Review

- 向量值函数在一点可微及微分的定义

$$f(x_0+\Delta x)-f(x_0)=\mathrm{A}\Delta x+\alpha,\ \lim_{\Delta x\to 0}\frac{\|\alpha\|_m}{\|\Delta x\|_n}=0$$

- $f=\left(f_1,f_2,\cdots,f_m\right)^{\mathrm{T}}:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^m$在$x_0$可微

$\Leftrightarrow n$元函数$f_i:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}$在$x_0$可微,$i=1,2,\cdots,m.$

•Chain Rule

$u = g(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, y = f(u) : g(\Omega) \subset \mathbb{R}^m \to \mathbb{R}^k,$

$g(x)$在$x_0 \in \Omega$可微, $f(u)$在$u_0 = g(x_0)$可微, 则

$$J(f \circ g)\big|_{x_0} = J(f)\big|_{u_0} \cdot J(g)\big|_{x_0},$$

即 $$\left.\frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0} = \left.\frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(u_1, u_2, \cdots, u_m)}\right|_{u_0} \cdot \left.\frac{\partial(u_1, u_2, \cdots, u_m)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0}.$$

$k = 1$时,

$$\frac{\partial y}{\partial x_i} = \frac{\partial y}{\partial u_1}\frac{\partial u_1}{\partial x_i} + \frac{\partial y}{\partial u_2}\frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial y}{\partial u_m}\frac{\partial u_m}{\partial x_i}, i = 1, 2, \cdots, n.$$

§6. 隐函数定理与反函数定理

曲线$x^2+y^2=1$在$(0,1)$的某个邻域中可表示为

$y=\sqrt{1-x^2}$, 且$y'(x)=\dfrac{-x}{\sqrt{1-x^2}}$; 在$(1,0)$的某个邻域

中可表示为$x=\sqrt{1-y^2}$, 且$x'(y)=\dfrac{-y}{\sqrt{1-y^2}}$.

Question: (1) $f(x,y)=0$何时确定隐函数$y=y(x)$?

(2)如何通过$f(x,y)$的性质研究隐函数$y=y(x)$的性质, 如连续性, 可微性?

(3)如何计算隐函数的(偏)导数和(全)微分?

1. 一个方程确定的隐函数

设$f(x,y)=0$, $f(x_0,y_0)=0$.若存在连续可微的隐函数$y=y(x)$, $y(x_0)=y_0$,满足

$$f(x,y(x))\equiv 0,$$

则上式两边对x求导,有

$$f'_x+f'_y\cdot y'(x)=0.$$

若$f'_y(x_0,y_0)\neq 0$,则在x_0的某个邻域中,

$$y'(x)=-\frac{\partial f(x,y)}{\partial x}\bigg/\frac{\partial f(x,y)}{\partial y}.$$

