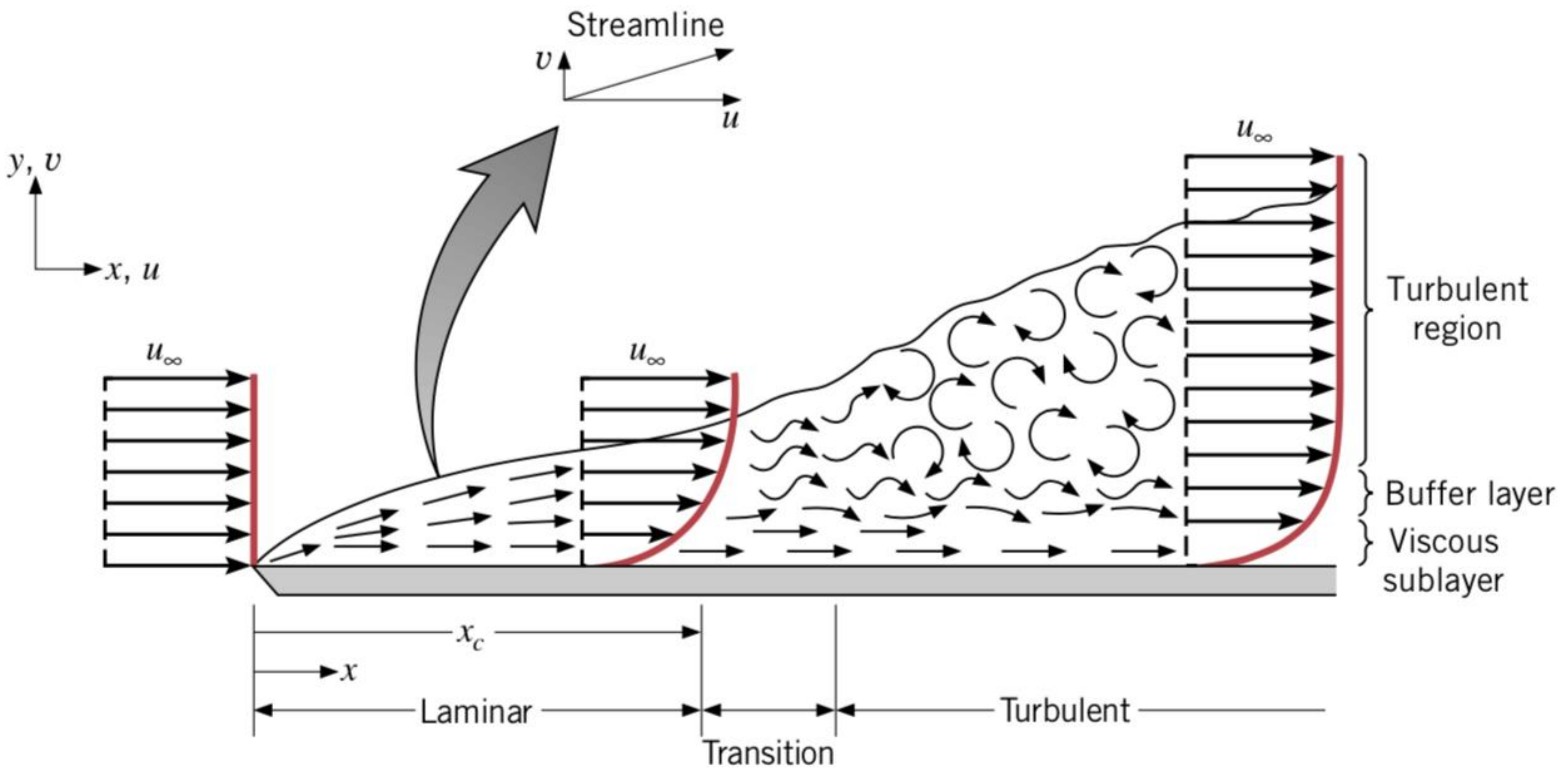
- ❑ Initially viscous forces are dominant
- ❑ As boundary layer grows and δ increases, the relative contribution of viscous forces decrease
- ❑ Not able to dissipate some disturbances in the flow and disturbances get amplified by inertial forces

Turbulent Flow:

- ❑ Viscous sublayer: viscous effects dominates and velocity profile is almost linear
- ❑ Buffer sublayer: viscous and turbulent effects are comparable
- ❑ Turbulent sublayer: dominated by turbulent effects and velocity profile is flat



