# Review

- $f(x)$以$2l$ 为周期,在$[-l,l]$上可积或广义绝对可积,则

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx,\ n = 0,1,2,\cdots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx,\ n = 1,2,\cdots$$

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- 定义内积$(f,g)\triangleq\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)g(x)dx$,则

$$\{\varphi_n\}=\left\{\sqrt{2}/2,\cos x,\sin x,\cdots,\cos nx,\sin nx,\cdots\right\}$$

为$\Re[-\pi,\pi]$中标准正交函数系.

- $$\left\|f(x)-\left(\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\right)\right\|$$

$$=\min_{\lambda_k,\mu_k\in\mathbb{R}}\left\|f(x)-\left(\frac{\lambda_0}{2}+\sum_{k=1}^{n}(\lambda_k\cos kx+\mu_k\sin kx)\right)\right\|$$

- Bessel不等式

$$f(x) \sim \frac{a_0}{\sqrt{2}} \frac{\sqrt{2}}{2} + \sum_{n=1}^{+\infty} \left( a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{+\infty} \left( a_n^2 + b_n^2 \right) \le \| f \|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)dx$$

- $f \in \Re[-\pi, \pi] \Rightarrow \begin{cases} \lim\limits_{n\to\infty} \int_{-\pi}^{\pi} f(x)\cos nx dx = 0, \\ \lim\limits_{n\to\infty} \int_{-\pi}^{\pi} f(x)\sin nx dx = 0. \end{cases}$

# §2. Fourier级数的收敛性

## 1. Riemann-Lebesgue引理

设$f$在$[a,b]$上可积或广义绝对可积,记为$f\in\Re[a,b]$.

上一节我们证明,若$f\in\Re[-\pi,\pi]$,则Fourier系数$a_n\to 0$, $b_n\to 0$,即

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx dx=0,$$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx dx=0.$$

这一结论可以推广为更一般的情形.

Lemma (Riemann-Lebesgue). $f\in\mathfrak{R}[a,b]$, 则

$$\lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x dx=0,\quad \lim_{\lambda\to\infty}\int_a^b f(x)\sin\lambda x dx=0.$$

Proof. 只证第一式, 第二式同理.

Case1. 设$f$在$[a,b]$上可积, 则$f$在$[a,b]$上有界, 即

$$\exists M>0, s.t. |f(x)|\le M, \forall x\in[a,b].$$

任意给定$\lambda>1$, 令$n=\lfloor\sqrt{\lambda}\rfloor$. $n$等分$[a,b]$:

$$x_i=a+(b-a)i/n,\quad i=0,1,2,\cdots,n.$$

$$\omega_i(f)=\sup\{f(\xi)-f(\eta):\xi,\eta\in[x_{i-1},x_i]\},\quad i=1,2,\cdots,n.$$

$f$在$[a,b]$上可积,则$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\omega_i(f)\Delta x_i=0$. 于是

$$\left|\int_a^b f(x)\cos\lambda x dx\right|=\left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\cos\lambda x dx\right|$$

$$\leq\left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\big(f(x)-f(x_i)\big)\cos\lambda x dx\right|+\left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x_i)\cos\lambda x dx\right|$$

$$\leq\sum_{i=1}^{n}\omega_i(f)\Delta x_i+\sum_{i=1}^{n}|f(x_i)|\left|\int_{x_{i-1}}^{x_i}\cos\lambda x dx\right|$$

$$\leq\sum_{i=1}^{n}\omega_i(f)\Delta x_i+\frac{2Mn}{\lambda}=\sum_{i=1}^{\lfloor\sqrt{\lambda}\rfloor}\omega_i(f)\Delta x_i+\frac{2M\lfloor\sqrt{\lambda}\rfloor}{\lambda}$$

$$\to 0,\text{当}\lambda\to+\infty\text{时}.$$

Case2. $f$在$[a,b]$上广义绝对可积,不妨设$a$为唯一的瑕点.

