§ 5.6

1.

(2) $$\int_0^{\pi}\sqrt{1-\sin 2x}\,dx = \int_0^{\pi}\sqrt{1-\cos(\frac{\pi}{2}-2x)}\,dx \qquad (令\ 2t=\frac{\pi}{2}-2x)$$

$$= \int_{\frac{\pi}{4}}^{-\frac{3\pi}{4}}\sqrt{1-\cos 2t}\,d(-t)$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\sqrt{2\sin^2 t}\,dt$$

$$= \sqrt{2}\left(\int_0^{\frac{\pi}{4}}\sin t\,dt - \int_{-\frac{3\pi}{4}}^{0}\sin t\,dt\right)$$

$$= 2\sqrt{2}$$

(11) $$\int_0^1 \frac{1}{(x+1)\sqrt{1+x^2}}dx = \int_0^{\frac{\pi}{4}}\frac{1}{(1+\tan\theta)\sqrt{1+\tan^2\theta}}\frac{1}{\cos^2\theta}d\theta \qquad (令\ x=\tan\theta)$$

$$= \int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos\theta+\sin\theta}$$

$$= \int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{2}\sin(\theta+\frac{\pi}{4})}$$

$$\overset{(i)}{=} \int_0^{\frac{\pi}{4}}\frac{d(\ln\tan\frac{\theta+\frac{\pi}{4}}{2})}{\sqrt{2}} = \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})$$ 直接用 $\frac{d(\ln\tan\frac{\theta}{2})}{d\theta}=\frac{1}{\sin\theta}$

$$\overset{(ii)}{=} \frac{1}{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dt}{\sin t} = \frac{1}{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sin t\,dt}{\sin^2 t} = \frac{1}{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d\cos t}{\cos^2 t-1}$$

$$= -\frac{\sqrt{2}}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{1+\cos t}+\frac{1}{1-\cos t}\right)d\cos t = -\frac{\sqrt{2}}{4}\left(\ln(1+\cos t)-\ln(1-\cos t)\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\sqrt{2}}{4}\ln\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2}\ln(1+\sqrt{2})$$

(14) $\int_0^{\ln 2} \sqrt{1+e^x}\, dx = \int_1^2 \frac{\sqrt{1+u}}{u}\, du$ （令 $u=e^x$）

$= \int_{\sqrt2}^{\sqrt3} \frac{2v^2}{v^2-1}\, dv$ （令 $v=\sqrt{1+u}$）

$= \int_{\sqrt2}^{\sqrt3} \left(2+\frac{1}{v-1}-\frac{1}{v+1}\right) dv$

$= 2(\sqrt3-\sqrt2) + \ln\frac{\sqrt3-1}{\sqrt2-1} - \ln\frac{\sqrt3+1}{\sqrt2+1}$

2.

(2) $\int_0^1 x^2 e^{-2x}\, dx = -\frac12 \int_0^1 x^2\, de^{-2x}$

$= -\frac12 (x^2 e^{-2x})\Big|_0^1 + \int_0^1 e^{-2x}\, x\, dx$

$= -\frac12 e^{-2} - \frac12 \int_0^1 x\, de^{-2x}$

$= -\frac12 e^{-2} - \frac12 (x e^{-2x})\Big|_0^1 + \frac12 \int_0^1 e^{-2x}\, dx$

$= -\frac12 e^{-2} - \frac12 e^{-2} - \frac14 (e^{-2}-1)$

$= \frac14 - \frac54 e^{-2}$

(4) $\int_0^{\sqrt3} x \arctan x\, dx = \frac12 \int_0^{\sqrt3} \arctan x\, dx^2$

$= \frac12 x^2 \arctan x \Big|_0^{\sqrt3} - \frac12 \int_0^{\sqrt3} \frac{x^2}{1+x^2}\, dx$

$= \frac{\pi}{2} - \frac12 \int_0^{\sqrt3} dx + \frac12 \int_0^{\sqrt3} d \arctan x$

$= \frac{2\pi}{3} - \frac{\sqrt3}{2}$

(8) $\int_0^{\frac{\pi}{2}} e^{2x} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} e^{2x} \cdot \frac{1-\cos 2x}{2} \, dx$

