Review

- $\int_a^b f(x)\mathrm{d}x = \mathrm{I}: \forall \varepsilon > 0, \exists \delta > 0,$ 当 $|T| < \delta$ 时，无论 $\xi_i \in [x_{i-1}, x_i]$ 如何取，都有
$$\left| \sum_{i=1}^{n} f(\xi_i)\Delta x_i - \mathrm{I} \right| < \varepsilon.$$

- Darboux上和 $U(f,T) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad M_i \triangleq \sup_{x\in[x_{i-1},x_i]} f(x),$

  Darboux下和 $L(f,T) = \sum_{i=1}^{n} m_i \Delta x_i, \qquad m_i \triangleq \inf_{x\in[x_{i-1},x_i]} f(x).$

  Riemann和 $\sigma(f,T,\{\xi_i\}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$

$$m(b-a) \le L(f,T) \le \sigma(f,T,\{\xi_i\}) \le U(f,T) \le M(b-a).$$

- 在T中加入k个新分点得到T_k, 则

$$0 \le U(f,T) - U(f,T_k) \le k|T|(M-m);$$

$$0 \le L(f,T_k) - L(f,T) \le k|T|(M-m).$$

- $L(f,T_1) \le U(f,T_2)$.

- Darboux上积分 : $\overline{\int_a^b} f(x)\mathrm{d}x = \inf\{U(f,T) : T 为[a,b]的分割\}$,

Darboux下积分 : $\underline{\int_a^b} f(x)\mathrm{d}x = \sup\{L(f,T) : T 为[a,b]的分割\}$.

- $L(f,T) \le \underline{\int_a^b} f(x)\mathrm{d}x \le \overline{\int_a^b} f(x)\mathrm{d}x \le U(f,T)$.

- f在$[a,b]$有界,则

$$f \in R[a,b];$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists [a,b]\text{的分割}T, s.t.\ U(f,T) - L(f,T) < \varepsilon;$$

$$\Leftrightarrow \underline{\int_a^b} f(x)\mathrm{d}x = \overline{\int_a^b} f(x)\mathrm{d}x.$$

- $[a,b]$上的可积函数类.

Remark. 修改有限个点处的函数值,不改变有界闭区间上函数的Riemann可积性.

Remark. 有界闭区间上的无界函数不是Riemann可积的. 我们将来会讨论无界函数的广义Riemann可积性.

§ 2.Riemann积分的性质

Prop1. (线性性质)

$$f,g\in R[a,b],\alpha,\beta\in\mathbb{R}\Rightarrow\alpha f+\beta g\in R[a,b],\text{且}$$

$$\int_a^b\big(\alpha f(x)+\beta g(x)\big)\mathrm{d}x=\alpha\int_a^b f(x)\mathrm{d}x+\beta\int_a^b g(x)\mathrm{d}x.$$

Proof. $\lim\limits_{|T|\to0}\sigma(\alpha f+\beta g,T,\{\xi_i\})$

$$=\lim_{|T|\to0}\alpha\sigma(f,T,\{\xi_i\})+\lim_{|T|\to0}\beta\sigma(g,T,\{\xi_i\}).\square$$

Prop2. (积分区间的可加性) $a<b<c$,则

$$f\in R[a,c]\Leftrightarrow f\in R[a,b]\,\&\,f\in R[b,c].$$

此时$\int_a^c f(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x.$

Proof. $\Leftarrow$:设$f\in R[a,b], f\in R[b,c].\forall\varepsilon>0,\exists[a,b]$的分割$T_1$, $[b,c]$的分割T_2, s.t.

$$U(f,T_1)-L(f,T_1)<\varepsilon,\quad U(f,T_2)-L(f,T_2)<\varepsilon.$$

合并T_1,T_2的分点得到$[a,c]$的分割T,则

$$\begin{aligned}&U(f,T)-L(f,T)\\&=U(f,T_1)-L(f,T_1)+U(f,T_2)-L(f,T_2)<2\varepsilon.\end{aligned}$$

$\Rightarrow$:设$f \in R[a,c]$.则$\forall \varepsilon > 0, \exists [a,c]$的分割$T_0$, s.t.

$$U(f,T_0) - L(f,T_0) < \varepsilon.$$

在T_0中添加分点b得到$[a,c]$的分割T,则

$$U(f,T) - L(f,T) \le U(f,T_0) - L(f,T_0) < \varepsilon.$$

T在$[a,b],[b,c]$上的限制分别记为T_1, T_2,则

$$U(f,T_i) - L(f,T_i) \le U(f,T) - L(f,T) < \varepsilon, \quad i = 1,2.$$

故$f \in R[a,b], f \in R[b,c]$.