Laminar and Turbulent Flow



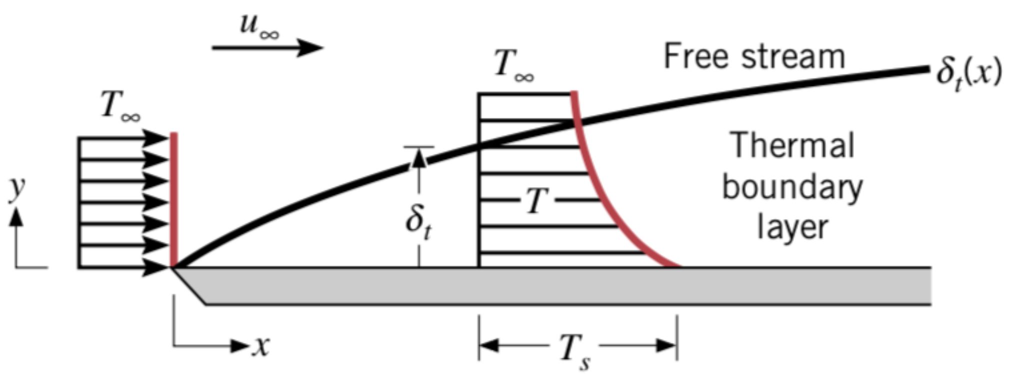
$$Re_x = \frac{\rho u_\infty x}{\mu}$$

$$Re_{x_c} \equiv \frac{\rho u_\infty x_c}{\mu} = 5 \times 10^5$$

For flow over a flat plate

Transition from laminar to turbulent flow depends on fluid-type, flow velocity, and surface geometry and roughness, among other things

Boundary Layers

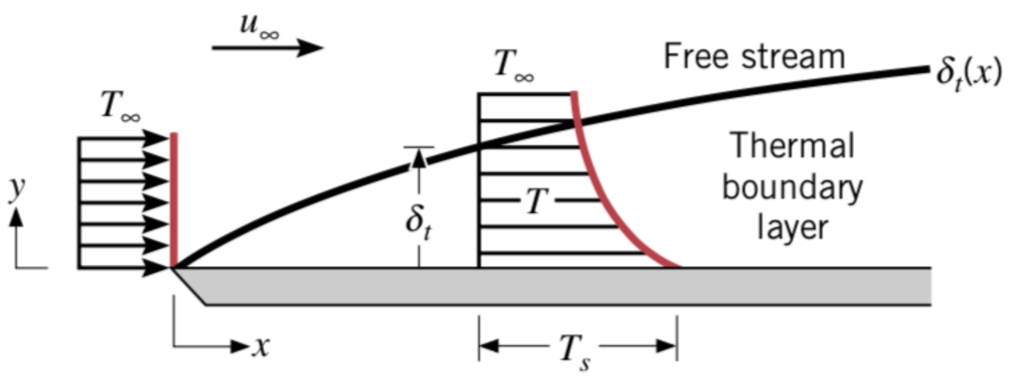


<u>Thermal Boundary Layer:</u>

Consider flow over a flat-plate:

- ❑ fluid particles @ $y = 0, T = T_S$
- ❑ These particles exchange energy with those in the adjoining fluid layer → temperature gradients → Thermal boundary layer
- ❑ Boundary layer thickness, δ_t, y for which: $\frac{T - T_S}{T_\infty - T_S} = 0.99$
- ❑ δ_t increases with x, i.e., thermal boundary layer develops.

Boundary Layers



Thermal Boundary Layer:

Consider flow over a flat-plate:

- Temperature gradients → heat transfer
- At the surface, no fluid motion → heat transfer only due to conduction:

$$q_s'' = -k_f \frac{\partial T}{\partial y}\Big|_{y=0}$$

- From Newton's law of cooling:

