Review

- Leibnitz判别法

$$a_n \searrow 0 \Rightarrow \sum_{n=1}^{+\infty} (-1)^n a_n \text{收敛.}$$

- Dirichlet判别法

$$\left.\begin{array}{l} \text{数列}\{a_n\}\text{单调趋于}0; \\ \left|\sum_{k=1}^{n} b_k\right| \le M, \forall n; \end{array}\right\} \Rightarrow \sum_{k=1}^{+\infty} a_n b_n \text{收敛.}$$

- Abel判别法

$$\left.\begin{array}{l}\text{数列}\{a_n\}\text{单调且有界,}\\ \sum\limits_{k=1}^{+\infty} b_k\text{收敛}\end{array}\right\} \Rightarrow \sum_{k=1}^{+\infty} a_n b_n\text{收敛.}$$

- Taylor展开在级数判敛中的应用

- 非负项级数的比较、比值判敛法不适用于一般项级数

清華大學

- 无穷和运算的结合律

(1) $\sum a_n$ 收敛 $\Rightarrow$

$$(a_1+\cdots+a_{n_1})+(a_{n_1+1}+\cdots+a_{n_2}) +\cdots+(a_{n_{k-1}+1}+\cdots+a_{n_k})+\cdots$$

收敛到同一和.

(2) $$(a_1+\cdots+a_{n_1})+(a_{n_1+1}+\cdots+a_{n_2}) +\cdots+(a_{n_{k-1}+1}+\cdots+a_{n_k})+\cdots$$

收敛, 且同一括号中各项有相同的正负号

$\Rightarrow \sum a_n$ 也收敛到同一和.

- 无穷和运算的交换律

Thm $\sum a_n$绝对收敛

$\Rightarrow$ 任意重排$\sum a_n'$也绝对收敛到同一和.

Thm (Riemann定理) $\sum a_n$条件收敛, 则

$\forall \lambda \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}, \exists$重排$\sum a_n', s.t., \sum a_n' = \lambda.$

§ 4. 无穷乘积

无穷乘积： $\prod_{1\le k<+\infty} p_n = p_1 p_2 \cdots p_n \cdots,$

部分乘积： $P_n = \prod_{1\le k\le n} p_k.$

Def.若 $\lim_{n\to\infty} P_n = P \ne 0$,则称 $\prod_{1\le k<+\infty} p_n$收敛,记为

$$\prod_{1\le k<+\infty} p_n = P;$$

若数列$\{P_n\}$发散,则称 $\prod_{1\le k<+\infty} p_n$发散.

Remark. $\prod_{1\le k<+\infty} p_n$收敛 $\Leftrightarrow \{P_n\}$收敛到非零实数.

Thm.(无穷乘积收敛的必要条件)

$$\prod_{1\le k<+\infty} p_n \text{收敛} \Rightarrow \lim_{n\to\infty} p_n = 1.$$

Proof. 设 $\prod_{1\le k<+\infty} p_n = P \neq 0$, 记 $P_n = \prod_{1\le k\le n} p_k$, 则 $\lim_{n\to\infty} P_n = P$.

$$\text{于是} \lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \frac{\lim_{n\to\infty} P_n}{\lim_{n\to\infty} P_{n-1}} = \frac{P}{P} = 1. \square$$

Corollary. $\prod_{1\le k<+\infty} (1+a_n)$收敛 $\Rightarrow \lim_{n\to\infty} a_n = 0.$

Thm. 设$p_n > 0, |a_n| < 1$, 则

$$\prod_{1\le k<+\infty} p_n \text{收敛} \Leftrightarrow \sum_{n=1}^{+\infty} \ln p_n \text{ 收敛}$$

$$\prod_{1\le k<+\infty} (1+a_n)\text{收敛} \Leftrightarrow \sum_{n=1}^{+\infty} \ln(1+a_n) \text{ 收敛}.$$

Proof. $\ln\left(\prod_{1\le k\le n} p_k\right) = \sum_{1\le k\le n} \ln p_k,\ \prod_{1\le k\le n} p_k = e^{\sum_{1\le k\le n} \ln p_k}$. □

Remark. $\prod_{1\le k<+\infty} p_n = e^{\sum_{1\le k<+\infty} \ln p_k}$.

例.证明 $\prod_{2\le n<+\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3}$.

Proof. $\frac{n^3-1}{n^3+1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} = \frac{n^2+n+1}{n(n+1)} \Big/ \frac{n^2-n+1}{n(n-1)}$

$$\sum_{n=2}^{m} \ln \frac{n^3-1}{n^3+1} = \sum_{n=2}^{m} \left(\ln \frac{n^2+n+1}{n(n+1)} - \ln \frac{n^2-n+1}{n(n-1)} \right)$$

$$= \ln \frac{m^2+m+1}{m(m+1)} - \ln \frac{3}{2}$$

$$\sum_{n=2}^{+\infty} \ln \frac{n^3-1}{n^3+1} = \ln \frac{2}{3}, \quad \prod_{2\le n<+\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3}. \square$$

例. $\sum_{n=1}^{+\infty} u_n^2 < +\infty$, 证明 $\prod_{1\le n<+\infty} \cos u_n$ 收敛.

Proof. $\sum_{n=1}^{+\infty} u_n^2 < +\infty$, 则 $\lim_{n\to\infty} u_n = 0$.

$\exists N$, 当 $n \ge N$ 时, $\cos u_n > 0$,

$$0 > \ln\cos u_n = \ln\sqrt{1-\sin^2 u_n} = \frac{1}{2}\ln(1-\sin^2 u_n)$$

$$\sim -\frac{1}{2}\sin^2 u_n \sim -\frac{1}{2}u_n^2,\quad n\to+\infty \text{时}.$$

$\sum_{n=N}^{+\infty} u_n^2 < +\infty$, 则 $\sum_{n=N}^{+\infty} \ln\cos u_n$ 收敛, 从而 $\prod_{1\le n<+\infty} \cos u_n$ 收敛.□

作业：习题5.4, No. 1, 2