Review

- 二重积分的几何与物理意义

$$\iint_D f(x,y)dxdy = \lim_{\lambda(T)\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta x_i \Delta y_j,$$

- 二重积分的性质

- Darboux上、下和与Riemann积分

$f\in B(D)$,则$f\in R(D)$的充分必要条件是:

$\lim\limits_{\lambda(T)\to 0} L(f,T)$与$\lim\limits_{\lambda(T)\to 0} U(f,T)$存在且相等, 且

$$\iint_D f(x,y)dxdy$$

$$=\lim_{\lambda(T)\to 0} L(f,T)=\lim_{\lambda(T)\to 0} U(f,T).$$

§2. 二重积分的计算

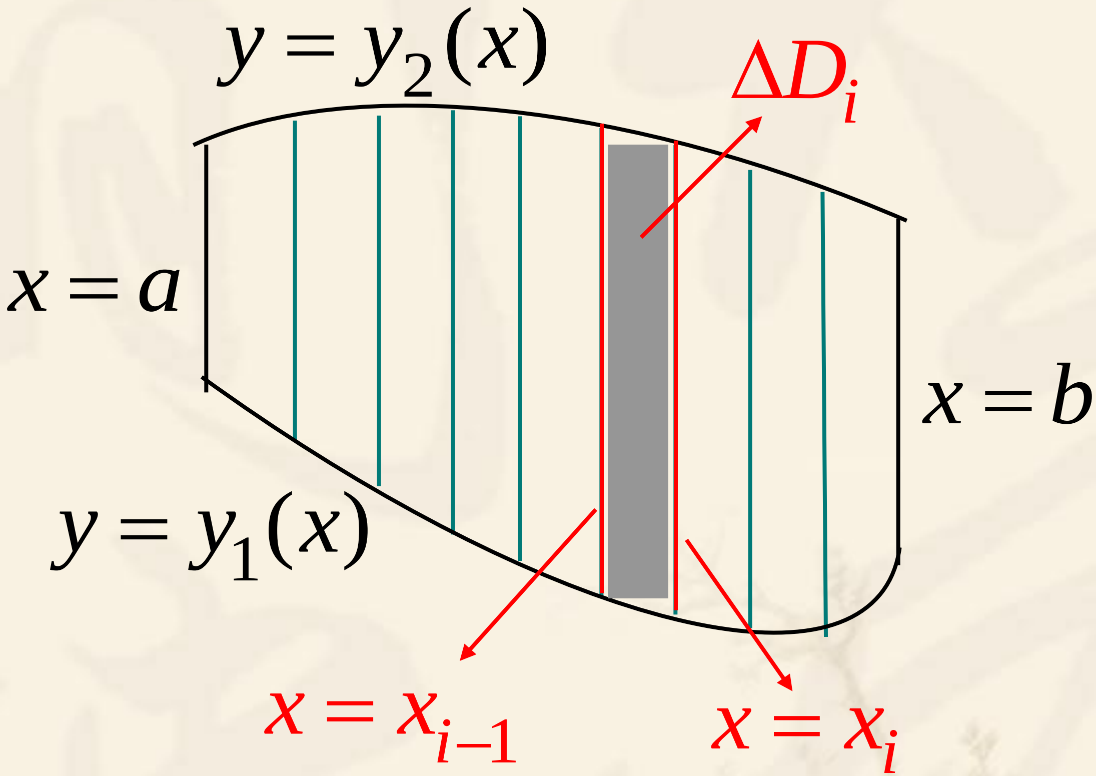
- 直角坐标下二重积分的计算及例题
- 极坐标下二重积分的计算及例题
- 补充例题

1. 用直角坐标系计算二重积分

$$S: z = f(x,y), (x,y) \in D.$$

换一个思路来计算以D为下底,以S为顶的曲顶柱体Ω的体积 $$V(\Omega) = \iint_D f(x,y)dxdy.$$

设$D=\left\{(x,y)\middle|a\le x\le b, y_1(x)\le y\le y_2(x)\right\}$.



• Step1.对D进行分划: $a=x_0<x_1<\cdots<x_n=b$,

将D分成平行于y轴的细条$\Delta D_1,\Delta D_2,\cdots,\Delta D_n$.

相应地,Ω被平行于OYZ平面的平面$x = x_i$切成薄片$\Delta\Omega_1, \Delta\Omega_2, \cdots, \Delta\Omega_n$.

- Step2.求近似和

曲顶柱体Ω中截面$x = x$的面积为

$$\mathrm{A}(x) = \int_{y_1(x)}^{y_2(x)} f(x, y)dy.$$

于是薄片$\Delta\Omega_i$的体积近似为

$$\mathrm{V}(\Delta\Omega_i) \approx \mathrm{A}(x_i)(x_{i+1} - x_i) = \mathrm{A}(x_i)\Delta x_i.$$

曲顶柱体的体积近似为

$$\mathrm{V}(\Omega) \approx \sum_{i=1}^{n} \mathrm{A}(x_i)\Delta x_i.$$

•Step3.取极限　当分划越来越细时,

$$\sum_{i=1}^{n} \mathrm{A}(x_i)\Delta x_i \to \mathrm{V}(\Omega).$$

综上,

$$\mathrm{V}(\Omega) = \int_a^b \mathrm{A}(x)dx = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)dy \right) dx,$$

即

$$\iint\limits_D f(x,y)dxdy = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)dy \right) dx$$

$$\triangleq \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)dy. \qquad (*)$$

Remark:等式后两项的意义是, 先固定x(视x为常数), 对变量y求定积分

$$\mathrm{A}(x)=\int_{y_1(x)}^{y_2(x)} f(x,y)dy,$$

再让x变起来, 对变量x求定积分

$$\int_a^b \mathrm{A}(x)dx.$$

正因为如此,(*)式右端的积分也称为先y后x的累次积分.

