4.2.3

(1) $x'(t)=3, y'(t)=6t, z'(t)=6t^2$

$$\begin{aligned}L&=\int_0^1\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt\\&=\int_0^1\sqrt{9+36t^2+36t^4}dt\\&=5\end{aligned}$$

(2) $x'(t)=-e^{-t}\sin t-e^{-t}\cos t$ ，$y'(t)=e^{-t}\cos t-e^{-t}\sin t$ ，$z'(t)=-e^{-t}$

$$\begin{aligned}L&=\int_0^{+\infty}\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt\\&=\sqrt{3}\int_0^{+\infty}e^{-t}dt\\&=\sqrt{3}\end{aligned}$$

注：求该类型的曲线积分线元需要对参数求导，即

$$ds=\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt$$

有同学没有求导，使用

$$ds=\sqrt{(x(t))^2+(y(t))^2+(z(t))^2}dt$$

进行计算，这是错误的。

4.2.4.

$$m=\int_{\sqrt{3}}^{\sqrt{15}}\rho(x)\sqrt{1+(y'(x))^2}dx=\int_{\sqrt{3}}^{\sqrt{15}}x^2\sqrt{1+\frac{1}{x^2}}dx=\frac{56}{3}$$

4.2.5. 圆柱面被 $z=0$ 平面截得圆 L 的参数方程为

$$x=a\cos\theta,\ y=a\sin\theta,\ z=0,\ 0\le\theta\le 2\pi$$

截出圆柱面面积为

$$
\begin{aligned}
S &= \oint_L \left(a + \frac{x^2}{a} - 0\right) ds \\
&= \int_0^{2\pi} (a + a\cos^2\theta)\sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2} d\theta \\
&= 3\pi a^2
\end{aligned}
$$

4.2.6. 摆线长度为

$$
\begin{aligned}
L &= \int_0^{\pi} \sqrt{(x'(t))^2 + (y'(t))^2} dt \\
&= \int_0^{\pi} \sqrt{(a - a\cos t)^2 + (a\sin t)^2} dt \\
&= 2a \int_0^{\pi} \sin\frac{t}{2} dt \\
&= 4a
\end{aligned}
$$

设质心坐标为 (X_c, Y_c)，记摆线为 S, 则

$$
\begin{aligned}
X_c &= \frac{1}{L}\int_S x(t) ds \\
&= \frac{1}{4a}\int_0^{\pi} a(t - \sin t)\sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t} dt \\
&= \frac{a}{2}\int_0^{\pi} (t - \sin t)\sin\frac{t}{2} dt \\
&= \frac{4}{3}a \\
Y_c &= \frac{1}{L}\int_S y(t) ds \\
&= \frac{1}{4a}\int_0^{\pi} a(1 - \cos t)\sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t} dt \\
&= \frac{a}{2}\int_0^{\pi} (1 - \cos t)\sin\frac{t}{2} dt \\
&= \frac{4}{3}a
\end{aligned}
$$

所以质心坐标为 $\left(\frac{4}{3}a, \frac{4}{3}a\right)$

4.5.1

(1) 首先写出球面参数方程

$$x = 2R\cos\theta\sin\theta\cos\varphi,\quad y = 2R\cos\theta\sin\theta\sin\varphi,\quad z = 2R\cos^2\theta$$

$$D_{\theta\varphi} = \{(\theta,\varphi)\,|0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le 2\pi\}$$

进行该坐标变换对应的 Jacobi 行列式为

$$\begin{aligned} C = \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi} \end{vmatrix} &= \begin{vmatrix} 2R\cos 2\theta\cos\varphi & -R\sin 2\theta\sin\varphi \\ 2R\cos 2\theta\sin\varphi & R\sin 2\theta\cos\varphi \end{vmatrix} \\ &= 2R^2\sin 2\theta\cos 2\theta \end{aligned}$$

所求积分为

$$\begin{aligned} \oiint_{S^+} dx\wedge dy &= +\iint\limits_{D_{\theta\varphi}} Cd\theta d\varphi \\ &= \int_0^{\frac{\pi}{2}} 2R^2\sin 2\theta\cos 2\theta d\theta \int_0^{2\pi} d\varphi \\ &= 0 \end{aligned}$$