$= \int_0^{\pi} e^{x} \cdot \frac{1-\cos x}{4} \, dx$

$= \frac{1}{4}(e^{\pi}-1) - \frac{1}{4}\int_0^{\pi} e^{x} \cos x \, dx$

注意到 $\int_0^{\pi} e^{x} \cos x \, dx = \int_0^{\pi} \cos x \, de^{x}$

$= -e^{\pi} - 1 + \int_0^{\pi} e^{x} \sin x \, dx$

$= -e^{\pi} - 1 + \int_0^{\pi} \sin x \, de^{x}$

$= -e^{\pi} - 1 - \int_0^{\pi} e^{x} \cos x \, dx$

故 $\int_0^{\pi} e^{x} \cos x \, dx = -\frac{1}{2}(e^{\pi}+1)$，从而 原式 $= \frac{3}{8}e^{\pi} - \frac{1}{8}$

3.

(1) $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 2x + \cos^2 2x\right) dx$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$

$= \frac{3}{16}\pi$

(4) $\int_0^{\pi} \cos^7 x \, dx = \int_0^{\pi} \sin^7\left(\frac{\pi}{2}-x\right) dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$

$= 0$ （因 $\sin^7 x$ 是奇函数）

(8) $\int_{-a}^{a} (1-x)\sqrt{a^2-x^2} \, dx = \int_{-a}^{a} \sqrt{a^2-x^2} \, dx$ （因 $x\sqrt{a^2-x}$ 是奇函数）

$= \frac{\pi}{2} a^2$ （由积分的几何含义）

5.

证明：$\int_0^{\pi} x f(\sin x) \, dx = \int_0^{\frac{\pi}{2}} x f(\sin x) \, dx + \int_{\frac{\pi}{2}}^{\pi} x f(\sin x) \, dx$

$= \int_0^{\frac{\pi}{2}} x f(\sin x) \, dx + \int_0^{\frac{\pi}{2}} (\pi - x) f(\sin x) \, dx$

$= \pi \int_0^{\frac{\pi}{2}} f(\sin x) \, dx$

$= \frac{\pi}{2} \int_0^{\pi} f(\sin x) \, dx$ ∎

8. $\int_0^1 x f(x) \, dx = \int_0^1 f(x) \, d\frac{x^2}{2}$

$= \frac{x^2}{2} f(x) \Big|_0^1 - \int_0^1 \frac{x^2}{2} \cdot 2x \cdot e^{-x^4} dx$

$= -\frac{1}{4} \int_0^1 e^{-x^4} \, dx^4$

$= -\frac{1}{4} (1 - e^{-1})$

11. $\int_{-a}^{a} f(x)\,dx = \frac{1}{2}\left(\int_{-a}^{a} f(x)\,dx + \int_{-a}^{a} f(x)\,dx\right)$

$$= \frac{1}{2}\left(\int_{-a}^{a} f(x)\,dx + \int_{-a}^{a} f(-x)\,dx\right)$$

用上述结论，有：

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+e^{-x}}\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\sin^2 x}{1+e^{-x}} + \frac{\sin^2 x}{1+e^{x}}\right)dx$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{\pi}{4}$$

13.

(1) 证明：记 $F(x) = x\int_0^x f(t)\,dt$，有 $F(0) = F(1) = 0$，且 $F(x)$在$(0,1)$可导。

因 $F(0) = F(1) = 0$，由 Rolle定理，$\exists\, \xi \in (0,1)$，s.t

$F'(\xi) = \xi f(\xi) + \int_0^{\xi} f(t)\,dt = 0$，即 $\int_0^{\xi} f(t)\,dt = -\xi f(\xi)$ ▨

(2) 证明：记 $G(x) = \int_0^x f(t)\,dt + x f(x)$，有 $G(0) = G(1) = 0$，且

$G(x)$在$(0,1)$可导。由 Rolle定理，$\exists\, \eta \in (0,1)$，s.t

$G'(\eta) = 2f(\eta) + \eta f(\eta) = 0$ ▨