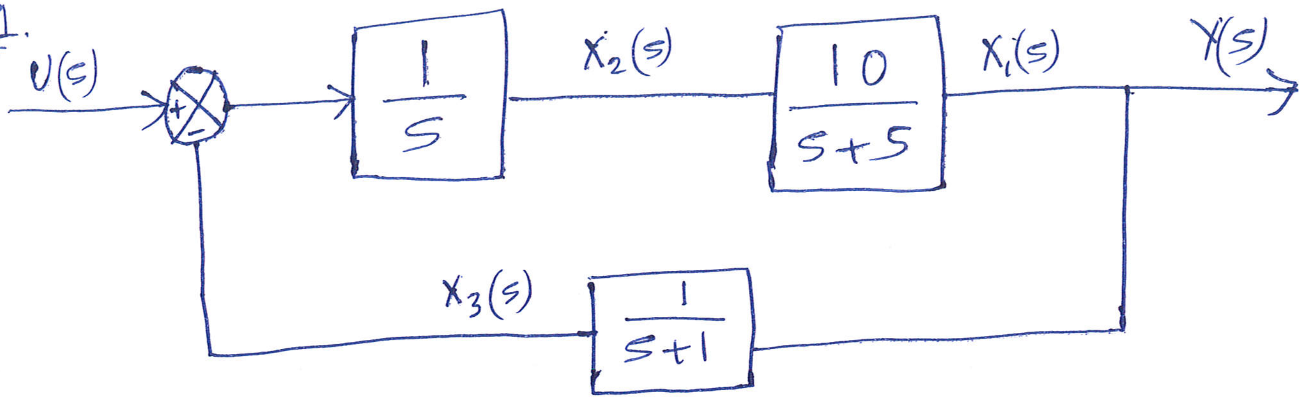
Sol 1.



$$\frac{X_1(s)}{X_2(s)} = \frac{10}{s+5} \Rightarrow s X_1(s) = -5 X_1(s) + 10 X_2(s)$$

$$\frac{X_2(s)}{U(s) - X_3(s)} = \frac{1}{s} \Rightarrow s X_2(s) = -X_3(s) + U(s)$$

$$\frac{X_3(s)}{X_1(s)} = \frac{1}{s+1} \Rightarrow s X_3(s) = -X_3(s) + X_1(s)$$

taking inverse L.T. of the three eqns. above we get

$$\dot{x}_1 = -5x_1 + 10x_2$$

$$\dot{x}_2 = -x_3 + u$$

$$\dot{x}_3 = -x_3 + x_1$$

also note that $y = x_1$

Putting these in matrix form we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & -10 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

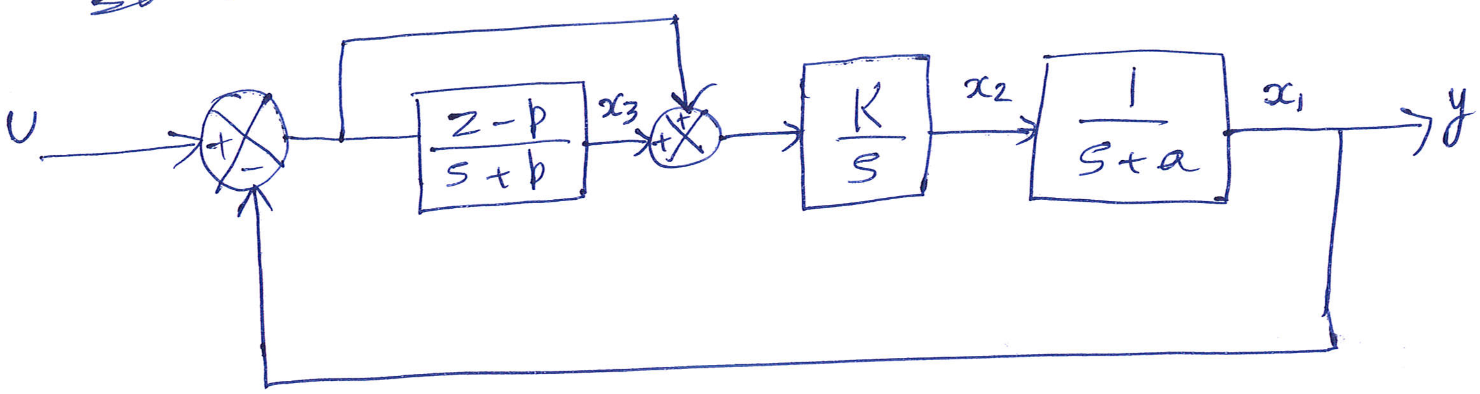
$$\underline{\dot{x}} = A \underline{x} + B u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Sol. 2. $\dfrac{s+z}{s+p} = 1 + \dfrac{z-p}{s+p}$

& $\dfrac{K}{s(s+a)} = \left(\dfrac{K}{s}\right)\left(\dfrac{1}{s+a}\right)$

So the block diagram can be re-drawn as



$$\frac{X_1(s)}{X_2(s)} = \frac{1}{s+a} \Rightarrow s X_1(s) = -a X_1(s) + X_2(s)$$

$$\frac{X_2(s)}{X_3(s) + U(s) - X_1(s)} = \frac{K}{s} \Rightarrow s X_2(s) = -K X_1(s) + K X_3(s) + K U(s)$$

$$\frac{X_3(s)}{U(s) - X_1(s)} = \frac{(z-p)}{s+p} \Rightarrow s X_3(s) = -(z-p) X_1(s) - p X_3(s) + (z-p) U(s)$$