Thm. 设F在$(x_0, y_0) \in \mathbb{R}^2$的某个邻域$W$中有定义,且

(1)$F(x_0, y_0) = 0,$

(2) $F(x, y) \in C^1(W)$,即F_x', F_y'在W中连续,

(3)$F_y'(x_0, y_0) \neq 0.$

则存在$\delta > 0$以及$I = (x_0 - \delta, x_0 + \delta)$上定义的函数

$y = y(x)$,满足

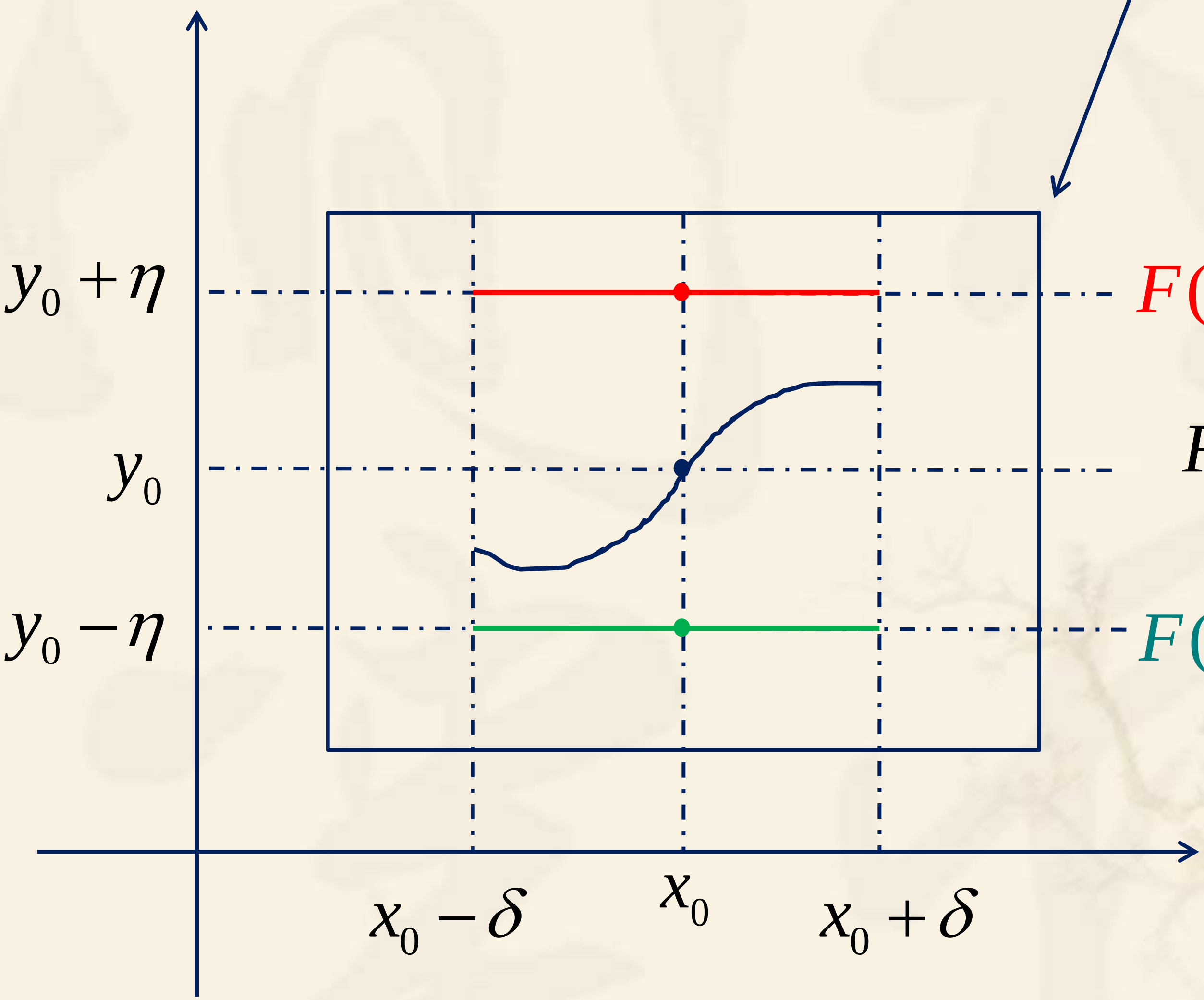
(1)$y(x_0) = y_0$,且$F(x, y(x)) \equiv 0, \forall x \in I,$

(2)$y = y(x) \in C^1(I)$,即$y'(x)$在I上连续,

(3)$$\frac{dy}{dx} = -\frac{\partial F(x,y)}{\partial x} \Bigg/ \frac{\partial F(x,y)}{\partial y}, \forall x \in I.$$

Remark:$F_y'(x_0, y_0) \neq 0$不是隐函数存在的必要条件.

$$F_y'(x_0, y_0) > 0 \Rightarrow F_y'(x, y) > 0, \forall (x, y) \in W_1$$



$$F(x, y_0 + \eta) > 0,$$
$$\forall |x - x_0| < \delta$$
$$\Uparrow$$
$$F(x_0, y_0 + \eta) > 0$$

$$F(x_0, y_0) = 0$$

$$F(x_0, y_0 - \eta) < 0$$
$$\Downarrow$$
$$F(x, y_0 - \eta) < 0,$$
$$\forall |x - x_0| < \delta$$

Proof. (1)先证隐函数的存在性.

