Review

Thm.(确界原理) 非空有上界的集合必有上确界.

Thm.(单调收敛原理) 单调有界列必收敛.

Thm.(闭区间套定理) 若闭区间列$[a_n, b_n]$满足条件:

(1) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n] (n = 1, 2, \cdots)$,

(2) $\lim\limits_{n\to\infty}(b_n - a_n) = 0$,

则$\exists!\, \xi \in \mathbb{R}, s.t.\ \xi \in \bigcap\limits_{n\geq 1}[a_n, b_n]; \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \xi$.

Thm.(Bolzano-Weirstrass定理) 有界列必有收敛子列.

Thm.(Cauchy收敛原理) 收敛列$\Leftrightarrow$ Cauchy列.

清華大學

§1. 函数的极限

$$N(x_0,\delta):=(x_0-\delta,x_0+\delta),$$

$$U(x_0,\delta):=N(x_0,\delta)\setminus\{x_0\}$$

Def.(函数在一点的极限) 设f在$U(x_0,\rho)$中有定义, $A\in\mathbb{R}$. 若$\forall\varepsilon>0,\exists\delta\in(0,\rho), s.t.$

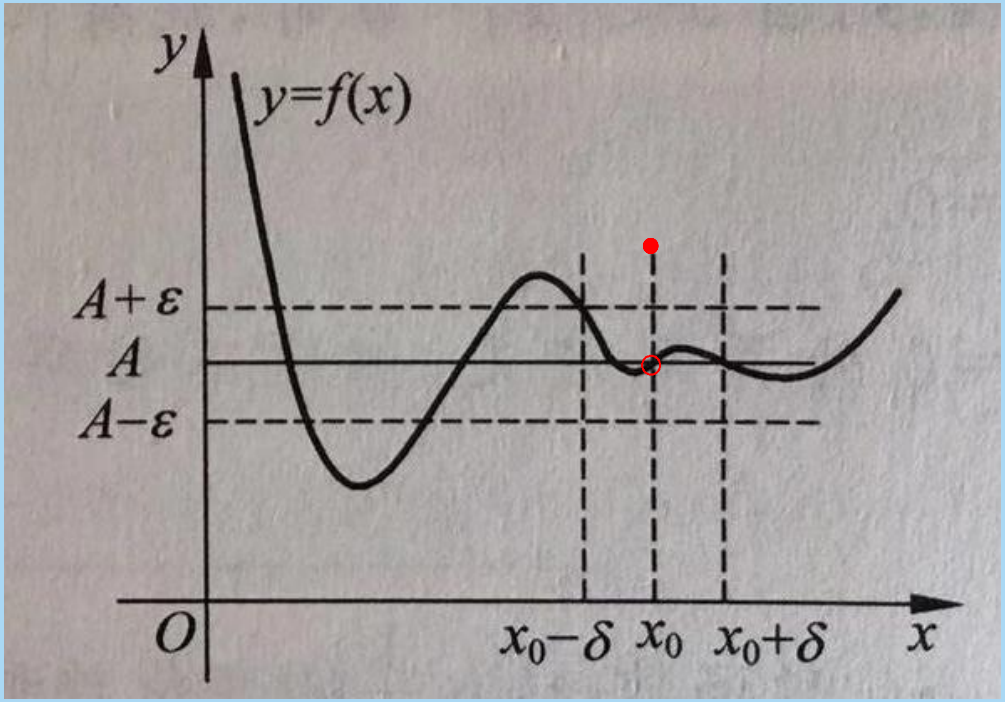
$$|f(x)-A|<\varepsilon,\quad \forall x\in U(x_0,\delta),$$

则称$f(x)$在点x_0处有极限A,或者当x趋于x_0时, $f(x)$趋于A.

记作 $\lim\limits_{x\to x_0} f(x)=A$, 或 $f(x)\to A(x\to x_0)$.

Remark. $\lim\limits_{x\to x_0} f(x)$与$f$在$x_0$的定义无关.

