Review

- 隐函数求导

Remark: 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

Remark: m个方程确定m个隐函数, 将某m个变量看成函数, 其它变量相互独立.

- 逆映射的Jacobi矩阵 $\dfrac{\partial(x,y)}{\partial(u,v)}=\left(\dfrac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$

§4. 空间曲面和曲线

曲线$y=f(x)$在x_0可导, 即

$$y-y_0=f'(x_0)(x-x_0)+o(x-x_0),$$

$x\to x_0$时.

则曲线$y=f(x)$在x_0的切线方程为

$$y-y_0=f'(x_0)(x-x_0).$$

以直代曲:

以全微分代替函数值的改变量.

类比曲线的情形,曲面$z=g(x,y)$在点(x_0,y_0)可微, 即

$$z-z_0=g'_x(x_0,y_0)(x-x_0)+g'_y(x_0,y_0)(y-y_0)$$

$$+o(\sqrt{(x-x_0)^2+(y-y_0)^2}),$$

当$(x,y)\to(x_0,y_0)$时.

则曲面$z=g(x,y)$在点(x_0,y_0)的切平面方程为:

$$z-z_0=g'_x(x_0,y_0)(x-x_0)+g'_y(x_0,y_0)(y-y_0).$$

1. 参数方程下空间曲线的切线

空间C^1曲线$L: r=r(t)=\big(x(t),y(t),z(t)\big)$

记 $\Delta x = x(t+\Delta t)-x(t),\quad \Delta y = y(t+\Delta t)-y(t),$

$\Delta z = z(t+\Delta t)-z(t),\quad \Delta r = r(t+\Delta t)-r(t).$

Def. $$r'(t)=\lim_{\Delta t\to 0}\frac{\Delta r}{\Delta t}=\lim_{\Delta t\to 0}\left(\frac{\Delta x}{\Delta t},\frac{\Delta y}{\Delta t},\frac{\Delta z}{\Delta t}\right)$$

$$=\big(x'(t),y'(t),z'(t)\big).$$

Def. 若$r'(t_0)\neq 0$,则称$r(t_0)$为曲线$L: r=r(t)$的正则点.

Question. 正则点的几何意义.（隐函数定理）

Remark1: (几何意义) T= $\dfrac{r'(t)}{\|r'(t)\|}$ 为曲线L在点$r(t)$处的单位切向量.

Remark2: L在$r(t_0)$处的切线方程为

$$(x(\tau), y(\tau), z(\tau))^{\mathrm{T}} = r(t_0) + \tau \cdot r'(t_0).$$

Remark3: (物理意义)设质点的位移为$r(t)$,则速度为$r'(t)$,加速度为$r''(t)$.

Remark4: $r'(t)$既反映了$r(t)$在长度上的变化,又反映了$r(t)$在方向上的变化.

2. 参数方程下曲面的切平面

设曲面S的参数方程为$r=r(u,v)$,即

$$S:\begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \end{cases}$$

Def. $r(u,v)$连续可微,称$r_0=r(u_0,v_0)=\left(x_0,y_0,z_0\right)$

$$=\left(x(u_0,v_0),y(u_0,v_0),z(u_0,v_0)\right)$$

为曲面S的正则点,若$\operatorname{rank}\dfrac{\partial(x,y,z)}{\partial(u,v)}=2.$

Question. 正则点的几何意义.(隐函数定理)

Question. 求曲面S在正则点r_0处的

- 切平面Π
- 法向量$\vec{n}$(即S在r_0处切平面Π的法向量)
- 法线(即S在r_0处切平面Π的法线)

若Π为S在r_0的切平面,则任取S上一条过r_0的光滑曲线ℓ, ℓ与Π在r_0必相切,即ℓ在r_0的切线落在平面Π上.考虑S上两条特殊的光滑(?)曲线

$$\ell_1 : r = r(u, v_0), \ell_2 : r = r(u_0, v).$$

$$\ell_1:\begin{cases}x=x(u,v_0)\\y=y(u,v_0)\\z=z(u,v_0)\end{cases}\quad \ell_2:\begin{cases}x=x(u_0,v)\\y=y(u_0,v)\\z=z(u_0,v)\end{cases}$$

ℓ_1在r_0的切向量为$r_u'(u_0,v_0)=\left(x_u',y_u',z_u'\right)\big|_{(u_0,v_0)}$,

ℓ_2在r_0的切向量为$r_v'(u_0,v_0)=\left(x_v',y_v',z_v'\right)\big|_{(u_0,v_0)}$.

