清華大学 数学作业纸
TSINGHUA UNIVERSITY

班级：数理-化学12 姓名：赵晨 编号：2021013257 科目：微积分A(2) 第 1 页

习题1.6

注意隐函数定理的表达式

4. $\frac{\partial u}{\partial x}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial u}}=\frac{2x\frac{\partial f}{\partial(u^2-x^2)}}{2u\frac{\partial f}{\partial(u^2-x^2)}+2u\frac{\partial f}{\partial(u^2-y^2)}+2u\frac{\partial f}{\partial(u^2-z^2)}}$

故 $\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=\frac{2\left[\frac{\partial f}{\partial(u^2-x^2)}+\frac{\partial f}{\partial(u^2-y^2)}+\frac{\partial f}{\partial(u^2-z^2)}\right]}{2u\left[\frac{\partial f}{\partial(u^2-x^2)}+\frac{\partial f}{\partial(u^2-y^2)}+\frac{\partial f}{\partial(u^2-z^2)}\right]}=\frac{1}{u}$

5. $\begin{cases}x=u+v\\ y=u-v\end{cases}\Rightarrow\begin{cases}u=\frac{1}{2}(x+y)\\ v=\frac{1}{2}(x-y)\end{cases}$ $z=u^2v^2=\frac{1}{16}(x^2-y^2)^2$ 能.

或者使用向量值函数对应的隐函数定理

$\frac{\partial z}{\partial x}=\frac{1}{16}\cdot 2(x^2-y^2)\cdot 2x=\frac{1}{4}x(x^2-y^2)$ $\frac{\partial z}{\partial y}=\frac{1}{4}y(y^2-x^2)$

7. $\begin{cases}x^2+y^2=\frac{1}{2}z^2\\ x+y+z=2\end{cases}\Rightarrow\begin{cases}2x\frac{\partial x}{\partial z}+2y\frac{\partial y}{\partial z}=z\\ \frac{\partial x}{\partial z}+\frac{\partial y}{\partial z}+1=0\end{cases}\Rightarrow\begin{cases}\frac{\partial x}{\partial z}=-\frac{z+2y}{2y-2x}\\ \frac{\partial y}{\partial z}=\frac{z+2x}{2y-2x}\end{cases}$

$\begin{cases}2\left(\frac{\partial x}{\partial z}\right)^2+2x\frac{\partial^2 x}{\partial z^2}+2\left(\frac{\partial y}{\partial z}\right)^2+2y\frac{\partial^2 y}{\partial z^2}=1\\ \frac{\partial^2 x}{\partial z^2}+\frac{\partial^2 y}{\partial z^2}=0\end{cases}\Rightarrow\begin{cases}\frac{(z+2y)^2+(z+2x)^2}{2(y-x)^2}+2x\frac{\partial^2 x}{\partial z^2}+2y\frac{\partial^2 y}{\partial z^2}=1\\ \frac{\partial^2 x}{\partial z^2}+\frac{\partial^2 y}{\partial z^2}=0\end{cases}$

$\Rightarrow\begin{cases}\frac{\partial^2 x}{\partial z^2}=\frac{(z+2y)^2+(z+2x)^2-2(y-x)^2}{4(y-x)^3}\\ \frac{\partial^2 y}{\partial z^2}=-\frac{(z+2y)^2+(z+2x)^2-2(y-x)^2}{4(y-x)^3}\end{cases}$

$\frac{dx}{dz}=0$ $\frac{dy}{dz}=-1$ $\frac{d^2x}{dz^2}=-\frac{1}{4}$ $\frac{d^2y}{dz^2}=\frac{1}{4}$

10. (1) $\begin{cases}u=e^{2x}\cos 2y\\ v=e^{2x}\sin 2y\end{cases}$ $Jg\circ f(x)=\begin{bmatrix}2e^{2x}\cos 2y & -2e^{2x}\sin 2y\\ 2e^{2x}\sin 2y & 2e^{2x}\cos 2y\end{bmatrix}=\begin{bmatrix}2e^2 & 0\\ 0 & 2e^2\end{bmatrix}$

$\begin{vmatrix}2e^2 & 0\\ 0 & 2e^2\end{vmatrix}=4e^4\neq 0$ 能

或者使用多变量复合求导法则

习题1.7

1. (6) $A=\begin{vmatrix}2u & 2v\\ 3u^2 & 3v^2\end{vmatrix}=\begin{vmatrix}2 & 4\\ 3 & 12\end{vmatrix}=12$ $B=\begin{vmatrix}3u^2 & 3v^2\\ 1 & 1\end{vmatrix}=\begin{vmatrix}3 & 12\\ 1 & 1\end{vmatrix}=-9$ $C=\begin{vmatrix}1 & 1\\ 2 & 4\end{vmatrix}=2$

切平面为 $12(x-3)-9(y-5)+2(z-9)=0$ 即 $12x-9y+2z-9=0$

法线为 $\frac{x-3}{12}=\frac{y-5}{-9}=\frac{z-9}{2}$

或者先求u,v方向的切线，再求叉积，注意计算

2. $\frac{\partial F}{\partial x}=\frac{2}{a^2}x_0=\frac{\partial F}{\partial y}=\frac{2}{b^2}y_0=\frac{\partial F}{\partial z}=\frac{2}{c^2}z_0$ 即 $\frac{x_0}{a^2}=\frac{y_0}{b^2}=\frac{z_0}{c^2}$