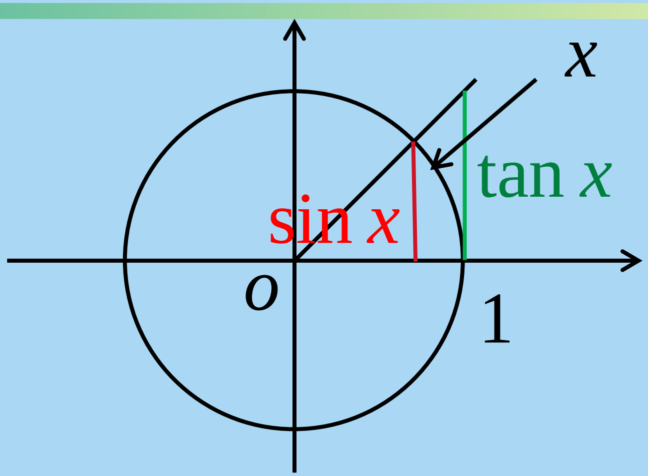
Question. $\lim\limits_{x\to x_0} f(x)=\mathrm{A}$的几何意义?



Question. 如何用$\varepsilon-\delta$语言描述$\lim\limits_{x\to x_0} f(x)\neq \mathrm{A}$?

$$|\sin x| \le |x|, \forall x \in \mathbb{R}.$$

$$|x| \le |\tan x|, \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$$



Ex. $\lim\limits_{x \to x_0} \cos x = \cos x_0.$

Proof. $\forall \varepsilon > 0, \exists \delta = \varepsilon$, 当$0 < |x - x_0| < \delta$时，有

$$|\cos x - \cos x_0| = \left| 2\sin\frac{x + x_0}{2}\sin\frac{x - x_0}{2} \right|$$

$$\le 2\left|\sin\frac{x - x_0}{2}\right| \le 2 \cdot \frac{|x - x_0|}{2} = |x - x_0| < \delta = \varepsilon. \square$$

Ex. $\lim\limits_{x \to x_0} \sin x = \sin x_0.$

Ex. $\lim_{x\to 0} x\sin\frac{1}{x} = \underline{\quad 0 \quad}.$ $\left|x\sin\frac{1}{x}\right| \le |x|.$

Ex. $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-x} = \underline{\quad -1 \quad}.$

分析: $\frac{x^2-3x+2}{x^2-x} = \frac{(x-1)(x-2)}{x(x-1)} = \frac{x-2}{x}, \forall x \ne 1.$

Proof. 当$|x-1| < \frac{1}{2}$时, $|x| > \frac{1}{2}$. $\underline{\forall\varepsilon > 0, \exists\delta} = \min\{\frac{1}{2}, \frac{\varepsilon}{4}\} \underline{> 0, s.t.}$

$$\underline{\left|\frac{x^2-3x+2}{x^2-x} - (-1)\right|} = 2\left|\frac{x-1}{x}\right| < 4|x-1| \underline{< \varepsilon,}$$

$\underline{\forall 0 < |x-1| < \delta}.\square$

Def.(右极限) 设f在$(x_0, x_0+\rho)$中有定义, $A\in\mathbb{R}$.若$\forall\varepsilon>0$,

$\exists\delta\in(0,\rho), s.t.$

$$|f(x)-A|<\varepsilon, \qquad \forall x_0<x<x_0+\delta,$$

则称$f(x)$在点x_0处有右极限A,或者当x趋于x_0^+时, $f(x)$趋于A.记作$\lim\limits_{x\to x_0^+} f(x)=A$,或$f(x)\to A(x\to x_0^+)$.

Def.(左极限) 设f在$(x_0-\rho, x_0)$中有定义, $A\in\mathbb{R}$.若$\forall\varepsilon>0$,

$\exists\delta\in(0,\rho), s.t.$

$$|f(x)-A|<\varepsilon, \qquad \forall x_0-\delta<x<x_0,$$

则称$f(x)$在点x_0处有左极限A,或者当x趋于x_0^-时, $f(x)$趋于A.记作$\lim\limits_{x\to x_0^-} f(x)=A$,或$f(x)\to A(x\to x_0^-)$.

Thm. $\lim\limits_{x\to x_0} f(x)=A \Leftrightarrow \lim\limits_{x\to x_0^+} f(x)=\lim\limits_{x\to x_0^-} f(x)=A.$

Proof. 略.