因$F_y'(x_0,y_0)\neq 0$,不妨设$F_y'(x_0,y_0)>0.F\in C^1(W)$,则 $\exists a,b>0,s.t.$ $F_y'(x,y)>0,\forall |x-x_0|<a,|y-y_0|<b.$ (*)

$F(x_0,y)$对y连续,由(*)及$F(x_0,y_0)=0$,给定$\eta\in(0,b)$,有

$$F(x_0,y_0-\eta)<0<F(x_0,y_0+\eta).$$

由F的连续性,$\exists\delta\in(0,a),s.t.$

$$F(x,y_0-\eta)<0<F(x,y_0+\eta),\ \forall |x-x_0|<\delta.$$

由(*)知,任意给定$|x-x_0|<\delta,F(x,y)$是y的增函数.结合连续函数的介值定理,$\forall |x-x_0|<\delta,\exists! y=y(x)\in(y_0-\eta,y_0-\eta),s.t.\ F(x,y)=0.$

(2)记(1)中构造的隐函数为$y = f(x)$，下证其连续性.

由(1)中证明知，不论$\eta > 0$取多小，都$\exists \delta > 0$，当$|x - x_0| < \delta$时，必有$|y - y_0| < \eta$. 因此$y = f(x)$在x_0连续.

任给$x_1 \in (x_0 - \delta, x_0 + \delta)$，记$y_1 = f(x_1)$，则$|y_1 - y_0| < \eta$，$F(x_1, y_1) = 0, F_y'(x_1, y_1) > 0$.即$F$在$(x_1, y_1)$与$(x_0, y_0)$满足相同的条件.由前面的证明，$F$在$(x_1, y_1)$的充分小邻域中确定了同一个隐函数$y = f(x)$，且$f$在$x_1$连续.

(3)最后证隐函数$y=y(x)$的可导公式及连续可微性.

任意给定$x\in(x_0-\delta,x_0+\delta)$,由隐函数的连续性, 当$\Delta x\to 0$时,$\Delta y=y(x+\Delta x)-y(x)\to 0.$由隐函数的定义及$F$的连续可微性知,

$$0=F(x+\Delta x,y(x)+\Delta y)-F(x,y(x))$$

$$=F_x'(x,y(x))\Delta x+F_y'(x,y(x))\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y,$$

其中,$\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_i=0,i=1,2.$

而$F_y'(x,y(x))>0$,于是有

$$\lim_{\Delta x\to\infty}\frac{\Delta y}{\Delta x}=-\lim_{\Delta x\to\infty}\frac{F_x'(x,y(x))+\varepsilon_1}{F_y'(x,y(x))+\varepsilon_2}$$

$$=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))},\qquad \forall\left|x-x_0\right|<\delta.$$

即 $$y'(x)=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))},\qquad \forall\left|x-x_0\right|<\delta.$$

由F的连续可微性知, $y'(x)$在$(x_0-\delta,x_0+\delta)$上连续.□

设$f(x_1, x_2, \cdots, x_n, y) = 0$, $f(x_1^0, x_2^0, \cdots, x_n^0, y_0) = 0$.

若存在连续可微的隐函数

$$y = y(x_1, x_2, \cdots, x_n),\ y_0 = y(x_1^0, x_2^0, \cdots, x_n^0),$$

满足 $f\left(x_1, x_2, \cdots, x_n, y(x_1, x_2, \cdots, x_n)\right) \equiv 0$,

则上式两边对x_i求偏导,有 $\dfrac{\partial f}{\partial x_i} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial x_i} = 0$.

