第十一次作业参考答案.

4.7： 3(5)，5(1)，6(2)，7(1)

5.2： 3，5，8(1)，9.

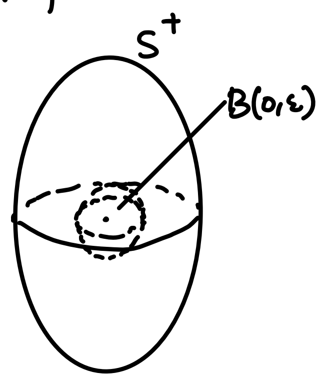
4.7.3(5).

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right)$$

$$=\frac{1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}-\frac{3}{2}x\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}\cdot 2x=\frac{1}{r^3}-\frac{3x^2}{r^5}$$

$$\operatorname{div}\vec{A}=\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)=\frac{3}{r^3}-\frac{3\left(x^2+y^2+z^2\right)}{r^5}=0$$

以原点O为球心，半径ε作一小球$B(0,\varepsilon)$，使其在椭球内部



$$原式=\oiint_{S^+\sqcup\overline{\partial B(0,\varepsilon)}}\vec{A}\cdot d\vec{S}+\oiint_{\partial B(0,\varepsilon)}\vec{A}\cdot d\vec{S}$$

$$=\iiint_{\Omega-B(0,\varepsilon)}\operatorname{div}\vec{A}\,dxdydz+\oiint_{\partial B(0,\varepsilon)}\vec{A}\cdot d\vec{S}$$

$$=\oiint_{\partial B(0,\varepsilon)}\frac{1}{\varepsilon^2}dS=\frac{1}{\varepsilon^2}\cdot 4\pi\varepsilon^2=4\pi$$

4.7.5(1) $\vec{V}=(y,z,x)$

$$\operatorname{rot}\vec{V}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}=(-1,-1,-1)$$

$$原式=\iint_{S^+}(-1,-1,-1)\,d\vec{S}$$

其中 S^+ 为平面 $x+y+z=0$ 与球内部的交，定向与 L^+ 相容.

故 S^+ 法向量为 $\dfrac{(1,1,1)}{\sqrt{3}}$.

原式 $= \iint_S -\sqrt{3}\, dS = -\sqrt{3}\pi R^2$.

4.7.6(2).

(i) 可以验证 $\mathrm{rot}\, \vec{v} = 0$:

$$\mathrm{rot}\, \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x+z)^2+y^2} & -\frac{z+x}{(x+z)^2+y^2} & \frac{y}{(x+z)^2+y^2} \end{vmatrix}$$

$$= \begin{bmatrix} \frac{1}{(x+z)^2+y^2} + \frac{-2y^2}{((x+z)^2+y^2)^2} + \frac{1}{(x+z)^2+y^2} - \frac{2(z+x)^2}{((x+z)^2+y^2)^2} \\ \frac{2(x+z)y}{((x+z)^2+y^2)^2} + \frac{-2y(x+z)}{((x+z)^2+y^2)^2} \\ -\frac{1}{(x+z)^2+y^2} + \frac{2(z+x)^2}{((x+z)^2+y^2)^2} - \frac{1}{(x+z)^2+y^2} + \frac{2y^2}{((x+z)^2+y^2)^2} \end{bmatrix}^T$$

$= (0,0,0)$.

由定理 4.7.3.，存在三元函数 $u(x,y,z)$ 使

$$du = \frac{1}{(x+z)^2+y^2}\left[y\,dx - (z+x)\,dy + y\,dz \right].$$

$$\int_{(1,1,1)}^{(x,y,z)} du = \int_{(1,1,1)}^{(x,1,1)} \frac{1}{(x+1)^2+1} dx + \int_{(x,1,1)}^{(x,y,1)} \frac{-(1+x)}{(x+1)^2+y^2} dy$$

$$+ \int_{(x,y,1)}^{(x,y,z)} \frac{y}{(x+z)^2+y^2} dz$$

$$= \arctan(x+1) - \arctan 2 - \arctan\left(\frac{y}{x+1}\right)$$

$$+ \arctan\left(\frac{1}{x+1}\right) + \arctan \frac{x+z}{y} - \arctan \frac{x+1}{y}$$

$$= \arctan \frac{x+z}{y} - \arctan 2 .$$

$$\therefore \quad u(x, y, z) = \arctan \frac{x+z}{y} + C.$$

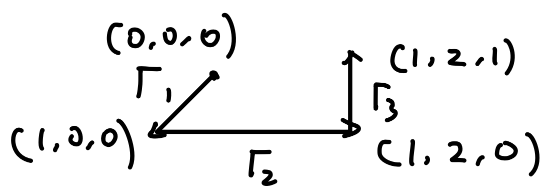
4.7.7.(1)

$$\text{rot } V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = (0, 0, 0)$$

∴ 曲线积分与路径无关.