$$q_s'' = h(T_s - T_\infty)$$

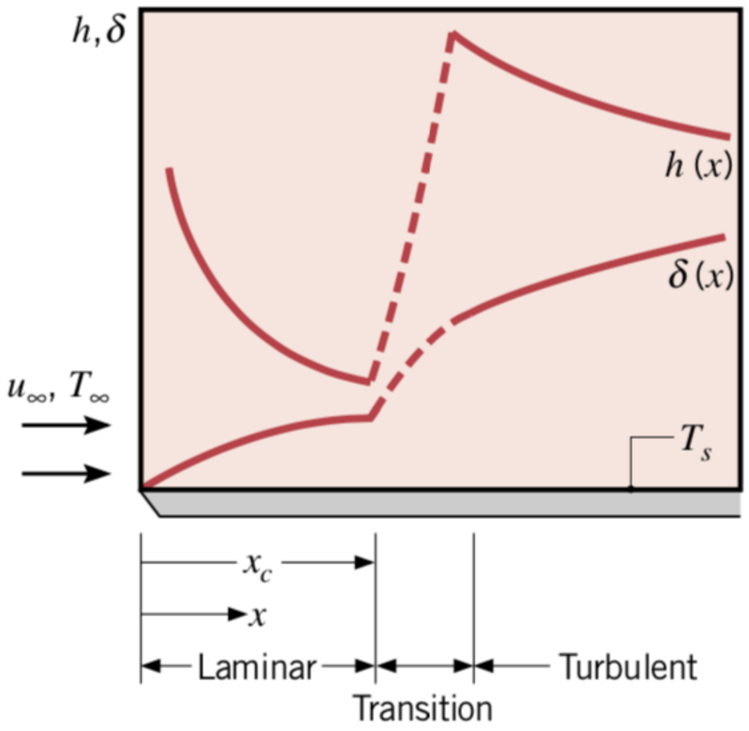
$$h = \frac{-k_f \frac{\partial T}{\partial y}\big|_{y=0}}{(T_s - T_\infty)}$$

Notice that k_f and $(T_s - T_\infty)$ are independent of x

→ as δ_t increase, magnitude of $\frac{\partial T}{\partial y}\big|_{y=0}$ must decrease

→ h decreases as boundary layer grows

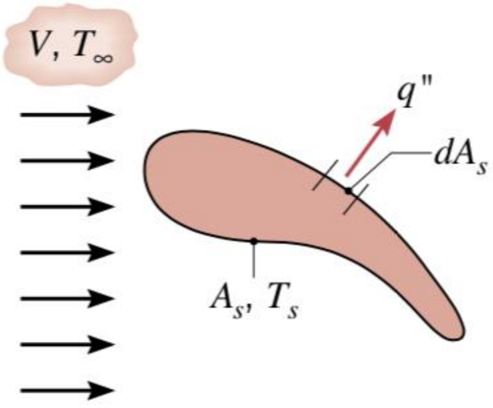
Thermal Boundary Layer



Laminar region: h decreases as boundary layer grows
Transition region: h increases suddenly due to onset of mixing
Turbulent region: h decreases as turbulent boundary layer grows

Due to large mixing, the importance of conduction is reduced in the turbulence region → the differences in δ and δ_t are much smaller in turbulent region than in the laminar region

Local and average coefficients



Total heat transfer rate:

$$q_{Total} = \int_{A_s} h\, dA_s (T_s - T_\infty)$$

$$\bar{h} \equiv \frac{q_{Total}}{A_s(T_s - T_\infty)}$$

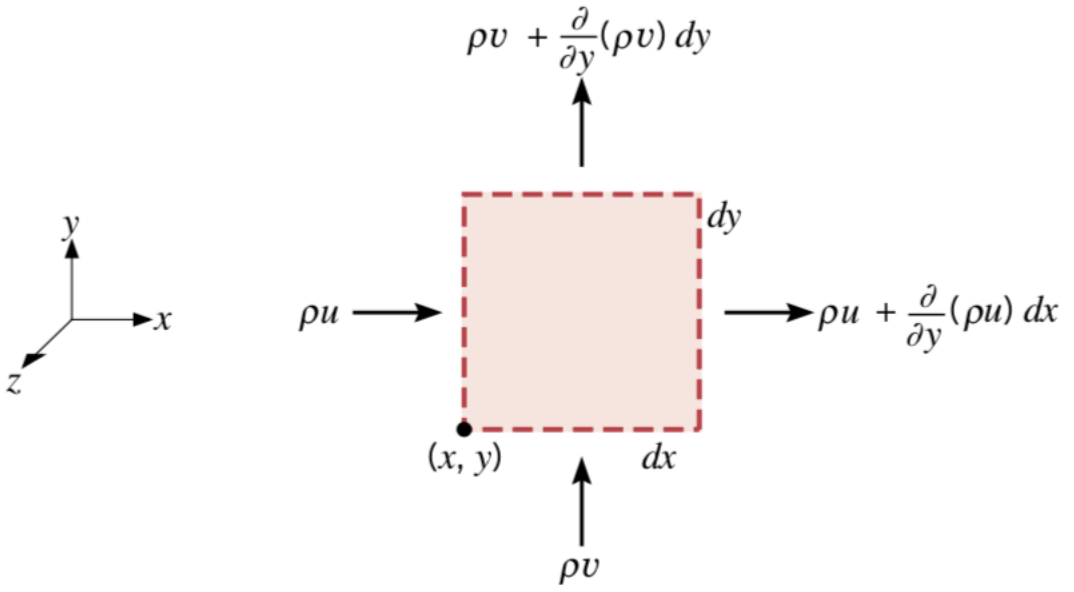
$$= \frac{\int_{A_s} h\, dA_s (T_s - T_\infty)}{A_s(T_s - T_\infty)}$$

