Review

定积分的计算

- $\int_a^b F'(x)\mathrm{d}x = F(x)\Big|_{x=a}^{b}$
- $\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t$

$f \in C[a,b], \varphi(\alpha)=a, \varphi(\beta)=b, a \le \varphi(t) \le b, \varphi \in C^1[\alpha,\beta].$

- $\int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\mathrm{d}x.$

带积分余项的Taylor公式

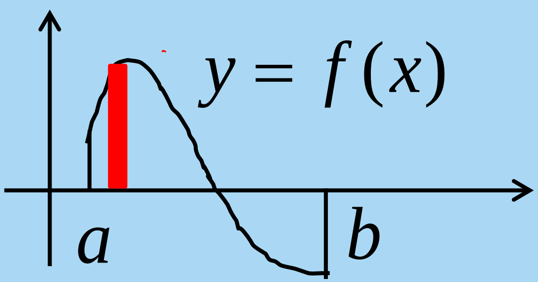
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\mathrm{d}t.$$

§ 7.积分的应用 --微元法

- 平面区域的面积
- 曲线的弧长
- 平面曲线的曲率
- 旋转体的体积
- 旋转面的面积
- 积分在物理中的应用

- 平面图形的面积

1) $y = f(x), y = 0, x \in [a,b].$ Riemann积分四部曲:



分割、取点、近似和、极限

$$\lim_{|T|\to 0}\sum_{i=1}^{n}|f(\xi_i)|\Delta x_i = S = \int_a^b |f(x)|\mathrm{d}x.$$

微元法: $[x, x+\Delta x]$对应窄条的面积 $\Delta S \approx |f(x)|\Delta x,$

$$f \in C[a,b] \Rightarrow \left|\Delta S - |f(x)|\Delta x\right| \le \left(\max_{x\le t\le x+\Delta x} f(t) - \min_{x\le t\le x+\Delta x} f(t)\right)\Delta x$$

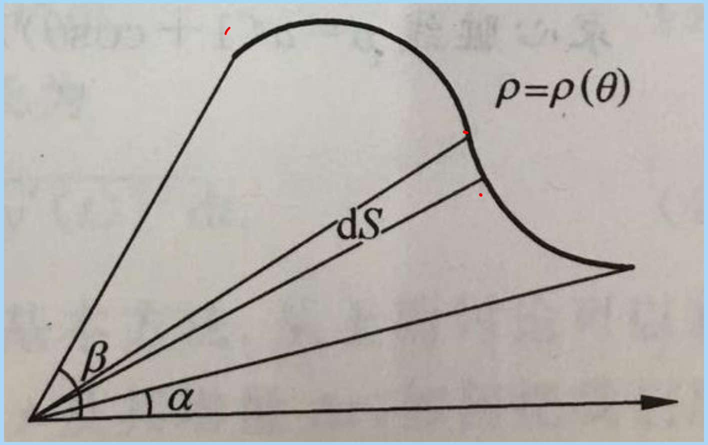
$$\Delta S = |f(x)|\Delta x + o(\Delta x), \Delta x \to 0.$$

$$\mathrm{d}S = |f(x)|\mathrm{d}x, \quad S = \int_a^b |f(x)|\mathrm{d}x.$$

2) $y = f_1(x), y = f_2(x), x \in [a,b].$

$$S = \int_a^b |f_1(x) - f_2(x)| \mathrm{d}x$$

3) $\rho = \rho(\theta), \theta \in [\alpha, \beta].$



（注意ρ和θ的几何意义！）

微元法: $\Delta S \approx \dfrac{1}{2}\rho^2(\theta)\Delta\theta$

$$\left(\rho(\theta) \in C[\alpha, \beta] \Rightarrow\right) \Delta S = \frac{1}{2}\rho^2(\theta)\Delta\theta + o(\Delta\theta), \Delta\theta \to 0.$$

$$\mathrm{d}S = \frac{1}{2}\rho^2(\theta)\mathrm{d}\theta, \qquad S = \int_\alpha^\beta \frac{1}{2}\rho^2(\theta)\mathrm{d}\theta.$$

Ex. 求 $x^{2/3}+y^{2/3}=a^{2/3}(a>0)$所围区域的面积.