选择路径: $(0,0,0) \longrightarrow (1,0,0) \longrightarrow (1,2,0) \longrightarrow (1,2,1)$

$\int_{\Gamma_1} = 0$, $\int_{\Gamma_2} = \int_0^2 1\, dy = 2$,



$\int_{\Gamma_3} = \int_0^1 (1+2)\, dz = 3$.

$\therefore$ 原积分 $= 0+2+3=5.$

补充题:

$$\iint_{\partial\Omega}(x+y+z)\vec{v}\cdot d\vec{S} = \iiint_{\Omega}\left(\nabla(x+y+z)\cdot\vec{v} + (x+y+z)\,\nabla\cdot\vec{v}\right)dxdydz$$

$$= \iiint_{\Omega}(1,1,1)\cdot\vec{v}\,dxdydz = \iiint_{\Omega}(P+Q+R)\,dxdydz$$

同时, 单位球面法向量 $\vec{n}=(x,y,z).$

$$\iint_{\partial\Omega}(x+y+z)\vec{v}\cdot d\vec{S} = \iint_{\partial\Omega}(x+y+z)\left((1,1,1)\cdot(x,y,z)\right)dS$$

$$= \iint_{\partial\Omega}(x+y+z)^2\,dS \xlongequal{\text{对称性}} \iint_{\partial\Omega}(x^2+y^2+z^2)\,dS$$

$$= \iint_{\partial\Omega}dS = 4\pi$$

$$\Rightarrow \iiint_{\Omega}(P+Q+R)\,dxdydz = 4\pi.$$

§ 5.2.

3. (1) $a_n = \dfrac{2^n}{\sqrt{n^n}} \geq 0,\quad \sqrt[n]{a_n} = \dfrac{2}{\sqrt{n}}$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{2}{\sqrt{n}} = 0 < 1.$$

$\therefore \sum\limits_{n=1}^{\infty}\dfrac{2^n}{\sqrt{n^n}}$ 收敛.

(2). $a_n = \frac{1}{3^n}\left(1+\frac{1}{n}\right)^{n^2}$, $\sqrt[n]{a_n} = \frac{1}{3}\left(1+\frac{1}{n}\right)^n$.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{e}{3} < 1.$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{3^n}\left(1+\frac{1}{n}\right)^{n^2}$ 收敛.

(3) $$\frac{a_n}{a_{n+1}} = \frac{(n+1)^p(\ln(n+1))^q(\ln\ln(n+1))^r}{n^p(\ln n)^q(\ln\ln n)^r}$$

$$= \left(1+\frac{1}{n}\right)^p\left(1+\frac{\ln(1+\frac{1}{n})}{\ln n}\right)^q\left(1+\frac{\ln\left(\frac{\ln(n+1)}{\ln n}\right)}{\ln\ln n}\right)^r$$

$$= \left(1+\frac{p}{n}+o\left(\frac{1}{n\ln n\ln\ln n}\right)\right)\left(1+\frac{q}{n\ln n}+o\left(\frac{1}{n\ln n\ln\ln n}\right)\right)$$

$$\times\left(1+\frac{r}{n\ln n\ln\ln n}+o\left(\frac{1}{n\ln n\ln\ln n}\right)\right)$$

$$= 1+\frac{p}{n}+\frac{q}{n\ln n}+\frac{r}{n\ln n\ln\ln n}+o\left(\frac{1}{n\ln n\ln\ln n}\right).$$

$p>1 \Rightarrow$ 收敛.

$p<1 \Rightarrow$ 发散.

$p=1,\ q>1 \Rightarrow$ 收敛

$p=1,\ q<1 \Rightarrow$ 发散.

$p=q=1,\ r>1 \Rightarrow$ 收敛,

$p=q=1,\ r\le 1 \Rightarrow$ 发散.

(4) $\lim\limits_{n\to\infty} \dfrac{a_n}{\frac{1}{n^{\frac{4}{3}}}} = \lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n}}{\sqrt[3]{n+1}} \dfrac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} = 2$

由 $\sum\limits_{n=1}^{+\infty} \dfrac{1}{n^{\frac{4}{3}}}$ 收敛，故 $\sum\limits_{n=1}^{+\infty} \dfrac{1}{\sqrt[3]{n+1}} \ln \dfrac{n+2}{n}$ 收敛.