若 $\left.\dfrac{\partial f}{\partial y}\right|_{(x_1^0, x_2^0, \cdots, x_n^0, y_0)} \neq 0$,则在$(x_1^0, x_2^0, \cdots, x_n^0)$的某邻域中,

$$y'_{x_i}(x_1, x_2, \cdots, x_n) = -\frac{f'_{x_i}(x_1, x_2, \cdots, x_n, y)}{f'_y(x_1, x_2, \cdots, x_n, y)}.$$

Thm. 设函数$F(x_1,x_2,\cdots,x_n,y)$在点$(x_1^0,x_2^0,\cdots,x_n^0,y_0)$ $\in\mathbb{R}^{n+1}$的某个邻域W中有定义, 且

$$(1)F(x_1^0,x_2^0,\cdots,x_n^0,y_0)=0,$$

$$(2)F(x_1,x_2,\cdots,x_n,y)\in C^1(W),$$

$$(3)\left.\frac{\partial F}{\partial y}\right|_{(x_1^0,x_2^0,\cdots,x_n^0,y_0)}\neq 0.$$

则存在点$(x_1^0,x_2^0,\cdots,x_n^0)\in\mathbb{R}^n$的一个邻域$U$,以及定义在$U$上的$n$元函数$y=y(x_1,x_2,\cdots,x_n)$,满足

(1) $y_0 = y(x_1^0, x_2^0, \cdots, x_n^0)$, 且当 $(x_1, x_2, \cdots, x_n) \in U$ 时,

$$F\left(x_1, x_2, \cdots, x_n, y(x_1, x_2, \cdots, x_n)\right) \equiv 0;$$

(2) $y = y(x_1, x_2, \cdots, x_n) \in C^1(U)$, 即 y'_{x_i} 在 U 中连续,

$i = 1, 2, \ldots \cdots, n$;

(3) $$y'_{x_i}(x_1, x_2, \cdots, x_n) = -\frac{F'_{x_i}(x_1, x_2, \cdots, x_n, y)}{F'_y(x_1, x_2, \cdots, x_n, y)}.$$

Remark: $F'_y(x_1^0, x_2^0, \cdots, x_n^0, y_0) \neq 0$不是隐函数存在的必要条件.

2. 方程组确定的隐函数

设 $\begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0, \end{cases}$ $\begin{cases} F(x_0,y_0,z_0)=0, \\ G(x_0,y_0,z_0)=0, \end{cases}$ 若存在连续可微的隐函数 $y=y(x), z=z(x), y_0=y(x_0), z_0=z(x_0)$,

满足

$$\begin{cases} F\big(x, y(x), z(x)\big)\equiv 0, \\ G\big(x, y(x), z(x)\big)\equiv 0, \end{cases}$$

则

$$\begin{cases} \dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial y}\cdot y'(x)+\dfrac{\partial F}{\partial z}\cdot z'(x)=0, \\ \dfrac{\partial G}{\partial x}+\dfrac{\partial G}{\partial y}\cdot y'(x)+\dfrac{\partial G}{\partial z}\cdot z'(x)=0. \end{cases}$$

即
$$\begin{pmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial G}{\partial x} \end{pmatrix} + \begin{pmatrix} \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z} \end{pmatrix} \begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = 0.$$

若 $\left.\dfrac{\partial(F,G)}{\partial(y,z)}\right|_{(x_0,y_0,z_0)}$ 可逆,则在x_0的某个邻域中,

$$\begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = -\left(\dfrac{\partial(F,G)}{\partial(y,z)}\right)^{-1} \begin{pmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial G}{\partial x} \end{pmatrix}.$$

Thm. 设函数$F(x,y,z)$和$G(x,y,z)$在$(x_0,y_0,z_0)\in\mathbb{R}^3$的某个邻域$W$中有定义, 且满足

(1) $F(x_0,y_0,z_0)=0, G(x_0,y_0,z_0)=0$,

(2) F,G在W中(1阶)连续可微,

(3) $\det\left.\dfrac{\partial(F,G)}{\partial(y,z)}\right|_{(x_0,y_0,z_0)}\neq 0.$

则存在$x_0\in\mathbb{R}$的某个邻域U,以及定义在U上的两个一元函数$y=y(x), z=z(x)$,满足

(1) $y(x_0)=y_0, z(x_0)=z_0$, 且当$x\in U$时,

$$F(x,y(x),z(x))\equiv 0, G(x,y(x),z(x))\equiv 0.$$

(2) $y=y(x), z=z(x)$在U上(1阶)连续可微.