则$\forall \varepsilon>0, \exists \delta>0$, s.t., $f$在$[a+\delta,b]$上可积, 且

$$\int_a^{a+\delta}|f(x)|dx<\varepsilon/2.$$

从而 $\left|\int_a^{a+\delta} f(x)\cos\lambda x dx\right| \le \int_a^{a+\delta}|f(x)|dx<\varepsilon/2,$

$$\lim_{\lambda\to+\infty}\int_{a+\delta}^b f(x)\cos\lambda x dx=0.$$

于是$\exists \Lambda>0$, 当$\lambda>\Lambda$时, $\left|\int_{a+\delta}^b f(x)\cos\lambda x dx\right|<\varepsilon/2$, 进而有

$$\left|\int_a^b f(x)\cos\lambda x dx\right| \le \left|\int_a^{a+\delta} f(x)\cos\lambda x dx\right| + \left|\int_{a+\delta}^b f(x)\cos\lambda x dx\right|$$

$$<\varepsilon/2+\varepsilon/2=\varepsilon,\quad \forall \lambda>\Lambda. \square$$

例. $\int_0^{+\infty}\frac{\sin t}{t}\mathrm{d}t=\frac{\pi}{2}.$

解: 由广义积分的Dirichlet判别法, $\int_0^{+\infty}\frac{\sin t}{t}\mathrm{d}t$ 收敛. 于是

$$\int_0^{+\infty}\frac{\sin t}{t}\mathrm{d}t=\lim_{\lambda\to+\infty}\int_0^{\lambda\pi}\frac{\sin t}{t}\mathrm{d}t$$

$$=\lim_{\lambda\to+\infty}\int_0^{\pi}\frac{\sin\lambda t}{t}\mathrm{d}t=\lim_{n\to+\infty}\int_0^{\pi}\frac{\sin(n+1/2)t}{t}\mathrm{d}t.$$

恒等式 $\frac{\sin\left(n+1/2\right)t}{2\sin\frac{t}{2}}=\frac{1}{2}+\sum_{k=1}^{n}\cos kt$ 两边在$[0,\pi]$上积分,得

$$\int_0^{\pi}\frac{\sin\left(n+1/2\right)t}{2\sin\frac{t}{2}}dt=\frac{\pi}{2}.$$

令$g(t)=\frac{1}{t}-\frac{1}{2\sin\frac{t}{2}}$，往证$\lim\limits_{n\to+\infty}\int_0^{\pi} g(t)\sin(n+1/2)t\mathrm{d}t=0$.

$$\lim_{t\to 0} g(t)=\lim_{t\to 0}\frac{2\sin\frac{t}{2}-t}{2t\sin\frac{t}{2}}=\lim_{t\to 0}\frac{\sin t-t}{2t\sin t}$$

$$=\lim_{t\to 0}\frac{\frac{1}{3!}t^3+o(t^3)}{2t\left(t-\frac{1}{3!}t^3+o(t^3)\right)}=0.$$

故$t=0$是$g(t)$的可去间断点.由Riemann-Lebesgue引理,

$$\lim_{n\to+\infty}\int_0^{\pi} g(t)\sin(n+1/2)t\mathrm{d}t=0.\square$$

## 2. Fourier级数前n项和的积分表示

$f \in \Re[-\pi,\pi]$,即$f$在$[-\pi,\pi]$上可积或广义绝对可积, 则

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n \cos nx + b_n \sin nx\right),$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx,\ \ k = 0,1,2,\cdots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx,\ \ k = 1,2,\cdots$$

Question. Fourier级数的逐点收敛性?

