STA 2503 Project–3 – The Heston Model

In this project you will investigate several aspects of the Heston model. You are given that an asset price $S = (S_t)_{t\geq 0}$ and it's variance factor $v = (v_t)_{t\geq 0}$ follow the Heston model. Specifically they satisfy the coupled SDE

$$dS_t = S_t \sqrt{v_t}\, dW_t^S, \quad \text{and} \quad dv_t = \kappa(\theta - v_t)\, dt + \eta \sqrt{v_t}\, dW_t^v, \tag{1}$$

where $W_t^{S,v}$ are risk-neutral Brownian motions with correlation ρ. Use the following base model parameters:

$$r = 0 \qquad S_0 = 1 \qquad \sqrt{v_0} = 30\% \qquad \kappa = 3 \qquad \sqrt{\theta} = 40\% \qquad \eta = 5 \qquad \rho = -0.5.$$

Throughout fix the number of simulations at 5,000, and the step size to be $\frac{1}{1000}$.

1. Estimate the implied volatility smiles (with confidence intervals) of a put/call options (use puts for strikes $K < S_0$ and calls for strikes $K \geq S_0$) for the collection of strikes and maturities in the data file, using Monte Carlo simulation via
   - Euler discretization of both $x_t = log(S_t)$ and v_t
   - Milstein discretization of both $x_t = log(S_t)$ and v_t
   - Mixing method using Milstein discretization of v_t

2. Next you will repeat Q1 but using control variates.

   A control variate is a technique used to reduce the variance of a Monte Carlo simulation when you have an analytic solution of a closely related model. The idea is as follows: suppose that X and $Y_1, \dots, Y_m$ are random variables (representing, e.g., the payoff of an option). Let $(X^{(n)}, Y_1^{(n)}, \dots, Y_m^{(n)})_{n=1,\dots,N}$ denote Monte Carlo simulations of X and $Y_1, \dots, Y_m$.

   If you wish to estimate $g = \mathbb{E}[X]$ but know the analytical result for $h_i = \mathbb{E}[Y_i]$, $i = 1, \dots, m$, then you can write an estimate of g, denoted $\hat{g}$ as follows

   $$\hat{g} = \tfrac{1}{N} \sum_{n=1}^{N} X^{(n)} + \sum_{i=1}^{m} \gamma_i \left(h_i - \tfrac{1}{N} \sum_{n=1}^{N} Y_i^{(n)} \right), \tag{2}$$

   where $(\gamma_i)_{i=1,\dots,N}$ are arbitrary constants. The usual Monte Carlo estimate is with $\gamma = 0$. If, however, there are correlations between X and Y_i the optimal choice for γ_i is not zero.

   The optimal γ_i can be found by introducing the random variable $H = X + \sum_{i=1}^{m} \gamma_i (h_i - Y_i)$ and choosing γ_i to minimize its variance. Note that $\mathbb{E}[H] = \mathbb{E}[X]$ by construction, and therefore one can view (2) as the MC estimate of $\mathbb{E}[H]$ and therefore $\mathbb{E}[X]$. The variance of this estimator is proportional to the variance of H and therefore minimizing the variance of H minimizes the variance of the MC estimator of $\mathbb{E}[H]$ and therefore $\mathbb{E}[X]$.

   (a) Determine the (model independent) expression for the optimal choice of γ_i

(b) The analytical formula for the Heston model when v_t is replaced by $\bar{v}_t := \mathbb{E}[v_t]$ can be derived in closed form – derive it.

(c) Determine the value of a contingent claim paying $\int_0^T v_s\,ds$ at maturity T. Use Milstein simulations to demonstrate your analytic calculation is correct for several choices of $T = \{0.1, \ldots, 1\}$.

(d) Determine the value of a contingent claim paying $\int_0^T v_s^2\,ds$ at maturity T. Use Milstein simulations to demonstrate your analytic calculation is correct for several choices of $T = \{0.1, \ldots, 1\}$.

(e) Simulate paths of the Heston model and simultaneously the Heston model where v_t is replaced by $\bar{v}_t$. For both simulations, stochastic v_t and deterministic $\bar{v}_t$ variance factor, use the same random variables, and use Milstein discretization.

(f) Now estimate implied volatilities using three options as control variates: (i) the deterministic $\bar{v}_t$ variance path option price, (ii) the claim $\int_0^T v_s\,ds$ paying, and (iii) the claim paying $\int_0^T v_s\,ds$.

(g) Finally, estimate implied volatilities using the mixing method and the three options as control variates: (i) the deterministic $\bar{v}_t$ variance path option price, (ii) the claim $\int_0^T v_s\,ds$ paying, and (iii) the claim paying $\int_0^T v_s\,ds$.