Remark:对称地, 若区域D具有如下形式:

$$D=\left\{(x,y)\middle| c\le y\le d, x_1(y)\le x\le x_2(y)\right\}.$$

则 $$\iint\limits_D f(x,y)dxdy=\int_c^d\left(\int_{x_1(y)}^{x_2(y)} f(x,y)dx\right)dy$$

$$\triangleq\int_c^d dy\int_{x_1(y)}^{x_2(y)} f(x,y)dx.$$

Remark:对于一般的区域D, 可以分成若干个具有以上两种形式的区域, 并将二重积分利用区域可加性化为累次积分来计算.

Thm. 设$f(x,y)$在有界闭区域D上连续,若

$$D=\{(x,y)\big|a\le x\le b,y_1(x)\le y\le y_2(x)\},$$

其中$y_1(x),y_2(x)\in C([a,b])$.则

$$\iint\limits_D f(x,y)dxdy=\int_a^b dx\int_{y_1(x)}^{y_2(x)}f(x,y)dy.$$

若$D=\{(x,y)\big|c\le y\le d,x_1(y)\le x\le x_2(y)\}$,

其中$x_1(y),x_2(y)\in C([c,d])$.则

$$\iint\limits_D f(x,y)dxdy=\int_c^d dy\int_{x_1(y)}^{x_2(y)}f(x,y)dx. \square$$

Remark:将二重积分化为累次积分计算时,选择不同的积分次序,难易程度可能相差很大.一般应根据被积函数和积分区域选择合适的累次积分次序.

例: 求$I=\iint_{x^2+y^2\le a^2} y^2\sqrt{a^2-x^2}\,dxdy.$

解:积分区域为$x\in[-a,a], y\in[-\sqrt{a^2-x^2},\sqrt{a^2-x^2}].$

$$I=\int_{-a}^{a}dx\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\sqrt{a^2-x^2}\,dy$$

$$=\int_{-a}^{a}\sqrt{a^2-x^2}\,dx\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\,dy$$

$$=\int_{-a}^{a}\sqrt{a^2-x^2}\left(\frac{1}{3}y^3\Big|_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}}\right)dx$$

$$=\frac{2}{3}\int_{-a}^{a}\left(a^2-x^2\right)^2dx=\frac{32}{45}a^5.\ \square$$

例: 求$I=\iint\limits_D \frac{x^2}{y^2}dxdy$,其中$D$由直线$y=2x, y=\frac{1}{2}x$

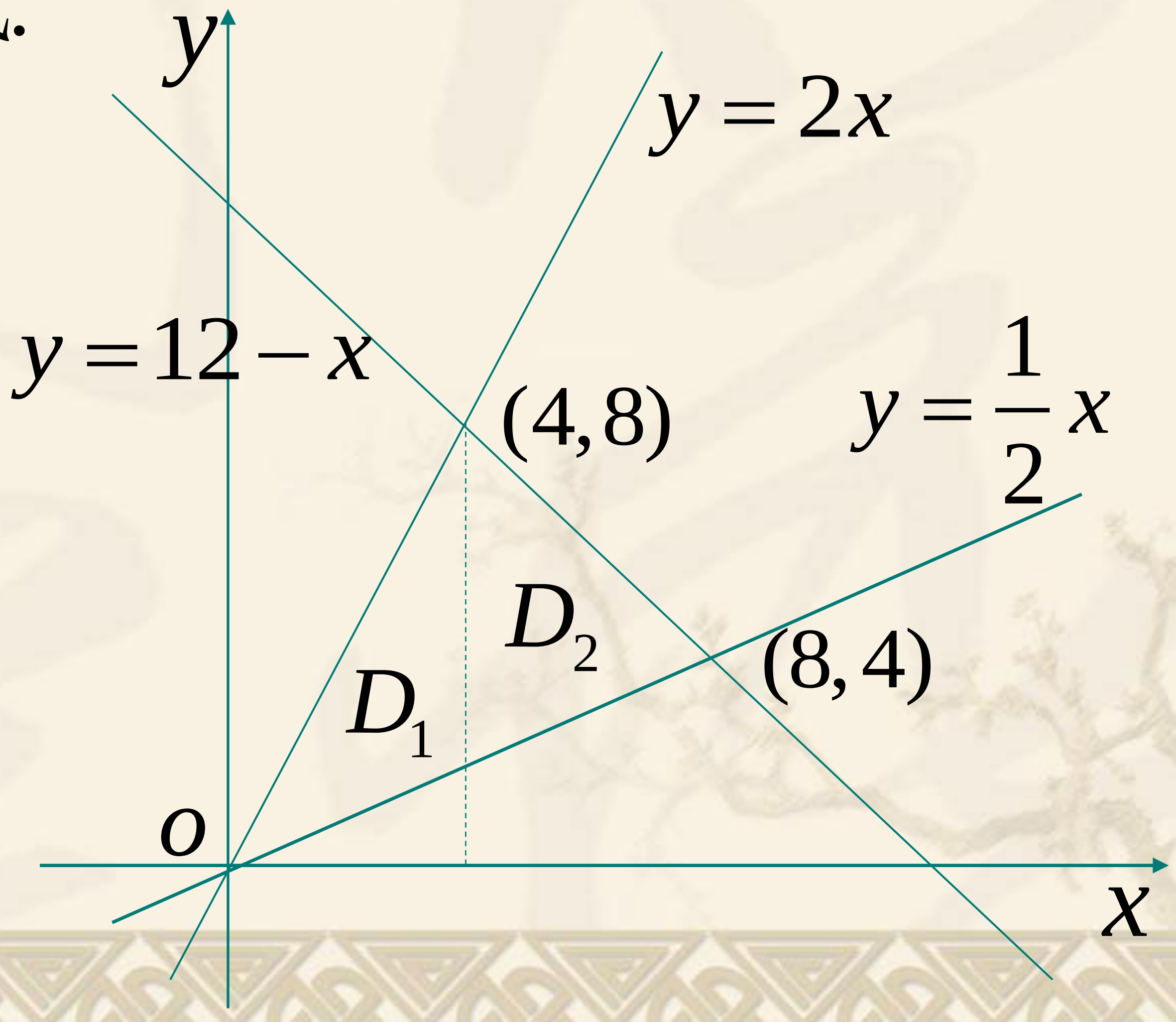
及$y=12-x$围成.

