微积分 A2 第 6 次作业参考答案

3.2

4. 设 $D \subset \mathbb{R}^2$ 为有界闭区域, 非负函数 $f(x,y) \in C(D)$, 证明: 若 $\iint_D f(x,\, y)\mathrm{d}x\mathrm{d}y = 0$, 则 $f(x,y) = 0, \forall (x,y) \in D$.

证: 用反证法: 假设 $\exists \left(x_0, y_0\right), \in D, f\left(x_0, y_0\right) \neq 0$, 因为 $f \geq 0$, 所以 $f\left(x_0, y_0\right) > 0$. 又因为 $f \in C(D), D$ 为有界闭集, 所以可得 $\forall \varepsilon, \exists \delta$,

$$\left|f\left(x_0, y_0\right) - f(x,y)\right| < \varepsilon, \forall (x,y) \in B\left(x_0, \delta\right) \cap D.$$

这里我们不妨设 $\varepsilon = \frac{f(x_0,y_0)}{2}, \exists \delta'$,

$$f(x,y) > \frac{\varepsilon}{2} > 0, \forall (x,y) \in B\left(x_0, \delta'\right) \cap D.$$

取 $B\left(x_0, \delta'\right) \cap D$ 区域内的一个有界闭区域 D', 则必然有 $\exists f_{\min,(x,y)\in D'}(x,y) = m > 0$.
根据积分中值定理, 有

$$\iint_{D'} f(x,y)\mathrm{d}x\mathrm{d}y \geq m\sigma\left(D'\right) > 0, \iint_{D\setminus D'} f(x,y)\mathrm{d}x\mathrm{d}y \geq 0,$$

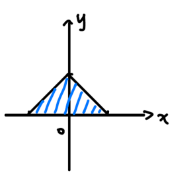
($\partial D'$ 为零面积集).
综上可得 $\iint_D f(x,y)\mathrm{d}x\mathrm{d}y > 0$, 与题矛盾, 则 $\forall (x,y) \in D, f(x,y) = 0$.

3.3

5. 在直角坐标系中画出下列积分的积分区域, 并交换积分次序.

(1) $\int_{-1}^0 \mathrm{d}x \int_0^{1+x} f(x,y)\mathrm{d}y + \int_0^1 \mathrm{d}x \int_0^{1-x} f(x,y)\mathrm{d}y$;

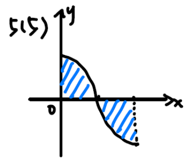
解:



$$
\begin{aligned}
&\int_{-1}^{0} \mathrm{d}x \int_{0}^{1+x} f(x,y)\mathrm{d}y + \int_{0}^{1} \mathrm{d}x \int_{0}^{1-x} f(x,y)\mathrm{d}y \\
=&\int_{0}^{1+x} dy \int_{-1}^{0} f(x,y)dx + \int_{0}^{1-x} dy \int_{0}^{1} f(x,y)dx \\
=&\int_{0}^{1} dy \int_{y-1}^{1-y} f(x,y)dx
\end{aligned}
$$

(5) $\int_0^\pi \mathrm{d}x \int_0^{\cos x} f(x,y)\mathrm{d}y$.

解:

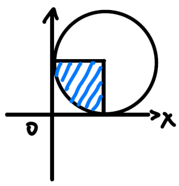


$$
\begin{aligned}
&\int_{0}^{\pi} dx \int_{0}^{\cos x} f(x,y)dy \\
=&\int_{0}^{1} dy \int_{0}^{\arccos y} f(x,y)dx - \int_{-1}^{0} dy \int_{\arccos y}^{\pi} f(x,y)dx.
\end{aligned}
$$

6. 计算下列二重积分.

(2) $\iint_D \frac{1}{\sqrt{2a-x}}\mathrm{d}x\mathrm{d}y, D=\{(x,y)\mid (x-a)^2+(y-a)^2\le a^2, 0\le x,y\le a\}$;

解:



$$
\begin{aligned}
&\iint_D \frac{1}{\sqrt{2a-x}}\mathrm{d}x\mathrm{d}y \\
=&\int_{0}^{a} dx \int_{a-\sqrt{x(2a-x)}}^{a+\sqrt{x(2a-x)}} \frac{1}{2\sqrt{2a-x}}dy \\
=&\int_{0}^{a} \frac{1}{\sqrt{2a-x}} \cdot \sqrt{x(2a-x)}dx \\
=&\int_{0}^{a} \sqrt{x}dx = \frac{2}{3}x^{\frac{1}{2}}\Big|_0^a = \frac{2}{3}a^{\frac{3}{2}}
\end{aligned}
$$

(7) $\iint_D \cos(x+y)\mathrm{d}x\mathrm{d}y, D=\{(x,y)\mid 0\le x,y\le \pi\}$;