至此,我们证明了$f\in R[a,c]\Leftrightarrow f\in R[a,b]\,\&\,f\in R[b,c]$.

下证 $\int_a^c f(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x$.

设T_1, T_2分别为$[a,b],[b,c]$上的分割,合并T_1, T_2的分点得到$[a,c]$上的分割T. 则 $|T|\to 0\Leftrightarrow |T_i|\to 0, i=1,2$, 且

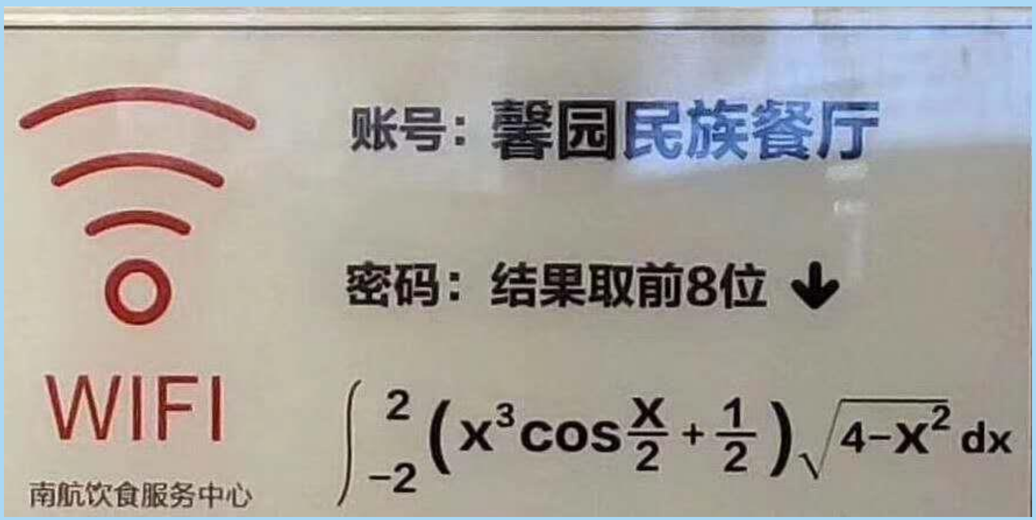
$$\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x$$

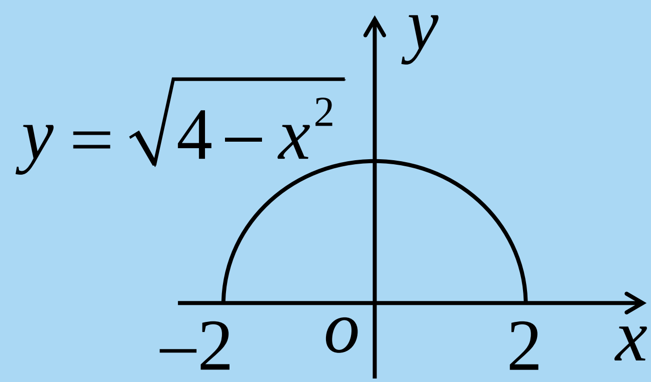
$$=\lim_{|T_1|\to 0}\sigma(f,T_1,\{\xi_i\})+\lim_{|T_2|\to 0}\sigma(f,T_2,\{\eta_i\})$$

$$=\lim_{|T|\to 0}\sigma(f,T,\{\xi_i\}\cup\{\eta_i\})=\int_a^c f(x)\mathrm{d}x.\ \square$$

Prop3. f 为 $[-a,a]$ 上的奇函数,则 $\int_{-a}^{a} f(x)\mathrm{d}x = 0$;

f 为 $[-a,a]$ 上的偶函数,则 $\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x$.





$$= 0 + \frac{1}{2}\int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x \quad = \pi = 3.1415926\cdots$$

Prop4. (单调性) $f,g\in R[a,b]$, 且 $f(x)\le g(x)$,则

$$\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x.$$

特别地, 若 $m\le f(x)\le M\,(a\le x\le b)$, 则

$$m(b-a)\le \int_a^b f(x)\mathrm{d}x \le M(b-a).$$

Proof. $g(x)-f(x)\ge 0(a\le x\le b)$, 则

$$\int_a^b g(x)\mathrm{d}x-\int_a^b f(x)\mathrm{d}x=\int_a^b (g(x)-f(x))\mathrm{d}x$$

$$=\lim_{|T|\to 0}\sigma(g-f,T,\{\xi_i\})\ \ge 0. \square$$

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Prop5.(积分估值) $f\in R[a,b]\Rightarrow|f|\in R[a,b]$, 且