Ex. $\operatorname{sgn}(x)=\begin{cases}1, & x>0,\\ 0, & x=0,\\ -1, & x<0,\end{cases}$

$\lim\limits_{x\to 0+}\operatorname{sgn}(x)\underline{\ =1\ }$, $\lim\limits_{x\to 0-}\operatorname{sgn}(x)\underline{\ =-1\ }$, $\lim\limits_{x\to 0}\operatorname{sgn}(x)$<u>不存在</u>.

Def. $\lim\limits_{x\to x_0} f(x)=+\infty$:

$$\forall M>0, \exists\delta>0, \text{使得} \forall x\in U(x_0,\delta), \text{有 } f(x)>M.$$

Question. 如何定义 $\lim\limits_{x\to x_0^+} f(x)=+\infty$, $\lim\limits_{x\to x_0^-} f(x)=+\infty$,

$\lim\limits_{x\to x_0} f(x)=-\infty$, $\lim\limits_{x\to x_0^+} f(x)=-\infty$, $\lim\limits_{x\to x_0^-} f(x)=-\infty$,

$\lim\limits_{x\to x_0} f(x)=\infty$, $\lim\limits_{x\to x_0^+} f(x)=\infty$, $\lim\limits_{x\to x_0^-} f(x)=\infty$?

Thm. $\lim\limits_{x\to x_0} f(x)=+\infty \Leftrightarrow \lim\limits_{x\to x_0^+} f(x)=\lim\limits_{x\to x_0^-} f(x)=+\infty.$

Ex. $\lim\limits_{x\to 0^-} e^{\frac{1}{x}}$ $\underline{\ =0\ }$, $\lim\limits_{x\to 0^+} e^{\frac{1}{x}}$ $\underline{\ =+\infty\ }$, $\lim\limits_{x\to 0} e^{\frac{1}{x}}$ <u>不存在</u>.

Def.(函数在无穷远点的极限)

(1)设$|x|>a$时f有定义,$\mathrm{A}\in\mathbb{R}$.若$\forall\varepsilon>0,\exists M>0,s.t.$

$$|f(x)-\mathrm{A}|<\varepsilon,\quad \forall |x|>M,$$

则称当x趋于∞时,$f(x)$有极限A.记作

$$\lim_{x\to\infty}f(x)=\mathrm{A},\ \text{或}\ f(x)\to\mathrm{A}(x\to\infty).$$

(2)设$x>a$时f有定义,$\mathrm{A}\in\mathbb{R}$.若$\forall\varepsilon>0,\exists M>0,s.t.$

$$|f(x)-\mathrm{A}|<\varepsilon,\quad \forall x>M,$$

则称当x趋于$+\infty$时,$f(x)$有极限A.记作

$$\lim_{x\to+\infty}f(x)=\mathrm{A},\ \text{或}\ f(x)\to\mathrm{A}(x\to+\infty).$$

(3)设$x<a$时f有定义, $\mathrm{A}\in\mathbb{R}$.若$\forall\varepsilon>0,\exists M>0, s.t.$

