Homework

王俊琪

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4.4

4.4.1

(1)

$$\int_{L+}\frac{x^2\,\mathrm{d}y-y^2\,\mathrm{d}x}{x^{\frac{5}{3}}+y^{\frac{5}{3}}}=\int_{\frac{\pi}{2}}^{0}\frac{3a^3\cos^7 t\sin^2 t+3a^3\sin^7 t\cos^2 t}{a^{\frac{5}{3}}(\sin^5 t+\cos^5 t)}\,\mathrm{d}t=a^{\frac{4}{3}}\int_{\frac{\pi}{2}}^{0}3\cos^2 t\sin^2 t\,\mathrm{d}t$$
$$=3a^{\frac{4}{3}}\int_{\frac{\pi}{2}}^{0}\frac{\sin^2 2t}{4}\,\mathrm{d}t=6a^{\frac{4}{3}}\int_{\frac{\pi}{2}}^{0}\frac{1-\cos 4t}{8}\,\mathrm{d}t=\frac{-3\pi}{16}a^{\frac{4}{3}}$$

(3)

$$\int_{L+}\frac{-y\,\mathrm{d}x+x\,\mathrm{d}y}{x^2+y^2}+b\,\mathrm{d}z=\int_0^{2\pi}\frac{-a\sin t(-a\sin t)+(a\cos t)a\cos xt}{a^2(\sin^2 t+\cos^2 t)}+b^2\,\mathrm{d}t=2\pi(1+b^2)$$

4.4.2

(3)

由对称性

$$\oint_{L+}\frac{\mathrm{d}x+\mathrm{d}y}{|x|+|y|}=0$$

(5)

设

$$x=\cos t, y=z=\frac{1}{\sqrt{2}}\sin t$$

$$\int_{L+}xyz\,\mathrm{d}z=\int_0^{2\pi}\cos t\frac{1}{2}\sin^2 t\frac{1}{\sqrt{2}}\cos t\,\mathrm{d}t=\frac{1}{2\sqrt{2}}\int_0^{2\pi}\sin^2 t\cos^2 t\,\mathrm{d}t=\frac{1}{2\sqrt{2}}\int_0^{2\pi}\frac{1-\cos 4t}{8}\,\mathrm{d}t=\frac{\sqrt{2}\pi}{16}$$

4.4.4

(1)

$$W = -x\,\mathrm{d}x + -y\,\mathrm{d}y$$

设

$$x = a\cos t, y = b\sin t$$

$$W = -\int_0^{\frac{\pi}{2}} a\cos t(-a\sin t) + b\sin t(b\cos t)\,\mathrm{d}t = -(-a^2+b^2)\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2t\,\mathrm{d}t$$

$$= \frac{-1}{2}(-a^2+b^2) = \frac{a^2-b^2}{2}$$

(2)

$$W = -\int_0^{2\pi} a\cos t(-a\sin t) + b\sin t(b\cos t)\,\mathrm{d}t = -(-a^2+b^2)\int_0^{2\pi} \frac{1}{2}\sin 2t\,\mathrm{d}t = 0$$

4.6

4.6.2

(2)

设

$$X = \frac{x+y}{x^2+y^2}, Y = \frac{y-x}{x^2+y^2}$$

则

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{y} = 0$$

且

$$V(x,y) = (X,Y)\text{在原点处不连续}$$

由(2), (3)中的曲线均包括了原点
　则在原点附近做一个小圆

$$x^2+y^2 = \epsilon^2(\epsilon > 0)$$

边界L_1以逆时针为正向

$$\int_{L+} \frac{(x+y)\,\mathrm{d}x + (y-x)\,\mathrm{d}y}{x^2+y^2} - \int_{L_1+} \frac{(x+y)\,\mathrm{d}x + (y-x)\,\mathrm{d}y}{x^2+y^2} = \int_D 0\,\mathrm{d}x\,\mathrm{d}y = 0$$

$$\int_{L+} \frac{(x+y)\,\mathrm{d}x + (y-x)\,\mathrm{d}y}{x^2+y^2} = \int_{L_1+} \frac{(x+y)\,\mathrm{d}x + (y-x)\,\mathrm{d}y}{x^2+y^2} =$$

$$= \int_0^{2\pi} \frac{\epsilon(\sin\theta+\cos\theta)(-\epsilon\sin\theta\,\mathrm{d}\theta) + (\epsilon\sin\theta - \epsilon\cos\theta)(\epsilon\cos\theta\,\mathrm{d}\theta)}{\epsilon^2} = \int_0^{2\pi}(-1)\,\mathrm{d}\theta = -2\pi$$