$$= \frac{1}{A_s}\int_{A_s} h\, dA_s$$

Governing Equations for Boundary Layers

- Conservation of Mass: Continuity Equation
- Conservation of Momentum: Momentum Equation
- Conservation of Energy: Energy Equation

Continuity Equation



- ❑ unit size in the z-direction, two-dimensional, steady flow
- ❑ for steady flow, the net rate at which mass enters the control volume (inflow - outflow) must equal zero

$$Rate\ of\ mass - inflow = \rho u dydz + \rho v dxdz$$

$$Rate\ of\ mass - outflow = \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right] dydz + \left[\rho v + \frac{\partial}{\partial y}(\rho v)dy\right] dxdz$$

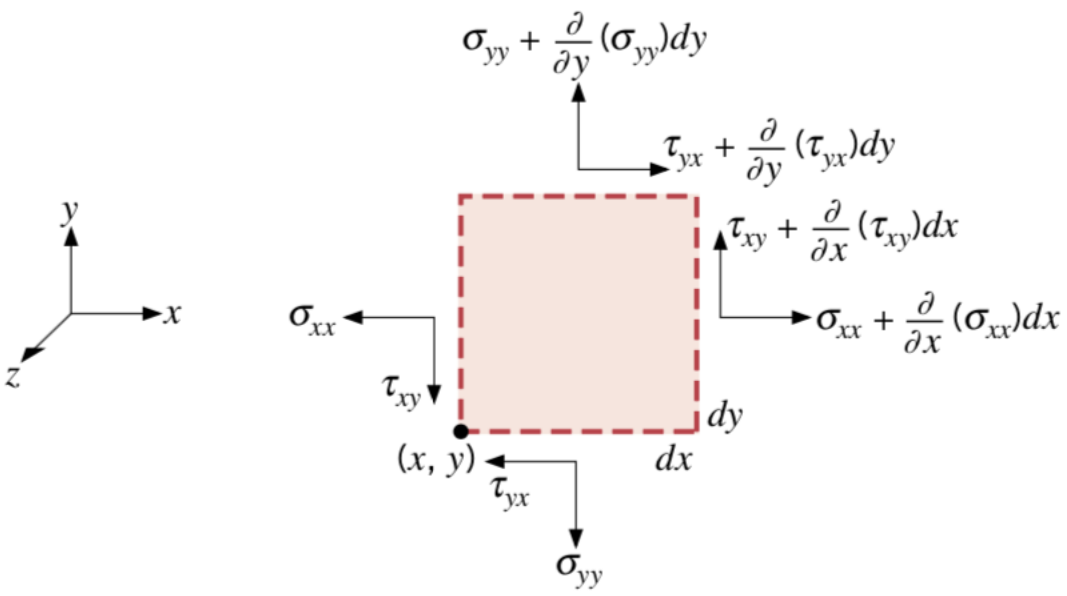
$$\rho u dydz + \rho v dxdz = \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right] dydz + \left[\rho v + \frac{\partial}{\partial y}(\rho v)dy\right] dxdz$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

For constant density:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum Equation



- Newton's Second Law: the sum of all forces must equal the rate of momentum change