$$\left|f(x)-\mathrm{A}\right|<\varepsilon,\quad \forall x<-M,$$

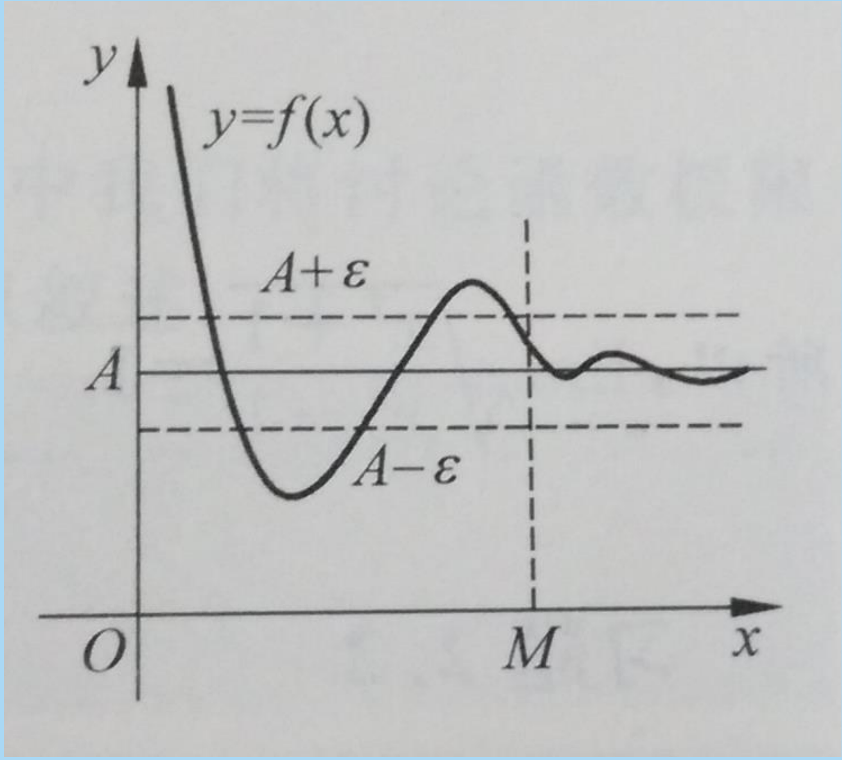
则称当x趋于$-\infty$时, $f(x)$有极限A.记作

$$\lim_{x\to-\infty} f(x)=\mathrm{A},\ \text{或}\ f(x)\to\mathrm{A}(x\to-\infty).$$

Question. 如何用$\varepsilon-\delta$语言描述$\lim\limits_{x\to+\infty} f(x)\neq\mathrm{A}$?

$$\exists\varepsilon>0,\forall M>0,\ \exists x>M, s.t.\left|f(x)-\mathrm{A}\right|>\varepsilon.$$

Question. $\lim\limits_{x\to+\infty} f(x) = A$的几何意义?



Ex. $\lim\limits_{x\to+\infty}\sqrt{\dfrac{x^2+1}{x^2-1}} = \underline{\quad 1 \quad}$

Proof. $\forall\varepsilon>0,\ \exists M=\max\{\sqrt{2},\dfrac{2}{\varepsilon}\}>0,\ s.t.$

$$\left|\sqrt{\frac{x^2+1}{x^2-1}}-1\right| = \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$=\frac{2}{\sqrt{x^2-1}\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)} < \frac{2}{|x|} < \varepsilon,\ \forall x>M.$$

故 $\lim\limits_{x\to+\infty}\sqrt{\dfrac{x^2+1}{x^2-1}}=1.$ □

Question. 如何定义

$$\lim_{x\to\infty} f(x)=\infty,\ \lim_{x\to+\infty} f(x)=\infty,\ \lim_{x\to-\infty} f(x)=\infty,$$

$$\lim_{x\to\infty} f(x)=-\infty,\ \lim_{x\to+\infty} f(x)=-\infty,\ \lim_{x\to-\infty} f(x)=-\infty,$$

$$\lim_{x\to\infty} f(x)=+\infty,\ \lim_{x\to+\infty} f(x)=+\infty,\ \lim_{x\to-\infty} f(x)=+\infty\,?$$

Remark. 函数极限的24种定义.

§ 2. 函数极限的性质

$$\lim_{x\to x_0} f(x),\quad \lim_{x\to x_0+} f(x),\quad \lim_{x\to x_0-} f(x),$$

$$\lim_{x\to\infty} f(x),\quad \lim_{x\to+\infty} f(x),\quad \lim_{x\to-\infty} f(x).$$

以 $\lim\limits_{x\to x_0} f(x)$ 为例叙述函数极限的性质, 其他情形类似.

Prop1. 若 $\lim\limits_{x\to x_0} f(x)$ 存在, 则极限值唯一.

Prop2. 若 $\lim\limits_{x\to x_0} f(x)$ 存在, 则 $\exists\delta>0, M>0, s.t.$

$$|f(x)|<M,\quad \forall x\in U(x_0,\delta).$$ (局部有界)

Prop3.(保序性) $\lim\limits_{x\to x_0} f(x) = \mathrm{A}, \lim\limits_{x\to x_0} g(x) = \mathrm{B}.$

(1)若$\mathrm{A} > \mathrm{B}$,则$\exists \delta > 0, s.t.$

$$f(x) > g(x), \quad \forall x \in U(x_0, \delta).$$

(2)若$\exists \delta > 0, s.t.$

$$f(x) \geq g(x), \quad \forall x \in U(x_0, \delta),$$

则$\mathrm{A} \geq \mathrm{B}.$

Prop4.(四则运算) $\lim\limits_{x\to x_0} f(x)=\mathrm{A}$, $\lim\limits_{x\to x_0} g(x)=\mathrm{B}$.

