5.4－5.5　　微积分作业.

5.4. 不定积分的概念与积分法.

3.(3) ① $a \neq e^{-1}$

$$\int a^x e^x dx = \int a^x de^x$$

$$= a^x e^x - \int e^x da^x = a^x e^x - \ln a \int a^x e^x dx$$

$$\Rightarrow \int a^x e^x dx = \frac{a^x e^x}{1+\ln a} + C.$$

② $a = e^{-1}$

$$\int a^x e^x dx = \int 1\, dx = x + C.$$

(7). $\int \left(\frac{4}{\sqrt{1-x^2}} + \sin x\right) dx = 4 \arcsin x - \cos x + C$

4. (1) $\int \frac{2x+1}{x^2+x+1} dx = \int \frac{d(x^2+x+1)}{x^2+x+1} = \ln(x^2+x+1) + C.$

(11) $\int \frac{1}{e^x + e^{-x}} dx = \int \frac{e^x dx}{e^{2x}+1} = \arctan e^x + C.$

5. (5). Sol 1: $\int \frac{2x+1}{\sqrt{4x-x^2}} dx = \int \frac{(2x-4)+5}{\sqrt{4x-x^2}} dx$

$$= -\int \frac{d(4x-x^2)}{\sqrt{4x-x^2}} + 5\int \frac{dx}{\sqrt{4x-x^2}}$$

$$= -2\sqrt{4x-x^2} + 5\int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= -2\sqrt{4x-x^2} + 5\arcsin\frac{x-2}{2} + C \qquad (i)$$

Sol 2: $4x - x^2 > 0 \Rightarrow x \in (0,4)$

故可令 $\sqrt{x} = 2\sin t$, $\sqrt{4-x} = 2\cos t$, $t \in (0, \frac{\pi}{2})$.

$$dx = 8\sin t\cos t\, dt$$

原式 $= \int \frac{8\sin^2 t + 1}{4\sin t\cos t} \cdot 8\sin t\cos t\, dt$

$$= \int 16\sin^2 t + 2\, dt$$

$$= \int 8(1-\cos 2t) + 2\, dt$$

$$= 10t - 4\sin 2t + C$$

$$= 10\arcsin\frac{\sqrt{x}}{2} - 2\sqrt{4x-x^2} + C \qquad (ii).$$

$$\left(\arcsin\left(\frac{x-2}{2}\right)\right)' = \frac{1}{\sqrt{1-\left(\frac{x-2}{2}\right)^2}} \cdot \frac{1}{2} = \frac{1}{\sqrt{4-(x-2)^2}} = \frac{1}{\sqrt{4x-x^2}}$$

$$2\left(\arcsin\frac{\sqrt{x}}{2}\right)' = \frac{2}{\sqrt{1-\frac{x}{4}}} \cdot \frac{1}{2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{4x-x^2}}$$

故 (i) = (ii).

5. (12) $\int \frac{\sin 2x}{1+\sin^4 x} dx = \int \frac{2\sin x\cos x\, dx}{1+\sin^4 x}$

$$= \int \frac{d\sin^2 x}{1+\sin^4 x} = \arctan(\sin^2 x) + C$$

6(1). $\int \underbrace{\frac{x^2}{\sqrt{a^2+x^2}}}\,dx = \int \frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}\,dx$

$$= \int \sqrt{a^2+x^2}\,dx - a^2 \int \frac{dx}{\sqrt{a^2+x^2}}$$

$$= x\sqrt{a^2+x^2} - \int \underbrace{\frac{x^2}{\sqrt{a^2+x^2}}}\,dx - a^2 \ln\left(x+\sqrt{a^2+x^2}\right).$$

$$\Rightarrow \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} - \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + C.$$

7(3). $\int x^2 \sin 2x\,dx = -\frac{1}{2}\int x^2\,d\cos 2x$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}\int \cos 2x\,dx^2$$

$$= -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}\int x\,d\sin 2x$$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