解法一: 曲线关于 x 轴和 y 轴对称,因此 $S=4\int_0^a y(x)\mathrm{d}x$.

第一象限中曲线有参数方程

$$x=a\sin^3 t, y=a\cos^3 t, t\in[0,\frac{\pi}{2}].$$

故

$$S=4\int_0^{\pi/2} a\cos^3 t\cdot 3a\sin^2 t\cos t\,\mathrm{d}t$$

$$=3a^2\int_0^{\pi/2}\sin^2 2t\cos^2 t\,\mathrm{d}t$$

$$=\frac{3}{4}a^2\int_0^{\pi/2}(1-\cos 4t)(1+\cos 2t)\,\mathrm{d}t=\frac{3}{8}\pi a^2.$$

解法二：曲线有参数方程$x = a\sin^3\theta, y = a\cos^3\theta, \theta \in [0, 2\pi]$.

$$\rho^2(\theta) = x^2(\theta) + y^2(\theta) = a^2(\sin^6\theta + \cos^6\theta)$$

$$= a^2(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta)$$

$$= a^2\left(1 - \frac{3}{4}\sin^2 2\theta\right)$$

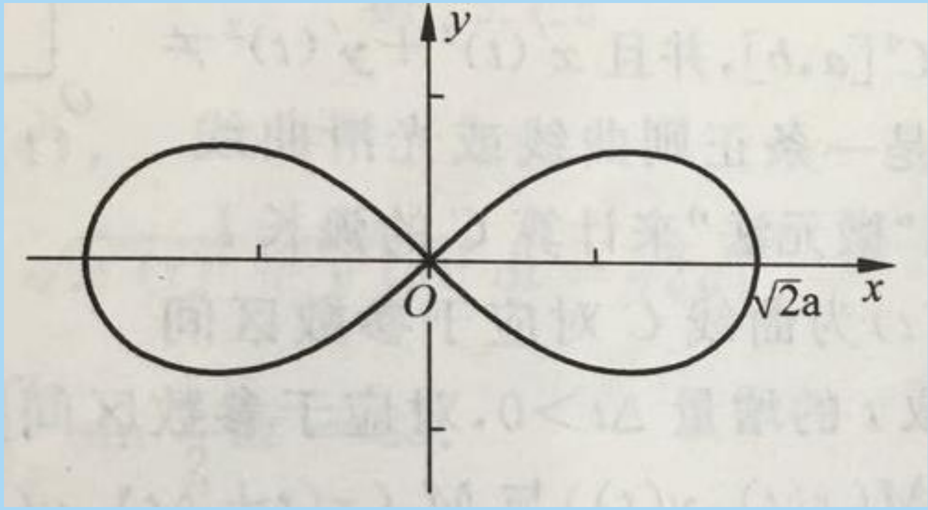
$$S \cancel{=} \frac{1}{2}\int_0^{2\pi} \rho^2(\theta)\,\mathrm{d}\theta = 2a^2\int_0^{\pi/2}\left(1 - \frac{3}{4}\sin^2 2\theta\right)\mathrm{d}\theta$$

$$= \pi a^2 - \frac{3}{4}a^2\int_0^{\pi/2}(1 - \cos 4\theta)\,\mathrm{d}\theta = \frac{5}{8}\pi a^2.$$

参数方程中的θ并非辐角！

Ex. 求双纽线$\rho^2=2a^2\cos 2\theta$所围区域的面积S.

解: $\rho(-\theta)=\rho(\theta)$, $\rho(\pi-\theta)=\rho(\theta)$, 故图像关于$x, y$轴对称.



