Review

- 三重积分化累次积分

（先一后二）

$$\Omega:\begin{cases} z_1(x,y)\le z\le z_2(x,y), \\ (x,y)\in D_{xy}, \end{cases}$$

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz=\iint\limits_{D_{xy}} dxdy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz.$$

（先二后一）

$$\Omega:\begin{cases} c \le z \le d, \\ (x,y)\in\Omega_z, \end{cases}$$

$$\iiint_\Omega f(x,y,z)dxdydz = \int_c^d dz \iint_{\Omega_z} f(x,y,z)dxdy.$$

- 投影法确定积分区域

- 三重积分的变量替换

$$u=u(x,y,z), v=v(x,y,z), w=w(x,y,z)$$

$$(x,y,z)\in\Omega \leftrightarrow (u,v,w)\in\Omega^*.$$

$$\iiint_\Omega f(x,y,z)dxdydz$$

$$=\iiint_{\Omega^*} f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right)$$

$$\cdot\left|\det\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw.$$

§5. 重积分的应用

- 曲面面积
- 质心
- 转动惯量
- 万有引力

原则：微元法

1. 曲面的面积

设曲面S有参数方程

$$x = x(u,v), y = y(u,v), z = z(u,v), (u,v) \in D,$$

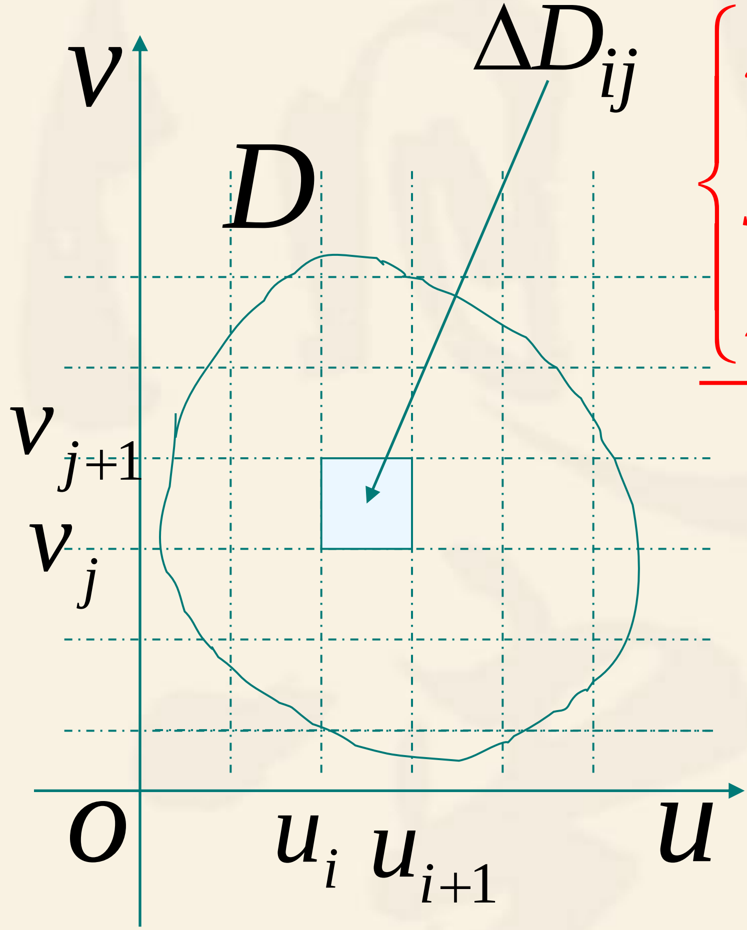
简记为

$$S : r = r(u,v), (u,v) \in D.$$

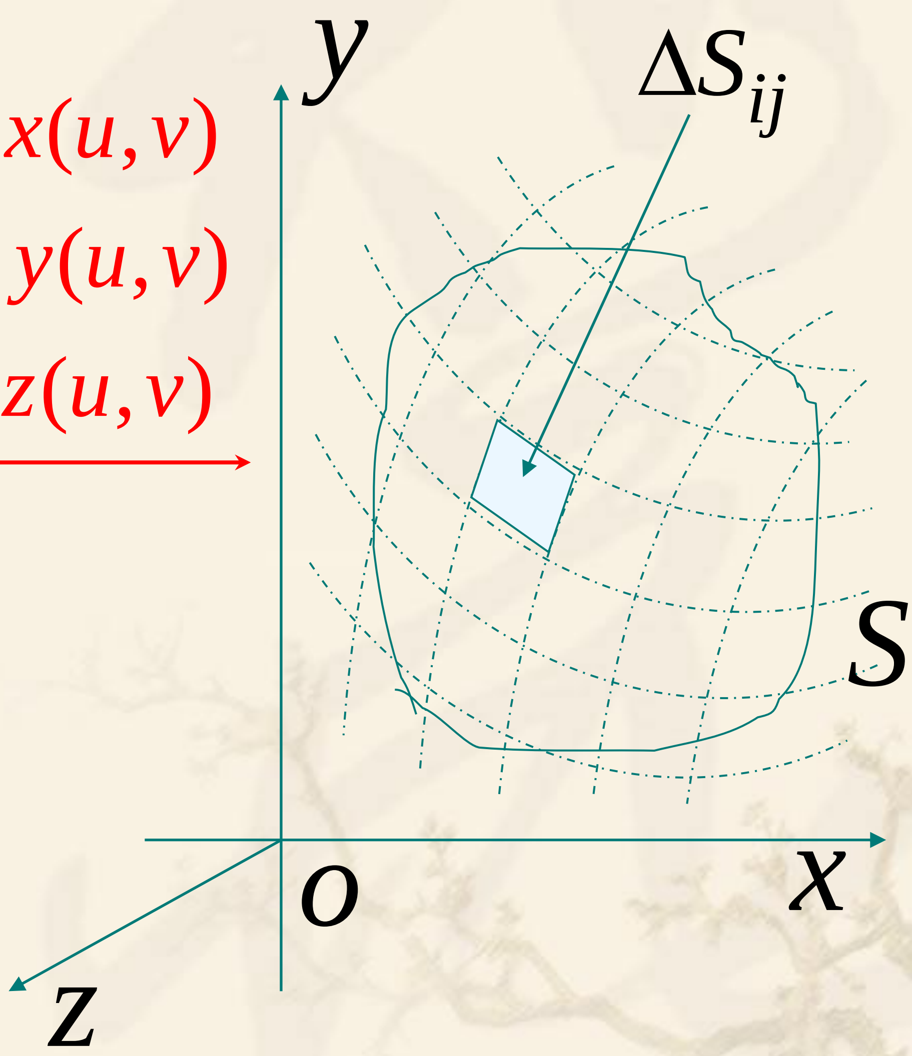
在ouv平面上,用平行于坐标轴的直线