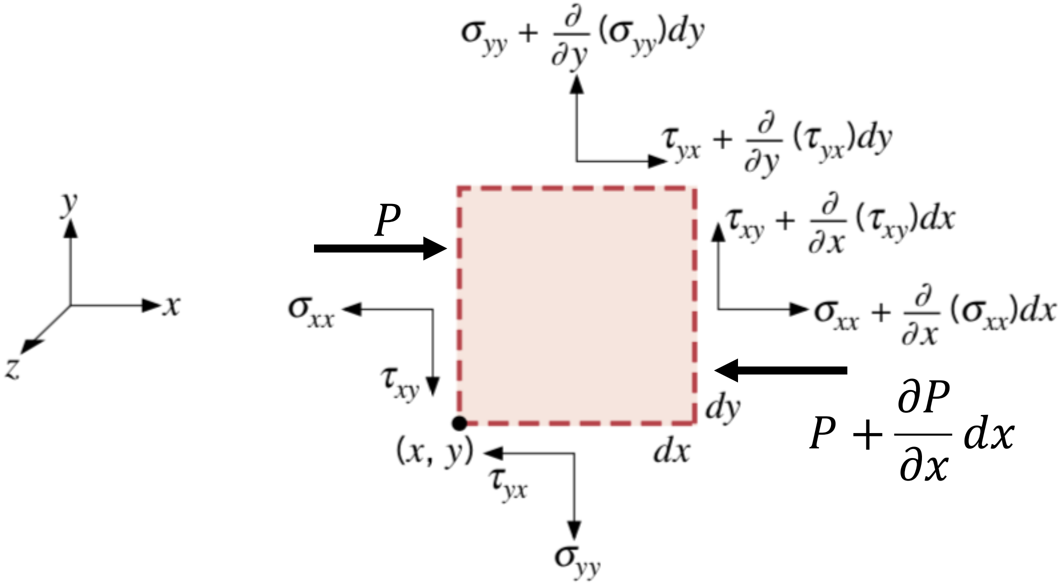
- Body forces: proportional to the volume of the body [such as gravity, electric, and magnetic forces]

- Surface forces: proportional to the surface area [such as pressure forces due to hydrostatic pressure and shear stresses due to viscous effects]

Surface forces: Pressure and viscous
Viscous forces: decomposed into two perpendicular comp: Normal and Shear stress

Note: Pressure and normal stress are different! Forces due to normal stress are arising from viscous effects related to velocity gradient and are zero for uniform velocity profile. While pressure forces are related to hydrostatic pressure and are non-zero even for uniform velocity profiles.

Momentum Equation

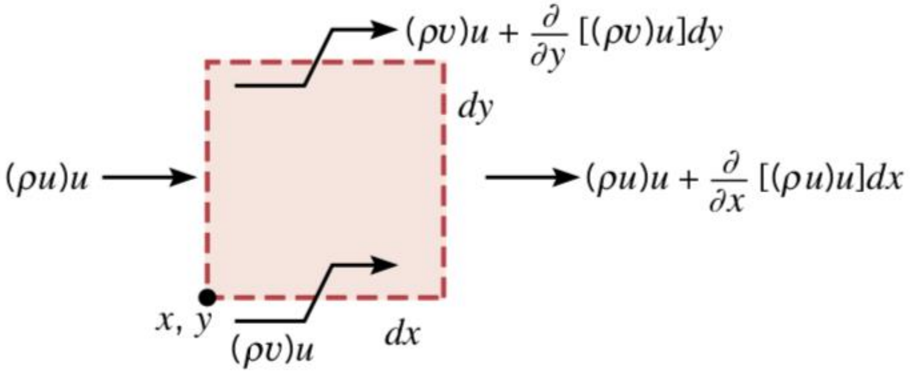


$$F_{x,pressure} = \left(P - \left[P + \frac{\partial P}{\partial x} dx\right]\right) dydz$$

$$F_{x,normal\,stress} = \left(-\sigma_{xx} + \left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right]\right) dydz$$

$$F_{x,shear\,stress} = \left(-\tau_{yx} + \left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right]\right) dxdz$$

$$F_{x,body} = X\ dxdydz$$



$Rate\ of\ change\ of\ x - memonetum$:

$$\left[(\rho u)u + \frac{\partial}{\partial x}[(\rho u)u]dx\right] dydz\ - (\rho u dydz)u$$

$$+ \left[(\rho v)u + \frac{\partial}{\partial y}[(\rho v)u]dy\right] dxdz - \ (\rho v)udxdz$$

$$\left[\frac{\partial}{\partial x}[(\rho u)u]\right] dxdydz + \left[\frac{\partial}{\partial y}[(\rho v)u]\right] dxdydz = -\frac{\partial P}{\partial x} dxdydz + \frac{\partial \sigma_{xx}}{\partial x} dxdydz + \frac{\partial \tau_{yx}}{\partial y} dxdydz + Xdxdydz$$