$\operatorname{rank}\dfrac{\partial(x,y,z)}{\partial(u,v)}=2$, $r_u'(u_0,v_0)$与$r_v'(u_0,v_0)$不平行, 则

Π过r_0, 由$r_u'(u_0,v_0)$与$r_v'(u_0,v_0)$张成, Π的方程为

$$\Pi: \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} = s\begin{pmatrix} x'_u \\ y'_u \\ z'_u \end{pmatrix}\Bigg|_{(u_0,v_0)} + t\begin{pmatrix} x'_v \\ y'_v \\ z'_v \end{pmatrix}\Bigg|_{(u_0,v_0)},$$

即 $$\begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} = \begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \\ z'_u & z'_v \end{pmatrix}\Bigg|_{(u_0,v_0)} \begin{pmatrix} s \\ t \end{pmatrix}.$$

即 $$r-r_0=\left.\frac{\partial(x,y,z)}{\partial(u,v)}\right|_{(u_0,v_0)}\begin{pmatrix}s\\t\end{pmatrix}.$$

故

•切平面Π: $\begin{cases}x-x_0=x'_u(u_0,v_0)t+x'_v(u_0,v_0)s\\y-y_0=y'_u(u_0,v_0)t+y'_v(u_0,v_0)s\\z-z_0=z'_u(u_0,v_0)t+z'_v(u_0,v_0)s\end{cases}$

•S在r_0的法向量为 $\vec{n}=\left.(r'_u\times r'_v)\right|_{(u_0,v_0)}$,

•S在r_0的法线为: $r(t)-r_0=t\vec{n}.$

Remark. $S: z = f(x, y)$可以看成以x, y为参数的曲面

$$x = x, y = y, z = f(x, y).$$

于是在点(x_0, y_0, z_0)处

$$r_x' = (1, 0, f_x'(x_0, y_0))^{\mathrm{T}}, r_y' = (0, 1, f_y'(x_0, y_0))^{\mathrm{T}}.$$

•切平面为 $\Pi: \begin{cases} x = x_0 + t \\ y = y_0 + s \\ z = z_0 + t f_x'(x_0, y_0) + s f_y'(x_0, y_0) \end{cases},$

即$z = z_0 + f_x'(x_0, y_0)(x - x_0) + f_y'(x_0, y_0)(y - y_0).$

•法向量

$$n = (1, 0, f_x'(x_0, y_0))^{\mathrm{T}} \times (0, 1, f_y'(x_0, y_0))^{\mathrm{T}}$$

$$= \det \begin{pmatrix} i & j & k \\ 1 & 0 & f_x'(x_0, y_0) \\ 0 & 1 & f_y'(x_0, y_0) \end{pmatrix}$$

$$= \begin{pmatrix} -f_x'(x_0, y_0) \\ -f_y'(x_0, y_0) \\ 1 \end{pmatrix}$$

例:求球面$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases}$ $\begin{pmatrix} 0 \le \varphi \le \pi \\ 0 \le \theta < 2\pi \end{pmatrix}$在$\varphi=\pi/6$,

$\theta=\pi/3$的切平面和法向量.

解: $r'_\theta = (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0)$

$r'_\varphi = (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi)$.