$$u = u_i (i = 1,2,\cdots,n), v = v_j (j = 1,2,\cdots,m)$$

将区域D分割成若干小矩形ΔD_{ij}.



$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$



ΔD_{ij}的顶点为$(u_i, v_j), (u_{i+1}, v_j), (u_i, v_{j+1}), (u_{i+1}, v_{j+1})$.

对应地,空间曲边四边形ΔS_{ij}的四个顶点为

$$P_{ij}(x(u_i, v_j), y(u_i, v_j), z(u_i, v_j)),$$

$$P_{i+1,j}(x(u_{i+1}, v_j), y(u_{i+1}, v_j), z(u_{i+1}, v_j)),$$

$$P_{i,j+1}(x(u_i, v_{j+1}), y(u_i, v_{j+1}), z(u_i, v_{j+1})),$$

$$P_{i+1,j+1}(x(u_{i+1}, v_{j+1}), y(u_{i+1}, v_{j+1}), z(u_{i+1}, v_{j+1})).$$

$$\overrightarrow{P_{ij}P_{i+1,j}} \approx (x'_u(u_i, v_j), y'_u(u_i, v_j), z'_u(u_i, v_j))\Delta u_i$$

$$= r'_u(u_i, v_j)\Delta u_i$$

$$\overrightarrow{P_{ij}P_{i,j+1}} \approx r'_v(u_i, v_j)\Delta v_j.$$

当分划很细时,空间曲面ΔS_{ij}可近似地看成以线段$P_{ij}P_{i+1,j}, P_{ij}P_{i,j+1}$为邻边的平行四边形. 于是

$$\Delta S_{ij} \approx \left\| r'_u(u_i, v_j) \times r'_v(u_i, v_j) \right\| \Delta u_i \Delta v_j$$

$$= \left| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix} \right|_{(u_i, v_j)} \Delta u_i \Delta v_j$$