$$2a^2\cos 2\theta=\rho^2\geq 0$$

$$\Rightarrow \cos 2\theta\geq 0$$

$$\Rightarrow 第一象限中\,\theta\in[0,\frac{\pi}{4}]$$

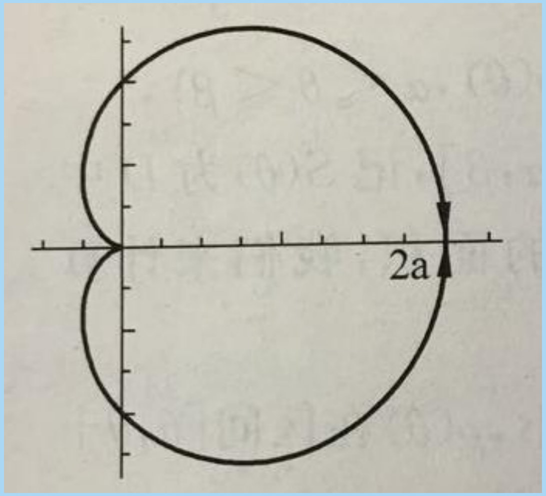
$$S=4S_1=4\int_0^{\pi/4}\frac{1}{2}\rho^2(\theta)\mathrm{d}\theta=4\int_0^{\pi/4}a^2\cos 2\theta\mathrm{d}\theta$$

$$=2a^2\sin 2\theta\Big|_0^{\pi/4}=2a^2.\square$$

Ex.求心脏线$\rho = a(1+\cos\theta)$所围区域的面积S.

解: $a(1+\cos\theta) = \rho \ge 0$

$$\Rightarrow \theta \in [-\pi, \pi].$$



$$S = \int_{-\pi}^{\pi} \frac{1}{2}\rho^2(\theta)\mathrm{d}\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\mathrm{d}\theta$$

$$= \int_0^{\pi} a^2(1+2\cos\theta+\frac{1+\cos 2\theta}{2})\mathrm{d}\theta$$

$$= \frac{3}{2}\pi a^2.\square$$

- 曲线的弧长 $L: x=x(t), y=y(t), z=z(t),\ t\in[\alpha,\beta],$

$$x(t), y(t), z(t)\in C^1[\alpha,\beta].$$

考虑 $[t,t+\Delta t]$ 对应的弧段

$$\Delta l \approx \sqrt{(x(t+\Delta t)-x(t))^2+(y(t+\Delta t)-y(t))^2+(z(t+\Delta t)-z(t))^2}$$

$$=\sqrt{(x'(\xi))^2+(y'(\eta))^2+(z'(\varsigma))^2}\,\Delta t$$

$$\approx \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,\Delta t.$$

弧长微元 $\mathrm{d}l=\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,\mathrm{d}t.$

$$l=\int_\alpha^\beta \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,\mathrm{d}t.$$

Remark. • 平面曲线$L: x=x(t), y=y(t), \alpha \le t \le \beta$,的弧长

$$l=\int_{\alpha}^{\beta}\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t.$$

• 曲线$L: y=f(x), a\le x\le b$,的弧长 $l=\int_a^b\sqrt{1+(f'(x))^2}\,\mathrm{d}x.$

• $\rho=\rho(\theta), \alpha\le\theta\le\beta$,的弧长（注意$\rho$和$\theta$的几何意义！）

$$x=\rho(\theta)\cos\theta, y=\rho(\theta)\sin\theta,$$

$$\sqrt{(x'(\theta))^2+(y'(\theta))^2}=\sqrt{(\rho'(\theta))^2+(\rho(\theta))^2}$$

$$l=\int_{\alpha}^{\beta}\sqrt{(\rho'(\theta))^2+(\rho(\theta))^2}\,\mathrm{d}\theta.$$

Ex.求心脏线$\rho = a(1+\cos\theta), -\pi \le \theta \le \pi$,的弧长$L$.