(3) $$\begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = -\left(\frac{\partial(F,G)}{\partial(y,z)}\right)^{-1}\begin{pmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial G}{\partial x} \end{pmatrix}.$$

Remark: $\left.\dfrac{\partial(F,G)}{\partial(y,z)}\right|_{(x_0,y_0,z_0)}$ 可逆不是隐函数存在必要条件.

设

$$F_i(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_m)=0, i=1,2,\cdots,m.$$

$$F_i(x_1^0,x_2^0,\cdots,x_n^0,y_1^0,y_2^0,\cdots,y_m^0)=0, i=1,2,\cdots,m.$$

若存在连续可微的隐函数

$$y_i=y_i(x_1,x_2,\cdots,x_n), i=1,2,\cdots,m.$$

满足

$$y_i(x_1^0,x_2^0,\cdots,x_n^0)=y_i^0, i=1,2,\cdots,m.$$

$$F_i(x_1,x_2,\cdots,x_n,y_1(x_1,x_2,\cdots,x_n),y_2(x_1,x_2,\cdots,x_n),$$

$$\cdots,y_m(x_1,x_2,\cdots,x_n))=0, \quad i=1,2,\cdots,m$$

则 $$\frac{\partial F_i}{\partial x_j}+\sum_{k=1}^{m}\frac{\partial F_i}{\partial y_k}\cdot\frac{\partial y_k}{\partial x_j}=0,\quad \begin{pmatrix} i=1,2,\cdots,m, \\ j=1,2,\cdots,n. \end{pmatrix}$$

即 $$\begin{pmatrix} \frac{\partial F_1}{\partial x_j} \\ \frac{\partial F_2}{\partial x_j} \\ \vdots \\ \frac{\partial F_m}{\partial x_j} \end{pmatrix}+\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \frac{\partial F_m}{\partial y_2} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial x_j} \\ \frac{\partial y_2}{\partial x_j} \\ \vdots \\ \frac{\partial y_m}{\partial x_j} \end{pmatrix}=0$$

$$j=1,2,\cdots,n.$$

若 $\left.\dfrac{\partial(F_1,F_2,\cdots,F_m)}{\partial(y_1,y_2,\cdots,y_m)}\right|_{(x_1^0,x_2^0,\cdots,x_n^0,y_1^0,y_2^0,\cdots,y_m^0)}$ 可逆,

则在 $(x_1^0,x_2^0,\cdots,x_n^0)$ 的某个邻域中,

$$\begin{pmatrix} \dfrac{\partial y_1}{\partial x_j} \\ \vdots \\ \dfrac{\partial y_m}{\partial x_j} \end{pmatrix} = -\left(\dfrac{\partial(F_1,F_2,\cdots,F_m)}{\partial(y_1,y_2,\cdots,y_m)} \right)^{-1} \begin{pmatrix} \dfrac{\partial F_1}{\partial x_j} \\ \vdots \\ \dfrac{\partial F_m}{\partial x_j} \end{pmatrix}$$

$$j=1,2,\cdots,n.$$

Thm. 设有m个$n+m$元函数

$$F_i(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = 0, i = 1, 2, \cdots, m.$$

它们在点$(x_1^0, x_2^0, \cdots, x_n^0, y_1^0, y_2^0, \cdots, y_m^0) \in \mathbb{R}^{n+m}$的某个邻域$W$中有定义, 且满足

(1)$F_i(x_1^0, x_2^0, \cdots, x_n^0, y_1^0, y_2^0, \cdots, y_m^0) = 0, i = 1, 2, \cdots, m.$