着手点: $S_n(x) = \dfrac{a_0}{2} + \displaystyle\sum_{k=1}^{n}\left(a_k \cos kx + b_k \sin kx\right).$

设$f$以$2\pi$为周期, 且$f\in\mathfrak{R}[-\pi,\pi]$.

$$S_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k\cos kx+b_k\sin kx\right)$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{2}+\sum_{k=1}^{n}\left(\cos kt\cos kx+\sin kt\sin kx\right)\right]dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos k(t-x)\right]dt$$

$$=\frac{1}{\pi}\int_{-\pi-x}^{\pi-x}f(x+\tau)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos k\tau\right]d\tau \qquad (\text{令}\tau=t-x)$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+\tau)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos k\tau\right]d\tau \quad (2\pi\text{周期性})$$

$$S_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\tau)\frac{\sin\left(n+\frac{1}{2}\right)\tau}{2\sin\frac{\tau}{2}}d\tau$$

$$=\frac{1}{\pi}\left(\int_{-\pi}^{0}+\int_{0}^{\pi}\right) f(x+\tau)\frac{\sin\left(n+\frac{1}{2}\right)\tau}{2\sin\frac{\tau}{2}}d\tau$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\frac{f(x+\tau)+f(x-\tau)}{2\sin\frac{\tau}{2}}\sin\left(n+\frac{1}{2}\right)\tau d\tau$$

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$$\forall\delta>0,\ g(\tau)\triangleq\frac{f(x_0+\tau)+f(x_0-\tau)}{2\sin\frac{\tau}{2}},\ g(\tau)\in\Re[\delta,\pi].$$

$$S_n(x_0)=\frac{1}{\pi}\int_0^{\pi}g(\tau)\sin\left(n+\frac{1}{2}\right)\tau d\tau$$

$$=\frac{1}{\pi}\int_0^{\delta}g(\tau)\sin\left(n+\frac{1}{2}\right)\tau d\tau+\frac{1}{\pi}\int_{\delta}^{\pi}g(\tau)\sin\left(n+\frac{1}{2}\right)\tau d\tau$$

$$\triangleq A_n(x_0)+B_n(x_0).$$

由Riemann-Lebesgue引理, $\lim\limits_{n\to+\infty}B_n(x_0)=0$. 因此, $S_n(x_0)$ 是否收敛仅仅取决于$f$在$x_0$附近的性质.

## 3. Fourier级数逐点收敛的判别法

**Thm (Dini判别法)** $f$以$2\pi$为周期, 且$f \in \Re[-\pi,\pi]$,

$x_0 \in [-\pi,\pi]$, 若$\exists s \in \mathbb{R}, \exists \delta > 0, s.t.$

$$\frac{f(x_0+t)+f(x_0-t)-2s}{t} \in \Re[0,\delta],$$

则$f$的Fourier级数在$x_0$点收敛于$s$.

**Proof.** $S_n(x_0) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \frac{f(x_0+t)+f(x_0-t)}{2\sin\frac{t}{2}} \sin\left(n+\frac{1}{2}\right)t\,dt,$

往证$S_n(x_0) \to s$.

为估计$\left|S_n(x_0)-s\right|$,应将$s$表示成相应的积分形式.恒等式

$$\frac{1}{2}+\sum_{k=1}^{n}\cos kt=\frac{\sin\left(n+1/2\right)t}{2\sin t/2}$$

两边在区间$[0,\pi]$上积分,得

$$\int_0^{\pi}\frac{\sin\left(n+1/2\right)t}{2\sin t/2}dt=\frac{\pi}{2},$$

$$s=\frac{1}{\pi}\int_0^{\pi}\frac{2s}{2\sin t/2}\cdot\sin\left(n+1/2\right)t\,dt.$$

$$g(t) \triangleq \frac{f(x_0+t)+f(x_0-t)-2s}{2\sin t/2},$$ 则

$$S_n(x_0)-s=\frac{1}{\pi}\int_0^{\pi} g(t)\sin\left(n+1/2\right)t\,dt \triangleq I_n(x_0).$$

