Review

- 隐函数求导、参数函数求导

- Thm. 设 $f(x)$ 与 $g(x)$ 在点 x 处有 n 阶导数, $c \in \mathbb{R}$, 则

$(1)(f+g)^{(n)}(x) = f^{(n)}(x) + g^{(n)}(x);$

$(2)(cf)^{(n)}(x) = c \cdot f^{(n)}(x);$

$(3)(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x).$(Leibniz公式)

§1. 微分中值定理

- (Fermat Thm) x_0是f的极值点, $f'(x_0)$存在, 则$f'(x_0) = 0.$

- $f, g \in C[a,b], f, g$在(a,b)可导, 则

  (Rolle)若$f(a) = f(b),$则$\exists \xi \in (a,b), s.t. f'(\xi) = 0.$

  (Lagrange)$\exists \xi \in (a,b), s.t. \ f'(\xi) = \dfrac{f(b) - f(a)}{b - a}.$

  (Cauchy)若$g'(t) \neq 0,$则$\exists \xi \in (a,b), s.t. \dfrac{f'(\xi)}{g'(\xi)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}.$

- (Darboux Thm)f在$[a,b]$上可导, $f'_+(a) \neq f'_-(b),$则对介于 $f'_+(a)$与$f'_-(b)$之间的任意实数$\lambda, \exists \xi \in (a,b), s.t. f'(\xi) = \lambda.$

Def. f 在 x_0 的邻域中有定义, 若 $\exists \rho > 0, s.t.$

$$f(x) \geq (\leq) f(x_0), \quad \forall x \in U(x_0, \rho),$$

则称 f 在 x_0 取得极小(大)值, 并称 x_0 是 f 的极小(大)值点.

若 $\exists \rho > 0, s.t.$

$$f(x) > (<) f(x_0), \quad \forall x \in U(x_0, \rho),$$

则称 f 在 x_0 取得严格极小(大)值, 并称 x_0 是 f 的严格极小(大)值点.

Question. 极值与最值的区别与联系?

Thm.(Fermat) x_0 是 f 的极值点, $f'(x_0)$ 存在, 则 $f'(x_0) = 0$.

Proof. 不妨设 x_0 是 f 的极小值点, 则 $\exists \rho > 0, s.t.$

$$f(x) \geq f(x_0), \quad \forall x \in U(x_0, \rho).$$

而 $f'(x_0)$ 存在, 由极限的保序性, 有

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \leq 0,$$

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

故 $\quad f'(x_0) = f'_-(x_0) = f'_+(x_0) = 0.$ □

Def. 若 $f'(x_0) = 0$, 则称 x_0 为 f 的驻点.

Thm.(Rolle) $f \in C[a,b]$, f 在 (a,b) 可导. 若 $f(a) = f(b)$, 则存在 $\xi \in (a,b)$, s.t. $f'(\xi) = 0$.

Proof. $f \in C[a,b]$, 则 f 在 $[a,b]$ 上有最大值 M 和最小值 m.

若 $f(a) = f(b) = M = m$, 则 f 在 $[a,b]$ 上恒为常数, 因而 $\forall x \in (a,b)$, 有 $f'(x) = 0$.

若 M 与 m 至少有一个不在端点取到, 不妨设 $f(a) < M$. 则 $\exists \xi \in (a,b)$, s.t. $f(\xi) = M$. 由 Fermat 定理, $f'(\xi) = 0$. □

Question. Rolle 定理的几何意义?

Thm.(Lagrange) $f \in C[a,b], f$ 在 (a,b) 可导, 则 $\exists \xi \in (a,b), s.t.$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Proof. 令 $h(x) = f(x) - \dfrac{f(b) - f(a)}{b - a} (x - a),$

则 $h \in C[a,b], h$ 在 (a,b) 可导, $h(a) = h(b) = f(a).$ 由 Rolle 定理,

$\exists \xi \in (a,b), s.t. \ h'(\xi) = f'(\xi) - \dfrac{f(b) - f(a)}{b - a} = 0. \square$

Question. Lagrange 中值定理的几何意义?

