第七次作业参考答案

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1 3.3节

13.(1)解：取$x = r\cos\theta, y = r\sin\theta$，计算知第一象限交点为$(a, \frac{\pi}{6})$. 由对称性，我们只算第一象限的面积，整体面积是其四倍。画示意图以及辅助计算知道(r,θ)的取值范围，则

$$S = 4\int_0^{\frac{\pi}{6}}\int_a^{a\sqrt{2\cos(2\theta)}} rdrd\theta = (\sqrt{3} - \frac{\pi}{3})a^2$$

14.(3)解：由对称性，我们只算第一象限的积分，整体积分是其四倍。原积分等于

$$S = 4\int_0^1 dx\int_0^{1-x} x^2 + y^2 dy = \frac{2}{3}$$

(注：也可以令$u = x + y, v = x - y$，原来的积分变成$\frac{1}{4}\int_{-1}^1\int_{-1}^1 u^2 + v^2 dudv$。)

14.(4)解：取$u = x - y^2, v = y$，画示意图以及辅助计算知道(u,v)的取值范围，计算$|\det\frac{\partial(u,v)}{\partial(x,y)}| = 1$，则原积分化为

$$\int_{1/3}^2\int_{-1-v}^{2v-2} ududv = -\frac{175}{54}$$

15.(1)解：取$u = a_1x + b_1y + c_1, v = a_2x + b_2y + c_2$，计算$|\det\frac{\partial(u,v)}{\partial(x,y)}| = |a_1b_2 - a_2b_1|$，因此$dudv = |a_1b_2 - a_2b_1|dxdy$. 原积分化为

$$\iint\limits_{u^2+v^2\leq 1}\frac{1}{|a_1b_2 - a_2b_1|}dudv = \frac{\pi}{|a_1b_2 - a_2b_1|}$$

(注：有同学忘了加绝对值，算出变换矩阵的行列式一定要加绝对值。)

16,(2)证明：令$u = xy, v = \frac{y}{x}$，计算$|\det\frac{\partial(u,v)}{\partial(x,y)}| = |\frac{2y}{x}|$，注意到积分区域内$y/x$大

于0，因此$dudv = \frac{2y}{x}dxdy = 2vdxdy$. 原积分化为

$$\int_1^2 du \int_1^4 \frac{f(u)}{2v}dv = \ln 2 \int_1^2 f(u)du$$

17，解：注意到原积分的区域关于x，y互换也是不变的，因此原积分等于

$$\frac{1}{2}\left(\iint\limits_{x^2+y^2\le R^2} \frac{af(x)+bf(y)}{f(x)+f(y)}dxdy + \iint\limits_{x^2+y^2\le R^2} \frac{af(y)+bf(x)}{f(y)+f(x)}dxdy\right) = \frac{a+b}{2}\pi R^2$$

18，解：令G(x,t)=$\int_{t^2}^{x^2} f(t,s)ds$，则$F(x) = \int_0^x G(x,t)dt$。由于f连续，积分号下可以求导，因此

$$F'(x) = G(x,x) + \int_0^x \frac{\partial}{\partial x}G(x,t)dt = \int_0^x 2xf(t,x^2)dt$$

2 3.4节

5(1),解：画出示意图可以知道积分区域为，$\{(x,y,z), 0 \le z \le xy, 0 \le y \le x \le 1\}$. 原积分等于

$$\iint\limits_{0\le y\le x\le 1} dxdy \int_0^{xy} xy^2z^3dz = \iint\limits_{0\le y\le x\le 1} \frac{x^5y^6}{4}dxdy = \frac{1}{4}\int_0^1 x^5 \int_0^x y^6dy = \frac{1}{364}$$

(4),解：由积分区域的对称性，我们可以得到

$$\iiint\limits_{\Omega} xdxdydz = 0, \iiint\limits_{\Omega} |y|dxdydz = \iiint\limits_{\Omega} |z|dxdydz$$

因此原积分等于

$$\iiint\limits_{\Omega} 2|y|dxdydz$$

由于8个卦限都是对称的，我们只用考虑x,y,z都大于0的区域，从而原积分等于

$$\iiint\limits_{0\le x,y,z\le 1, x+y+z\le 1} 16ydxdydz = \iint\limits_{0\le x,y\le 1, 0\le x+y\le 1} 16y(1-x-y)dxdy = \frac{2}{3}$$

6,解：

$$\int_0^1 dx\int_0^x dy\int_0^y \frac{\cos z}{(1-z)^2}dz = \iiint\limits_{0\le z\le y\le x\le 1}\frac{\cos z}{(1-z)^2}dxdydz$$
$$= \int_0^1 \frac{\cos z}{(1-z)^2}dz\int_z^1 dy\int_y^1 dx$$
$$= \int_0^1 dz\int_z^1 dy\int_y^1 \frac{\cos z}{(1-z)^2}dx = \int_0^1 \frac{\cos z}{2}dz = \frac{\sin 1}{2}$$