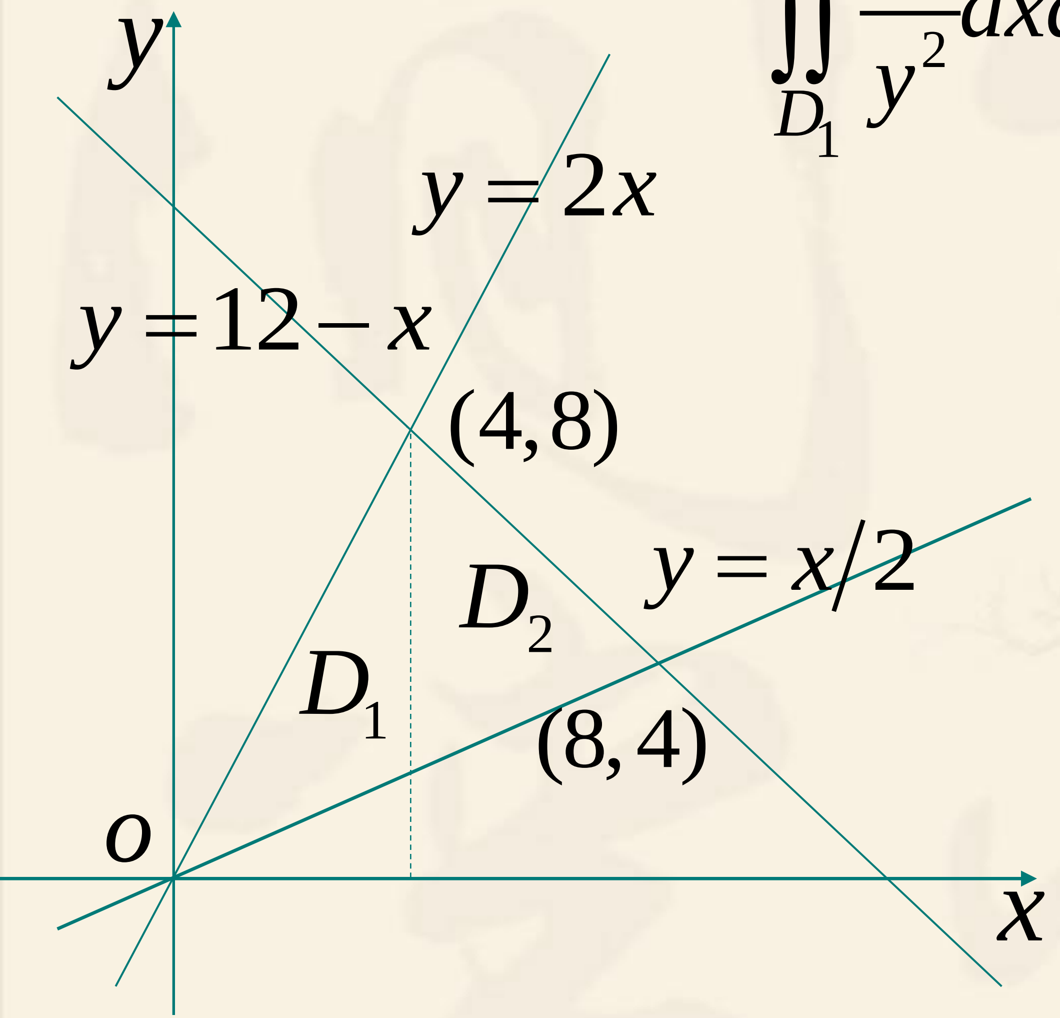
解: 如图,

区域D可

以分成D_1,

D_2两部分.





$$\iint_{D_1}\frac{x^2}{y^2}dxdy=\int_0^4 dx\int_{\frac{1}{2}x}^{2x}\frac{x^2}{y^2}dy$$

$$=\int_0^4\left(-\frac{x^2}{y}\Bigg|_{y=\frac{1}{2}x}^{y=2x}\right)dx$$

$$=\int_0^4 x^2\left(\frac{2}{x}-\frac{1}{2x}\right)dx$$

$$=12,$$

$$\iint\limits_{D_2} \frac{x^2}{y^2} dxdy = \int_4^8 dx \int_{\frac{1}{2}x}^{12-x} \frac{x^2}{y^2} dy$$

$$= \int_0^4 x^2 \left(\frac{2}{x} - \frac{1}{12-x} \right) dx = 120 - 144 \ln 2.$$