Momentum Equation

$$\left[\frac{\partial}{\partial x}[(\rho u)u]\right]dxdydz + \left[\frac{\partial}{\partial y}[(\rho v)u]\right]dxdydz = -\frac{\partial P}{\partial x}dxdydz + \frac{\partial \sigma_{xx}}{\partial x}dxdydz + \frac{\partial \tau_{yx}}{\partial y}dxdydz + Xdxdydz$$

$$\xrightarrow{rearrange} \quad \frac{\partial}{\partial x}[(\rho u)u] + \frac{\partial}{\partial y}[(\rho v)u] = \frac{\partial(\sigma_{xx}-P)}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X$$

$$\xrightarrow{chain-rule} \quad \rho u\frac{\partial u}{\partial x} + u\frac{\partial}{\partial x}(\rho u) + \rho v\frac{\partial u}{\partial y} + u\frac{\partial}{\partial y}(\rho v) = \frac{\partial(\sigma_{xx}-P)}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X$$

$$\xrightarrow{rearrange} \quad \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + u\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right] = \frac{\partial(\sigma_{xx}-P)}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X$$

$$\xrightarrow{\substack{using \\ continuity \\ equation}} \quad \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial(\sigma_{xx}-P)}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X$$

Momentum Equation

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{\partial(\sigma_{xx}-P)}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+X$$

$$\xrightarrow{similarly\ for\ y}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{\partial(\sigma_{yy}-P)}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+Y$$

<u>Boundary layer assumptions</u>:

$\delta \ll L \qquad \mathrm{v} \ll \mathrm{u}$

$$\frac{\partial v}{\partial x} \ll \frac{\partial v}{\partial y} \approx \frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$$

$$\sigma_{xx} \propto \frac{\partial u}{\partial x} \qquad \sigma_{yy} \propto \frac{\partial v}{\partial y} \qquad \tau_{yx}=\mu\frac{\partial u}{\partial y} \qquad \tau_{xy}=\mu\frac{\partial v}{\partial x}$$

(no body forces)

x – direction: $\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}$

y – direction: $-\frac{\partial P}{\partial y}=0$

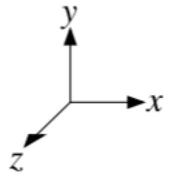
$\rightarrow P=P(x)$

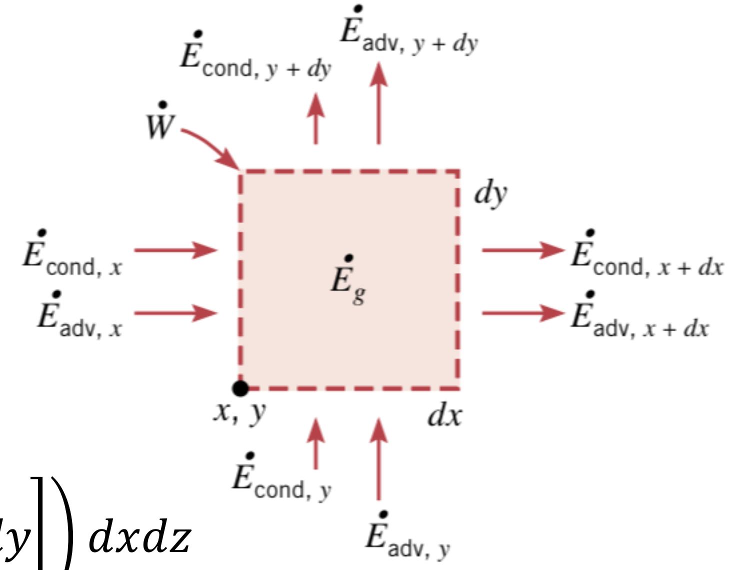
$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dP}{dx}+\mu\frac{\partial^2 u}{\partial y^2}$$

$$\xrightarrow[flat\ plate]{free-stream,} u=V, v=0 \rightarrow \frac{dP}{dx}=0 \rightarrow P=const.$$

Energy Equation

$$\dot{E}_{in} + \dot{W} = \dot{E}_{out}$$





$$\dot{E}_{cond,x} - \dot{E}_{cond,x+dx} + \dot{E}_{cond,y} - \dot{E}_{cond,y+dy}$$

$$= -\left(k\frac{\partial T}{\partial x} - \left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]\right)dydz - \left(k\frac{\partial T}{\partial y} - \left[k\frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)dy\right]\right)dxdz$$

$$= \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]dxdydz$$

$$\dot{E}_{adv,x} - \dot{E}_{adv,x+dx} + \dot{E}_{adv,y} - \dot{E}_{adv,y+dy}$$

$$= \left[\rho u(h + ke + pe) - \left(\rho u(h + ke + pe) + \frac{\partial}{\partial x}(\rho u(h + ke + pe))dx\right)\right]dydz$$