7(11). $\int \frac{\arcsin e^x}{e^x}\,dx = -\int \arcsin e^x\,de^{-x}$

$$= -\frac{\arcsin e^x}{e^x} + \int e^{-x}\,d\arcsin e^x$$

$$= -\frac{\arcsin e^x}{e^x} + \int e^{-x}\frac{e^x}{\sqrt{1-e^{2x}}}\,dx \qquad (*)$$

(Sol: 1) $= -\frac{\arcsin e^x}{e^x} - \int \frac{e^x}{\sqrt{1-e^{2x}}}\,de^{-x}$

$$= -\frac{\arcsin e^x}{e^x} - \int \frac{1}{\sqrt{e^{-2x}-1}}\,de^{-x}$$

$$= -\frac{\arcsin e^x}{e^x} - \ln\left(e^{-x}+\sqrt{e^{-2x}-1}\right) + C. \qquad (i).$$

(Sol 2) 此外，在(*)后，可以有

$$= -\frac{\arcsin e^x}{e^x} + \int \frac{1}{\sqrt{1-e^{2x}}}\,dx$$

令 $t = e^x$，$t \in (0,1)$

$$\int \frac{1}{\sqrt{1-e^{2x}}}\,dx = \int \frac{1}{t\sqrt{1-t^2}}\,dt$$

令 $t = \sin m$，$m \in (0, \frac{\pi}{2})$

$$\int \frac{1}{t\sqrt{1-t^2}}\,dt = \int \frac{1}{\sin m \cos m}\,d\sin m = \int \frac{dm}{\sin m}$$

$$= \ln\tan\frac{m}{2} = \ln\tan\left(\frac{\arcsin t}{2}\right)$$

$$\Rightarrow \int \frac{\arcsin e^x}{e^x}\,dx = -\frac{\arcsin e^x}{e^x} + \ln\tan\left(\frac{\arcsin e^x}{2}\right) + C$$

(ii).

可以证明 (i), (ii) 式相等，这是因为：($t = e^x$).

$$\tan\left(\frac{\arcsin t}{2}\right) = \frac{\sin\left(\frac{1}{2}\arcsin t\right)}{\cos\left(\frac{1}{2}\arcsin t\right)} \cdot \frac{\cos\left(\frac{1}{2}\arcsin t\right)}{\cos\left(\frac{1}{2}\arcsin t\right)}$$

$$= \frac{\sin(\arcsin t)}{1+\cos(\arcsin t)} = \frac{t}{1+\sqrt{1-t^2}}$$

$$\Rightarrow \ln\tan\left(\frac{\arcsin t}{2}\right) = -\ln\tan^{-1}\left(\frac{\arcsin t}{2}\right)$$

$$= -\ln \frac{1+\sqrt{1-t^2}}{t} = -\ln\left(t^{-1}+\sqrt{t^{-2}-1}\right)$$

$$= -\ln\left(e^{-x}+\sqrt{e^{-2x}+1}\right)$$

故 (i) = (ii). $\square$.

习题 5.5. 有理函数与三角函数的不定积分.

1. (2) $$\int \frac{1}{x(1+x^2)}dx = \int \frac{1}{x}dx - \int \frac{x}{1+x^2}dx$$

$$= \ln|x| - \frac{1}{2}\ln(1+x^2) + C.$$

(6) $$\frac{1}{1+x^3} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$$

$$\int \frac{1}{x+1}dx = \ln|x+1| + C$$

$$\int \frac{x-2}{x^2-x+1}dx = \frac{1}{2}\int \frac{d(x^2-4x)}{x^2-x+1}$$

$$= \frac{1}{2}\left(\int \frac{d(x^2-x+1)}{x^2-x+1} - 3\int \frac{dx}{x^2-x+1}\right)$$

$$= \frac{1}{2}\ln(x^2-x+1) - \frac{3}{2}\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \frac{1}{2}\ln(x^2-x+1) - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\arctan\left[(x-\tfrac{1}{2})/\tfrac{\sqrt{3}}{2}\right] + C$$

$$= \frac{1}{2}\ln(x^2-x+1) - \sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

$$\Rightarrow \int \frac{1}{1+x^3}dx = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