解: $x = \rho(\theta)\cos\theta = a(1+\cos\theta)\cos\theta,$

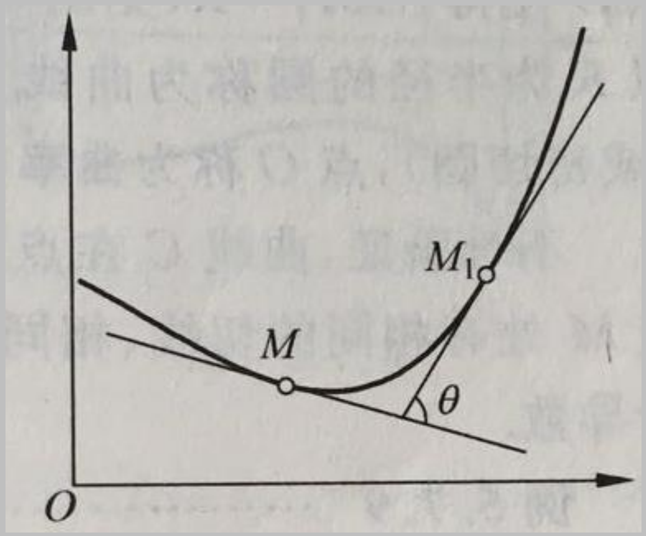
$$y = \rho(\theta)\sin\theta = a(1+\cos\theta)\sin\theta, \quad \theta \in [-\pi, \pi].$$

$$L = \int_{-\pi}^{\pi} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\,\mathrm{d}\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2}\,\mathrm{d}\theta$$

$$= 2a\int_{-\pi}^{\pi} \cos\frac{\theta}{2}\,\mathrm{d}\theta \quad = 4a\sin\frac{\theta}{2}\bigg|_{-\pi}^{\pi} \quad = 8a.\square$$

- 平面曲线的曲率 $L:\ x=x(t), y=y(t),\quad t\in[\alpha,\beta].$



$$M(x(t),y(t)), M_1(x(t+\Delta t), y(t+\Delta t)),$$

$$\sigma = MM_1 = \int_t^{t+\Delta t}\sqrt{(x'(\tau))^2+(y'(\tau))^2}\,\mathrm{d}\tau$$

$$\theta = \arctan\frac{y'(t+\Delta t)}{x'(t+\Delta t)} - \arctan\frac{y'(t)}{x'(t)}$$

曲率 $k \triangleq \lim\limits_{\Delta t\to 0}\dfrac{|\theta|}{\sigma} = \lim\limits_{\Delta t\to 0}\dfrac{|\theta/\Delta t|}{\sigma/\Delta t} = \dfrac{\left|\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\arctan\dfrac{y'(t)}{x'(t)}\right)\right|}{\sqrt{(x'(t))^2+(y'(t))^2}}$

$$= \frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left((x'(t))^2+(y'(t))^2\right)^{3/2}}.$$

Remark. $y=f(x)$在点x处的曲率为$k=\dfrac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$.

Ex. 求$x=R\cos t, y=R\sin t, 0\le t\le 2\pi$的曲率.

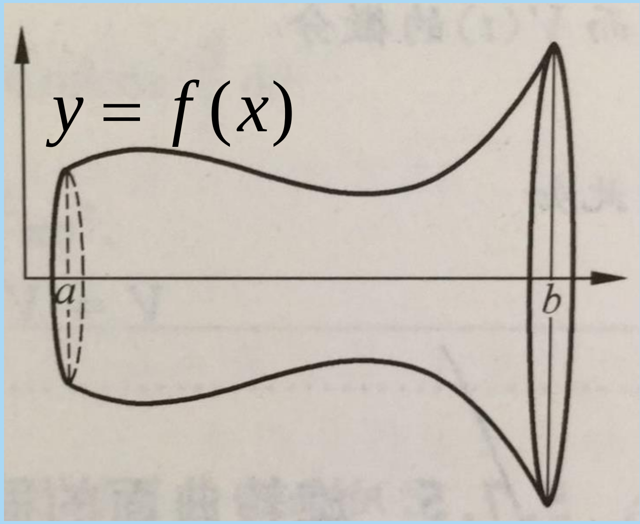
解: $k=\dfrac{|x'(t)y''(t)-x''(t)y'(t)|}{\left((x'(t))^2+(y'(t))^2\right)^{3/2}}=\dfrac{1}{R}$. □

Def. 曲线C在点M处的曲率k的倒数$R=1/k$称为C在点M的曲率半径. C的凹侧与C相切的半径为R的圆称为C在点M的曲率圆(密切圆), 曲率圆的圆心称为曲率中心.

Remark. 曲线的曲率圆与曲线在切点处有相同的切线、曲率与二阶导数.

- 旋转体的体积

曲线 $y=f(x), a\le x\le b$, 绕 x 轴旋转得旋转体 Ω. 求 $V(\Omega)$.