$$\left|\int_a^b f(x)\mathrm{d}x\right|\le\int_a^b|f(x)|\mathrm{d}x.$$

Proof. $U(|f|,T)-L(|f|,T)\le U(f,T)-L(f,T)$.□

Question. $|f|\in R[a,b]\overset{?}{\Rightarrow} f\in R[a,b]$

$$f=\begin{cases}1, & x\in\mathbb{Q}\bigcap[0,1],\\ -1, & x\in[0,1]\setminus\mathbb{Q}.\end{cases}\qquad f\notin R[0,1].$$

$$|f|=1,\forall x\in[0,1],\quad |f|\in R[0,1].$$

Prop6. $f, g \in R[a,b] \Rightarrow fg \in R[a,b]$.

Proof. 1)设 $f \geq 0, g \geq 0$. 记$M = \sup\limits_{a \leq x \leq b} f(x), N = \sup\limits_{a \leq x \leq b} g(x)$.

任给$T: a = x_0 < x_1 < \cdots < x_n = b$, 记

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x), \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x),$$

$$N_i = \sup_{x_{i-1} \leq x \leq x_i} g(x), \quad n_i = \inf_{x_{i-1} \leq x \leq x_i} g(x).$$

$$U(fg,T) - L(fg,T) \leq \sum_{i=1}^{n} (M_i N_i - m_i n_i) \Delta x_i$$

$$= \sum_{i=1}^{n} M_i (N_i - n_i) \Delta x_i + \sum_{i=1}^{n} n_i (M_i - m_i) \Delta x_i$$

$$\leq M\left[U(g,T) - L(g,T)\right] + N\left[U(f,T) - L(f,T)\right].$$

2)对任意 f, 记$f^{\pm}(x)=\dfrac{1}{2}\left[|f(x)|\pm f(x)\right]$,则

$$f^{\pm}\geq 0,\quad f^{\pm}\in R[a,b],\quad f=f^{+}-f^{-},$$

于是 $$fg=(f^{+}-f^{-})(g^{+}-g^{-})$$
$$=f^{+}g^{+}-f^{+}g^{-}-f^{-}g^{+}+f^{-}g^{-}\in R[a,b].\square$$

Prop7. $f\in R[a,b],|f|\geq\lambda>0\Rightarrow\dfrac{1}{f}\in R[a,b].$

Proof. 仿上可证. 略. $\square$

Prop8. $f \in R[a,b],\ f \geq 0 \Rightarrow \sqrt{f} \in R[a,b].$

Proof. $\left(U(\sqrt{f},T)-L(\sqrt{f},T)\right)^2 = \left(\sum_{i=1}^{n}(\sqrt{M_i}-\sqrt{m_i})\sqrt{\Delta x_i}\sqrt{\Delta x_i}\right)^2$

$$\leq \sum_{i=1}^{n}(\sqrt{M_i}-\sqrt{m_i})^2\Delta x_i \cdot \sum_{i=1}^{n}\Delta x_i$$

$$=(b-a)\sum_{i=1}^{n}(M_i-2\sqrt{M_i m_i}+m_i)\Delta x_i$$

$$\leq (b-a)\sum_{i=1}^{n}(M_i-m_i)\Delta x_i=(b-a)\left(U(f,T)-L(f,T)\right). \square$$

Ex. 设$f \in C[a,b]$, $f(x) \geq 0$, $\int_a^b f(x)\mathrm{d}x = 0$. 求证: $f(x) \equiv 0$.

Proof. 反证. 设$f(x)$在不恒为0, 则$\exists x_0 \in [a,b]$, $s.t.\ f(x_0) > 0$. 不妨设$x_0 \in (a,b)$. $f \in C[a,b]$, 则$\exists \delta > 0$, $s.t.\ N_\delta(x_0) \subset [a,b]$, 且

$$f(x) > f(x_0)/2 > 0, \quad \forall x \in [x_0 - \delta, x_0 + \delta].$$

而$f(x) \geq 0$, 于是

$$0 = \int_a^b f(x)\mathrm{d}x = \int_a^{x_0-\delta} f(x)\mathrm{d}x + \int_{x_0-\delta}^{x_0+\delta} f(x)\mathrm{d}x + \int_{x_0+\delta}^{b} f(x)\mathrm{d}x$$

$$\geq 0 + \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2}\mathrm{d}x + 0 \geq f(x_0)\delta > 0,$$ 矛盾.□