2(4). 令 $t=\tan x$, $\sin 2x = \dfrac{2t}{1+t^2}$, $dt = \dfrac{1}{\cos^2 x}dx$

$$dx = \cos^2 x \, dt = \frac{1+\cos 2x}{2}dt = \frac{1+\frac{1-t^2}{1+t^2}}{2}dt$$

$$= \frac{1}{1+t^2}dt$$

$$\int \frac{1+\tan x}{\sin 2x}dx = \int \frac{1+t}{\left(\frac{2t}{1+t^2}\right)} \cdot \frac{1}{1+t^2}dt = \int \frac{1+t}{2t}dt$$

$$= \frac{1}{2}\ln|t| + \frac{1}{2}t + c = \frac{1}{2}\ln|\tan x| + \frac{1}{2}\tan x + c$$

2(7). 令 $t=\tan x$, $x=\arctan t$, $dx = \dfrac{1}{1+t^2}dt$

$$\int \frac{\sin x}{\sin x+\cos x}dx = \int \frac{\tan x}{\tan x+1} \cdot dx = \int \frac{t}{1+t} \cdot \frac{1}{1+t^2}dt$$

$$= \int \left[\frac{-\frac{1}{2}}{1+t} + \frac{\frac{1}{2}t+\frac{1}{2}}{1+t^2}\right]dt = -\frac{1}{2}\ln|t+1| + \frac{1}{2}\int \frac{t+1}{1+t^2}dt$$

$$= -\frac{1}{2}\ln|t+1| + \frac{1}{4}\ln(t^2+1) + \frac{1}{2}\arctan t$$

$$= -\frac{1}{2}\ln|\tan x+1| - \frac{1}{4}\ln\cos^2 x + \frac{1}{2}x + C. \qquad (i)$$

$$= -\frac{1}{2}\ln|\sin x+\cos x| + \frac{1}{2}x + C \qquad (ii).$$

3(3). $$\int x\sqrt{x+2}\,dx = \int (x+2)\sqrt{x+2}\,dx - 2\int\sqrt{x+2}\,dx$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$$

3(6). $\int x\sqrt{\frac{1+x}{1-x}}\,dx$

Sol 1: 三角换元

令 $x = \sin t$, $dx = \cos t\, dt$

$$原式 = \int x\frac{1+x}{\sqrt{1-x^2}}dx = \int \sin x\cdot\frac{1+\sin t}{\cos t}\cdot\cos t\, dt$$

$$= \int(\sin t + \sin^2 t)dt = -\cos t + \int\frac{1-\cos 2t}{2}dt$$

$$= -\cos t + \frac{1}{2}t - \frac{1}{4}\sin 2t + C$$

$$= -\sqrt{1-x^2} + \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C \qquad (;)$$

Sol 2: 令 $t^2 = \frac{1+x}{1-x} \Rightarrow x = \frac{t^2-1}{1+t^2}$,

$$dx = \frac{2t(1+t^2)-(t^2-1)\cdot 2t}{(1+t^2)^2}dt = \frac{4t}{(1+t^2)^2}dt$$

$$原式 = \int\frac{t^2-1}{1+t^2}\cdot t\cdot\frac{4t}{(1+t^2)^2}dt$$

$$= \int\left[\frac{4}{1+t^2} + \frac{-12}{(1+t^2)^2} + \frac{8}{(1+t^2)^3}\right]dt$$

令 $I_k = \int\frac{1}{(1+t^2)^k}dt$, $k\in\mathbb{N}^*$. 由课本 P155 例 5.4.20.

$$I_1 = \arctan t + C$$

$$I_2 = \frac{1}{2}\left[t(t^2+1)^{-1} + I_1\right] = \frac{1}{2}\cdot\frac{t}{t^2+1} + \frac{1}{2}\arctan t + C$$

$$I_3 = \frac{1}{4}\left[t(t^2+1)^{-2} + 3\cdot I_2\right]$$

$$= \frac{1}{4}\cdot\frac{t}{(t^2+1)^2} + \frac{3}{4}\cdot\left(\frac{1}{2}\frac{t}{t^2+1} + \frac{1}{2}\arctan t\right) + C$$

$$= \frac{t}{4(t^2+1)^2} + \frac{3}{8}\cdot\frac{t}{t^2+1} + \frac{3}{8}\arctan t + C.$$