于是$\displaystyle\iint\limits_{D} \frac{x^2}{y^2} dxdy = \iint\limits_{D_1} \frac{x^2}{y^2} dxdy + \iint\limits_{D_2} \frac{x^2}{y^2} dxdy$

$$= 132 - 144 \ln 2. \quad \square$$

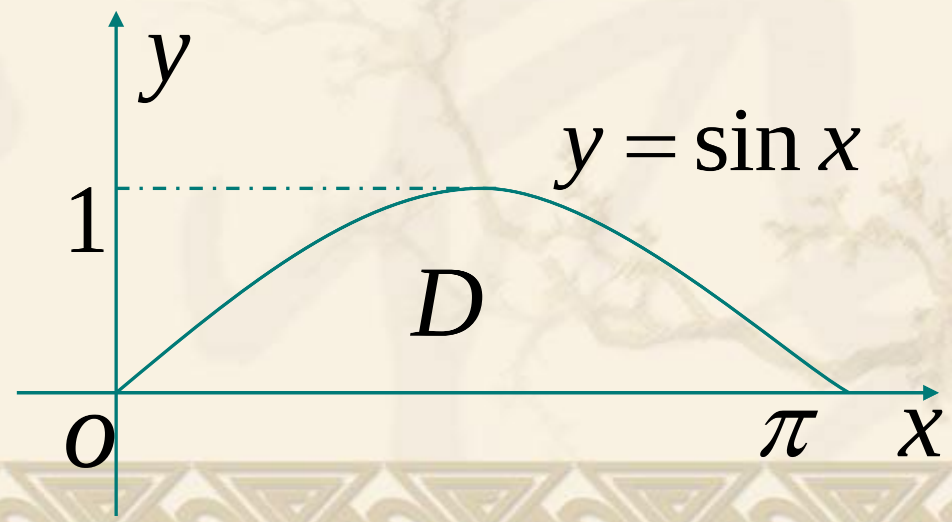
例: 求$I=\int_0^1 dy\int_{\arcsin y}^{\pi-\arcsin y} x dx.$

分析:按所给积分次序,内层积分容易求出,但再积分就困难了.所以尝试交换积分次序.

解: $I=\int_0^{\pi} xdx\int_0^{\sin x} dy$

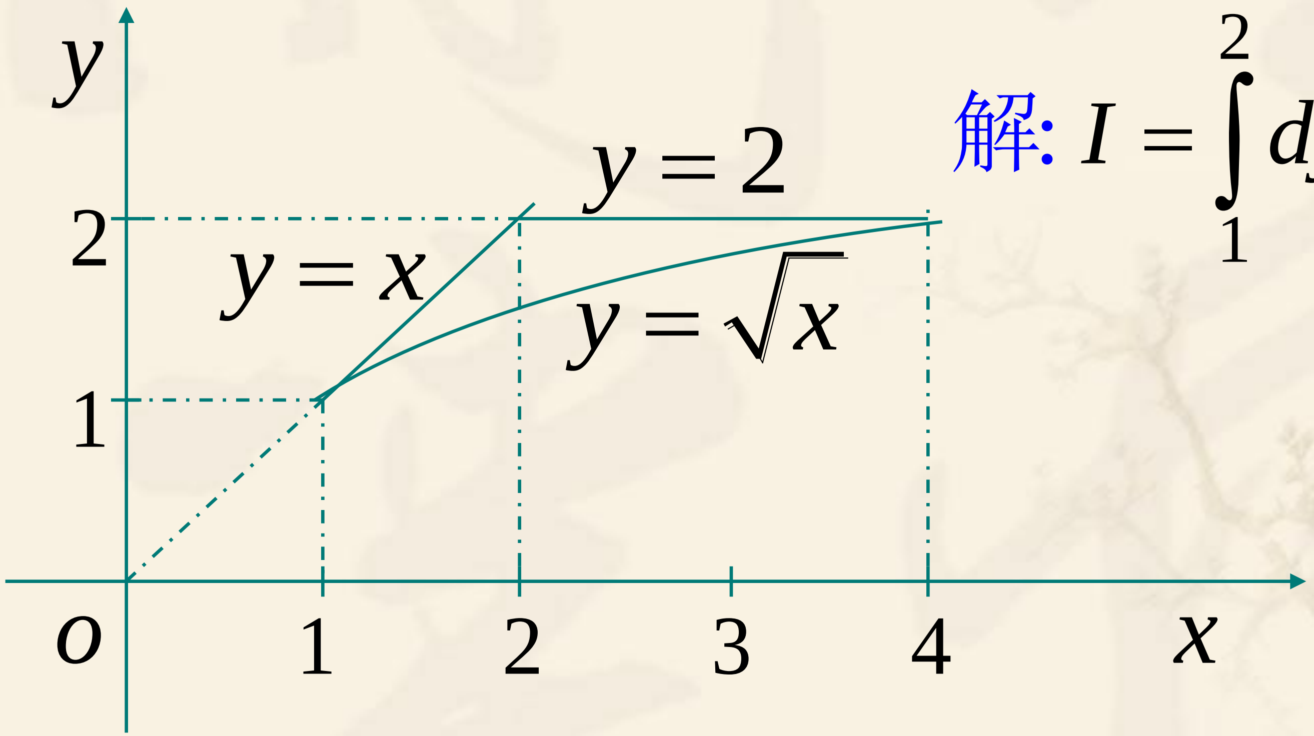
$=\int_0^{\pi} x\sin x dx=-\int_0^{\pi} xd\cos x$