(1) $\forall c\in\mathbb{R}$, $\lim\limits_{x\to x_0} cf(x)=c\mathrm{A}$;

(2) $\lim\limits_{x\to x_0}\big(f(x)\pm g(x)\big)=\mathrm{A}\pm\mathrm{B}$;

(3) $\lim\limits_{x\to x_0}\big(f(x)\cdot g(x)\big)=\mathrm{AB}$;

(4) $\mathrm{B}\neq 0$时, $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\mathrm{A}}{\mathrm{B}}$.

Remark. A, B可取 $+\infty,-\infty$或∞,只要右端运算有意义.

Ex. $\lim\limits_{x\to x_0}\tan x=\tan x_0$, $\lim\limits_{x\to x_0}\cot x=\cot x_0$,

$\lim\limits_{x\to x_0}\sec x=\sec x_0$, $\lim\limits_{x\to x_0}\csc x=\csc x_0$.

条件?

Prop5.(夹挤原理)若

$$\left.\begin{array}{l} f(x)\le g(x)\le h(x),\quad \forall x\in U(x_0,\rho) \\ \lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} h(x)=\mathrm{A} \end{array}\right\}\Rightarrow \lim_{x\to x_0} g(x)=\mathrm{A}.$$

Ex. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1.$ $\qquad \dfrac{0}{0}$型极限

Proof. $\forall 0<|x|<\dfrac{\pi}{2}$, 有 $|\sin x|\le|x|\le|\tan x|$,

$$\cos x\le\frac{|\sin x|}{|x|}\le 1,\qquad \cos x\le\frac{\sin x}{x}=\frac{|\sin x|}{|x|}\le 1.$$

而 $\lim\limits_{x\to 0}\cos x=1$, 由夹挤原理, $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1.\square$

Prop6.(单调收敛原理)

(1) f在(a,b)上的单增有上界, 则 $\lim\limits_{x\to b^-} f(x)=\sup\limits_{a<x<b} f(x)$;

(2) f在(a,b)上的单减有下界, 则 $\lim\limits_{x\to b^-} f(x)=\inf\limits_{a<x<b} f(x)$;

(3) f在(a,b)上的单增有下界, 则 $\lim\limits_{x\to a^+} f(x)=\inf\limits_{a<x<b} f(x)$;

(4) f在(a,b)上的单减有上界, 则 $\lim\limits_{x\to a^+} f(x)=\sup\limits_{a<x<b} f(x)$.

Proof. 只证(1), 其它情形同理可证. $\{f(x): x\in(a,b)\}$ 非空有上界, 从而有上确界

$$\mathrm{A}=\sup\{f(x): x\in(a,b)\}\in\mathbb{R}.$$

由上确界的定义,

$$\forall \varepsilon > 0, \exists x_1 \in (a,b), s.t.\quad f(x_1) > \mathrm{A} - \varepsilon,$$

且

$$f(x) \le \mathrm{A}, \quad \forall x \in (a,b).$$

$f \uparrow$,则$\forall x \in (x_1, b)$,有

$$\mathrm{A} - \varepsilon < f(x_1) \le f(x) \le \mathrm{A}.$$

故$\lim\limits_{x \to b^-} f(x) = \mathrm{A}.\square$

Corollary.(a,b)上的单调函数在每一点处左右极限都存在.