解:

$$
\begin{aligned}
&\iint_D \cos(x+y)\mathrm{d}x\mathrm{d}y \\
=&\int_0^{\pi} dx \int_0^{\pi} \cos(x+y)dy = \int_0^{\pi} dx \cdot \sin(x+y)\Big|_0^{\pi} \\
=&\int_0^{\pi} [\sin(x+\pi)-\sin x)dx = 2\cos x|_0^{\pi} = -4
\end{aligned}
$$

(9) $\int_D y^2\mathrm{d}x\mathrm{d}y$, D 由 $\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t), \end{cases}$ $0\le t\le 2\pi$ 以及 x 轴围成;

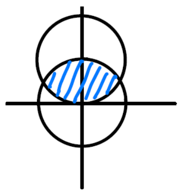
解:

$$
\begin{aligned}
&\int_D y^2\mathrm{d}x\mathrm{d}y \\
=&\int_0^{2\pi a}\mathrm{d}x\int_0^{a(1-\cos t)} y^2\mathrm{d}y = \int_0^{2\pi a}\frac{a^3}{3}(1-\cos t)^3\mathrm{d}x = \int_0^{2\pi}\frac{a^4}{3}(1-\cos t)^4\mathrm{d}t \\
=&\frac{a^4}{3}\left(\frac{\cos^3 x\sin x}{4} - 8\sin x + \frac{4}{3}\sin^3 x + \frac{27}{8}\cos x\sin x + \frac{35}{8}x\right)\Big|_0^{2\pi} \\
=&\frac{35}{12}\pi a^4.
\end{aligned}
$$

11. 画出下列积分区域的图形, 并将二重积分 $\iint_D f(x,y)\mathrm{d}x\mathrm{d}y$ 化为极坐标下的累次积分.

(1) $D=\{(x,y)\mid x^2+y^2\le 1, x^2+(y-1)^2\le 1\}$;

解:



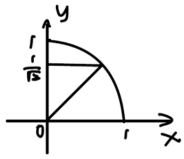
$$
\begin{aligned}
&\iint_D f(x,y)\mathrm{d}x\mathrm{d}y \\
=&\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}\mathrm{d}\theta\int_0^1 f(r\cos\theta, r\sin\theta)r\mathrm{d}r + \\
&\int_0^{\frac{\pi}{6}}\mathrm{d}\theta\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)r\mathrm{d}r + \int_{\frac{5\pi}{6}}^{\pi}\mathrm{d}\theta\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)r\mathrm{d}r
\end{aligned}
$$

注意: 利用极坐标计算时, 容易漏算左右两则弓形部分.

12. 计算下列二重积分.

(6) $\int_0^{\frac{1}{\sqrt{2}}}\mathrm{d}y\int_0^y \mathrm{e}^{-x^2-y^2}\mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^1\mathrm{d}y\int_0^{\sqrt{1-y^2}}\mathrm{e}^{-x^2-y^2}\mathrm{d}x$;

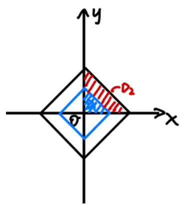
解:



$$
\begin{aligned}
&\int_0^{\frac{1}{\sqrt{2}}}\mathrm{d}y\int_0^{y}\mathrm{e}^{-x^2-y^2}\mathrm{d}x+\int_{\frac{1}{\sqrt{2}}}^{1}\mathrm{d}y\int_0^{\sqrt{1-y^2}}\mathrm{e}^{-x^2-y^2}\mathrm{d}x\\
=&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\rho\sin\theta}\rho\mathrm{e}^{-\rho^2}\mathrm{d}\rho+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_{\rho\sin\theta}^{1}\rho\mathrm{e}^{-\rho^2}\mathrm{d}\rho\\
=&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{1}\rho\mathrm{e}^{-\rho^2}\mathrm{d}\rho\\
=&\frac{\pi}{8}-\frac{\pi}{8e}.
\end{aligned}
$$

(7) $\iint_D f(x,y)\mathrm{d}x\mathrm{d}y, f(x,y)=\begin{cases}1, & |x|+|y|\leq 1,\\ 2, & 1<|x|+|y|\leq 2,\end{cases}\ |x|+|y|\leq 2\}$

解：



$$
\begin{aligned}
&\iint_D f(x,y)\mathrm{d}x\mathrm{d}y\\
=&4\left(\int_0^{\frac{\pi}{2}}\ \mathrm{d}\theta\int_0^{\frac{1}{\sin\theta+\cos\theta}}\rho\mathrm{d}\rho+\int_0^{\frac{\pi}{2}}\ \mathrm{d}\theta\int_{\frac{1}{\sin\theta+\cos\theta}}^{\frac{2}{\sin\theta+\cos\theta}}2\rho\mathrm{d}\rho\right)\\
=&4\left(\left.\frac{-\cos 2x}{4\sin 2x+4}\right|_0^{\frac{\pi}{2}}+\left.\frac{-3\cos 2x}{2\sin 2x+2}\right|_0^{\frac{\pi}{2}}\right)\\
=&4\cdot\frac{7}{2}=14.
\end{aligned}
$$