Remark. $f \in C[a,b], f$ 在 (a,b) 可导, 则

(1) $\exists \xi \in (a,b), s.t. \quad f(b) - f(a) = f'(\xi)(b-a).$

(2) $\forall x, x_0 \in [a,b], \exists$ 介于 x 与 x_0 之间的 $\xi, s.t.$

$$f(x) - f(x_0) = f'(\xi)(x - x_0).$$

(3) $\forall x_0, x_0 + \Delta x \in [a,b], \exists \theta \in (0,1), s.t.$

$$f(x_0 + \Delta x) - f(x_0) = f'(x_0 + \theta \Delta x) \cdot \Delta x.$$

Thm.(Cauchy) $f, g \in C[a,b], f, g$ 在 (a,b) 可导, 且 $\forall t \in (a,b),$

有 $g'(t) \neq 0.$ 则存在 $\xi \in (a,b), s.t. \dfrac{f'(\xi)}{g'(\xi)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}.$

Proof. 由Lagrange中值定理, $\exists \eta \in (a,b), s.t.$

$$g(b) - g(a) = g'(\eta)(b-a) \neq 0.$$

令 $h(x) = f(x) - \dfrac{f(b) - f(a)}{g(b) - g(a)} \, (g(x) - g(a)),$ 则 $h \in C[a,b],$

h在(a,b)可导, $h(a) = h(b) = f(a).$ 由Rolle定理,

$$\exists \xi \in (a,b), s.t. \ h'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} \, g'(\xi) = 0. \square$$

Question.　Cauchy中值定理的几何意义？

Remark. Lagrange中值定理是Cauchy中值定理的特殊情形.

Thm. f 在 $[a,b]$ 上可导.

(1)若 $f'_+(a)f'_-(b) < 0$, 则 $\exists \xi \in (a,b), s.t. f'(\xi) = 0.$

(2)若 $f'_+(a) \neq f'_-(b)$, 则对介于 $f'_+(a)$ 与 $f'_-(b)$ 之间的任意实数 $\lambda, \exists \xi \in (a,b), s.t. f'(\xi) = \lambda.$ (Darboux)

Proof.(1)不妨设

$$f'_+(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a} > 0, \quad f'_-(b) = \lim_{x \to b-} \frac{f(x) - f(b)}{x - b} < 0.$$

由极限的保序性, $\exists x_1, x_2 \in (a,b), s.t. f(x_1) > f(a), f(x_2) > f(b).$

于是 f 在 $[a,b]$ 上的最大值在某点 $\xi \in (a,b)$ 处取得. 由 Fermat 定理, $f'(\xi) = 0.$

$(2) f'_+(a) \neq f'_-(b)$, 则对介于$f'_+(a)$与$f'_-(b)$之间的任意λ, 令

$$g(x) = f(x) - \lambda x.$$

则 $g'_+(a) g'_-(b) = (f'_+(a) - \lambda)(f'_-(b) - \lambda) < 0.$

由(1)中结论, $\exists \xi \in (a,b), s.t. g'(\xi) = 0,$ 即$f'(\xi) = \lambda.$ □

Corollary. 导函数没有第一类间断点. （思考题）

Ex. 若$f'(x) \equiv 0, x \in (a,b) \Leftrightarrow f(x)$在$(a,b)$上为常数.

Proof. $\Leftarrow$: 显然.

$\Rightarrow$: $\forall x_1, x_2 \in (a,b),$ 由Lagrange中值定理, $\exists$介于x_1, x_2之间的$\xi,$

s.t. $f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2) = 0.$ 故f在(a,b)上为常数. □

Ex. $\dfrac{x}{1+x} < \ln(1+x) < x \quad (x > -1, x \neq 0).$

Proof. 由Lagrange中值定理,$\forall x > -1, \exists \theta \in (0,1), s.t.$

$$\ln(1+x) = \ln(1+x) - \ln 1 = \dfrac{x}{1+\theta x}.$$

当$x > 0$时,

$$1 < 1+\theta x < 1+x, \quad \dfrac{1}{1+x} < \dfrac{1}{1+\theta x} < 1, \quad \dfrac{x}{1+x} < \dfrac{x}{1+\theta x} < x.$$