Prop7. $\left.\begin{array}{l}\lim\limits_{x\to x_0} g(x)=u_0\\ \lim\limits_{u\to u_0} f(u)=\mathrm{A}\\ g(x)\neq u_0,\forall x\neq x_0\end{array}\right\}\Rightarrow \lim\limits_{x\to x_0} f(g(x))=\mathrm{A}=\lim\limits_{u\to u_0} f(u).$

(复合函数的极限)

Proof. $\lim\limits_{u\to u_0} f(u)=\mathrm{A}$, 则$\forall\varepsilon>0,\exists\delta>0, s.t.$

$$|f(u)-\mathrm{A}|<\varepsilon,\quad \forall 0<|u-u_0|<\delta.$$

对此$\delta>0$, 因 $g(x)\neq u_0,\forall x\neq x_0$, $\lim\limits_{x\to x_0} g(x)=u_0,\exists\eta>0, s.t.$

$$0<|g(x)-u_0|<\delta,\quad \forall 0<|x-x_0|<\eta,$$

从而 $$|f(g(x))-\mathrm{A}|<\varepsilon,\quad \forall 0<|x-x_0|<\eta.$$

由函数极限定义,有 $\lim\limits_{x\to x_0} f(g(x))=\mathrm{A}.\square$

Remark. 复合函数的极限运算可以理解为函数极限运算的变量替换法.

Question. 条件"$g(x) \neq u_0, \forall x \neq x_0$"是否可去?反例?

否. 反例：$f(u)=\begin{cases}1 & u\neq 0\\ 0 & u=0\end{cases}$, $\lim\limits_{u\to 0} f(u)=1$,

$$g(x)=x\sin\frac{1}{x}, \qquad g\left(\frac{1}{k\pi}\right)=0, \forall k\in\mathbb{Z}\setminus\{0\},$$

$$f(g(x))=\begin{cases}1 & x\neq 0, \dfrac{1}{k\pi}\\ 0 & x=\dfrac{1}{k\pi}\end{cases}, \quad \lim_{x\to 0} f(g(x))\text{不存在}.$$

Question. 条件“$g(x) \neq u_0, \forall x \neq x_0$”何时可去?

Remark. $$\left.\begin{array}{l} \lim\limits_{x\to x_0} g(x) = u_0 \\ \lim\limits_{u\to u_0} f(u) = f(u_0) \end{array}\right\} \Rightarrow \lim_{x\to x_0} f(g(x)) = f(u_0)$$

$$= \lim_{u\to u_0} f(u) = f(\lim_{x\to x_0} g(x)).$$

Remark. $$\left.\begin{array}{l} \lim\limits_{x\to x_0} g(x) = \pm\infty \\ \lim\limits_{u\to \pm\infty} f(u) = \mathrm{A} \end{array}\right\} \Rightarrow \lim_{x\to x_0} f(g(x)) = \mathrm{A}$$

$$= \lim_{u\to \pm\infty} f(u).$$

Thm. f在$U(x_0,\rho)$中有定义,则以下命题等价：

(1)$\forall\varepsilon>0,\exists\delta>0,\forall x,y\in U(x_0,\delta)$,有$\left|f(x)-f(y)\right|<\varepsilon$;

(2)$\exists \mathrm{A}\in\mathbb{R}$,对$U(x_0,\rho)$中任意收敛到$x_0$的点列$\{x_n\}$,有

$$\lim_{n\to\infty} f(x_n)=\mathrm{A};$$

(3) $\lim\limits_{x\to x_0} f(x)=\mathrm{A}.$

Remark. (1) $\Leftrightarrow$ (3) (函数极限的Cauchy收敛原理)

Remark. (2) $\Leftrightarrow$ (3) (用数列的极限来研究函数的极限)

Proof. $(1)\Rightarrow(2)$: 设 $x_n\in U(x_0,\rho),\lim\limits_{n\to\infty}x_n=x_0.\ \forall\varepsilon>0$,由(1),

$$\exists\delta>0,\forall x,y\in U(x_0,\delta),\text{有}\left|f(x)-f(y)\right|<\varepsilon.$$

对此 δ,因 $\lim\limits_{n\to\infty}x_n=x_0,\exists N,s.t.\quad x_n\in U(x_0,\delta),\quad\forall n>N.$

于是 $$\left|f(x_n)-f(x_m)\right|<\varepsilon,\qquad\forall n,m>N.$$

故 $\{f(x_n)\}$ 为Cauchy列,收敛, $\exists \mathrm{A}\in\mathbb{R},s.t.\lim\limits_{n\to\infty}f(x_n)=\mathrm{A}.$

设 $\lim\limits_{n\to\infty}y_n=x_0$,同理 $\lim\limits_{n\to\infty}f(y_n)=\mathrm{B}$. 只要证 $\mathrm{A}=\mathrm{B}$ 即可.

