August 16, 2023

IE 621: Probability and Stochastic Processes 1

Assignment 1

1. Suppose we throw three red and two white dice, and dice of the same color are indistinguishable. Describe the probability space corresponding to this (random) experiment.

2. Let $\Omega = \{1, 2, 3, \cdots\}$ and let $\mathcal{F}$ be the set of all subsets of Ω. For $n = 1, 2, \cdots$, let

$$P_{n,k} = \begin{cases} \frac{1}{2^{k+1}} + \frac{1}{2n} & \text{for } 1 \leq k \leq n \\ \frac{1}{2^{k+1}} & \text{for } k > n \end{cases}$$

   and define $\mathbb{P}_n(A) = \sum_{k \in A} P_{n,k}$. Show that $\mathbb{P}_n$ is a probability distribution of $(\Omega, \mathcal{F})$.

3. You toss an ordinary coin repeatedly, recording the outcome of each toss. You do this until you have seen either two Heads or three Tails in total and then you stop.
   (a) Write down the sample space.
   (b) Write down the event "you toss the coin exactly four times" as a subset of the sample space.

4. There are m teachers and n children with $m \geq n$. Each teacher gives one (random) child a chocolate. What is the probability that every child gets at least one chocolate? (**Hint:** Use Inclusion-Exclusion principle.)

5. Let $A_1, A_2, \cdots, A_n \in \mathcal{F}$. For $i = 1, 2, \cdots, n$, let E_i denote the set of outcomes which lie exactly i of $A_1, A_2, \cdots, A_n$. Let $A = \bigcup_{i=1}^n A_i$.
   (a) Show that $\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(E_i)$.
   (b) If E_1 is not empty, show that $\mathbb{P}(E_1) = \sum_{i=1}^n \mathbb{P}(A_i)$.

6. A middle row on a plane seats m people. Assume that n of them order chicken and the remaining $m - n$ pasta. The flight attendant returns with the meals, but has forgotten who ordered what and discovers that they are all asleep, so she puts the meals in front of them at random. What is the probability that at least one of them receives correct order?

7. Pick an integer in $[1, 1000]$ at random. Compute the probability that it is divisible neither by 12 nor by 15.

8. Flip a fair coin. If you toss Heads, roll 1 die. If you toss Tails, roll 2 dice. Compute the probability that you roll exactly one 6.

9. Roll a die, then select at random, without replacement, as many cards from the deck as the number shown on the die. What is the probability that you get at least one Ace?

10. An engineer is confronted with 6 cables each of which needs to be plugged into a specific socket. The engineer has forgotten to label the cables and sockets. If they are plugged in randomly, how many ways could they all be wrong? How many ways could exactly half of them be wrong?

11. I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.
    (a) If the probability of rain is p, what is the probability that I get wet?
    (b) Current estimates show that $p = 0.6$ in Mumbai. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

12. We have a fair coin and an unfair coin, which always comes out Heads. Choose one at random, toss it twice. It comes out Heads both times. What is the probability that the coin is fair?

13. You decide to take part in a roulette game, starting with a capital of ₹C_0. At each round of the game you gamble ₹1000. You lose this money if the roulette gives an even number, and you double it (so receive ₹2000) if the roulette gives an odd number. Suppose the roulette is fair, i.e. the probabilities of even and odd outcomes are exactly $\frac{1}{2}$. What is the probability that you will leave the casino broke?

14. You have 16 balls, 3 blue, 4 green, and 9 red. You also have 3 urns. For each of the 16 balls, you select an urn at random and put the ball into it. (Urns are large enough to accommodate any number of balls.)
    (a) What is the probability that no urn is empty?
    (b) What is the probability that each urn contains 3 red balls?
    (c) What is the probability that each urn contains all three colors?

15. Let Ω be a sample space containing n sample points. We reandomly select r independent subsets $A_1, A_2, \cdots, A_r \subseteq \Omega$. All A_i's are chosen so that all 2^n choices are equally likely. Compute (in a simple closed form) the probability that the A_i's are pairwise disjoint.

16. An item is defective (independently of other items) with probability 0.3. You have a method of testing whether the item is defective, but it does not always give you correct answer. If the tested item is defective, the method detects the defect with probability 0.9 (and says it is good with probability 0.1). If the tested item is good, then the method says it is defective with probability 0.2 (and gives the right answer with probability 0.8). A box contains 3 items. You have tested all of them and the tests detect no defects. What is the probability that none of the 3 items is defective?

17. Three people are going to play a cooperative game. They are allowed to strategize before the game begins, but once it starts they cannot communicate with one another. The game goes as follows. A fair coin is tossed for each player to determine whether that player will receive a red hat or a blue hat, but the color of the hat (and result of the coin toss) is not revealed to the player. Then the three players are allowed to see one another, so each player sees the other two players' hats, but not her own. Simultaneously, each player must either guess at the color of her own hat or 'pass'. They win if nobody guesses incorrectly and at least one person guesses correctly (so they can't all pass).

    The players would like to maximize their probability of winning, so the question is what should their strategy be? A naive strategy is for them to agree in advance that two people will pass and one person (designated in advance) will guess either red or blue. This strategy gives them a 50% chance of winning, but it is not optimal. Devise a strategy that gives the players a greater probability of winning.

18. A woman has 2 children, one of whom is a boy born on a Tuesday. What is the probability that both children are boys? (You may assume that boys and girls are equally likely and independent, and that children are equally likely to born on any given day of the week.)

19. There are two boxes: Box 1 and Box 2. Box 1 contains 4 red balls and 5 black balls. Box 2 has 12 red balls and 15 black balls. One of the two boxes is picked at random, and then a ball is picked at random from that box.
    (a) Is the color of the ball independent of which box is chosen? Explain.
    (b) What if instead there were 12 red balls and 14 black balls in Box 2? Explain.
    (c) Suppose again that Box 1 contains 4 red balls and 5 black balls and Box 2 has 12 red balls and 14 black balls. Given that a red ball is selected, what is the chance that it came from Box 1?

20. Two treatments for a disease are tested on a group of 390 patients. Treatment A is given to 160 patients of whom 100 are men and 60 are women; 20 of these men and 40 of these women recover. Treatment B is given to 230 patients of whom 210 are men and 20 are women; 50 of these men and 15 of these women recover.
    (a) For which of A and B is there a higher probability that a patient chosen randomly from among those given that treatment recovers? Express this as an inequality between two conditional probabilities.
    (b) For which of A and B is there a higher probability that a man chosen randomly from among those given that treatment recovers? Express this as an inequality between two conditional probabilities.
    (c) For which of A and B is there a higher probability that a woman chosen randomly rrom among those given that treatment recovers? Express this as an inequality between two conditional probabilities .
    (d) Compare the inequality in part (a) with the inequalities in part (b) and (c). Are you surprised by the result?