& $Y(s) = X(s)$

Taking inverse Laplace Transform of these eqns. and writing them in matrix form we get.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -a & 1 & 0 \\ -K & 0 & K \\ -(z-p) & 0 & -p \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ K \\ z-p \end{bmatrix} U$$

$$\dot{\underline{x}} = A \, \underline{x} + B \, U$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Soln. 3. Taking L.T.,

$$H(s) = -\frac{1}{(s+1)^2} + \frac{2}{s+1} = \frac{2s+1}{(s+1)^2}$$

Soln 4.

$$m_1 \ddot{x}_1 = -c_1 \dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) - k_1 x_1 + k_2(x_2 - x_1) + F_1$$

$$m_2 \ddot{x}_2 = -c_2(\dot{x}_2 - \dot{x}_1) - c_3 \dot{x}_2 - k_2(x_2 - x_1) - k_3 x_2 + F_2$$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -c_1 - c_2 & c_2 \\ c_2 & -c_2 - c_3 \end{bmatrix}}_{-\Theta} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix}}_{-K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

define the state vector as $X = \begin{bmatrix} x_1 & x_2 & \dot{x}_1 & \dot{x}_2 \end{bmatrix}^T$ we get

$$\dot{X} = \underbrace{\left[\begin{array}{c|c} O_{2\times2} & I_{2\times2} \\ \hline -M^{-1}K & -M^{-1}\Theta \end{array}\right]}_{A} X + \underbrace{\left[\begin{array}{c} O_{2\times2} \\ \hline M^{-1} I_{2\times2} \end{array}\right]}_{B} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

Note: in the solution you should simplify $M^{-1}K$, $M^{-1}\Theta$ etc.

Soln. 5. **Linearity** given $u_1 \rightarrow y_1$

and $u_2 \rightarrow y_2$

show $\alpha_1 u_1 + \alpha_2 u_2 \rightarrow \alpha_1 y_1 + \alpha_2 y_2$

Time Invariance Given $u(t) \longrightarrow y(t)$

show $u(t-\tau) \longrightarrow y(t-\tau)$

(a) $\dot{y}(t) = a\, u(t) + b$

$$\dot{y}_1(t) = a\, u_1(t) + b_1$$

$$\dot{y}_2(t) = a\, u_2(t) + b_1$$

$$\alpha_1 \dot{y}_1(t) + \alpha_2 \dot{y}_2(t) = a(\alpha_1 u_1 + \alpha_2 u_2) + \underbrace{(\alpha_1 + \alpha_2)}\, b$$

this should be b for linearity

Not Linear.

$$\dot{y}(t) = a\, u(t) + b$$

using another variable $t' = t - \tau$ $\qquad \dfrac{dt}{dt'} = 1 = \dfrac{dt'}{dt}$

$$\frac{d}{dt'} y(t') = a\, u(t') + b$$

$$\Rightarrow \frac{d}{dt} y(t') \left(\cancel{\frac{dt}{dt'}}\right)^{1} = a\, u(t') + b$$

$$\Rightarrow \frac{d}{dt} y(t-\tau) = a\, u(t-\tau) + b$$

$$\Rightarrow u(t-\tau) \longrightarrow y(t-\tau)$$

Time Invariant

(b) $\dot{y} + ay = btu$

$\dot{y}_1 + ay_1 = btu_1$

$\dot{y}_2 + ay_2 = btu_2$

$\alpha_1 \dot{y}_1 + \alpha_2 \dot{y}_2 + a(\alpha_1 y_1 + \alpha_2 y_2) = bt(\alpha_1 u_1 + \alpha_2 u_2)$

$\Rightarrow \quad \cancel{\alpha_1 y_1 + \alpha_2 y_2} \quad \alpha_1 u_1 + \alpha_2 u_2 \longrightarrow \alpha_1 y_1 + \alpha_2 y_2$

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using a new variable $t' = t - \tau$ $\quad \dfrac{dt}{dt'} = 1$

$$\frac{d\, y(t')}{dt'} + a\, y(t') = bt'\, u(t')$$

$$\frac{d}{dt} y(t-\tau) \cancel{\left(\frac{dt}{dt'}\right)}^{1} + a\, y(t-\tau) = b(t-\tau)\, u(t-\tau)$$

$$\Rightarrow \frac{d\, y(t-\tau)}{dt} + a\, y(t-\tau) = bt\, u(t-\tau) \underbrace{- b\tau\, u(t-\tau)}_{\text{extra term.}}$$

$u(t) \rightarrow y(t)$

does not imply $u(t-\tau) \rightarrow y(t-\tau)$

$\Rightarrow$ Time variant.

(c) $\dot{y}(t) + \dot{y}^2(t) = u(t)$

Similarly show Non-Linear

Time-Invariant

Soln: 6 (a) $m\ddot{x} + c\dot{x} + kx = U$

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + cs + k}$$

(b) $X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ $\quad \dot{X} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} X + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} U$

$$Y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} X$$

taking Laplace transform with zero initial conditions.

$$\left\{ sI - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \right\} X(s) = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} U(s)$$

$$X(s) = \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{c}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} U(s)$$

$$\frac{Y(s)}{U(s)} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{c}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\text{Compute the inverse \& simplify and show.}} U(s)$$

(c) Yes BIBO stable because for +ive values of m, c, k poles are always in LHP as proved in one of the previous tutorials.