当$-1 < x < 0$时,

$$1 > 1+\theta x > 1+x, \quad \dfrac{1}{1+x} > \dfrac{1}{1+\theta x} > 1, \quad \dfrac{x}{1+x} < \dfrac{x}{1+\theta x} < x. \square$$

Remark. $\dfrac{1}{n+1} < \ln(1+\dfrac{1}{n}) < \dfrac{1}{n} \quad (n \in \mathbb{N}).$

Ex. f 在 $[0, +\infty)$ 上可导, $f'(x)$ 单调递减, $f(0) = 0$, 则

$$f(x_1 + x_2) \leq f(x_1) + f(x_2) \quad (x_1, x_2 > 0).$$

Proof. 不妨设 $0 < x_1 \leq x_2$, 由 $f(0) = 0$, 得

$$f(x_1 + x_2) - f(x_1) - f(x_2)$$

$$= f(x_1 + x_2) - f(x_2) - (f(x_1) - f(0))$$

$$= f'(x_2 + \theta_1 x_1) x_1 - f'(\theta_2 x_1) x_1 \quad (0 < \theta_1, \theta_2 < 1)$$

$$\leq 0 \quad (f' \text{单调递减}). \square$$

Ex. $f \in C[a,b]$, f 在 (a,b) 上可导, $f(x) + f'(x) < \varepsilon \ (a < x < b)$ $f(a) < \varepsilon$. 证明: $f(x) < \varepsilon \ (a < x < b)$.

证法一: 用 Lagrange 中值定理. 令 $h(x) = e^x(f(x) - \varepsilon)$, 则 $\forall x \in (a,b), \exists \xi \in (a,x), s.t.$

$$h(x) = h(a) + e^\xi(f(\xi) + f'(\xi) - \varepsilon)(x - a) < h(a) < 0,$$

故 $f(x) < \varepsilon$.□

证法二: 用 Cauchy 中值定理. $\forall x \in (a,b), \exists \xi \in (a,x), s.t.$

$$\frac{e^x f(x) - e^a f(a)}{e^x - e^a} = \frac{e^\xi(f(\xi) + f'(\xi))}{e^\xi} < \varepsilon,$$

故 $e^x f(x) < e^a f(a) + \varepsilon(e^x - e^a) < \varepsilon e^x$, 从而 $f(x) < \varepsilon$.□

Ex. $x^4 + 2x^3 + 6x^2 - 4x - 5 = 0$ 恰有两个不同的实根.

Proof. 令 $f(x) = x^4 + 2x^3 + 6x^2 - 4x - 5$,则

$$\lim_{x \to \pm\infty} f(x) = +\infty.$$

于是 $\exists a < 0 < b, s.t. f(a) > 0, f(b) > 0.$ 而 $f(0) = -5 < 0,$

由介值定理, $f(x) = 0$ 至少有两个相异实根.

假设 $f(x) = 0$ 至少有 3 个相异实根. 由 Rolle 定理, $f'(x)$

至少有 2 个相异实根, $f''(x)$ 至少有 1 个实根. 但

$$f''(x) = 12x^2 + 12x + 12 > 0,$$

矛盾. 故 $f(x) = 0$ 恰有两个相异实根.□

Ex. 求 $\lim\limits_{x \to e} \dfrac{\tan x^x - \tan e^x}{e^{x^x} - e^{e^x}}$.

解: 令 $f(x) = \tan x, g(x) = e^x$, 由 Cauchy 中值定理, 存在 ξ_x

介于 x^x 与 e^x 之间, 使得

$$\frac{\tan x^x - \tan e^x}{e^{x^x} - e^{e^x}} = \frac{f(x^x) - f(e^x)}{g(x^x) - g(e^x)} = \frac{f'(\xi_x)}{g'(\xi_x)} = \frac{\sec^2 \xi_x}{e^{\xi_x}},$$

且 $\lim\limits_{x \to e} \xi_x = e^e$, 因此

$$\lim_{x \to e} \frac{\tan x^x - \tan e^x}{e^{x^x} - e^{e^x}} = \lim_{x \to e} \frac{\sec^2 \xi_x}{e^{\xi_x}} = \frac{\sec^2 e^e}{e^{e^e}}. \ \square$$

Ex. $f, g \in C[a,b], f, g$ 在 (a,b) 内可导, $f(a) = f(b) = 0$. 证明:

$\exists \xi \in (a,b), s.t. \ f'(\xi) + g'(\xi) f(\xi) = 0.$

Proof. 令 $h(x) = f(x) e^{g(x)}$, 则 h 在 (a,b) 内可导, $h(a) = f(b) = 0$.