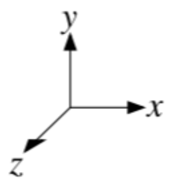
$$+ \left[\rho v(h + ke + pe) - \left(\rho v(h + ke + pe) + \frac{\partial}{\partial y}(\rho v(h + ke + pe))dy\right)\right]dxdz$$

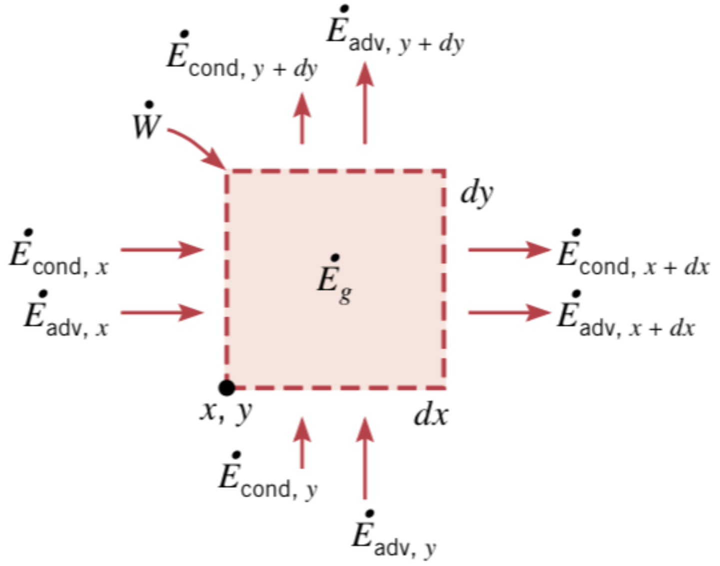
[$ke + pe$: much small compared to h (P – work already accounted for in h)]

$$= -\left(\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h)\right)dxdydz$$

Energy Equation

$$\dot{E}_{in} + \dot{W} = \dot{E}_{out}$$





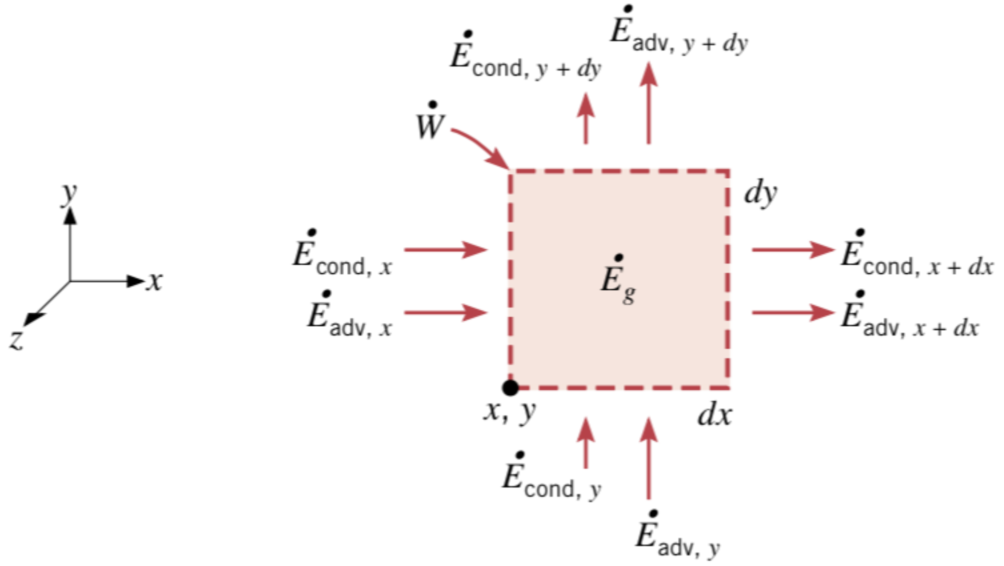
$$\left[\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h)\right] dxdydz - \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right] dxdydz = \dot{W}$$

$$\xrightarrow{chain-rule} \left[\rho u\frac{\partial h}{\partial x} + h\frac{\partial(\rho u)}{\partial x} + \rho v\frac{\partial h}{\partial y} + h\frac{\partial(\rho v)}{\partial y}\right] dxdydz - \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right] dxdydz = \dot{W}$$

$$\xrightarrow{rearrange} \left[\rho u\frac{\partial h}{\partial x} + \rho v\frac{\partial h}{\partial y} + h\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right]\right] dxdydz - \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right] dxdydz = \dot{W}$$

$$\xrightarrow[\text{equation}]{\substack{using\\continuity}} \left[\rho u\frac{\partial h}{\partial x} + \rho v\frac{\partial h}{\partial y}\right] dxdydz - \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right] dxdydz = \dot{W}$$