当$\varphi=\pi/6, \theta=\pi/3$时,

$$(x, y, z) = (a/4, \sqrt{3}a/4, \sqrt{3}a/2),$$

$$r'_\varphi = (\sqrt{3}a/4, 3a/4, -a/2),$$

$$r'_\theta = (-\sqrt{3}a/4, a/4, 0).$$

$$\vec{n} // \left(r'_\varphi \times r'_\theta\right) = \det\begin{pmatrix} i & j & k \\ \sqrt{3}a/4 & 3a/4 & -a/2 \\ -\sqrt{3}a/4 & a/4 & 0 \end{pmatrix}$$

$$\vec{n} // \left(1/8, \sqrt{3}/8, \sqrt{3}/4\right).$$

切平面方程为

$$(x - a/4, y - \sqrt{3}a/4, z - \sqrt{3}a/2) \cdot \vec{n} = 0,$$

即

$$x + \sqrt{3}y + 2\sqrt{3}z - 4a = 0 \ . \square$$

3. 一般方程下曲面的切平面

设$S:F(x,y,z)=0$, $F(x_0,y_0,z_0)=0$.求曲面S在点$r_0=(x_0,y_0,z_0)$处的法线和切平面.

$S:z=f(x,y)$是$F(x,y,z)=0$确定的隐函数,则

$$f'_x(x_0,y_0)=-\left.\frac{F'_x}{F'_z}\right|_{r_0},\ f'_y(x_0,y_0)=-\left.\frac{F'_y}{F'_z}\right|_{r_0}.$$

•S在r_0的法向量为

$$\vec{n}=(-f'_x(x_0,y_0),-f'_y(x_0,y_0),1)^{\mathrm{T}}=\left.\left(\frac{F'_x}{F'_z},\frac{F'_y}{F'_z},1\right)^{\mathrm{T}}\right|_{r_0}$$

即$\vec{n}\,//\,\mathrm{grad}F(x_0,y_0,z_0)$.

•S在r_0的切平面方程为

$$(r-r_0)\cdot \mathrm{grad}F(r_0)=0$$

即

$$(x-x_0)F'_x(r_0)+(y-y_0)F'_y(r_0)+(z-z_0)F'_z(r_0)=0.$$

•S在r_0的法线方程为 $r=r_0+t\cdot \mathrm{grad}F(r_0)$

$$即\begin{cases} x=x_0+F'_x(r_0)t \\ y=y_0+F'_y(r_0)t \\ x=z_0+F'_z(r_0)t. \end{cases}$$

例:球面S_1:$x^2+y^2+z^2=R^2$与锥面S_2:$x^2+y^2=a^2z^2$

正交(即交点处的法向量相互垂直).

证明:记$F(x,y,z)=x^2+y^2+z^2-R^2$,

$$G(x,y,z)=x^2+y^2-a^2z^2.$$

交点(x,y,z)处S_1与S_2的法向量分别为

$$\mathrm{grad}F(x,y,z)=(2x,2y,2z)$$

$$\mathrm{grad}G(x,y,z)=(2x,2y,-2a^2z).$$

而 $\mathrm{grad}F(x,y,z)\cdot\mathrm{grad}G(x,y,z)=4(x^2+y^2-a^2z^2)$

$=0$,故S_1与S_2正交. □

例:设f可微.求证曲面$S: f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right)=0$上任意一点处的切平面通过一定点.

证明:记$F(x,y,z) \triangleq f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right)$.则曲面$S$在点$(x_0, y_0, z_0)$的法向量为

$$\vec{n} = \mathrm{grad} F(x_0, y_0, z_0)$$

$$= \left(\frac{f_1'}{z_0-c}, \frac{f_2'}{z_0-c}, \frac{a-x_0}{(z_0-c)^2} f_1' + \frac{b-y_0}{(z_0-c)^2} f_2'\right)^{\mathrm{T}}$$

S在点(x_0, y_0, z_0)的切平面方程为

$$(x - x_0)\frac{f_1'}{z_0 - c} + (y - y_0)\frac{f_2'}{z_0 - c}$$

$$+(z - z_0)\frac{a - x_0}{(z_0 - c)^2} f_1' + (z - z_0)\frac{b - y_0}{(z_0 - c)^2} f_2' = 0.$$