$=-x\cos x\Big|_{x=0}^{\pi}+\int_0^{\pi}\cos x dx=\pi.$□



例: $I=\int_1^2 dx\int_{\sqrt{x}}^{x}\sin\frac{\pi x}{2y}dy+\int_2^4 dx\int_{\sqrt{x}}^{2}\sin\frac{\pi x}{2y}dy.$

分析: 里层积分困难, 考虑交换积分次序.

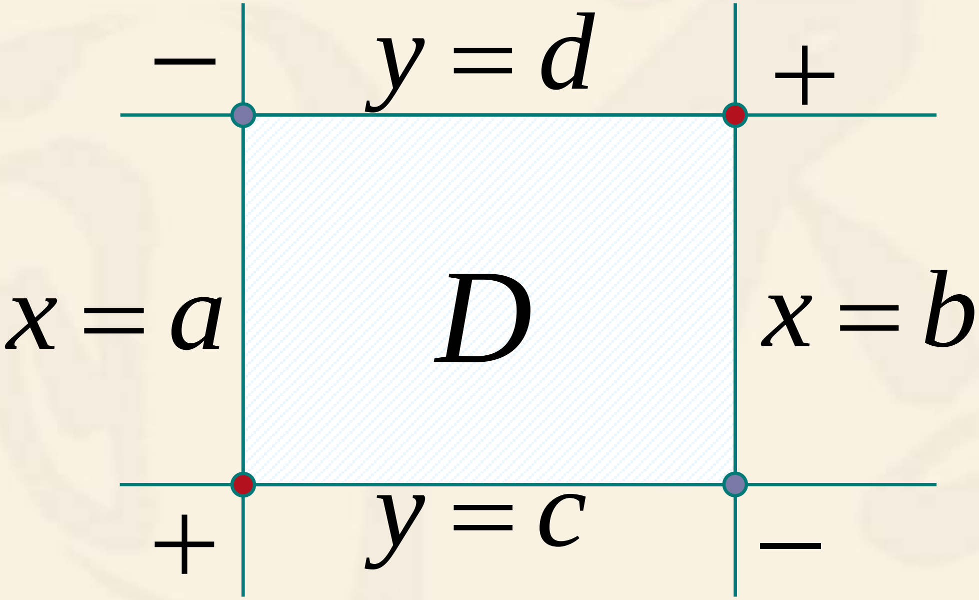


解: $I=\int_1^2 dy\int_{y}^{y^2}\sin\frac{\pi x}{2y}dx$

$$I=\int_1^2 dy\int_y^{y^2}\sin\frac{\pi x}{2y}dx$$

$$=\frac{2}{\pi}\int_1^2 y\left(\cos\frac{\pi}{2}-\cos\frac{\pi y}{2}\right)dy$$

$$=-\frac{2}{\pi}\int_1^2 y\cos\frac{\pi y}{2}dy=4(2+\pi)/\pi^3 .\square$$



例: 设 $\dfrac{\partial^2 f}{\partial x \partial y}$ 在 $D=[a,b]\times[c,d]$ 上可积, 则

$$\iint_D \frac{\partial^2 f}{\partial x \partial y} dxdy = f(b,d) - f(b,c) - f(a,d) + f(a,c).$$

证明: $$\iint_D \frac{\partial^2 f}{\partial x \partial y} dxdy = \int_c^d dy \int_a^b \frac{\partial^2 f}{\partial x \partial y} dx$$

$$= \int_c^d \left[\left. \frac{\partial f(x,y)}{\partial y} \right|_{x=a}^{b} \right] dy$$

$$= \int_c^d \frac{\partial f(b,y)}{\partial y} dy - \int_c^d \frac{\partial f(a,y)}{\partial y} dy$$