即 $\Delta S_{ij} \approx \sqrt{A^2 + B^2 + C^2}\Delta u_i \Delta v_j$，其中

$$A = \det \frac{\partial(y,z)}{\partial(u,v)}\bigg|_{(u_i,v_j)}, \quad B = \det \frac{\partial(z,x)}{\partial(u,v)}\bigg|_{(u_i,v_j)},$$

$$C = \det \frac{\partial(x,y)}{\partial(u,v)}\bigg|_{(u_i,v_j)}.$$

•曲面$S: x = x(u,v), y = y(u,v), z = z(u,v), (u,v) \in D$，的面积为

$$\iint_D \|r'_u \times r'_v\| dudv = \iint_D \sqrt{A^2 + B^2 + C^2} dudv.$$

•若曲面S的方程为$z = f(x,y),(x,y)\in D$,则

$$S: x = x, y = y, z = f(x,y), (x,y)\in D.$$

$$r'_x \times r'_y = \det\begin{pmatrix} i & j & k \\ 1 & 0 & f'_x \\ 0 & 1 & f'_y \end{pmatrix} = (-f'_x, -f'_y, 1)$$

$$A = \det\begin{pmatrix} 0 & f'_x \\ 1 & f'_y \end{pmatrix} = -f'_x, B = -f'_y, C = 1.$$

曲面S的面积为$\iint_D \sqrt{1+f'^2_x+f'^2_y}\,dxdy$.

例: 求球面$S: x^2+y^2+z^2=R^2$的面积.

解: 球面S的参数方程为

$$x=R\sin\varphi\cos\theta, y=R\sin\varphi\sin\theta, z=R\cos\varphi,$$

$$(0\le\varphi\le\pi, 0\le\theta\le 2\pi)$$

则 $$A=\det\frac{\partial(y,z)}{\partial(\varphi,\theta)}=\det\begin{pmatrix} R\cos\varphi\sin\theta & R\sin\varphi\cos\theta \\ -R\sin\varphi & 0 \end{pmatrix}$$

$$=R^2\sin^2\varphi\cos\theta,$$

$$B=\det\frac{\partial(z,x)}{\partial(\varphi,\theta)}=\det\begin{pmatrix} -R\sin\varphi & 0 \\ R\cos\varphi\cos\theta & -R\sin\varphi\sin\theta \end{pmatrix}$$

$$=R^2\sin^2\varphi\sin\theta,$$

$$C = \det\frac{\partial(x,y)}{\partial(\varphi,\theta)} = \det\begin{pmatrix} R\cos\varphi\cos\theta & -R\sin\varphi\sin\theta \\ R\cos\varphi\sin\theta & R\sin\varphi\cos\theta \end{pmatrix}$$

$$= R^2\sin\varphi\cos\varphi,$$

球面S的面积为

$$\iint\limits_S dS = \iint\limits_{\substack{0\le\varphi\le\pi \\ 0\le\theta\le 2\pi}} \sqrt{A^2+B^2+C^2}\,d\varphi d\theta$$

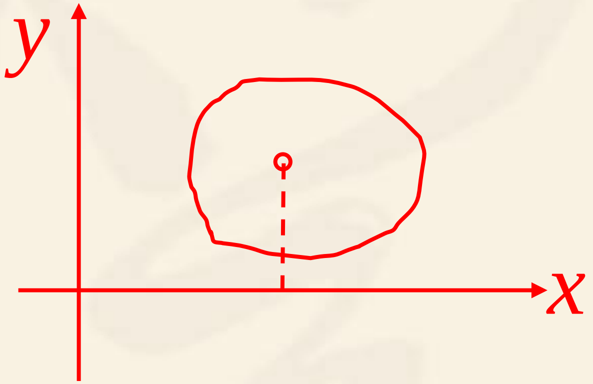
$$= \iint\limits_{\substack{0\le\varphi\le\pi \\ 0\le\theta\le 2\pi}} R^2\sin\varphi d\varphi d\theta$$

$$= R^2\int_0^{\pi}\sin\varphi d\varphi\int_0^{2\pi}d\theta = 4\pi R^2 \square$$

2. 物体的质心

关于x轴的力矩微元为

$y\,\mu(x,y)dxdy$



•平板D的质心$(\bar{x},\bar{y})$

平板密度$\mu(x,y)$

平板质量$M=\iint_D \mu(x,y)dxdy$

平板关于x轴的静力矩为$M\bar{y}=\iint_D y\mu(x,y)dxdy$

故 $\bar{y}=\dfrac{\iint_D y\mu(x,y)dxdy}{\iint_D \mu(x,y)dxdy}$，$\bar{x}=\dfrac{\iint_D x\mu(x,y)dxdy}{\iint_D \mu(x,y)dxdy}$