(2) 继续采用 (1) 中坐标变换

$$\begin{aligned} \oiint_{S^+} zdx\wedge dy &= +\iint\limits_{D_{\theta\varphi}} z(\theta,\varphi)Cd\theta d\varphi \\ &= \int_0^{\frac{\pi}{2}} \left(2R\cos^2\theta\right) 2R^2\sin 2\theta\cos 2\theta d\theta \int_0^{2\pi} d\varphi \\ &= \frac{4}{3}\pi R^3 \end{aligned}$$

(3) 同理

$$
\begin{aligned}
\oiint_{S^+} z^2 dx \wedge dy &= + \iint\limits_{D_{\theta\varphi}} z(\theta,\varphi)^2 C d\theta d\varphi \\
&= \int_0^{\frac{\pi}{2}} \left(2R\cos^2\theta\right)^2 2R^2 \sin 2\theta \cos 2\theta d\theta \int_0^{2\pi} d\varphi \\
&= \frac{8}{3}\pi R^4
\end{aligned}
$$

4.5.5

首先写出球面的参数方程

$$
x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta
$$
$$
D_{\theta\varphi} = \{(\theta,\varphi)|0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}
$$

进行该坐标变换对应的 Jacobi 行列式为

$$
\begin{aligned}
A &= \begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ -\sin\theta & 0 \end{vmatrix} = \sin^2\theta\cos\varphi \\
B &= \begin{vmatrix} \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} -\sin\theta & 0 \\ \cos\theta\cos\varphi & -\sin\theta\sin\varphi \end{vmatrix} = \sin^2\theta\sin\varphi \\
C &= \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \end{vmatrix} = \sin\theta\cos\theta
\end{aligned}
$$

流量为

$$
\begin{aligned}
Q &= \iint_S \mathbf{V} \cdot d\mathbf{S} \\
&= + \iint\limits_{D_{\theta\varphi}} \left(V_x A + V_y B + V_z C\right) d\theta d\varphi \\
&= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \left(\sin^4\theta\cos^2\varphi\sin\varphi + \sin^3\theta\cos\theta\sin^2\varphi + \sin^2\theta\cos^2\theta\cos\varphi\right) \\
&= \frac{3\pi}{16}
\end{aligned}
$$

注：在计算积分时可以利用被积函数和积分区域的轮换对称性，即 $x \to y, y \to z, z \to x$ 积分结果不变。可以验证

$$\iint\limits_{D_{\theta\varphi}} V_x A d\theta d\varphi = \iint\limits_{D_{\theta\varphi}} V_y B d\theta d\varphi = \iint\limits_{D_{\theta\varphi}} V_z C d\theta d\varphi = \frac{\pi}{16}$$

4.5.7

坐标变换对应的 Jacobi 行列式为

$$\begin{aligned}
A &= \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} \sin v & u\cos v \\ 0 & a \end{vmatrix} = a\sin v \\
B &= \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & a \\ \cos v & -u\sin v \end{vmatrix} = -a\cos v \\
C &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u
\end{aligned}$$

所以积分变为

$$\begin{aligned}
I &= +\iint\limits_{D_{uv}} \left(u^2\sin^2 v\,(a\sin v) + a^2v^2\,(-a\cos v) + u^2u\right) dudv \\
&= a\int_0^1 u^2du\int_0^{2\pi}\sin^3 v dv - a^3\int_0^1 du\int_0^{2\pi} v^2\cos v dv \\
&+ \int_0^1 u^3du\int_0^{2\pi} dv \\
&= \frac{\pi}{2} - 4\pi a^3
\end{aligned}$$

注：有同学没有注意到 3 个 Jacobi 行列式需要乘到对应的体元上，计算出积分结果为

$$\begin{aligned}
I &= +\iint\limits_{D_{uv}} \left(u^2\,(a\sin v) + u^2\sin^2 v\,(-a\cos v) + a^2v^2u\right) dudv \\
&= \frac{4\pi^3}{3}a^2
\end{aligned}$$

是错误的。该题体元的顺序与书上不同，应该写成

$$\iint\limits_{S^+} Z dx \wedge dy + X dy \wedge dz + Y dz \wedge dx = \pm \iint\limits_{D_{uv}} (ZC + XA + YB) du dv$$