微元法: $[x, x+\Delta x]$对应的薄片体积

$$\left(f\in C[a,b]\Rightarrow\right)\Delta V=\pi f^2(x)\Delta x+o(\Delta x), \Delta x\to 0.$$

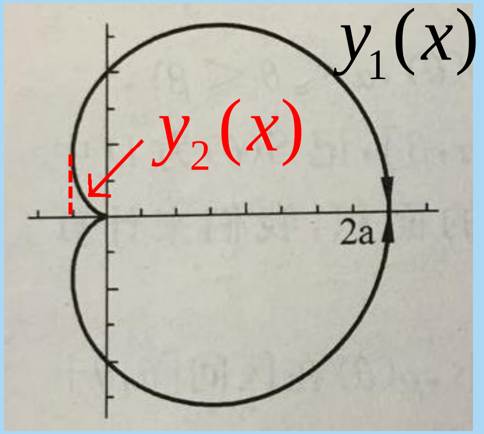
$$V(\Omega)=\pi\int_a^b f^2(x)\mathrm{d}x.$$

Remark. $x=f(y), c\le y\le d$, 绕 y 轴旋转得旋转体Ω的体积

$$V(\Omega)=\pi\int_c^d f^2(y)\mathrm{d}y.$$

Ex.求心脏线$\rho = a(1+\cos\theta)$绕 x 轴旋转所得旋转体的体积.

解:心脏线关于 x 轴对称,因此只需考虑$0 \le \theta \le \pi$.



$$x = a(1+\cos\theta)\cos\theta$$

$$= a\left((\cos\theta + \frac{1}{2})^2 - \frac{1}{4}\right) \ge -\frac{1}{4}a.$$

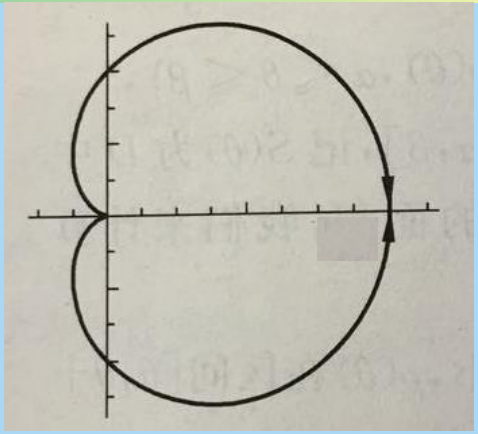
$$V = \int_{-a/4}^{2a} \pi y_1^2(x)\,dx - \int_{-a/4}^{0} \pi y_2^2(x)\,dx$$

$$= \pi\int_{2\pi/3}^{0} y_1^2(\theta)x'(\theta)d\theta - \pi\int_{2\pi/3}^{\pi} y_2^2(\theta)x'(\theta)d\theta$$

$$= \pi\int_{\pi}^{0} a^2(1+\cos\theta)^2\sin^2\theta\cdot a(\cos\theta+\cos^2\theta)'d\theta = \frac{8}{3}\pi a^3. \square$$

(2)曲线 $x=x(t), y=y(t), \alpha \le t \le \beta, x(t),$

$y(t) \in C^1[\alpha,\beta]$,绕 x 轴旋转得旋转体 Ω.



$$\Delta x = x'(t)\Delta t + o(\Delta t), \Delta t \to 0,$$

$\Delta t > 0$ 不能保证 $\Delta x \ge (\le) 0$.

微元法: $[t, t+\Delta t]$对应薄片的(有向)体积

$$\Delta V = \pi y^2(t)\Delta x + o(\Delta x), \Delta x \to 0$$

$$= \pi y^2(t) x'(t)\Delta t + o(\Delta t), \Delta t \to 0$$

$$V(\Omega) = \left| \pi \int_\alpha^\beta y^2(t) x'(t) \mathrm{d}t \right|.$$

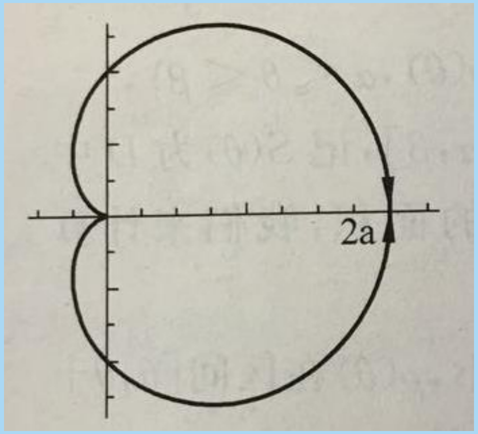
Remark.上例的计算在参数形式下可简化.