7,(2)解：取$x = r\cos\varphi, y = r\sin\varphi\cos\theta, z = r\sin\varphi\sin\theta$. 满足$x \ge 0, y^2+z^2 \le x^2 \le R^2-y^2-z^2$，我们得到$0\le r\le R, 0\le\varphi\le\frac{\pi}{4}, 0\le\theta<2\pi$。因此原积分等于

$$\int_0^{\frac{\pi}{4}}d\varphi\int_0^{2\pi}d\theta\int_0^R r^4\sin\varphi dr = \frac{(2-\sqrt{2})\pi R^5}{5}$$

(注：有同学漏掉了$x\ge 0$的条件，导致结果是答案的两倍。)

(5)解：取$z = r\cos\varphi, x = r\sin\varphi\cos\theta, y = r\sin\varphi\sin\theta$. 满足$x^2+y^2+z^2\le 4, x^2+y^2+(z-2)^2\le 4$，我们得到$0\le r\le 2, 0\le r\le 4\cos\varphi, 0\le\varphi<\frac{\pi}{2}, 0\le\theta<\frac{\pi}{2}$。因此原积分等于

$$\int_0^{\frac{\pi}{3}}d\varphi\int_0^{\pi/2}d\theta\int_0^2 r^5\sin^3\varphi\cos\varphi\sin\theta\cos\theta dr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}d\varphi\int_0^{\pi/2}d\theta\int_0^{4\cos\varphi} r^5\sin^3\varphi\cos\varphi\sin\theta\cos\theta dr$$
$$= \frac{3}{4}+\frac{1024}{3}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^7\phi\sin^3\phi d\phi = \frac{3}{4}+\frac{1024}{3}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^7\phi(1-\cos^2\phi)(-d\cos\phi)$$
$$= \frac{3}{4}+\frac{1024}{3}\int_0^{\frac{1}{2}}x^7-x^9dx = \frac{3}{4}+\frac{2}{15} = \frac{53}{60}$$

(注：有同学没看到$x\ge 0, y\ge 0$。)

8.(1)解：取$x = ar\sin\varphi\cos\theta, y = br\sin\varphi\sin\theta, z = cr\cos\varphi$. $0\le r\le 1, 0\le\varphi<\pi, 0\le\theta<2\pi$。原积分等于

$$\int_0^{\pi}d\varphi\int_0^{2\pi}d\theta\int_0^1 abc\sqrt{1-r^2}r^2\sin\varphi dr = \frac{abc\pi^2}{4}$$

(2)解：先考虑$y>0$，(注：这里有个很常见的错误，在做变换的时候$y>0$，和$y<0$分别是两个情况，容易漏掉。)取$u=\frac{z}{y^2}, v=\frac{z}{x}, w=z$, 由题目条件可以得到$1\le u\le 4, 1\le v\le 2, 0\le w\le 3$. 计算$|\det\frac{\partial(u,v,w)}{\partial(x,y,z)}| = |\frac{2z^2}{y^3x^2}| = \frac{2u^{\frac{3}{2}}v^2}{w^{\frac{3}{2}}}$. $x^2=\frac{w^2}{v^2}$。原积分等于$y>0$情况下的两倍，等于

$$\int_1^4 u^{-\frac{3}{2}}du\int_1^2 v^{-4}\int_0^3 w^{\frac{7}{2}} = \frac{21\sqrt{3}}{4}$$

另外一种方法，同样的，考虑$y > 0$，由对称性，原积分等于

$$2\int_0^3 dz\int_{\frac{\sqrt{z}}{2}}^{\sqrt{z}} dy\int_{\frac{z}{2}}^{z} x^2dx = \frac{21\sqrt{3}}{4}$$

9.(7)取$u = a_{11}x + a_{12}y + a_{13}z, v = a_{21}x + a_{22}y + a_{23}z, w = a_{31}x + a_{32}y + a_{33}z$, $u^2 + v^2 + w^2 \le r^2$, 计算$|\det \frac{\partial(u,v,w)}{\partial(x,y,z)}| = |det(A)|$. 因此原积分等于

$$\iiint\limits_{u^2+v^2+w^2\le r^2} \frac{1}{|det(A)|}dudvdw = \frac{4\pi r^3}{3|det(A)|}$$

11.解：由于f连续，因此$\lim_{r\to 0^+}\sup_{x^2+y^2+z^2\le r}|f(x,y,z) - f(0,0,0)| = 0$.

$$\begin{aligned}
&\lim_{r\to 0^+}\left|\frac{1}{r^3}\iiint\limits_{x^2+y^2+z^2\le r^2} f(x,y,z)dxdydz - \frac{4\pi f(0,0,0)}{3}\right| \\
\le &\lim_{r\to 0^+}\frac{1}{r^3}\iiint\limits_{x^2+y^2+z^2\le r^2} |f(x,y,z) - f(0,0,0)|dxdydz \\
\le &\frac{4\pi}{3}\lim_{r\to 0^+}\sup_{x^2+y^2+z^2\le r}|f(x,y,z) - f(0,0,0)| = 0
\end{aligned}$$

因此极限为$\frac{4\pi f(0,0,0)}{3}$.