Remark. 设$f,g\in C[a,b]$, $f(x)\ge g(x)$, $f\ne g$, 则

$$\int_a^b f(x)\mathrm{d}x>\int_a^b g(x)\mathrm{d}x.$$

Thm.(Cauchy不等式) $f,g\in R[a,b]$, 则

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2\le\int_a^b f^2(x)\mathrm{d}x\cdot\int_a^b g^2(x)\mathrm{d}x.$$

Proof. 令$A=\int_a^b f^2(x)\mathrm{d}x, B=\int_a^b f(x)g(x)\mathrm{d}x, C=\int_a^b g^2(x)\mathrm{d}x.$

则 $0\le\int_a^b\left[tf(x)+g(x)\right]^2\mathrm{d}x=At^2+2Bt+C,\quad \forall t\in\mathbb{R}.$

故 $(2B)^2-4AC\le 0.$ □

证法二: $\left(\sigma(fg,T,\{\xi_i\})\right)^2\le\sigma(f^2,T,\{\xi_i\})\cdot\sigma(g^2,T,\{\xi_i\}).$

Thm.(积分第一中值定理) $f\in C[a,b], g\in R[a,b]$, g不变号,

则$\exists\xi\in[a,b], s.t.$ $\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$

特别地, $g(x)\equiv 1$时, $\int_a^b f(x)\mathrm{d}x = f(\xi)(b-a).$

Proof.不妨设$g\geq 0$.记f在$[a,b]$上的最大值与最小值为M, m,则

$$m\int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\mathrm{d}x \leq M\int_a^b g(x)\mathrm{d}x.$$

因此, $\exists\lambda\in[m,M]$, s.t. $\int_a^b f(x)g(x)\mathrm{d}x = \lambda\int_a^b g(x)\mathrm{d}x.$

由连续函数的介值定理, $\exists\xi\in[a,b]$, s.t. $f(\xi)=\lambda$,

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x. \ \square$$

Question. g变号, 反例?

$$\text{提示:构造}\int_a^b g(x)\mathrm{d}x=0,\int_a^b f(x)g(x)\mathrm{d}x>0.$$

Ex. $f\in R[a,b],\ f\geq 0,\int_a^b xf(x)\mathrm{d}x=0,$ 则

$$\int_a^b x^2 f(x)\mathrm{d}x\leq -ab\int_a^b f(x)\mathrm{d}x.$$

Proof. $(x-a)(x-b)f(x)\leq 0,\forall x\in[a,b].$

$$0\geq\int_a^b (x-a)(x-b)f(x)\mathrm{d}x$$

$$=\int_a^b x^2 f(x)\mathrm{d}x-(a+b)\int_a^b xf(x)\mathrm{d}x+ab\int_a^b f(x)\mathrm{d}x$$

$$=\int_a^b x^2 f(x)\mathrm{d}x+ab\int_a^b f(x)\mathrm{d}x.\square$$

Ex. $x \to 0$时, $f(x) = \int_{\sin x}^{x} \ln(1+e^t)\mathrm{d}t$ 与x^p是同阶无穷小, 则

$p =$ ________, 此时$\lim\limits_{x\to 0} f(x)/x^p =$ ________.

解: 由积分中值定理, 存在介于x与$\sin x$之间的ξ, s.t.

$$f(x) = (x - \sin x)\ln(1+e^{\xi}) = \left(\frac{x^3}{3!} + o(x^3)\right)\ln(1+e^{\xi}), x \to 0.$$

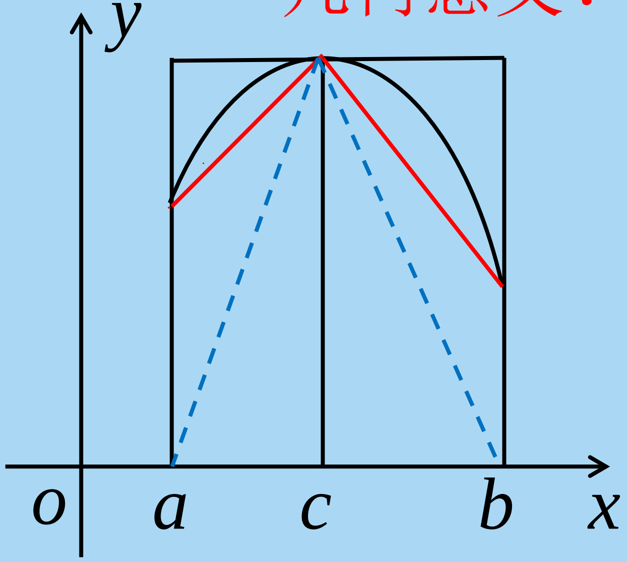
$f(x)$与x^p是同阶无穷小, 则

$$\lim_{x\to 0}\frac{f(x)}{x^p} = \ln 2 \cdot \lim_{x\to 0}\frac{\frac{x^3}{3!} + o(x^3)}{x^p}$$ 存在且非0,