$\Rightarrow$ 原式 $= 4I_1 - 12I_2 + 8I_3$

$$= 4\arctan t - 6\frac{t}{t^2+1} - 6\arctan t + \frac{2t}{(t^2+1)^2} + 3\cdot\frac{t}{t^2+1}$$
$$+3\arctan t + C$$

$$= \arctan t - 3\frac{t}{t^2+1} + \frac{2t}{(t^2+1)^2} + C$$

$$= \arctan\sqrt{\frac{1+x}{1-x}} - 3\cdot\frac{\sqrt{\frac{1+x}{1-x}}}{\frac{1+x}{1-x}+1} + \frac{2\sqrt{\frac{1+x}{1-x}}}{\left(\frac{1+x}{1-x}+1\right)^2} + C$$

$$= \arctan\sqrt{\frac{1+x}{1-x}} - \frac{3}{2}\sqrt{1-x^2} + \frac{1}{2}\cdot\sqrt{1-x^2}(1-x) + C$$

$$= \arctan\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2}\left(1+\frac{1}{2}x\right) + C \qquad \text{(ii)}$$

由于 $\left(\arctan\sqrt{\frac{1+x}{1-x}}\right)' = \frac{1}{1+\frac{1+x}{1-x}}\cdot\frac{1}{2}\frac{1}{\sqrt{\frac{1+x}{1-x}}}\cdot\frac{(1-x)+(1+x)}{(1-x)^2}$

$$= \frac{1-x}{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1+x}{1-x}}}\cdot\frac{2}{(1-x)^2} = \frac{1}{2\sqrt{1-x^2}} = \left(\frac{1}{2}\arcsin x\right)'$$

故 (i) = (ii).

4(6). $\int \frac{1+\sin x}{1+\cos x} e^x dx = \int \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} e^x dx$

$= \int \left[\left(\tan\frac{x}{2}\right)' + \tan\frac{x}{2}\right] e^x dx = e^x \cdot \tan\frac{x}{2} + C$

4(8). $\int \frac{x\ln x}{(x^2+1)^2} dx = -\frac{1}{2}\int \ln x \, d\,(x^2+1)^{-1}$

$= -\frac{1}{2}\cdot\frac{\ln x}{(x^2+1)} + \frac{1}{2}\int \frac{1}{x^2+1}\cdot\frac{1}{x} dx$

$= -\frac{\ln x}{2(x^2+1)} + \frac{1}{2}\int\left[\frac{-x}{x^2+1} + \frac{1}{x}\right] dx$

$= -\frac{\ln x}{2(x^2+1)} - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\ln x + C.$

4(9) $\int \frac{\arctan x}{x^2(1+x^2)} dx = \int \arctan x \left(\frac{1}{x^2} - \frac{1}{1+x^2}\right) dx$

$= \int \arctan x \, d\left(-\frac{1}{x}\right) - \int \arctan x \, d \arctan x$

$= -\frac{1}{x}\arctan x + \int \frac{1}{x}\cdot\frac{1}{1+x^2} dx - \frac{1}{2}\arctan^2 x$

$= -\frac{1}{x}\arctan x - \frac{1}{2}\arctan^2 x + \frac{1}{2}\int \frac{1}{x^2}\cdot\frac{dx^2}{1+x^2} dx$

$= -\frac{1}{x}\arctan x - \frac{1}{2}\arctan^2 x + \frac{1}{2}\int \frac{1}{x^2}dx^2 - \frac{1}{2}\int\frac{1}{1+x^2}dx^2$

$= -\frac{1}{x}\arctan x - \frac{1}{2}\arctan^2 x + \frac{1}{2}\ln x^2 - \frac{1}{2}\ln(1+x^2)$

$= -\frac{1}{x}\arctan x - \frac{1}{2}\arctan^2 x + \ln|x| - \frac{1}{2}\ln(1+x^2)$