$$= \left. f(b,y) \right|_{y=c}^{d} - \left. f(a,y) \right|_{y=c}^{d}$$

$$= f(b,d) - f(b,c) - f(a,d) + f(a,c). \square$$

2. 用极坐标系计算二重积分

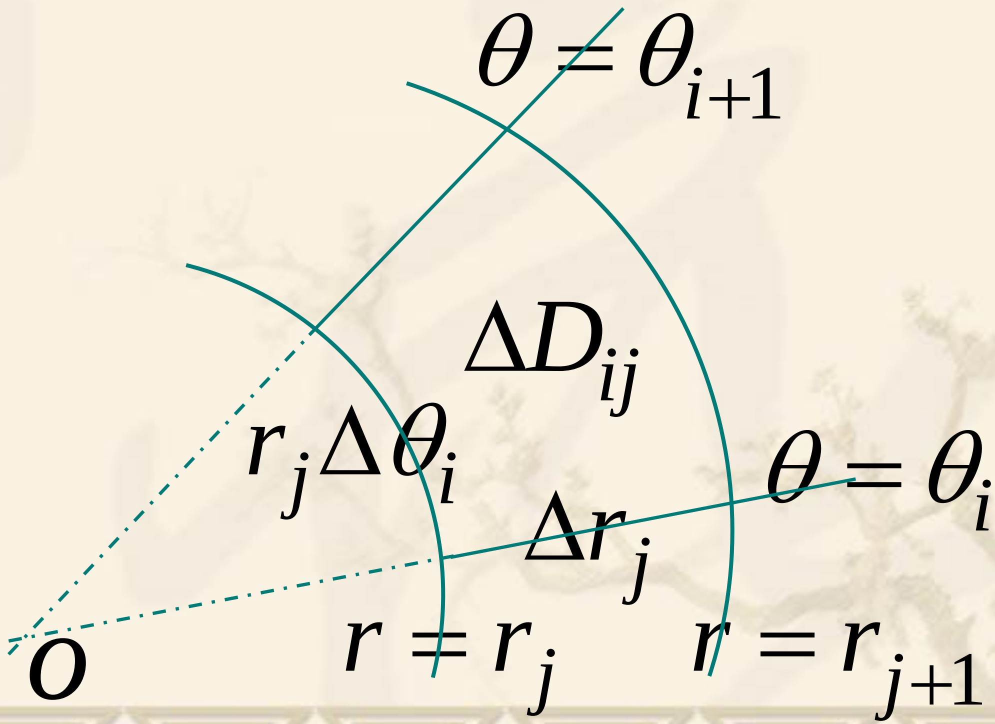
在直角坐标系下将二重积分化为累次积分来计算,如果被积区域D的形状不好,或者被积函数的表达式比较复杂,那么累次积分的计算将很复杂,甚至可能计算不出结果来.

再换一个思路来计算以D为底,以曲面 $S: z = f(x,y), (x,y) \in D$为顶的曲顶柱体的$\Omega$体积$\mathrm{V}(\Omega) = \iint\limits_{D} f(x,y)dxdy.$

用过原点的射线$\theta=\theta_i(i=1,2,\cdots,n)$和以原点为圆心的同心圆$r=r_j(j=1,2,\cdots,m)$对区域$D$作分划.

忽略位于区域D边界的那些不规则的小区域,考虑由$\theta=\theta_i,\theta=\theta_{i+1},r=r_j$和$r=r_{j+1}$围成的曲边四边形$\Delta D_{ij}$. 当$\Delta r_j=r_{j+1}-r_j$, $\Delta\theta_i=\theta_{i+1}-\theta_i$很小时,$\Delta D_{ij}$近似为矩形,边长分别为$\Delta r_j$和$r_j\Delta\theta_i$.

$$\sigma(\Delta D_{ij})\approx r_j\Delta\theta_i\Delta r_j$$



于是$\mathrm{V}(\Omega) \approx \sum_{1\le i\le n, 1\le j\le m} \sigma(\Delta D_{ij}) f(r_j \cos\theta_i, r_j \sin\theta_i)$

$$\approx \sum_{1\le i\le n, 1\le j\le m} r_j \Delta\theta_i \Delta r_j f(r_j \cos\theta_i, r_j \sin\theta_i).$$

当分划越来越细时,有.

$$\sum_{i,j} r_j \Delta\theta_i \Delta r_j f(r_j \cos\theta_i, r_j \sin\theta_i) \to \mathrm{V}(\Omega).$$

设E是原积分区域D在极坐标下的表示, 即

$$E = \{(r,\theta) \mid (r\cos\theta, r\sin\theta) \in D\}.$$

则 $\mathrm{V}(\Omega) = \iint_E f(r\cos\theta, r\sin\theta) r dr d\theta.$

即 $\iint_D f(x,y) dx dy = \iint_E f(r\cos\theta, r\sin\theta) r dr d\theta.$

Remark:于是在极坐标系下面积微元为$d\sigma = rdrd\theta.$

若$E = \{(r,\theta) \mid \alpha \le \theta \le \beta, r_1(\theta) \le r \le r_2(\theta)\}$,则

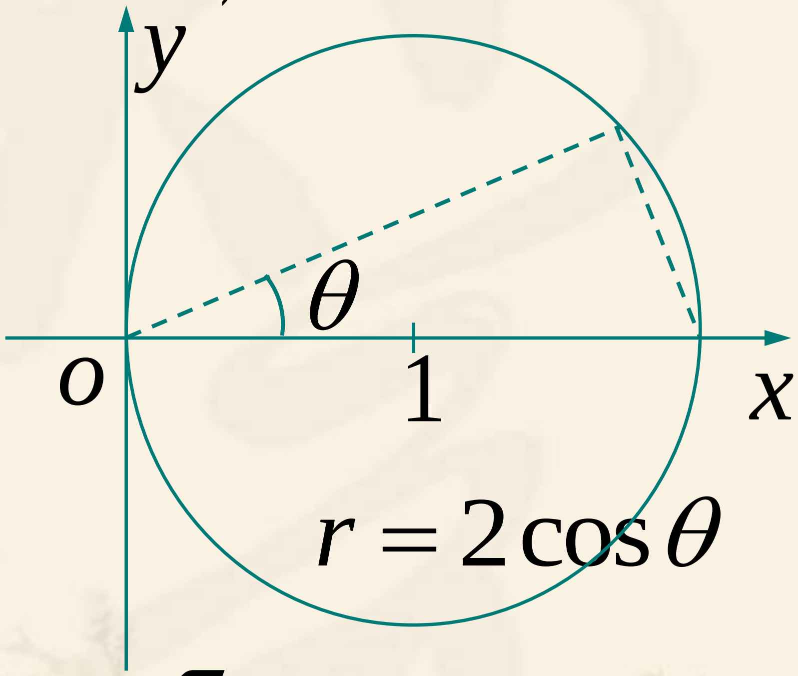
$$\iint\limits_E f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$= \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r dr.$$

于是,我们可以将二重积分化为极坐标下的累次积分来计算.