由 Rolle 定理, $\exists \xi \in (a,b), s.t.$

$$h'(\xi) = (f'(\xi) + g'(\xi) f(\xi)) e^{g(\xi)} = 0.$$

$$f'(\xi) + g'(\xi) f(\xi) = 0. \square$$

Remark. 辅助函数.

Question. 什么辅助函数求导可出现

(1) $f'(x) + g'(x)f(x)$?　　　　(2) $f''(x) + 2f'(x) + f(x)$?

(3) $f''(x) - 2f'(x) + f(x)$?　　(4) $f''(x) - f(x)$?

(5) $f''(x) - f'(x)$?　　　　　　　$\cdots$

(1) $\left(f(x)e^{g(x)} \right)' = (f'(x) + f(x)g'(x))e^{g(x)}$,

(2)(3) $(f(x)e^{\pm x})'' = \left((f'(x) \pm f(x))e^{\pm x} \right)'$

$\qquad\qquad\qquad = (f''(x) \pm 2f'(x) + f(x))e^{\pm x}$,

(4) $\left(e^{\pm x}(f'(x) \mp f(x)) \right)' = e^{\pm x}(f''(x) - f(x))$

(5) $(f'(x) - f(x))' = f''(x) - f'(x)$.

$\qquad (e^{-x}f'(x))' = e^{-x}\left(f''(x) - f'(x) \right)$.

Ex. f 在 $[0,1]$ 上二阶可导, $f(0) = f(1) = 0$, 则 $\exists \xi \in (0,1), s.t.$

$$f''(\xi) = 2f'(\xi)/(1-\xi).$$

分析: $\left(a(x)e^{b(x)}\right)' = \left(a'(x) + a(x)b'(x)\right)e^{b(x)}.$

取 $a(x) = f'(x)$, 则 $b'(x) = -2/(1-x)$, $b(x) = 2\ln(1-x).$

Proof. $f(0) = f(1) = 0$, 则 $\exists \eta \in (0,1), s.t. f'(\eta) = 0.$

令 $h(x) = (1-x)^2 f'(x)$, 则 h 在 $[0,1]$ 上可导, $h(\eta) = h(1) = 0,$

由 Rolle 定理, $\exists \xi \in (\eta,1) \subset (0,1), s.t.$

$$h'(\xi) = (1-\xi)^2 f''(\xi) - 2(1-\xi)f'(\xi) = 0.$$

$$f''(\xi) = 2f'(\xi)/(1-\xi). \square$$

$$\left(\alpha(x)e^{\beta(x)}\right)' = (\alpha'(x)+\alpha(x)\beta'(x))e^{\beta(x)}$$

Ex. f 在$[0,1]$上二阶可导, $f'(0)=0$. 则$\exists\xi\in(0,1), s.t.$

$$f'(\xi)-(\xi-1)^2 f''(\xi)=0.$$

Proof. 令 $g(x)=\begin{cases} e^{1/(x-1)}f'(x), & x\neq 1 \\ 0, & x=1 \end{cases}$, 则 $g\in C[0,1], g$ 在$(0,1)$

上可导, $g(0)=e^{-1}f'(0)=0=g(1),$

$$g'(x)=\left(f''(x)-\frac{1}{(x-1)^2}f'(x)\right)e^{1/(x-1)}, x\in(0,1).$$

由Rolle定理, $\exists\xi\in(0,1), s.t. g'(\xi)=0.$ 从而有

$$f''(\xi)-\frac{1}{(\xi-1)^2}f'(\xi)=0. \ \square$$

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Ex. f, g 在 $[a,b]$ 上二阶可导, $f(a) = f(b) = g(a) = g(b) = 0$,

$g''(x) \neq 0 \, (x \in (a,b))$. 则 $\exists \xi \in (a,b), s.t. \dfrac{f(\xi)}{g(\xi)} = \dfrac{f''(\xi)}{g''(\xi)}$.

