Homework

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3.2

3.2.4

Proof. 用反证法，假设对于任意的 $(x,y) \in D$,$f(x,y) = 0$ 不成立，则存在 (x_0,y_0)，满足 $f(x_0,y_0) > 0$。由函数的连续性可知存在 $D_r = \{(x,y)|(x-x_0)^2 + (y-y_0)^2 \le r^2\}$，使得对于 D_r 中的任意 (x,y)，均有 $f(x,y) > 0$。从而有 $\iint_{D_r} f(x,y)\,\mathrm{d}x\,\mathrm{d}y > 0$。又因为

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D_r} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D/D_r} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

而 $\iint_{D/D_r} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \ge 0, \iint_{D_r} f(x,y)\,\mathrm{d}x\,\mathrm{d}y > 0$ 从而 $\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y > 0$, 矛盾！ □

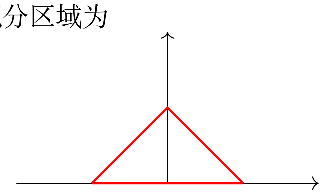
3.3

3.3.5

(1)

$$\int_{-1}^{0} \mathrm{d}x \int_0^{1+x} f(x,y)\,\mathrm{d}y + \int_0^1 \mathrm{d}x \int_0^{1-x} f(x,y)\,\mathrm{d}y$$

积分区域为



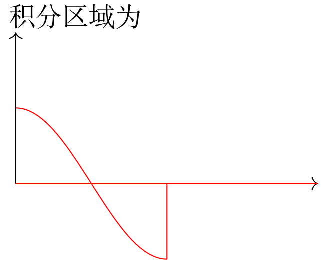
交换积分次序后

$$\int_0^1 \mathrm{d}y \int_{y-1}^{1-y} f(x,y)\,\mathrm{d}x$$

(2)

$$\int_0^{\pi} \mathrm{d}x \int_0^{\cos x} f(x,y)\,\mathrm{d}y$$

积分区域为



交换积分次序后

$$\int_0^{1} \mathrm{d}y \int_0^{\arccos y} f(x,y)\,\mathrm{d}x - \int_{-1}^{0} \mathrm{d}y \int_{\arccos y}^{\pi} f(x,y)\,\mathrm{d}x$$

3.3.6

(2)

$$\iint_D \frac{1}{\sqrt{2a-x}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \frac{\mathrm{d}x}{\sqrt{2a-x}} \int_0^{a-\sqrt{2ax-x^2}} \mathrm{d}y$$

$$= \int_0^a \frac{\mathrm{d}x}{\sqrt{2a-x}}(a-\sqrt{2ax-x^2})\,\mathrm{d}x = \int_0^a \frac{a}{\sqrt{2a-x}} - \sqrt{x}\,\mathrm{d}x$$

$$= (-2a\sqrt{2a-x} - \frac{2}{3}x^{\frac{2}{3}})|_0^a = (-2\sqrt{2} - \frac{8}{3})a^{\frac{2}{3}}$$

(7)

$$\iint_D \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\pi} \mathrm{d}y \int_0^{\pi} \cos(x+y)\,\mathrm{d}y$$

$$= \int_0^{\pi} \sin(x+y)|_0^{\pi}\,\mathrm{d}y = \int_0^{\pi} (-2\sin y)\,\mathrm{d}y = 2\cos y|_0^{\pi} = -4$$

(9)

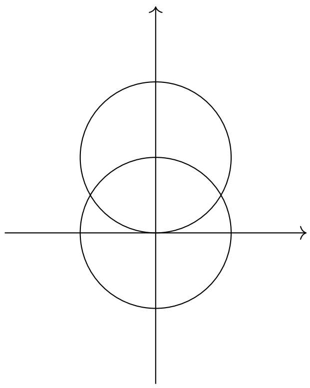
$$\iint_D y^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi a} \mathrm{d}x \int_0^{y(x)} y^2\,\mathrm{d}y = \frac{1}{3}\int_0^{2\pi a} y(x)^3\,\mathrm{d}x$$

$$= \frac{a^4}{3}\int_0^{2\pi} (1-\cos t)^4\,\mathrm{d}t = \frac{16a^4}{3}\int_0^{2\pi} \sin^8\frac{t}{2}\,\mathrm{d}t = \frac{32a^4}{3}\int_0^{\pi} \sin^8 z\,\mathrm{d}z$$

$$= \frac{64a^4}{3}\int_0^{\frac{\pi}{2}} \sin^8 z\,\mathrm{d}z = \frac{64a^4}{3}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{35\pi a^4}{12}$$

3.3.11

(1)



两圆相交区域为积分区域

3.3.12

(6)

注意到

$$\int_0^{\frac{1}{\sqrt{2}}} \mathrm{d}y \int_0^y e^{-x^2-y^2}\,\mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^1 \mathrm{d}y \int_0^{\sqrt{1-y^2}} e^{-x^2-y^2}\,\mathrm{d}x$$

积分区域为一个半径为 1，圆心角为 $\frac{\pi}{4}$ 的扇形

从而可以考虑极坐标

$$\int_0^{\frac{1}{\sqrt{2}}} \mathrm{d}y \int_0^y e^{-x^2-y^2}\,\mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^1 \mathrm{d}y \int_0^{\sqrt{1-y^2}} e^{-x^2-y^2}$$

$$= \frac{1}{8}\int_0^{2\pi} \mathrm{d}\theta \int_0^1 re^{-r^2}\,\mathrm{d}r = \frac{1}{8}\int_0^{2\pi} (-\frac{1}{2})(\frac{1}{e}-1)\,\mathrm{d}\theta = \frac{\pi}{8}(1-\frac{1}{e})$$

(7)

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 4\iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + 4\iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

这里

$$D_1 = \{(x,y)|\ 0 \le x+y \le 1\}, D_2 = \{(x,y)|\ 1 < x+y \le 2\}$$

$$\iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{0\le x+y\le 1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_0^{1-y} 1\,\mathrm{d}x = \int_0^1 (1-y)\,\mathrm{d}y = \frac{1}{2}$$

$$\iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{1<x+y\le 2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_0^{1-y} 2\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_{1-y}^{2-y} 2\,\mathrm{d}x = 3$$

从而 $\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 14$