例: 求$I=\iint_{x^2+y^2\le 2x}\left(y+\sqrt{x^2+y^2}\right)dxdy.$

解:积分区域关于OX轴对称,



故 $\displaystyle\iint_{x^2+y^2\le 2x} ydxdy=0,$

$$I=\iint_{x^2+y^2\le 2x}\sqrt{x^2+y^2}\,dxdy.$$

极坐标下,积分区域为$\{-\frac{\pi}{2}\le\theta\le\frac{\pi}{2},r^2\le 2r\cos\theta\}.$

故

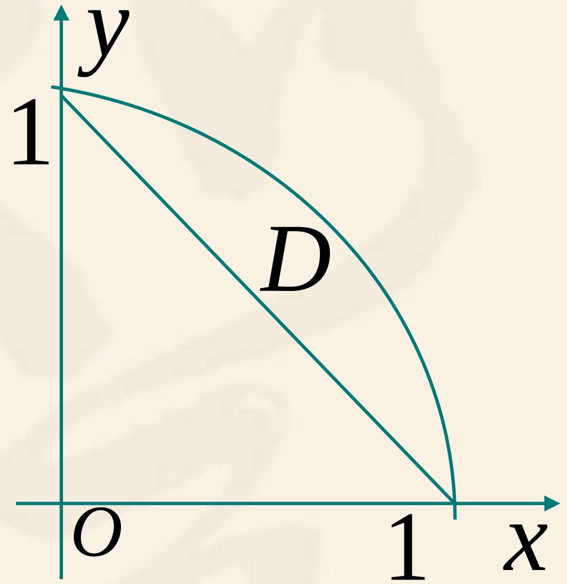
$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^2dr.$$

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{0}^{2\cos\theta}r^2dr=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\left(\frac{1}{3}r^3\right)\Bigg|_{r=0}^{2\cos\theta}$$

$$=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\theta d\theta=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2\theta)d\sin\theta$$

$$=\frac{8}{3}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{32}{9}.\ \square$$

例:求$I=\iint\limits_{x^2+y^2\le 1,x+y>1}\frac{x+y}{x^2+y^2}dxdy.$



解:极坐标下积分区域为

$$0\le\theta\le\frac{\pi}{2},\frac{1}{\sin\theta+\cos\theta}\le r\le 1.$$

$$I=\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{1}{\sin\theta+\cos\theta}}^{1}\frac{r\sin\theta+r\cos\theta}{r^2}\cdot rdr$$

$$=\int_0^{\frac{\pi}{2}}(\sin\theta+\cos\theta-1)d\theta=2-\frac{\pi}{2}.\ \square$$

例:求Poisson积分$I=\int_{-\infty}^{+\infty}e^{-x^2}dx.$

解:令$I(R)=\int_{-R}^{+R}e^{-x^2}dx$,则$\mathrm{I}(R)>0.$

$$I^2(R)=\int_{-R}^{+R}e^{-x^2}dx\int_{-R}^{+R}e^{-y^2}dy$$

$$=\iint_{-R\le x,y\le R}e^{-(x^2+y^2)}dxdy$$

于是,$\iint_{x^2+y^2\le R^2}e^{-(x^2+y^2)}dxdy\le \mathrm{I}^2(R)$

$$\le\iint_{x^2+y^2\le 2R^2}e^{-(x^2+y^2)}dxdy$$

而$\iint_{x^2+y^2\le R^2} e^{-(x^2+y^2)}dxdy = \int_0^{2\pi} d\theta \int_0^R re^{-r^2}dr$

$$= 2\pi \cdot \left(-\frac{1}{2} e^{-r^2} \right)\Bigg|_{r=0}^{R} = \pi(1 - e^{-R^2}).$$

同理, $\iint_{x^2+y^2\le 2R^2} e^{-(x^2+y^2)}dxdy = \pi(1 - e^{-2R^2}).$

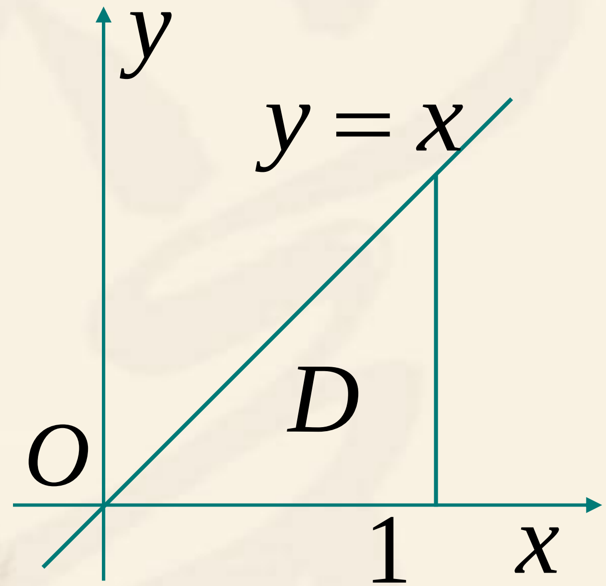
所以 $\pi(1 - e^{-R^2}) \le \mathrm{I}^2(R) \le \pi(1 - e^{-2R^2}).$