Proof. 令 $h(x) = f'(x)g(x) - f(x)g'(x)$. 由 $f(a) = f(b) = g(a)$

$= g(b) = 0$, 得 $h(a) = h(b) = 0$. 由 Rolle 定理, $\exists \xi \in (a,b), s.t.$

$$h'(\xi) = f''(\xi)g(\xi) - f(\xi)g''(\xi) = 0.$$

若 $g(\xi) = 0$, 因 $g(a) = g(b) = 0$, 由 Roll 定理, $\exists \xi_1 \in (a, \xi)$,

$\xi_2 \in (\xi, b), s.t. \, g'(\xi_1) = g'(\xi_2) = 0. \, \exists \xi_3 \in (\xi_1, \xi_2), s.t. \, g''(\xi_3) = 0$,

与 $g''(x) \neq 0 \, (x \in (a,b))$ 矛盾. 故 $g(\xi) \neq 0, \dfrac{f(\xi)}{g(\xi)} = \dfrac{f''(\xi)}{g''(\xi)}$. $\square$

Ex. f, g 在 $[a,b]$ 上可导, $g'(x) \neq 0 (x \in (a,b))$. 则 $\exists \xi \in (a,b), s.t.$

$$\frac{f(a) - f(\xi)}{g(\xi) - g(b)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof. 令 $h(x) = \big(f(a) - f(x)\big)\big(g(x) - g(b)\big)$, 则 $h(a) = h(b) = 0$,

$$h'(x) = [g(x) - g(b)]'[f(a) - f(x)]$$
$$+ [f(a) - f(x)]'[g(x) - g(b)].$$

由 Rolle 定理, $\exists \xi \in (a,b), s.t. h'(\xi) = 0$, 即

$$g'(\xi)[f(a) - f(\xi)] - f'(\xi)[g(\xi) - g(b)] = 0$$

$$\Leftrightarrow \frac{f(a) - f(\xi)}{g(\xi) - g(b)} = \frac{f'(\xi)}{g'(\xi)}. \square$$

Ex. f 在$[0,1]$上可导, $f(0) = 0, f(x) > 0 (0 < x < 1)$, 则

$$\exists \xi \in (0,1), s.t. \ \frac{2f'(\xi)}{f(\xi)} = \frac{f'(1-\xi)}{f(1-\xi)}.$$

Proof. 令$h(x) = f^2(x)f(1-x)$. 由$f(0) = 0$得$h(0) = h(1) = 0$.

由Rolle定理, $\exists \xi \in (0,1), s.t.$

$$h'(\xi) = 2f(\xi)f'(\xi)f(1-\xi) - f^2(\xi)f'(1-\xi) = 0.$$

$$2f'(\xi)f(1-\xi) - f(\xi)f'(1-\xi) = 0.$$

又 $f(x) > 0 (0 < x < 1)$, 故 $\dfrac{2f'(\xi)}{f(\xi)} = \dfrac{f'(1-\xi)}{f(1-\xi)}. \ \square$

Ex. f 在 $[0,1]$ 上二阶可导, $f(x) > 0 (x \in [0,1]), f'(0) = f'(1) = 0,$

则 $\exists \xi \in (0,1), s.t. f(\xi)f''(\xi) - 2(f'(\xi))^2 = 0.$

Proof. 令 $h(x) = f'(x)/f^2(x)$. 由 $f'(0) = f'(1) = 0,$ 得

$$h(0) = h(1) = 0.$$

由 Rolle 定理, $\exists \xi \in (0,1), s.t.$

$$h'(\xi) = \frac{f''(\xi)f^2(\xi) - 2f(\xi)(f'(\xi))^2}{f^4(\xi)} = 0.$$

$$f(\xi)f''(\xi) - 2(f'(\xi))^2 = 0. \square$$

作业：习题4.1
　　No. 2,5,11,12,14