Homework

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3.5

3.5.1

(3)

令

$$z = \frac{x^2+y^2}{a} = 2a - \sqrt{x^2+y^2}$$

解得 $x^2+y^2=a^2$ 交线在 xOy 面的投影为 $D=\{(x,y)|x^2+y^2\leq a^2\}$

$$S_1 = \iint_D \sqrt{1+((\frac{x^2+y^2}{a})'_x)^2+((\frac{x^2+y^2}{a})'_y)^2}\,\mathrm{d}x\,\mathrm{d}y = \iint_D \sqrt{1+\frac{4x^2}{a^2}+\frac{4y^2}{a^2}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^{2\pi}\mathrm{d}\theta\int_0^a \frac{\sqrt{a^2+4r^2}}{a}r\,\mathrm{d}r(x=r\cos\theta, y=r\sin\theta) = \frac{5\sqrt{5}-1}{6}a^2\pi$$

$$S_2 = \iint_D \sqrt{1+((\frac{x}{\sqrt{x^2+y^2}})'_x)^2+((\frac{y}{\sqrt{x^2+y^2}})'_y)^2}\,\mathrm{d}x\,\mathrm{d}y = \iint_D \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\pi a^2$$

$$S = S_1+S_2 = \frac{5\sqrt{5}-1}{6}\pi a^2+\sqrt{2}\pi a^2 = \frac{5\sqrt{5}+6\sqrt{2}-1}{6}a^2\pi$$

3.5.8

设球面方程 $x^2+y^2+z^2=a^2$

对于上表面的一个点 (x,y,z) 它到水面的距离为 $h-z=h-\sqrt{a^2-x^2-y^2}$
设 $D=\{(x,y)|x^2+y^2\leq a^2\}$ 则

$$F_1 = \iint_D \rho_{水}\sigma g(h-\sqrt{a^2-x^2-y^2})\,\mathrm{d}x\,\mathrm{d}y = \rho_{水}\sigma g(ha^2\pi - \iint_D(\sqrt{a^2-x^2-y^2})\,\mathrm{d}x\,\mathrm{d}y$$
$$= \rho_{水}\sigma g(ha^2\pi - \int_0^{2\pi}\mathrm{d}\theta\int_0^a\sqrt{a^2-r^2}r\,\mathrm{d}r) = \rho_{水}\sigma g(ha^2\pi-\frac{2}{3}\pi a^3)$$

对于下表面的一个点 (x,y,z) 它到水面的距离为 $h+z=h+\sqrt{a^2-x^2-y^2}$ 设 $D=\{(x,y)|x^2+y^2\le a^2\}$ 则

$$F_1=\iint_D \rho_{水}\sigma g(h+\sqrt{a^2-x^2-y^2})\,\mathrm{d}x\,\mathrm{d}y=\rho_{水}\sigma g(ha^2\pi+\iint_D(\sqrt{a^2-x^2-y^2})\,\mathrm{d}x\,\mathrm{d}y$$

$$=\rho_{水}\sigma g(ha^2\pi+\int_0^{2\pi}\mathrm{d}\theta\int_0^a\sqrt{a^2-r^2}r\,\mathrm{d}r)=\rho_{水}\sigma g(ha^2\pi+\frac{2}{3}\pi a^3)$$

4.3

4.3.1

(4)

注意到由于该区域关于 xOz 平面对称，则

$$\iint_S(xy+yz+zx)\,\mathrm{d}S=\iint_S(zx)\,\mathrm{d}S$$

$$\iint_S zx\,\mathrm{d}S=\iint_S zx(\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,\mathrm{d}x\,\mathrm{d}y=\sqrt{2}\iint_{x^2+y^2\le 2ax}x\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$$

设

$$x=r\cos\theta,y=r\sin\theta$$

则

$$0\le r\le 2a\cos\theta$$

$$\sqrt{2}\iint_{x^2+y^2\le 2ax}x\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2a\cos\theta}r^3\cos\theta\,\mathrm{d}r=4\sqrt{2}a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5\theta\,\mathrm{d}\theta=\frac{64\sqrt{2}}{15}a^4$$

4.3.6

由于对称性。$M_x=M_y=M_z$ 假设质量密度为 1

$$M_x=\iint_S x\,\mathrm{d}S=\int_0^a\mathrm{d}y\int_0^{\sqrt{a^2-y^2}}x\frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x=\int_0^a-a\sqrt{a^2-y^2}\,\mathrm{d}y=a^3\int_0^{\frac{\pi}{2}}\sin^2\theta\,\mathrm{d}\theta=\frac{\pi a^3}{4}$$

$$M=\frac{\pi a^3}{2},x=\frac{M_x}{M}=\frac{a}{2}=y=z$$

从而质心在 $(\frac{a}{2},\frac{a}{2},\frac{a}{2})$ 上

对于上半球面，易知 $z=\frac{2M_z}{2M}=\frac{a}{2},y=0,x=0$

从而质心在 $(0,0,\frac{a}{2})$ 上

4.3.10

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$\mathrm{d}S = \sqrt{1 + \frac{c^2\frac{x^2}{a^4}}{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} + \frac{c^2\frac{y^2}{b^4}}{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\,\mathrm{d}x\,\mathrm{d}y = \frac{c\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\,\mathrm{d}x\,\mathrm{d}y$$

$$\text{切平面为}\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = 1$$

$$L(x,y,z) = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

$$\iint_S L(x,y,z)\,\mathrm{d}S = 8\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} \frac{c}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} = 8\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 abc\frac{r}{\sqrt{1-r^2}}\,\mathrm{d}r = 4\pi abc$$