由夹挤原理, $\lim\limits_{R\to+\infty} \mathrm{I}^2(R) = \pi.$

故$I = \lim\limits_{R\to\infty} \mathrm{I}(R) = \sqrt{\pi}.$ □

3. 补充例题

$*$例: 求$\mathrm{I}=\int_0^1 \frac{\ln(1+x)}{(2-x)^2}dx.$



解:

$$\mathrm{I}=\int_0^1 \frac{1}{(2-x)^2}\left(\int_0^x \frac{1}{1+y}dy\right)dx$$

$$=\int_0^1 \frac{1}{(2-x)^2}dx\int_0^x \frac{1}{1+y}dy$$

$$=\int_0^1 \frac{1}{1+y}dy\int_y^1 \frac{1}{(2-x)^2}dx$$ (交换积分次序)

$$=\int_0^1 \frac{(1-y)dy}{(1+y)(2-y)}$$

$$=\frac{2}{3}\int_0^1 \frac{dy}{1+y}+\frac{1}{3}\int_0^1 \frac{dy}{2-y}=\frac{1}{3}\ln 2. \square$$

Remark:将一元函数的定积分化成二重积分计算,有时候可能会更简单.

$*$例: $\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx.$

证明:记$D=[a,b]\times[a,b]$.

$$0\le\iint_D[f(x)g(y)-f(y)g(x)]^2\,dxdy$$

$$=\iint_D f^2(x)g^2(y)dxdy+\iint_D f^2(y)g^2(x)dxdy$$

$$-2\iint_D f(x)f(y)g(x)g(y)dxdy$$

$$=2\int_a^b f^2(x)dx\int_a^b g^2(y)dy$$

$$-2\int_a^b f(x)g(x)dx\int_a^b f(y)g(y)dy$$

$$=2\int_a^b f^2(x)dx\int_a^b g^2(x)dx-2\left(\int_a^b f(x)g(x)dx\right)^2.\square$$

*例: $f(x)\in C[0,1],\ f>0,\ f\downarrow$.求证

$$\frac{\int_0^1 xf^2(x)dx}{\int_0^1 xf(x)dx}\le\frac{\int_0^1 f^2(x)dx}{\int_0^1 f(x)dx}.$$

证明: 只要证 $\mathrm{I}=\int_0^1 xf^2(x)dx\int_0^1 f(x)dx$

$$-\int_0^1 xf(x)dx\int_0^1 f^2(x)dx\le 0.$$

定积分与积分变量所用字母无关, 故

$$I=\int_0^1 xf^2(x)dx\int_0^1 f(y)dy-\int_0^1 xf(x)dx\int_0^1 f^2(y)dy$$

即 $I=\iint_{0\le x,y\le 1} xf^2(x)f(y)dxdy$

$$-\iint_{0\le x,y\le 1} xf(x)f^2(y)dxdy$$

$$=\iint_{0\le x,y\le 1} xf(x)f(y)\left[f(x)-f(y)\right]dxdy$$

由于积分区域关于直线$y=x$对称,

$$I=\iint_{0\le x,y\le 1} yf(x)f(y)\left[f(y)-f(x)\right]dxdy$$

两式相加,由$f>0,\ f\downarrow$,得

$$2I=\iint_{0\le x,y\le 1}(x-y)f(x)f(y)\left[f(x)-f(y)\right]dxdy$$

$$\le 0. \square$$

*例: 设$D=\{(x,y)\mid 0\le x,y\le 1\}$, $z=f(x,y)$
$\in C^2(D)$.若

$$\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|\le 4,\quad \forall (x,y)\in D$$

$$f(x,y)\equiv f_x'(x,y)\equiv 0,\quad \forall (x,y)\in \partial D,$$

则$\left|\iint\limits_D f(x,y)dxdy\right|\le 1.$

证明: $\iint_D f(x,y)dxdy = \int_0^1 dy \int_0^1 f(x,y)dx$

(分部积分) $= \int_0^1 \left[x f(x,y) \Big|_{x=0}^{1} - \int_0^1 x \frac{\partial f(x,y)}{\partial x} dx \right] dy$

$$\boxed{0} = -\int_0^1 dy \int_0^1 x \frac{\partial f}{\partial x} dx = -\int_0^1 x dx \int_0^1 \frac{\partial f}{\partial x} dy$$

(分部积分) $= -\int_0^1 x \left[y \frac{\partial f}{\partial x} \Big|_{y=0}^{1} - \int_0^1 y \frac{\partial^2 f}{\partial x \partial y} dy \right] dx$

$$= \int_0^1 x\,dx \int_0^1 y \frac{\partial^2 f}{\partial x \partial y} dy = \iint_D xy \frac{\partial^2 f}{\partial x \partial y} dxdy$$

于是

$$\left| \iint_D f(x, y) dxdy \right| = \left| \iint_D xy \frac{\partial^2 f}{\partial x \partial y} dxdy \right|$$

$$\leq \iint_D \left| xy \frac{\partial^2 f}{\partial x \partial y} \right| dxdy \leq 4 \iint_D xydxdy$$

$$= 4 \int_0^1 x\,dx \int_0^1 y\,dy = 1. \square$$

作业:

习题3.3 No.5,6,11.