(5). $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sin\left(\dfrac{\pi}{4}+\dfrac{1}{n}\right) = \dfrac{\sqrt{2}}{2} < 1$

$\Rightarrow \sum\limits_{n=1}^{+\infty} \left(\sin\left(\dfrac{\pi}{4}+\dfrac{1}{n}\right)\right)^n$ 收敛

(6) $\dfrac{a_{n+1}}{a_n} = \dfrac{\ln((n+1)!)}{\ln(n!)} \cdot \dfrac{n!}{(n+1)!} = \left(\dfrac{\ln(n+1)}{\ln(n!)} + 1\right) \dfrac{1}{n+1}$

$\therefore \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 0$， $\sum\limits_{n=1}^{\infty} \dfrac{\ln(n!)}{n!}$ 收敛.

(7) $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} e^{-\frac{n^2+1}{n^2+n}} = e^{-1} < 1.$

$\therefore \sum\limits_{n=1}^{\infty} e^{-\frac{n^2+1}{n+1}}$ 收敛.

(8) Stirling 公式： $n! \sim \sqrt{2n\pi}\left(\dfrac{n}{e}\right)^n \quad (n\to\infty)$

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}\ \ln n}{n}$

$= \lim\limits_{n\to\infty} \dfrac{\left(\sqrt[2n]{2n\pi} \cdot \frac{n}{e} + o(1)\right) \ln n}{n}$

$= \lim\limits_{n\to\infty} \dfrac{\ln n}{e} = \infty.$

$\therefore \sum_{n=1}^{\infty} \frac{(\ln^n n)\, n!}{n^n}$ 发散.

(9) $0 \le \frac{3n-1}{2^n+2^{-n}} < \frac{3n-1}{2^n} < \frac{3n}{2^n} \qquad (\forall\ n \ge 1)$

$$\lim_{n\to\infty} \sqrt[n]{\frac{3n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{3n}}{2} = \frac{1}{2}$$

$\Rightarrow \sum_{n=1}^{\infty} b_n$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} \frac{3n-1}{2^n+2^{-n}}$ 收敛.

(10) $0<a<1$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+a^n} = 1.$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ 必发散.

$a=1$: $a_n \equiv \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} a_n$ 发散.

$a>1$: $0 \le \frac{1}{1+a^n} < \frac{1}{a^n}$ 且 $\sum_{n=1}^{\infty} \frac{1}{a^n}$ 收敛.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ 收敛,

综上: $\begin{cases} 0<a\le 1 : & \text{发散.} \\ a>1 : & \text{收敛.} \end{cases}$

5. $\{nu_n\}$有界 $\Rightarrow$ 存在常数 $c>0$ 使得对任意 $n\ge 1$ 均有

$u_n \cdot n \le c$

$\Rightarrow 0 \le \frac{u_n}{n} = \frac{u_n\cdot n}{n^2} \le \frac{C}{n^2}.$

又 $\sum_{n=1}^{\infty} \frac{C}{n^2}$ 收敛，故 $\sum_{n=1}^{\infty} \frac{u_n}{n}$ 收敛.

8(1). $a_n := \frac{\sqrt{n!}}{(1+\sqrt{1})(1+\sqrt{2})\cdots(1+\sqrt{n})}$

$$\frac{a_n}{a_{n+1}} = \frac{\sqrt{n!}}{\sqrt{(n+1)!}}\cdot(1+\sqrt{n+1}) = \frac{1+\sqrt{n+1}}{\sqrt{n+1}} = \frac{1}{\sqrt{n+1}}+1$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty}\frac{n}{\sqrt{n+1}} = +\infty$$

$\therefore \sum_{n=1}^{\infty} a_n$ 收敛.

9(1) $\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right) < \frac{1}{n}$.

$$\Rightarrow \quad 0 \le \frac{1}{\sqrt{n}} - \sqrt{\ln\frac{n+1}{n}} \le \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

注意到 $\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}} = \frac{1}{\sqrt{n(n+1)}\cdot(\sqrt{n+1}+\sqrt{n})} \le \frac{1}{2n\sqrt{n}}$

且 $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ 收敛.

故 $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \sqrt{\ln\frac{n+1}{n}}\right)$ 收敛

9(2). 令 $a_n = n^{\frac{1}{n^2+1}} - 1$

$$(1+a_n)^{n^2+1} = n \ge \binom{n^2+1}{2} a_n^2 = \frac{n^2(n^2+1)}{2} a_n^2$$

$$\Rightarrow \quad a_n^2 \le \frac{2}{n(n^2+1)} \le \frac{2}{n^3}$$

$$\Rightarrow \quad a_n \le \frac{\sqrt{2}}{n^{\frac{3}{2}}}$$

又 $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ 收斂， $\sum_{n=1}^{\infty} \left(n^{\frac{1}{n^2+1}} - 1\right)$ 收斂.