已知 $h(t) \triangleq \dfrac{f(x_0+t)+f(x_0-t)-2s}{t} \in \Re[0,\delta],\ f \in \Re[0,\pi],$

则 $h(t) \in \Re[0,\pi],$

$$J_n(x_0) \triangleq \frac{1}{\pi}\int_0^{\pi} h(t)\sin\left(n+1/2\right)t\,dt,$$

由Riemann-Lebesgue引理, $\lim\limits_{n\to\infty} J_n(x_0)=0.$

欲证 $\lim\limits_{n\to\infty} S_n(x_0)=s$, 只要证 $\lim\limits_{n\to\infty} I_n(x_0)=\lim\limits_{n\to\infty} J_n(x_0).$

$$I_n(x_0)-J_n(x_0)=\frac{1}{\pi}\int_0^{\pi}\big(g(t)-h(t)\big)\sin\big(n+1/2\big)t\,dt$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{2\sin t/2}-\frac{1}{t}\right)\big(f(x_0+t)+f(x_0-t)-2s\big)\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$w(t)\triangleq\frac{1}{2\sin t/2}-\frac{1}{t}=\frac{t-2\sin t/2}{2t\sin t/2},$$

补充定义$w(0)=0$，则$w\in C[0,\pi]$. 从而

$$w(t)\big(f(x_0+t)+f(x_0-t)-2s\big)\in\Re[0,\pi],$$

由Riemann-Lebesgue引理，$\lim\limits_{n\to\infty}\big(I_n(x_0)-J_n(x_0)\big)=0.\Box$

**Def.** 记$f$在$x_0$的左、右极限为$f(x_0-0), f(x_0+0)$, 若$\exists\delta>0$, $L>0, \alpha>0, s.t.$

$$\begin{rcases} |f(x_0+t)-f(x_0+0)|\le Lt^{\alpha}, \\ |f(x_0-t)-f(x_0-0)|\le Lt^{\alpha}, \end{rcases} \quad \forall t\in(0,\delta),$$

则称$f$在$x_0$附近满足广义$\alpha$阶Lipschitz条件.

**Remark.** 令$s=\dfrac{1}{2}[f(x_0+0)+f(x_0-0)]$,

$$g(t)=\frac{f(x_0+t)+f(x_0-t)-2s}{t}, \quad \forall t\in(0,\delta).$$

则$|g(t)|\le \dfrac{|f(x_0+t)-f(x_0+0)|}{t}+\dfrac{|f(x_0-t)-f(x_0-0)|}{t}$.

$f$在$x_0$附近满足广义$\alpha$阶Lipschitz条件, 则$\exists \delta>0, L>0, s.t.$

$$|g(t)| \le L t^{\alpha-1}, \forall t \in (0,\delta).$$

从而 $$\int_0^{\delta} |g(t)| dt \le \int_0^{\delta} L t^{\alpha-1} dt = \frac{L}{\alpha} t^{\alpha} \Big|_{t=0}^{\delta} < +\infty, \quad (\alpha>0)$$

$g(t) \in \Re[0,\delta]$. 由Dini判别法得以下定理:

**Thm (Lipschitz判别法)** $f$以$2\pi$为周期, 且$f \in \Re[-\pi,\pi]$, $x_0 \in [-\pi,\pi]$, $f$在$x_0$附近满足广义$\alpha$阶Lipschitz条件, 则$f$的Fourier级数在$x_0$收敛于$\frac{1}{2}\left[f(x_0+0)+f(x_0-0)\right]$.