因而$p = 3$, $\lim\limits_{x\to 0}\frac{f(x)}{x^p} = \frac{\ln 2}{6}$. □

Ex. $f\ge 0,\ f''\le 0\Rightarrow \max\limits_{a\le x\le b} f(x)\le \dfrac{2}{b-a}\displaystyle\int_a^b f(x)\mathrm{d}x.$

几何意义？



Proof.不妨设$f(c)=\max\limits_{a\le x\le b} f(x), c\in(a,b).$

$f''\le 0$,则 f 上凸,因此,$\forall x\in[a,c]$,有

$$f(x)\ge f(a)+\frac{f(c)-f(a)}{c-a}(x-a).$$

两边在$[a,c]$上积分,再利用$f\ge 0$,得

$$\int_a^c f(x)\mathrm{d}x\ge \frac{1}{2}(c-a)(f(a)+f(c))\ \ge \frac{1}{2}(c-a)f(c).$$

同理,$\displaystyle\int_c^b f(x)\mathrm{d}x\ge \frac{1}{2}(b-c)f(c)$, 从而

$$\int_a^b f=\int_a^c f+\int_c^b f\ge \frac{1}{2}(b-a)f(c)=\frac{1}{2}(b-a)\max_{a\le x\le b} f(x).\ \square$$

Ex. f在$[0,1]$可导, $f(1)=4\int_0^{1/4} e^{1-x^3} f(x)\mathrm{d}x$.则$\exists\xi\in(0,1), s.t.$

$$f'(\xi)=3\xi^2 f(\xi).$$

Proof. 由积分第一中值定理, $\exists\eta\in[0,1/4]\subset[0,1), s.t.$

$$f(1)=4\int_0^{1/4} e^{1-x^3} f(x)\mathrm{d}x=e^{1-\eta^3} f(\eta).$$

令$g(x)=e^{1-x^3} f(x)$,则 $g(\eta)=g(1)=f(1)$,

$$g'(x)=e^{1-x^3}(f'(x)-3x^2 f(x)).$$

由Rolle定理, $\exists\xi\in(\eta,1)\subset(0,1), s.t. g'(\xi)=0$, 从而

$$f'(\xi)=3\xi^2 f(\xi).\square$$

Ex. $f\in C[a,b]$, 则

$$\left(\int_a^b f(x)\cos x\mathrm{d}x\right)^2+\left(\int_a^b f(x)\sin x\mathrm{d}x\right)^2\le\left(\int_a^b |f(x)|\mathrm{d}x\right)^2.$$

Proof. 由积分估值及Cauchy不等式,

$$\left(\int_a^b f(x)\cos x\mathrm{d}x\right)^2\le\left(\int_a^b \sqrt{|f(x)|}\cdot\sqrt{|f(x)|}\,|\cos x|\mathrm{d}x\right)^2$$

$$\le\int_a^b |f(x)|\mathrm{d}x\cdot\int_a^b |f(x)|\cos^2 x\mathrm{d}x.$$

$$\left(\int_a^b f(x)\sin x\mathrm{d}x\right)^2\le\int_a^b |f(x)|\mathrm{d}x\cdot\int_a^b |f(x)|\sin^2 x\mathrm{d}x.$$

两式相加即证所需结论. □

Ex. $\lim\limits_{n\to+\infty}\int_0^1 x^n \mathrm{d}x = 0.$

Proof. 令$a_n=\int_0^1 x^n \mathrm{d}x$, 则$a_n$单调下降有下界0, $\lim\limits_{n\to+\infty} a_n$存在.

$\forall \varepsilon\in(0,1)$,

$$0\le \int_0^1 x^n \mathrm{d}x=\int_0^{1-\varepsilon} x^n \mathrm{d}x+\int_{1-\varepsilon}^1 x^n \mathrm{d}x \le (1-\varepsilon)^{n+1}+\varepsilon.$$

因此, $0\le \lim\limits_{n\to+\infty}\int_0^1 x^n \mathrm{d}x\le\varepsilon$. 由$\varepsilon$的任意性, $\lim\limits_{n\to+\infty}\int_0^1 x^n \mathrm{d}x=0$. □

Question. $\lim\limits_{n\to+\infty}\int_0^1 x^n \mathrm{d}x=\lim\limits_{n\to+\infty}\xi^n=0$, 是否正确?

错误. $\xi=\xi_n$不是常数, ξ_n^n不一定趋于0.

作业：习题5.2

No.6(1),7(2),9,10