Energy Equation



$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial y}\right) + \frac{\dot{W}}{dxdydz}$$

$\dot{W}$: accounts for work done by body and viscous forces
(work done by pressure forces is already accounted for)

Viscous work: typically negligible compared to other terms

$$\xrightarrow[]{\substack{constant\ properties, \\ incompressible\ flow, \\ negligible\ body\ forces}} \quad \rho c_p \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

Using $h = c_p T$, where
c_p: *heat capacity*

$$\rho c_p \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] + \mu\Phi \qquad \Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2$$

(needed for high-speed viscous flows like flow of oil in bearings)

Functional/Non-dimensional Solutions

$$x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad u^* = \frac{u}{V} \quad v^* = \frac{v}{L} \quad T^* = \frac{T - T_s}{T_\infty - T_s} \quad P^* = \frac{P}{\rho V^2}$$

$$Continuity: \ \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$Momentum: \left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*}\right) = -\frac{dP^*}{dx^*} + \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial {y^*}^2}$$

$$Energy: \left[u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*}\right] = \frac{1}{Re_L Pr} \frac{\partial^2 T^*}{\partial {y^*}^2}$$

$Re_L = \frac{VL}{\nu}$; Reynolds Number

$\mathrm{Pr} = \frac{\nu}{\alpha}$; Prandtl Number

Boundary Conditions: $u^*(0, y^*) = 1; \quad u^*(x^*, 0) = 0; \quad u^*(x^*, \infty) = 1;$

$v^*(0, y^*) = 0; \quad v^*(x^*, 0) = 0; \quad v^*(x^*, \infty) = 0;$

$T^*(0, y^*) = 1; \quad T^*(x^*, 0) = 0; \quad T^*(x^*, \infty) = 1;$

Functional/Non-dimensional Solutions

$$u^* = f_1\left(x^*, y^*, \frac{dP^*}{dx^*}, Re_L\right)$$

$$Re_L = \frac{VL}{\nu}$$; Reynolds Number

$$T^* = g_1\left(x^*, y^*, \frac{dP^*}{dx^*}, Re_L, Pr\right)$$

$$\mathrm{Pr} = \frac{\nu}{\alpha}$$; Prandtl Number

$$C_f \equiv \frac{\tau_s}{\frac{\rho V^2}{2}} = \frac{\mu \frac{\partial u}{\partial y}\big|_{y=0}}{\frac{\rho V^2}{2}} = \frac{\frac{\mu V}{L}\frac{\partial u^*}{\partial y^*}\big|_{y^*=0}}{\frac{\rho V^2}{2}} = \frac{2}{Re_L}\frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0} = f\left(x^*, y^*, \frac{dP^*}{dx^*}, Re_L\right)$$

$$h = \frac{-k_f \frac{\partial T}{\partial y}\big|_{y=0}}{(T_s - T_\infty)} = -\frac{\frac{k_f(T_\infty - T_s)}{L}\frac{\partial T^*}{\partial y^*}\big|_{y^*=0}}{(T_s - T_\infty)} = \frac{k_f}{L}\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0}$$

Nu: Non-dimensional T-gradient at surface

$$Nu \equiv \frac{hL}{k_f} = \frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} = g\left(x^*, y^*, \frac{dP^*}{dx^*}, Re_L, Pr\right)$$

Nu to thermal boundary layer is what C_f to velocity boundary layer

Significance of non-dimensional Parameters

$$Re = \frac{inertial\ forces}{viscous\ forces} = \frac{change\ in\ momentum\ of\ fluid}{viscous\ forces} \xrightarrow[\substack{order\ of\ magnitue \\ analysis}]{} \approx \frac{\rho V^2 A}{\mu \frac{V}{L} A} = \frac{\rho V L}{\mu}$$

Large Re → viscous forces less dominant → inertial forces amplify fluid disturbances → turbulent flow
At a fixed location on surface, if Re increase → viscous forces less dominant

$\mathrm{Pr}\ (= \nu/\alpha)$: measure of the relative effectiveness of momentum and energy transport by diffusion in the velocity and thermal boundary layers, respectively.

For laminar flow: $\frac{\delta}{\delta_t} \propto Pr^n, \quad n > 0$

Typical ranges of Prandtl numbers for common fluids

Fluid	Pr
Liquid metals	0.004–0.030
Gases	0.7–1.0
Water	1.7–13.7
Light organic fluids	5–50
Oils	50–100,000
Glycerin	2000–100,000