(3)

与（2）同理，答案相同

(5)

注意到

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{y} = 0$$

且

$$V(x,y) = (X,Y)\text{在原点之外均连续}$$

从而可以选取合适的D,使得$\int_{L+}\frac{(x+y)\mathrm{d}x+(y-x)\mathrm{d}y}{x^2+y^2}$在$D$内积分与路径$L$无关

取

$$C = (-\pi,\pi), D = (\pi,\pi), A = (-\pi,-\pi), B = (\pi,-\pi)$$

则从A到C积分为

$$\int_{-\pi}^{\pi}\frac{y+\pi}{y^2+\pi^2}\,\mathrm{d}y = \frac{1}{2}\ln(\pi^2+y^2) + tan^{-1}\frac{y}{\pi}|_{-\pi}^{\pi} = \frac{\pi}{2}$$

则从C到D积分为

$$\int_{-\pi}^{\pi}\frac{x+\pi}{x^2+\pi^2}\,\mathrm{d}x = \frac{\pi}{2}$$

从D到B积分为

$$\int_{-\pi}^{\pi}\frac{y-\pi}{y^2+\pi^2}\,\mathrm{d}y = \frac{1}{2}\ln(\pi^2+y^2) - tan^{-1}\frac{y}{\pi}|_{\pi}^{-\pi} = \frac{\pi}{2}$$

从而从A到B的积分为

$$\int_{L+}\frac{(x+y)\,\mathrm{d}x + (y-x)\,\mathrm{d}y}{x^2+y^2} = \int_{-\pi}^{\pi}\frac{y+\pi}{y^2+\pi^2}\,\mathrm{d}y + \int_{-\pi}^{\pi}\frac{x+\pi}{x^2+\pi^2}\,\mathrm{d}x + \int_{-\pi}^{\pi}\frac{y-\pi}{y^2+\pi^2}\,\mathrm{d}y = \frac{3\pi}{2}$$

4.6.3

(2)

设

$$X = 2xy + 3x\sin x, Y = x^2 - ye^y$$

则

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{y} = 0$$

且$V = (X,Y)$连续，从而积分

$$\int_{L+}(2xy + 3x\sin x)\,\mathrm{d}x + (x^2 - ye^y)\,\mathrm{d}y$$

与路径L无关

从而取点

$$A=(0,0),B=(0,2a),C=(\pi a,2a)$$

从而从A到B的积分为

$$\int_0^{2a}-ye^y\,\mathrm{d}y=(1-y)e^y|_0^{2a}=(1-2a)e^2a-1$$

B到C的积分为

$$\int_0^{\pi a}4ax+3x\sin x\,\mathrm{d}x=2ax^2+3\sin x-3x\cos x|_0=2\pi a^3+3\sin\pi a-3\pi a\cos\pi a$$

从而从A到C的积分为

$$\int_{L+}(2xy+3x\sin x)\,\mathrm{d}x+(x^2-ye^y)\,\mathrm{d}y=(1-2a)e^2a-1+2\pi a^3+3\sin\pi a-3\pi a\cos\pi a$$

4.6.4

(2)

设

$$x=\rho\cos\theta,y=\rho\sin\theta$$

则

$$\rho^4=a^2\rho^2\cos 2\theta$$

$$\cos\theta>0,\theta\in[-\frac{\pi}{4},\frac{\pi}{4}]\cup[\frac{3\pi}{4},\frac{5\pi}{4}]$$

$$\rho=a\sqrt{\cos 2\theta}$$

从而

$$x=a\cos\theta\sqrt{\cos 2\theta},y=a\sin\theta\sqrt{\cos 2\theta}$$

$$S=\frac{1}{2}\oint_{\partial D}x\,\mathrm{d}y+y\,\mathrm{d}x=$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a\cos\theta\sqrt{\cos 2\theta}(a\cos\theta\sqrt{\cos 2\theta}-a\sin\theta\frac{\sin 2\theta}{\sqrt{\cos 2\theta}})-a\sin\theta\sqrt{\cos 2\theta}(-a\sin\theta\sqrt{\cos 2\theta}-a\cos\theta\frac{\sin 2\theta}{\sqrt{\cos 2\theta}})$$