**Corollary.** $f$以$2\pi$为周期,且

(1) $f$在$[-\pi,\pi]$上的不连续点和不可微点至多有限多个;

(2)在每一不连续点$\xi$处具有第一类间断, 即左、右极限

$f(\xi-0)$与$f(\xi+0)$都存在;

(3)在每一不可微点(包括不连续点)$\eta$,以下两极限存在

$$\lim_{t\to 0+}\frac{f(\eta+t)-f(\eta+0)}{t},\ \lim_{t\to 0+}\frac{f(\eta-t)-f(\eta-0)}{-t},$$

则$f$在每一点$x_0$的Fourier级数收敛于

$$\frac{1}{2}[f(x_0+0)+f(x_0-0)].$$

Proof. 在所给条件下, $\forall x_0$, 不论$x_0$是否为$f$的可微点,以下两极限存在

$$\lim_{t\to 0+}\frac{f(x_0+t)-f(x_0+0)}{t},$$

$$\lim_{t\to 0+}\frac{f(x_0-t)-f(x_0-0)}{-t}.$$

因而$f$在$x_0$点满足$\alpha=1$的Lipschitz条件

$$\left|f(x_0\pm t)-f(x_0\pm 0)\right|\le Lt, \forall t\in(0,\delta).$$

由Lipschitz判别法,命题得证.□

Remark. 两极限

$$\lim_{t\to 0+}\frac{f(x_0+t)-f(x_0+0)}{t},$$

$$\lim_{t\to 0+}\frac{f(x_0-t)-f(x_0-0)}{-t}$$

也称为广义单侧导数. 若$x_0$是$f$的连续点,则广义单侧导数就是单侧导数.

Def. 称$f$在$[a,b]$上分段可微, 若$\exists[a,b]$的一个分割

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

使得$\forall i = 1, 2, \cdots, n$,

(1) $f$在$(x_{i-1}, x_i)$内可微;

(2) $f$在端点$x_{i-1}, x_i$处右、左极限分别存在;

(2) $f$在端点$x_{i-1}, x_i$处右、左广义导数分别存在.

前面的推论可以等价地描述为:

Corollary. $f$以$2\pi$为周期,且在$[-\pi,\pi]$上分段连续可微,则$f$在每一点$x_0$的Fourier级数收敛于

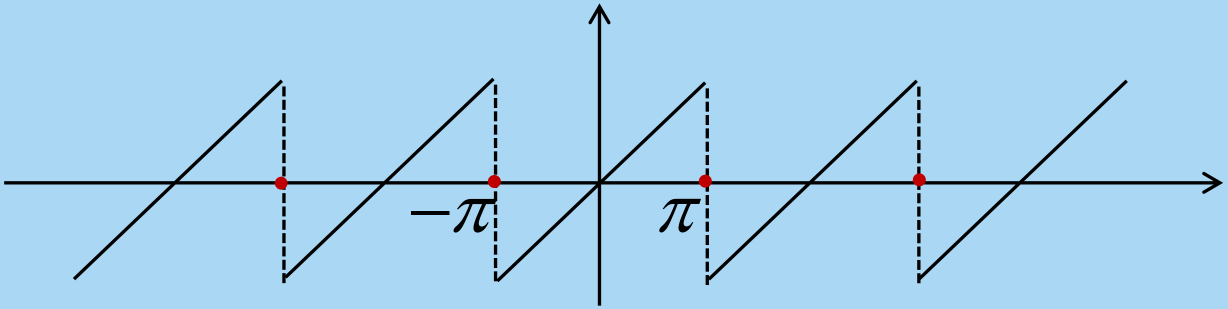
$$\frac{1}{2}[f(x_0+0)+f(x_0-0)].$$

特别地,若$f$在$x_0$连续,则$f$在$x_0$的Fourier级数收敛于$f(x_0)$.

例. 将$f(x)=\cos^2 x$展开成$2\pi$周期的Fourier级数.