构造点列 $\{z_n\}$: $z_{2n-1}=x_n,\ z_{2n}=y_n$, 则 $\lim\limits_{n\to\infty}z_n=x_0,\{f(z_n)\}$

收敛, 且 $\mathrm{A}=\lim\limits_{n\to\infty}f(z_{2n-1})=\lim\limits_{n\to\infty}f(z_n)=\lim\limits_{n\to\infty}f(z_{2n})=\mathrm{B}.$

$(2)\Rightarrow(3):$

$$\text{设}\lim_{x\to x_0} f(x)\neq \mathrm{A}.\text{则}\exists\varepsilon_0>0,\forall n\in\mathbb{N},\exists x_n\in U(x_0,\frac{1}{n}),s.t.$$

$$\left|f(x_n)-\mathrm{A}\right|>\varepsilon_0.$$

此时,$\lim\limits_{n\to\infty}x_n=x_0$,但$\lim\limits_{n\to\infty}f(x_n)\neq\mathrm{A}$,与(2)矛盾.

$(3)\Rightarrow(1):$略.□

Remark. $x_n \neq x_0, y_n \neq x_0, \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = x_0$,则

- $\lim\limits_{n\to\infty} f(x_n) = \mathrm{A} \neq \mathrm{B} = \lim\limits_{n\to\infty} f(y_n) \Rightarrow \lim\limits_{x\to x_0} f(x)$不存在;
- $\lim\limits_{n\to\infty} f(x_n)$不存在 $\Rightarrow \lim\limits_{x\to x_0} f(x)$不存在.

Ex. Dirichlet函数 $D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$

$\lim\limits_{x\to x_0^-} D(x)$ __不存在__, $\lim\limits_{x\to x_0^+} D(x)$ __不存在__, $\lim\limits_{x\to x_0} D(x)$ __不存在__,

$\forall x_0 \in \mathbb{R}.$

Ex. $\lim_{x\to 0}\sin\frac{1}{x}$ __不存在__.

Proof. $x_n=\frac{1}{2n\pi}, y_n=\frac{1}{\left(2n+\frac{1}{2}\right)\pi}$,

$$\lim_{n\to+\infty}x_n=\lim_{n\to+\infty}y_n=0,$$

而 $\lim_{n\to+\infty}\sin\frac{1}{x_n}=0$, $\lim_{n\to+\infty}\sin\frac{1}{y_n}=1$, 故 $\lim_{x\to 0}\sin\frac{1}{x}$ 不存在. □

Ex.(1) $\lim\limits_{x\to x_0} e^x = e^{x_0}$, (2) $\lim\limits_{x\to x_0} \ln x = \ln x_0 \,(x_0 > 0)$.

Proof. $\forall \{x_n\}, x_n \to x_0$, 有 $\lim\limits_{n\to\infty} e^{x_n} = e^{x_0}, \lim\limits_{n\to\infty} \ln x_n = \ln x_0$.□

Thm. $\lim\limits_{x\to x_0} u(x) = a, \lim\limits_{x\to x_0} v(x) = b, a^b$有意义, 则 $\lim\limits_{x\to x_0} u(x)^{v(x)} = a^b$.

Proof. $\lim\limits_{x\to x_0} u(x)^{v(x)} = \lim\limits_{x\to x_0} e^{v(x)\ln u(x)}$

$$= e^{\lim\limits_{x\to x_0}\left(v(x)\ln u(x)\right)} = e^{\lim\limits_{x\to x_0} v(x)\cdot \lim\limits_{x\to x_0} \ln u(x)} = e^{b\ln a} = a^b.$$ □

Remark. $\lim\limits_{x\to x_0} \sqrt{u(x)} = \sqrt{\lim\limits_{x\to x_0} u(x)}$, $\lim\limits_{x\to x_0} a^{u(x)} = a^{\lim\limits_{x\to x_0} u(x)}, \cdots$

