Homework

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4.2

4.2.3

(1)

$$\int_{L+} \mathrm{d}L = \int_0^1 \sqrt{3^2 + (6t)^2 + (6t^2)^2}\,\mathrm{d}t = 3\int_0^1 (1 + 2t^2)\,\mathrm{d}t = 5$$

(2)

$$\int_{L+} \mathrm{d}L = \int_0^{+\inf} \sqrt{e^{-2t}(\sin t + \cos t)^2 + e^{-2t}(\sin t - \cos t)^2 + e^{-2t}}\,\mathrm{d}t = \int_0^{\inf} \sqrt{3}e^{-t}\,\mathrm{d}t = \sqrt{3}$$

4.2.4

$$m = \int_{\sqrt{3}}^{\sqrt{15}} x^2 \sqrt{1 + \frac{1}{x^2}} = \int_{\sqrt{3}}^{\sqrt{15}} x\sqrt{x^2+1} = \frac{(x^2+1)^{\frac{3}{2}}}{3}|_{\sqrt{3}}^{\sqrt{15}} = \frac{56}{3}$$

4.2.5

设

$$x = a\cos t, y = a\sin t$$

$$\int_L a + \frac{x^2}{a}\,\mathrm{d}L = \int_0^{2\pi} a(a + \frac{a^2\cos t}{a})\,\mathrm{d}t = \int_0^{2\pi} a^2(1 + \cos^2 t) = 3\pi a^2$$

4.2.6

$$M = \int_L \mathrm{d}L = a\int_0^{\pi} \sqrt{a^2(t - \sin t)^2 + a^2(1 - \cos t)^2}\,\mathrm{d}t = a\int_0^{\pi} a\sqrt{2 - 2\cos t}\,\mathrm{d}t = 4a^2$$

$$M_x = \int_L a(1-\cos t)\,\mathrm{d}L = \int_0^{\pi} a(1-\cos t)\sqrt{a^2(t - \sin t)^2 + a^2(1 - \cos t)^2}\,\mathrm{d}t = 4a^2\int_0^{\pi} (\sin\frac{t}{2})^{\frac{3}{2}} = \frac{16a^2}{3}$$

$$M_y = \int_L a(t-\sin t)\,\mathrm{d}L = \int_0^{\pi} a(t-\sin t)\sqrt{a^2(t - \sin t)^2 + a^2(1 - \cos t)^2}\,\mathrm{d}t = a^2\int_0^{\pi} 2\sin\frac{t}{2}(t-\sin t)\,\mathrm{d}t$$

$$= a^2\int_0^{\pi} 2t\sin\frac{t}{2} - 4\sin^2\frac{t}{2}\cos\frac{t}{2}\,\mathrm{d}t = 2a^2\int_0^{\frac{\pi}{2}} 4x\sin x - 4\sin^2 x\cos\mathrm{d}x = \frac{16a^2}{3}$$

从而质心为

$$(\frac{M_x}{M}, \frac{M_y}{M}) = (\frac{4}{3}, \frac{4}{3})$$

4.5

4.5.1

(1)

$$\oiint_{S+} \mathrm{d}x\wedge\mathrm{d}y = \iint_{S_1+} \mathrm{d}x\wedge\mathrm{d}y + \iint_{S_2+} \mathrm{d}x\wedge\mathrm{d}y = \iint_{D_{xy}} \mathrm{d}x\,\mathrm{d}y - \iint_{D_{xy}} \mathrm{d}x\,\mathrm{d}y = 0$$

(2)

$$\oiint_{S+} z\,\mathrm{d}x\wedge\mathrm{d}y = \iint_{D_{xy}} R + \sqrt{R^2-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_{xy}} R - \sqrt{R^2-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \iint_{D_{xy}} 2\sqrt{R^2-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = 8\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R \sqrt{R^2-r^2}r\,\mathrm{d}r = 4\pi\int_0^R \sqrt{R^2-r^2}r\,\mathrm{d}r = \frac{4\pi R^3}{3}$$

(3)

$$\oiint_{S+} z^2\,\mathrm{d}x\wedge\mathrm{d}y = \iint_{D_{xy}} (R+\sqrt{R^2-x^2-y^2})^2\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_{xy}} (R-\sqrt{R^2-x^2-y^2})^2\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \iint_{D_{xy}} 4R\sqrt{R^2-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = 16R\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R \sqrt{R^2-r^2}r\,\mathrm{d}r = 8R\pi\int_0^R \sqrt{R^2-r^2}r\,\mathrm{d}r = \frac{8\pi R^4}{3}$$

4.5.5

流量为

$$\iint_{S+} xy\,\mathrm{d}y\wedge\mathrm{d}z + yz\,\mathrm{d}z\wedge\mathrm{d}x + zx\,\mathrm{d}x\wedge\mathrm{d}y = -\iint_{S+} xy\,\mathrm{d}y\,\mathrm{d}z + yz\,\mathrm{d}z\,\mathrm{d}x + zx\,\mathrm{d}x\,\mathrm{d}y$$

设

$$x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta, 0 < \theta, \varphi < \frac{\pi}{2}$$

从而有

$$-\iint_{S+} xy\,\mathrm{d}y\,\mathrm{d}z + yz\,\mathrm{d}z\,\mathrm{d}x + zx\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\frac{\pi}{2}} \sin^4\theta\sin\varphi\cos^2\varphi + \sin^3\theta\sin^2\varphi\cos\theta + \sin^2\theta\cos^2\theta\cos\varphi\,\mathrm{d}\varphi =$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}\sin^4\theta + \frac{\pi}{4}\sin^3\theta\cos\theta + \sin^2\theta\cos^2\theta\,\mathrm{d}\theta = \frac{\pi}{16} + \frac{\pi}{16} + \frac{\pi}{16} = \frac{3\pi}{16}$$

4.5.7

设

$$\frac{D(x,y)}{D(u,v)} = u, \frac{D(y,z)}{D(u,v)} = a\sin v, \frac{D(z,x)}{D(u,v)} = -a\cos v$$

从而

$$\iint_{S+} x^2 + y^2 \,\mathrm{d}x\wedge\mathrm{d}y + y^2 \,\mathrm{d}y\wedge\mathrm{d}z + z^2 \,\mathrm{d}z\wedge\mathrm{d}x = \int_0^{2\pi} \mathrm{d}v \int_0^1 (u^3 + u^2\sin^3 v - a^3 v^2 \cos v)\,\mathrm{d}u$$

$$= \int_0^{2\pi} \mathrm{d}v \frac{1}{4} - a^3 v^2 \cos v) = \frac{\pi}{2} - 4a^3\pi$$