解: $f(x)=\cos^2 x=\dfrac{1}{2}+\dfrac{1}{2}\cos 2x,\quad x\in\mathbb{R}.$

例. $f(x)=x$在$(0,\pi)$上的正弦Fourier级数为



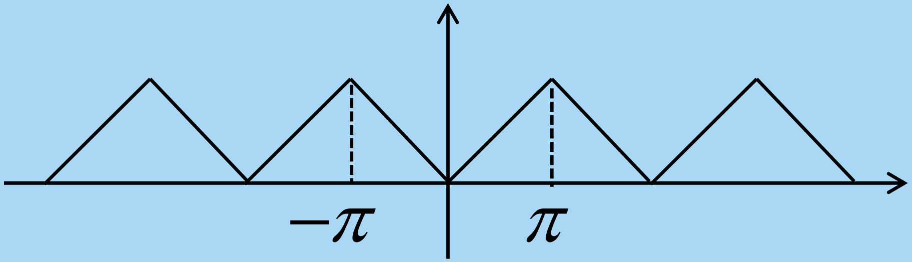

$$f(x)\sim\sum_{k=1}^{+\infty}(-1)^{k+1}\frac{2}{k}\sin kx,$$

$$\sum_{k=1}^{+\infty}(-1)^{k+1}\frac{2}{k}\sin kx=\begin{cases}x, & x\in(-\pi,\pi),\\ 0, & x=\pm\pi.\end{cases}$$

令$x=\dfrac{\pi}{2}$,得

$$1-\frac{1}{3}+\frac{1}{5}+\cdots+(-1)^{n+1}\frac{1}{2n-1}+\cdots=\frac{\pi}{4}.\square$$

例. $f(x)=x$在$(0,\pi)$上的余弦Fourier级数为




$$f(x)\sim\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{\cos(2k+1)x}{(2k+1)^2}.$$

$$x=f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{\cos(2k+1)x}{(2k+1)^2},\quad \forall x\in[-\pi,\pi].$$

令$x=\dfrac{\pi}{4}$,得

$$1-\left(\frac{1}{3^2}+\frac{1}{5^2}\right)+\left(\frac{1}{7^2}+\frac{1}{9^2}\right)+\cdots$$

$$+(-1)^n\left(\frac{1}{(4n-1)^2}+\frac{1}{(4n+1)^2}\right)+\cdots=\frac{\sqrt{2}}{16}\pi^2.\square$$

例. $f(x)=x^2, \forall x\in[-1,1], f$以$T=2$为周期.

$$f(x)\sim\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

$$x^2 = f(x) = \frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos n\pi x,\quad \forall x\in[-1,1].$$

令$x=1$,得

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}.$$

令$x=0$,得

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\frac{(-1)^{n+1}}{n^2}+\cdots=\frac{\pi^2}{12}.\square$$

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**Thm (Dirichlet判别法)** 周期为$2\pi$的函数$f$在$[-\pi,\pi]$上分段单调且有界, 则$f$的Fourier级数在任意一点$x_0$收敛于

$$\frac{1}{2}\left[f(x_0+0)+f(x_0-0)\right].$$

**Proof.** $S_n(x_0)=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\frac{f(x_0+t)+f(x_0-t)}{2\sin t/2}\sin\left(n+\frac{1}{2}\right)t\,dt,$

只要证 $\displaystyle\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\frac{f(x_0\pm t)}{2\sin t/2}\sin\left(n+\frac{1}{2}\right)t\,dt=\frac{1}{2}f(x_0\pm 0).$

注意到 $\displaystyle\int_0^{\pi}\frac{\sin(n+1/2)t}{2\sin t/2}dt=\frac{\pi}{2}$,只要证

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\frac{f(x_0\pm t)-f(x_0\pm 0)}{2\sin t/2}\sin\left(n+\frac{1}{2}\right)tdt=0.$$

$$w(t)\triangleq\begin{cases}\dfrac{1}{2\sin t/2}-\dfrac{1}{t}, & t\neq 0\\ 0, & t=0\end{cases},$$ 则 $w\in C[0,\pi]$. 而 $f$ 以 $2\pi$ 为周期, 在 $[-\pi,\pi]$ 上分段单调且有界, 则 $f\in\Re[x_0-\pi,x_0+\pi]$,