Question. $x \to +\infty$时, $x^b, a^x, \ln x, x^x$的增长速度? $(a>1, b>0)$

Ex. $\lim\limits_{x\to+\infty} \dfrac{\log_a x}{x^b} = 0 \ (a>1, b>0).$

Proof. $$0 < \frac{\ln x}{x} \le \frac{\ln\left(\lfloor x\rfloor + 1\right)}{\lfloor x\rfloor} \le \frac{\ln 2}{\lfloor x\rfloor} + \frac{\ln\lfloor x\rfloor}{\lfloor x\rfloor}, \quad \forall x > 1.$$

$$\lim_{x\to+\infty}\left(\frac{\ln 2}{\lfloor x\rfloor} + \frac{\ln\lfloor x\rfloor}{\lfloor x\rfloor}\right) = \lim_{x\to+\infty}\frac{\ln 2}{\lfloor x\rfloor} + \lim_{x\to+\infty}\frac{\ln\lfloor x\rfloor}{\lfloor x\rfloor} = 0.$$

由夹挤原理, $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x} = 0.$

$$\lim_{x\to+\infty}\frac{\log_a x}{x^b} = \lim_{y\to+\infty}\frac{\log_a y^{1/b}}{y} = \frac{1}{b\ln a}\lim_{y\to+\infty}\frac{\ln y}{y} = 0. \square$$

Remark. $\lim\limits_{x\to 0^+} x^b \log_a x = 0\ (a>1, b>0).$

Ex. $\lim\limits_{x\to+\infty} \dfrac{x^b}{a^x} = 0\ (a>1, b>0).$

Proof. $0 < \dfrac{x^b}{a^x} \le \dfrac{\left(\lfloor x\rfloor+1\right)^b}{a^{\lfloor x\rfloor}} \le \dfrac{\left(2\lfloor x\rfloor\right)^b}{a^{\lfloor x\rfloor}} \le \dfrac{2^b\lfloor x\rfloor^b}{a^{\lfloor x\rfloor}}, \forall x>1.$

$\lim\limits_{x\to+\infty} \dfrac{2^b\lfloor x\rfloor^b}{a^{\lfloor x\rfloor}} = 2^b \lim\limits_{x\to+\infty} \dfrac{\lfloor x\rfloor^b}{a^{\lfloor x\rfloor}} = 0.$ 由夹挤原理, $\lim\limits_{x\to+\infty} \dfrac{x^b}{a^x} = 0.\square$

Ex. $\lim\limits_{x\to+\infty} \dfrac{a^x}{x^x} = 0\ (a>0, a\ne 1).$

Proof. $\lim\limits_{x\to+\infty} \dfrac{a^x}{x^x} = \lim\limits_{x\to+\infty} e^{x(\ln a-\ln x)} = e^{(+\infty)\cdot(-\infty)} = e^{-\infty} = 0.\square$

Ex. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e.$

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Proof. $\left(1+\dfrac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor}\le\left(1+\dfrac{1}{x}\right)^{x}\le\left(1+\dfrac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1},\ \forall x>1.$

$$\lim_{x\to+\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1}=\lim_{x\to+\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor}\lim_{x\to+\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)=e,$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor}=\frac{\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor+1}}{\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{\lfloor x\rfloor+1}\right)}=e,$$

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由夹挤原理, $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e.$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{x\to-\infty}\left(\frac{x}{1+x}\right)^{-x}$$

$$=\lim_{x\to-\infty}\left(1+\frac{1}{-(1+x)}\right)^{-(x+1)}\cdot\lim_{x\to-\infty}\frac{x}{1+x}=\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y}=e.$$

综上, $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e.\square$

Remark. $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}=e.$ 常用以处理1^{∞}型极限.

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1^{∞}型极限

Ex. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}$.

Proof. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \lim\limits_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{x^2}}$

$$= \lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{-2\sin^2\frac{x}{2}}\cdot\frac{-2\sin^2\frac{x}{2}}{x^2}}$$

$$= \left\{\lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{-2\sin^2\frac{x}{2}}}\right\}^{-\frac{1}{2}\lim\limits_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} = e^{-\frac{1}{2}}. \square$$

作业：

习题2.2 No. 3(4)(7),7,8

习题2.3 No. 6(8)(11)(13)(14)(17),
7(4)(12),8(2)(6),9(1).