可见所有的切平面都过定点(a, b, c). $\square$

例: 求$\lambda > 0$,使以下两曲面相切:

$$S_1 : xyz = \lambda, \qquad S_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

解: 设S_1与S_2在点(x, y, z)相切, 则两曲面在(x, y, z)的切平面的法向量平行,即

$$(yz, xz, xy) // \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right).$$

于是存在$\mu \in \mathbb{R}$, $s.t.$

$$yz = \mu \frac{x}{a^2}, xz = \mu \frac{y}{b^2}, xy = \mu \frac{z}{c^2}.$$

用x, y, z分别乘各等式, 得

$$xyz=\mu\frac{x^2}{a^2}=\mu\frac{y^2}{b^2}=\mu\frac{z^2}{c^2} \qquad (*)$$

于是 $3xyz=\mu\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right).$

点(x,y,z)在两曲面上,因此$3\lambda=\mu.$

注意到$xyz=\lambda$,由$(*)$式得

$$x^2=a^2xyz/\mu=a^2\lambda/\mu=a^2/3,$$

$$y^2=b^2xyz/\mu=b^2\lambda/\mu=b^2/3,$$

$$z^2=c^2xyz/\mu=c^2\lambda/\mu=c^2/3.$$

故$\lambda=\sqrt{x^2y^2z^2}=\sqrt{3}abc/9.$ □

4. 一般方程表示的空间曲线的切线

曲线L是曲面S_1与S_2的交线,L: $\begin{cases} (S_1:) F(x,y,z)=0 \\ (S_2:) G(x,y,z)=0. \end{cases}$

求L在点$r_0=(x_0,y_0,z_0)$处的切线.

L在点r_0处的切线必落在S_1,S_2在点r_0的切平面上.

因而L在r_0的切向量$\vec{v}$与S_1,S_2在点r_0的法向量垂直.

于是, $\vec{v}=\mathrm{grad}F(r_0)\times\mathrm{grad}G(r_0)$,

L在点$r_0=(x_0,y_0,z_0)$处的切线方程为

$$r-r_0=t\left(\mathrm{grad}F(r_0)\times\mathrm{grad}G(r_0)\right).$$

例:求曲线 $\begin{cases} x^2+y^2+z^2-6=0, \\ z-x^2-y^2=0 \end{cases}$ 在点$M_0(1,1,2)$处的切线方程.

解:令$F(x,y,z)=x^2+y^2+z^2-6,$

$$G(x,y,z)=z-x^2-y^2.$$

则$\mathrm{grad}F(1,1,2)=(2,2,4)^{\mathrm{T}},\mathrm{grad}G(1,1,2)=(-2,-2,1)^{\mathrm{T}}.$

曲线在点$M_0(1,1,2)$的切向量为

$$\vec{v}=\mathrm{grad}F(M_0)\times\mathrm{grad}G(M_0)=(10,-10,0)^{\mathrm{T}}.$$

曲线在点M_0的切线方程为 $\begin{cases} x=1+10t, \\ y=1-10t, \\ z=2. \end{cases}$ □

4. 总结

曲面的切平面与法线:

曲面方程	点	法向量
$r = r(u,v)$	$r_0 = r(u_0, v_0)$	$(r_u' \times r_v')\big\|_{(u_0,v_0)}$
$z = f(x,y)$	(x_0, y_0, z_0) $z_0 = f(x_0, y_0)$	$(-f_x', -f_y', 1)^{\mathrm{T}}\big\|_{(x_0,y_0)}$
$F(x,y,z) = 0$	$r_0 = (x_0, y_0, z_0)$	$\mathrm{grad}F(r_0)$

曲线的切向量:

曲线方程	点	切向量
$r = r(t)$	$r_0 = r(t_0)$	$r'(t_0) = \big(x'(t_0), y'(t_0), z'(t_0)\big)$
$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$	$r_0 = (x_0, y_0, z_0)$	$\mathrm{grad}F(r_0)\times \mathrm{grad}G(r_0)$

作业:习题1.7

No.1(5)(6),2,3,5,6