$$+\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}a\cos\theta\sqrt{\cos 2\theta}(a\cos\theta\sqrt{\cos 2\theta}-a\sin\theta\frac{\sin 2\theta}{\sqrt{\cos 2\theta}})-a\sin\theta\sqrt{\cos 2\theta}(-a\sin\theta\sqrt{\cos 2\theta}-a\cos\theta\frac{\sin 2\theta}{\sqrt{\cos 2\theta}})$$

$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a^2\cos 2\theta\,\mathrm{d}\theta+\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}a^2\cos 2\theta\,\mathrm{d}\theta=a^2$$

4.6.8

(2)

Proof.

$$\oint_{\partial D} v\frac{\partial u}{\partial \mathbf{n}}\,\mathrm{d}l = \oint_{\partial D} v\frac{\partial u}{\partial x}\,\mathrm{d}y - v\frac{\partial u}{\partial y}\,\mathrm{d}x = \iint_D \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_D \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\,\mathrm{d}x\,\mathrm{d}y + \iint_D v\frac{\partial^2 u}{\partial x} + v\frac{\partial^2 u}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_D v\Delta u\,\mathrm{d}x\,\mathrm{d}y + \iint_D \nabla u \nabla v\,\mathrm{d}x\,\mathrm{d}y$$

□

(3)

Proof.

$$LHS = \oint_{\partial D} v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\,\mathrm{d}l = \oint_{\partial D} v(\frac{\partial u}{\partial x}\,\mathrm{d}y - \frac{\partial u}{\partial y}\,\mathrm{d}x) - u(\frac{\partial v}{\partial x}\,\mathrm{d}y - \frac{\partial v}{\partial y}\,\mathrm{d}x)$$

$$= \oint_{\partial D} (v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})\,\mathrm{d}y + (u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y})\,\mathrm{d}x =$$

$$\iint_D \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - u\frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2}$$

$$= \iint_D v\Delta u - u\Delta v\,\mathrm{d}x\,\mathrm{d}y = RHS$$

□

4.6.9

$$\mathrm{d}l = (\mathrm{d}x, \mathrm{d}y)$$

$$\mathbf{n} = (\frac{\mathrm{d}y}{\mathrm{d}l}, -\frac{\mathrm{d}x}{\mathrm{d}l})$$

$$\cos < \mathbf{n}, \mathbf{i} >= \frac{\mathrm{d}y}{\mathrm{d}l}, \cos < \mathbf{n}, \mathbf{j} >= -\frac{\mathrm{d}x}{\mathrm{d}l}$$

从而

$$\oint_L (x\cos < \mathbf{n}, \mathbf{i} > + y\cos < \mathbf{n}, \mathbf{j} >)\,\mathrm{d}l = \oint_L x\,\mathrm{d}y - y\,\mathrm{d}x = 2S_L$$

这里S_L为L所围成的面积

4.6.11

(5)

$$(1-\frac{\sin^2 y}{x^2})\,\mathrm{d}x+\frac{x\sin 2y}{x^2}\,\mathrm{d}y=0$$

由于

$$\frac{\partial(1-\frac{\sin^2 y}{x^2})}{\partial y}=\frac{\partial\frac{x\sin 2y}{x^2}}{\partial x}$$

从而存在$u(x,y)$,使得

$$\mathrm{d}u=(1-\frac{\sin^2 y}{x^2})\,\mathrm{d}x+\frac{\sin 2y}{x}\,\mathrm{d}y$$

$$\frac{\partial u}{\partial x}=1-\frac{\sin^2 y}{x^2}$$

$$u=x+\frac{\sin^2 y}{x}+C(y)$$

$$\frac{\partial u}{\partial y}=\frac{\sin 2y}{x}$$

$$u=\frac{-1}{2}\frac{\cos 2y}{x}+C(x)=\frac{\sin^2 y}{x}+C(x)$$

从而

$$u=x+\frac{\sin^2 y}{x}+C=C_1$$

从而

$$x+\frac{\sin^2 y}{x}=Constant$$