(2)$F_1, F_2, \cdots, F_m$在W中(1阶)连续可微,

(3)$\det \left.\dfrac{\partial(F_1, F_2, \cdots, F_m)}{\partial(y_1, y_2, \cdots, y_m)}\right|_{(x_1^0, x_2^0, \cdots, x_n^0, y_1^0, y_2^0, \cdots, y_m^0)} \neq 0.$

则存在$(x_1^0, x_2^0, \cdots, x_n^0) \in \mathbb{R}^n$的某个邻域$U \in \mathbb{R}^n$, 以及定义在$U$上的$m$个$n$元函数$y_i = y_i(x_1, x_2, \cdots, x_n), i = 1,$

$2,\cdots,m,$满足

(1) $y_i(x_1^0,x_2^0,\cdots,x_n^0)=y_i^0, i=1,2,\cdots,m,$且

$$F_i(x_1,x_2,\cdots,x_n,y_1(x_1,x_2,\cdots,x_n),y_2(x_1,x_2,\cdots,x_n),$$

$$\cdots,y_m(x_1,x_2,\cdots,x_n))\equiv 0,$$

$$\forall(x_1,x_2,\cdots,x_n)\in U,\ i=1,2,\cdots,m.$$

(2) $y_i(x_1,x_2,\cdots,x_n)(i=1,2,\cdots,m)$在$U$上$q$阶连续可微,

(3) $$\frac{\partial(y_1,y_2,\cdots y_m)}{\partial(x_1,x_2,\cdots x_n)}$$

$$=-\left(\frac{\partial(F_1,F_2,\cdots,F_m)}{\partial(y_1,y_2,\cdots,y_m)}\right)^{-1}\cdot\frac{\partial(F_1,F_2,\cdots F_m)}{\partial(x_1,x_2,\cdots x_n)}.$$

Remark: $\left.\dfrac{\partial(F_1,F_2,\cdots,F_m)}{\partial(y_1,y_2,\cdots,y_m)}\right|_{(x_1^0,x_2^0,\cdots,x_n^0,y_1^0,y_2^0,\cdots,y_m^0)}$ 可逆不是隐函数存在的必要条件.

Remark: 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

Remark: m个方程确定m个隐函数, 将某m个变量看成函数, 其它变量相互独立.

例. φ可微, $x^2+z^2=y\varphi\left(\dfrac{z}{y}\right)$确定隐函数$z=z(x,y)$.求$z'_x, z'_y$.

解: 视$x^2+z^2=y\varphi(z/y)$中$z=z(x,y)$为隐函数. 两边分别对x, y求偏导, 有

$$2x+2zz'_x=y\varphi'(z/y)\cdot\frac{1}{y}z'_x,$$

$$2zz'_y=\varphi(z/y)+y\varphi'(z/y)\cdot\frac{1}{y^2}(yz'_y-z).$$

求解得

$$z'_x=\frac{2x}{\varphi'(z/y)-2z},\ z'_y=\frac{y\varphi(z/y)-\varphi'(z/y)}{2yz-y\varphi'(z/y)}.\square$$

例. $u = f(x, y, z)$有连续偏导数, 且$z = z(x, y)$由方程 $xe^x - ye^y = ze^z$所确定, 求du.

分析: 在已知条件下 u是(x, y)的二元函数.

解: 方程$xe^x - ye^y = ze^z$两边分别对x, y求偏导, 有

$$e^x + xe^x = \frac{\partial z}{\partial x} e^z + z \frac{\partial z}{\partial x} e^z,$$

$$-e^y - ye^y = \frac{\partial z}{\partial y} e^z + z \frac{\partial z}{\partial y} e^z.$$

求解得 $$\frac{\partial z}{\partial x} = \frac{1+x}{1+z} e^{x-z}, \frac{\partial z}{\partial y} = \frac{-(1+y)}{1+z} e^{y-z}.$$