Ex.求心脏线$\rho = a(1+\cos\theta)$绕 x 轴旋转所得旋转体的体积.

解:心脏线关于 x 轴对称,因此只需考虑$0 \le \theta \le \pi$.

$$x = a(1+\cos\theta)\cos\theta$$

$$y = a(1+\cos\theta)\sin\theta$$



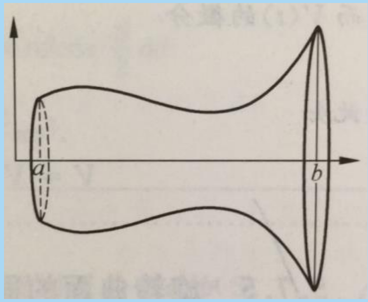
$$V = \left|\int_0^{\pi} \pi y^2(\theta)\, x'(\theta)\mathrm{d}\theta\right|$$

$$= \left|\pi\int_0^{\pi} a^2(1+\cos\theta)^2\sin^2\theta \cdot a(\cos\theta+\cos^2\theta)'\mathrm{d}\theta\right|$$

$$= \frac{8}{3}\pi a^3. \square$$

- 旋转面的面积

(1) 曲线$y=f(x), a\le x\le b, f\in C^1[a,b]$,
绕 x 轴旋转得旋转面面积



Question.用圆柱侧面积近似旋转面面积微元:

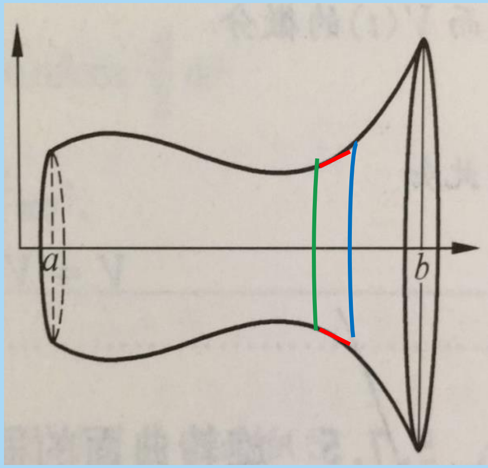
$[x, x+\Delta x]$对应的面积微元$\mathrm{d}S=2\pi f(x)\mathrm{d}x$? ×

- 曲线是几乎垂直于x轴的直线段?
- 曲线高频振动,曲面褶皱很多? $f(x)=x^4\sin^2\dfrac{1}{x}, f(0)=0.$

$\Delta S(0)\ne 2\pi f(0)\Delta x+o(\Delta x), \Delta x\to 0$ 时.

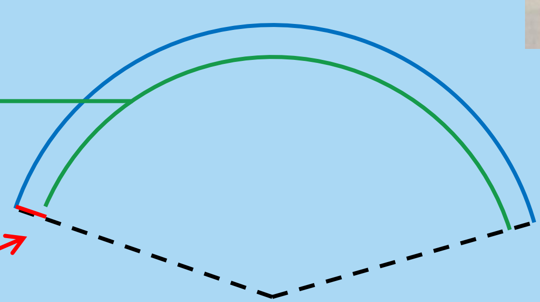
Question.用圆台侧面积近似旋转面面积微元? √

曲线 $y=f(x)$上$[x,x+\Delta x]$对应弧段σ的弧长微元为 $\sqrt{1+(f'(x))^2}\mathrm{d}x,$



平展 $[x,x+\Delta x]$ 对应的旋转面：

$2\pi|f(x)|$



$\sqrt{1+(f'(x))^2}\mathrm{d}x$

旋转面面积微元 $\mathrm{d}S=2\pi|f(x)|\sqrt{1+(f'(x))^2}\mathrm{d}x.$

$$S=2\pi\int_a^b|f(x)|\sqrt{1+(f'(x))^2}\mathrm{d}x.$$

(2)曲线$x=x(t), y=y(t), \alpha\le t\le\beta, x(t), y(t)\in C^1[a,b]$,绕 x 轴旋转得旋转面Σ的面积$S(\Sigma)$.