•空间物体Ω的质心$(\bar{x}, \bar{y}, \bar{z})$

密度$\mu(x, y, z)$, 质量$\mathrm{M} = \iiint_{\Omega} \mu(x, y, z) dxdydz$

Ω关于yz平面的静力矩为

$$\mathrm{M}\bar{x} = \iiint_{\Omega} x\mu(x, y, z) dxdydz$$

故

$$\bar{x} = \frac{\iiint_{\Omega} x\mu(x, y, z) dxdydz}{\iiint_{\Omega} \mu(x, y, z) dxdydz},$$

$$\bar{y} = \frac{\iiint_{\Omega} y\mu(x, y, z) dv}{\iiint_{\Omega} \mu(x, y, z) dv}, \bar{z} = \frac{\iiint_{\Omega} z\mu(x, y, z) dv}{\iiint_{\Omega} \mu(x, y, z) dv}.$$

3. 转动惯量

- 位于(x, y, z)处质量为m的质点, 绕x, y, z轴的转动惯量分别为$m(y^2+z^2), m(z^2+x^2), m(x^2+y^2)$.
- $\Omega \subset \mathbb{R}^3$, 密度$\rho(x, y, z)$, 绕坐标轴的转动惯量为

$$J_x = \iiint_\Omega (y^2+z^2)\rho(x, y, z)dxdydz,$$

$$J_y = \iiint_\Omega (z^2+x^2)\rho(x, y, z)dxdydz,$$

$$J_z = \iiint_\Omega (x^2+y^2)\rho(x, y, z)dxdydz.$$

Question. Ω绕直线l的转动惯量?

4. 万有引力

- 位于$P(x,y,z)$, $P_0(x_0,y_0,z_0)$的两质点, 质量分别为m, m_0.记$r=\|PP_0\|$, $\overrightarrow{P_0P}$与x,y,z正半轴夹角为α,β,γ, m对m_0的万有引力的大小为$\dfrac{kmm_0}{r^2}$, 引力沿x,y,z轴的分量分别为

$$F_x=\frac{kmm_0}{r^2}\cos\alpha=\frac{kmm_0(x-x_0)}{r^3},$$

$$F_y=\frac{kmm_0}{r^2}\cos\beta=\frac{kmm_0(y-y_0)}{r^3},$$

$$F_z=\frac{kmm_0}{r^2}\cos\beta=\frac{kmm_0(z-z_0)}{r^3}.$$

- 密度为$\rho(x,y,z)$的物体Ω对$P_0(x_0,y_0,z_0)\notin\Omega$处质量为$m_0$的质点的万有引力:

$$F_x=\iiint_\Omega \frac{km_0(x-x_0)\rho(x,y,z)dxdydz}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

$$F_y=\iiint_\Omega \frac{km_0(y-y_0)\rho(x,y,z)dxdydz}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

$$F_z=\iiint_\Omega \frac{km_0(z-z_0)\rho(x,y,z)dxdydz}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

例: 半径为R,质量为M的均匀球体$x^2+y^2+z^2\le R^2$对点$P(0,0,a)\ (a>R)$处质量为m的质点的引力.

解: $$F_x=\iiint_\Omega \frac{kmx\rho dxdydz}{\left(\sqrt{x^2+y^2+(a-z)^2}\right)^3}=0,\ F_y=0.$$

$$F_z=\iiint_\Omega \frac{km(a-z)\rho dxdydz}{\left(\sqrt{x^2+y^2+(a-z)^2}\right)^3},\quad \frac{4}{3}\pi R^3\rho=M.$$

在柱坐标系$x=r\cos\theta, y=r\sin\theta, z=z$下,

$$F_z=\int_{-R}^{R}dz\int_0^{2\pi}d\theta\int_0^{\sqrt{R^2-z^2}}\frac{km(a-z)\rho rdr}{(\sqrt{r^2+(a-z)^2})^3}$$

$$F_z = \int_{-R}^{R} dz \int_0^{2\pi} d\theta \int_0^{\sqrt{R^2-z^2}} \frac{km(a-z)\rho r dr}{(\sqrt{r^2+(a-z)^2})^3}$$

$$= k\pi m\rho \int_{-R}^{R} (a-z) dz \int_0^{\sqrt{R^2-z^2}} \frac{dr^2}{(\sqrt{r^2+(a-z)^2})^3}$$

$$= 2k\pi m\rho \int_{-R}^{R} \left(1 - \frac{a-z}{\sqrt{R^2+a^2-2az}}\right) dz$$

$$= \frac{4k\pi m\rho R^3}{3a^2} = \frac{kMm}{a^2}.\square$$

(分部积分)

Question. 密度分别为$\rho_1(x, y, z), \rho_2(x, y, z)$的两物体$\Omega_1, \Omega_2$之间的万有引力？

作业：习题3.5 No.1,9