STA 2503 Project–4 – Stochastic Interest Rates and Swaptions

In this project you will investigate several aspects of the stochastic interest rates. Let $r = (r_t)_{t\geq 0}$ denote the short rate of interest and suppose that r satisfies the SDE

$$dr_t = \alpha(\theta_t - r_t)\,dt + \sigma\,dW_t^1, \qquad \text{and} \tag{1a}$$

$$d\theta_t = \beta(\phi - \theta_t)\,dt + \eta\,dW_t^2, \tag{1b}$$

where $W^{1,2} = (W_t^{1,2})_{t\geq 0}$ are independent risk-neutral Brownian motions, and $\theta = (\theta_t)_{t\geq 0}$ denotes the long-run interest rate which itself is stochastic. This is sometimes called a two-factor interest rate model.

For base parameters use the following:

$$r_0 = 2\%, \quad \alpha = 3, \quad \sigma = 1\%, \quad \theta_0 = 3\%, \quad \beta = 1, \quad \phi = 5\%, \quad \eta = 0.5\%,$$

1. This is an affine model, therefore writing T-maturity bond price process as $P(T) = (P_t(T))_{t\in[0,T]}$, there exist deterministic functions $A_t(T)$, $B_t(T)$ and $C_t(T)$ such that

   $$P_t(T) = \exp\{A_t(T) - B_t(T)\,r_t - C_t(T)\,\theta_t\}\ .$$

   Determine the functions A, B and C by either solving for the distribution of $\int_0^T r_u du$ or using the PDE approach and draw the term structure of interest rates with the base parameters.

2. Use an Euler-scheme (which in this model is identical to the Milstein-scheme) to generate simulations of risk-neutral interest rate paths, and use the paths to obtain Monte Carlo estimate of bond yields (with confidence bands) to compare with analytical formula you derived in Q1 using the base parameters.

3. Investigate what the various parameters in the model do to the term structure.

4. Using a bond of maturity T_1 as a numeraire asset, it is possible to determine an analytic expression for the price of a bond option paying $(P_{T_1}(T_2) - K)_+$ at T_1 where $K = P_0(T_2)/P_0(T_1)$. Derive this formula, and use Monte Carlo simulation to check your results for a collection of strikes $\alpha\,K$, for $\alpha = 0.7, 0.75, \ldots, 1.3$.

5. Suppose we have an IRS with tenure structure $\tau = \{3, 3.25, \ldots, 6\}$ where 3 is the first reset date (no payment) and the first payment is at 3.25 and every 0.25 after that. Determine the Black implied volatility of a swaption with strike equal to today's swap-rate. The black implied volatility is the volatility in an LSM model that makes the LSM price equal the price you obtained. How does this change as a function of strike?