$P\left(\frac{a^2}{\sqrt{a^2+b^2+c^2}},\frac{b^2}{\sqrt{a^2+b^2+c^2}},\frac{c^2}{\sqrt{a^2+b^2+c^2}}\right)$ 或 $\left(\frac{-a^2}{\sqrt{a^2+b^2+c^2}},\frac{-b^2}{\sqrt{a^2+b^2+c^2}},\frac{-c^2}{\sqrt{a^2+b^2+c^2}}\right)$

利用内积转化“和坐标轴成等角”这一条件

3、$F(x,y,z)=x^2+2y^2+3z^2-21=0$

$$\frac{\partial F}{\partial x}=2x_0 \quad \frac{\partial F}{\partial y}=4y_0 \quad \frac{\partial F}{\partial z}=6z_0$$

切平面为 $2x_0(x-x_0)+4y_0(y-y_0)+6z_0(z-z_0)=0$

注意多解

$$\begin{cases}\frac{2x_0}{1}=\frac{4y_0}{4}=\frac{6z_0}{6}\\ x_0^2+2y_0^2+3z_0^2=21\end{cases} \Rightarrow \begin{cases}x_0=1\\ y_0=2\\ z_0=2\end{cases} 或 \begin{cases}x_0=-1\\ y_0=-2\\ z_0=-2\end{cases}$$

切平面为 $x+4y+6z-21=0$ 或 $x+4y+6z+21=0$

5. $\begin{cases}F(x,y,z)=x^2+y^2+z^2-6=0\\ G(x,y,z)=x+y+z=0\end{cases}$

先求F,G两个隐函数确定的曲面的法向，切线即为两个法向的叉积，注意计算

$$\frac{\partial(F,G)}{\partial(x,y,z)}=\begin{bmatrix}2x & 2y & 2z\\ 1 & 1 & 1\end{bmatrix} \quad \left.\frac{\partial(F,G)}{\partial(x,y,z)}\right|_P=\begin{bmatrix}2 & -4 & 2\\ 1 & 1 & 1\end{bmatrix}$$

$$\left.\frac{D(F,G)}{D(y,z)}\right|_P=\begin{vmatrix}-4 & 2\\ 1 & 1\end{vmatrix}=-6 \quad \left.\frac{D(F,G)}{D(z,x)}\right|_P=\begin{vmatrix}2 & 2\\ 1 & 1\end{vmatrix}=0 \quad \left.\frac{D(F,G)}{D(x,y)}\right|_P=\begin{vmatrix}2 & -4\\ 1 & 1\end{vmatrix}=6$$

切线为 $\frac{x-1}{-1}=\frac{y+2}{0}=\frac{z-1}{1}$

法平面为 $x-z=0$

6. $x'(t_0)=-a\sin t_0 \quad y'(t_0)=a\cos t_0 \quad z'(t_0)=b$

切线的方向向量为 $T=(-a\sin t_0, a\cos t_0, b)$

直接由定义计算即可

记z轴的方向向量为 $\vec{e}=(0,0,1)$

切线与z轴夹角 $\cos\theta=\frac{T\cdot\vec{e}}{|T||\vec{e}|}=\frac{b}{\sqrt{a^2+b^2}}$ 为定值，即夹角恒定

习题 1.8

2.(2) $z=f(x,y)=\frac{\cos x}{\cos y}$ $\quad Jf(x,y)=\left(-\frac{\sin x}{\cos y}, \frac{\cos x\sin y}{\cos^2 y}\right)$ $\quad Jf(0,0)=(0,0)$

$$H(x,y)=\begin{bmatrix}-\frac{\cos x}{\cos y} & -\frac{\sin x\sin y}{\cos^2 y}\\ -\frac{\sin x\sin y}{\cos^2 y} & \frac{\cos x(1+\sin^2 y)}{\cos^3 y}\end{bmatrix} \quad H(0,0)=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$

注意Taylor展开式中二阶项的系数

$$z=f(x,y)=1+0+\frac{1}{2!}(x,y)\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\binom{x}{y}=1+\frac{1}{2}(-x^2+y^2)$$

(3) $z=f(x,y)=e^{-x}\ln(1+y)$ $\quad Jf(x,y)=\left(-e^{-x}\ln(1+y), \frac{e^{-x}}{1+y}\right)$ $\quad Jf(0,0)=(0,1)$

$$H(x,y)=\begin{bmatrix}e^{-x}\ln(1+y) & -\frac{e^{-x}}{1+y}\\ -\frac{e^{-x}}{1+y} & -\frac{e^{-x}}{(1+y)^2}\end{bmatrix} \quad H(0,0)=\begin{bmatrix}0 & -1\\ -1 & -1\end{bmatrix}$$

$$z=f(x,y)=0+(0,1)\binom{x}{y}+\frac{1}{2!}(x,y)\begin{bmatrix}0 & -1\\ -1 & -1\end{bmatrix}\binom{x}{y}=y-xy-\frac{1}{2}y^2$$