由Riemann-Lebesgue引理,

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}w(t)\big(f(x_0\pm t)-f(x_0\pm 0)\big)\sin\left(n+\frac{1}{2}\right)tdt=0.$$

故往证: $$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\frac{f(x_0\pm t)-f(x_0\pm 0)}{t}\sin\left(n+\frac{1}{2}\right)tdt=0.$$

$\forall x_0, \exists \eta > 0, s.t.\, f(x_0 \pm t)$在$t \in (0,\eta)$上单调.

$$\int_0^{\pi} \frac{f(x_0 \pm t) - f(x_0 \pm 0)}{t} \sin\left(n + \frac{1}{2}\right) t\,dt$$

$$= \left(\int_0^{\delta} + \int_{\delta}^{\pi}\right) \frac{f(x_0 \pm t) - f(x_0 \pm 0)}{t} \sin\left(n + \frac{1}{2}\right) t\,dt$$

$$\triangleq A_{n,\delta}(x_0) + B_{n,\delta}(x_0)$$

$\delta \in (0,\eta)$时, 由积分第二中值定理, $\exists \xi \in (0,\delta), s.t.$

$$A_{n,\delta}(x_0) = \left(f(x_0 \pm \delta) - f(x_0 \pm 0)\right) \int_{\xi}^{\delta} \frac{\sin(n+1/2)t}{t}\,dt$$

$$\int_{\xi}^{\delta}\frac{\sin\left(n+1/2\right)t}{t}dt=\int_{(n+1/2)\xi}^{(n+1/2)\delta}\frac{\sin\tau}{\tau}d\tau$$

因为$\displaystyle\int_{0}^{+\infty}\frac{\sin\tau}{\tau}d\tau=\pi/2$,积分收敛,因此$\exists M>0,\ s.t.$

$$\left|\int_{0}^{v}\frac{\sin\tau}{\tau}d\tau\right|<M,\forall v\in\mathbb{R}.$$

从而

$$\left|\int_{(n+1/2)\xi}^{(n+1/2)\delta}\frac{\sin\tau}{\tau}d\tau\right|<2M,\forall\delta,\xi>0.$$

$$\forall\varepsilon>0,\exists\delta\in(0,\eta),s.t.\left|f(x_0\pm\delta)-f(x_0\pm0)\right|<\varepsilon/4M,$$

此时,

$$\left|A_{n,\delta}(x_0)\right|<\varepsilon/2.$$

对此$\varepsilon$及$\delta$,由Liemann-Lebesgue引理, $\lim\limits_{n\to\infty} B_{n,\delta}(x_0)=0.$

因此, $\exists N\geq 1, s.t.$

$$\left|B_{n,\delta}(x_0)\right|<\varepsilon/2, \forall n>N.$$

因此
$$\left|\int_0^{\pi}\frac{f(x_0\pm t)-f(x_0\pm 0)}{t}\sin\left(n+\frac{1}{2}\right)tdt\right|$$
$$\leq\left|A_{n,\delta}(x_0)\right|+\left|B_{n,\delta}(x_0)\right|$$
$$<\varepsilon/2+\varepsilon/2=\varepsilon, \qquad \forall n>N$$

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\frac{f(x_0\pm t)-f(x_0\pm 0)}{t}\sin\left(n+\frac{1}{2}\right)tdt=0.\square$$

# 4. Fourier级数的均方收敛性,Parseval等式

$$(f,g) \triangleq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx, \quad \forall f,g \in R[-\pi,\pi].$$

$$\varphi_0(x) = 1\big/\sqrt{2},$$

$$\varphi_{2n-1}(x) = \cos nx, \quad n = 1,2,\cdots$$

$$\varphi_{2n}(x) = \sin nx, \qquad n = 1,2,\cdots$$

$\{\varphi_k(x)\}$为$R[-\pi,\pi]$中标准正交函数系.