微元法: $[t,t+\Delta t]$对应段弧σ的弧长为

$$\mathrm{d}l=\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t.$$

σ 绕 x 轴旋转所得曲面面积

$$\Delta S\approx 2\pi|y(t)|\sqrt{(x'(t))^2+(y'(t))^2}\,\Delta t$$

$$\Delta S=2\pi|y(t)|\sqrt{(x'(t))^2+(y'(t))^2}\,\Delta t+o(\Delta t), \Delta t\to 0.$$

$$S=2\pi\int_\alpha^\beta|y(t)|\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t.$$

Ex. 求心脏线 $\rho = a(1+\cos\theta), a>0, 0\le\theta\le 2\pi$ 绕 x 轴旋转一周所得旋转面的面积 S.

解: 心脏线关于 x 轴对称, 只需考虑 $0\le\theta\le\pi$.

$$x=\rho\cos\theta,\ y=\rho\sin\theta.$$

$$S = 2\pi\int_0^{\pi}|y(\theta)|\sqrt{(x'(\theta))^2+(y'(\theta))^2}\,\mathrm{d}\theta$$

$$= 2\pi\int_0^{\pi}\rho\sin\theta\sqrt{(\rho'(\theta))^2+(\rho(\theta))^2}\,\mathrm{d}\theta$$

$$= 2\pi\int_0^{\pi}a(1+\cos\theta)\sin\theta\cdot 2a\cos\frac{\theta}{2}\,\mathrm{d}\theta$$

$$=16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,\mathrm{d}\theta = \frac{32\pi a^2}{5}.\ \square$$

- 积分在物理中的应用(功, 质量, 质心, 引力)

Ex. $C: x = x(t), y = y(t), a \le t \le b$. C上点$M(x(t), y(t))$处密度为$\rho(t)$. 求C的质量m与重心坐标$(\bar{x}, \bar{y})$.

分析: 平面质点系$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$的质量分别为$m_1, m_2, \cdots, m_n$. 其重心坐标$(\bar{x}, \bar{y})$为

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \quad \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}.$$

对C进行分割, 近似为有限个质点.

解: 分析$[t,t+\Delta t]$对应的段弧.

弧长微元 $\mathrm{d}l=\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t$

质量微元 $\mathrm{d}m=\rho(t)\mathrm{d}l=\rho(t)\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t.$

因而 $$m=\int_a^b\rho(t)\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t.$$

$$\bar{x}=\frac{\int_a^b x(t)\rho(t)\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t}{m},$$

$$\bar{y}=\frac{\int_a^b y(t)\rho(t)\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t}{m}.\square$$

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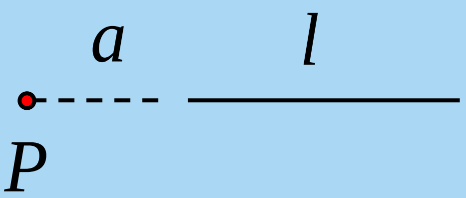
Ex.质量为 M 长度为 l 的均匀细杆,对其延长线上距离 a 处质量为 m 的质点 P 的引力 F.



解:取P为坐标原点,细杆所在直线为 x 轴.

考虑细杆上一小段 $[x, x+\Delta x]$,视之为质点, 其质量为

$\frac{M}{l}\Delta x$,它对P的引力为$\Delta F \approx k\frac{Mm\Delta x}{lx^2}$,$k$为引力常数, 故

$$F = k\int_a^{a+l}\frac{Mm}{lx^2}\mathrm{d}x = -\frac{kMm}{lx}\bigg|_a^{a+l} = \frac{kMm}{a(a+l)}.\square$$

作业：习题5.7

No.2(5), 3(1), 7(4), 8(2), 9(2).