于是,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= \left(f_x' + f_z' z_x' \right) dx + \left(f_y' + f_z' z_y' \right) dy$$

$$= \left(f_x' + \frac{1+x}{1+z} e^{x-z} f_z' \right) dx + \left(f_y' - \frac{1+y}{1+z} e^{y-z} f_z' \right) dy. \square$$

Remark: $du = f_x' dx + f_y' dy + f_z' dz$

$$= f_x' dx + f_y' dy + f_z' (z_x' dx + z_y' dy)$$

$$= \left(f_x' + f_z' z_x' \right) dx + \left(f_y' + f_z' z_y' \right) dy.$$

例. $u = f\left(x-ut, y-ut, z-ut\right), g(x,y,z)=0$, 求$u'_x, u'_y$.

分析: 五个变量x, y, z, t, u, 两个方程, 确定两个隐函数$z = z(x,y,t) = z(x,y), u = u(x,y,t)$.

解:视$g(x,y,z)=0$中$z = z(x,y)$, 两边对x, y求偏导,有

$$\begin{cases} g'_x + g'_z z'_x = 0, \\ g'_y + g'_z z'_y = 0, \end{cases} \Rightarrow \begin{cases} z'_x = -g'_x / g'_z, \\ z'_y = -g'_y / g'_z. \end{cases}$$

视$u = f\left(x-ut, y-ut, z-ut\right)$中$z = z(x,y)$为隐函数,

两边分别对x, y求偏导, 有

$$u'_x = (1-tu'_x) f'_1 + (-tu'_x) f'_2 + (z'_x - tu'_x) f'_3,$$

$$u'_y = (-tu'_y) f'_1 + (1-tu'_y) f'_2 + (z'_y - tu'_y) f'_3.$$

求解得

$$u'_x = \frac{f_1' + f_3' z_x'}{1 + t(f_1' + f_2' + f_3')} = \frac{f_1' g_z' - f_3' g_x'}{[1 + t(f_1' + f_2' + f_3')] g_z'}$$

$$u'_y = \frac{f_2' + f_3' z_y'}{1 + t(f_1' + f_2' + f_3')} = \frac{f_2' g_z' - f_3' g_y'}{[1 + t(f_1' + f_2' + f_3')] g_z'}. \square$$

3. 逆映射定理

Thm. (逆映射的微分) $f:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^n$可微, $x_0\in\Omega$.若 $J(f)\big|_{x_0}$ 可逆, 则存在$y_0=f(x_0)$的某个邻域U, 使得U上定义了映射$y=f(x)$的逆映射$x=f^{-1}(y)$, $x_0=f^{-1}(y_0)$, 且$x=f^{-1}(y)$在y_0可微,

$$\left.\frac{\partial(x_1,x_2,\cdots,x_n)}{\partial(y_1,y_2,\cdots,y_n)}\right|_{y_0}=\left(\left.\frac{\partial(y_1,y_2,\cdots,y_n)}{\partial(x_1,x_2,\cdots,x_n)}\right|_{x_0}\right)^{-1},$$

即$J(f^{-1})\big|_{y_0}=\left(J(f)\big|_{x_0}\right)^{-1}$.

Proof: 令向量值函数$F(x,y)=f(x)-y$,则

$$F(x_0,y_0)=0, \text{且} \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} = J(f)\big|_{x_0} \text{可逆.}$$

由隐函数定理, 存在$y_0=f(x_0)$的邻域U及U上定义的函数$x=x(y)$, $x(y_0)=x_0$,满足

$$F(x(y),y)\equiv 0,$$

即

$$f\big(x(y)\big)=y,\ x=f^{-1}(y).$$

记$\mathbb{R}^n$中恒等映射为I, 则

$$\mathrm{I}(x)=x=f^{-1}\circ f(x),\quad \forall x\in\mathbb{R}^n.$$

由链式法则, $\mathrm{I}=J(f^{-1})\big|_{y_0}\cdot J(f)\big|_{x_0}$. □

作业： 习题1.6 No. 4,5,7,9.