Thm. $f$以$2\pi$为周期, 在$[-\pi,\pi]$上可积,$\{\varphi_k\}$如前,

$c_k=(f,\varphi_k)$,则

(1) $f$的Fourier级数在$[-\pi,\pi]$上均方收敛到$f(x)$,即

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\left(f(x)-\sum_{k=0}^{n}c_k\varphi_k(x)\right)^2dx=0;$$

(2) $\sum_{k=0}^{\infty}c_k^2=\|f\|^2$.(Parseval等式)

这个定理可以翻译为:

Thm. $f$以$2\pi$为周期, 在$[-\pi,\pi]$上可积,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=0}^{\infty}\left(a_n \cos nx + b_n \sin nx\right).$$

(1) $f$的Fourier级数在$[-\pi,\pi]$上均方收敛到$f(x)$,即

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\left(f(x)-S_n(x)\right)^2 dx = 0,$$

其中$S_n(x) = \dfrac{a_0}{2} + \displaystyle\sum_{k=0}^{n}\left(a_k \cos kx + b_k \sin kx\right)$;

(2) $\dfrac{a_0^2}{2} + \displaystyle\sum_{n=1}^{+\infty}\left(a_n^2 + b_n^2\right) = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$.(Parseval等式)

**Proof.** 证明思路如下:

Step1. Fejer和 $\sigma_n(x) \triangleq \frac{1}{n}\sum_{k=0}^{n-1} S_k(x)$.

若 $f \in C[-\pi,\pi]$, 则 $\sigma_n(x)$ 在 $[-\pi,\pi]$ 上一致收敛到 $f(x)$.

Step2. $f \in R[-\pi,\pi], \forall \varepsilon > 0, \exists g \in C[-\pi,\pi], s.t. \| f - g \| < \varepsilon$.

Step3. 由前两步的结论知: 任何函数 $f \in R[-\pi,\pi]$ 都可以用三角多项式(即 $1, \cos x, \sin x, \cdots, \cos nx, \sin nx, \cdots$ 的有限线性组合)均方逼近, 使得均方误差小于预先给定的正数 $\varepsilon$.

Step4. 已知$\{1,\cos x,\sin x,\cdots,\cos nx,\sin nx,\cdots\}$的前$n+1$项线性组合中, $f$的Fourier级数的部分和$S_n(x)$是$f$的最佳均方逼近. 于是由Step3的结论知: $\forall f\in R[-\pi,\pi]$, $f$的Fourier级数在$[-\pi,\pi]$上均方收敛到$f(x)$, 即$\|f-S_n\|\to 0.$

Step5. 在

$$\left\|f-\sum_{k=0}^{n}c_k\varphi_k\right\|^2=\|f\|^2-\sum_{k=0}^{n}c_k^2$$

中令$n\to\infty$,得Parseval等式.□

例. $x^{1000}=\frac{a_0}{2}+\sum_{n=0}^{+\infty}(a_n\cos nx+b_n\sin nx),\ x\in(-\pi,\pi)$. 求$\sum_{n=1}^{+\infty}a_n^2$.

解：由$Passeval$等式，

$$\frac{a_0^2}{2}+\sum_{n=1}^{+\infty}a_n^2+\sum_{n=1}^{+\infty}b_n^2=\frac{1}{\pi}\int_{-\pi}^{\pi}x^{2000}dx=\frac{\pi^{2000}}{2001}.$$

$x^{1000}$为偶函数, 所以$b_n=0,\forall n.$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^{1000}dx=\frac{\pi^{1000}}{1001},$$

$$\sum_{n=1}^{+\infty}a_n^2=\frac{\pi^{2000}}{2001}-\frac{a_0^2}{2}-\sum_{n=1}^{+\infty}b_n^2=2\pi^{2000}\left(\frac{1}{2001}-\frac{1}{1001^2}\right).\ \square$$

作业:习题7.1 No.3.

习题7.2 No.1-4.