习题1.4

2. (1) $f(x,y)=\sqrt{|x|}\cos y$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{\sqrt{|x|}}{x}$$ 不存在

$\Rightarrow$ 在 $(0,0)$ 不可微.

(3) $$f(x,y)=\begin{cases}\dfrac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-(0+0)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{(x^2+y^2)^2}$$. 取 $$\lim_{\substack{x\to 0\\ y=kx}}\frac{x^2y^2}{(x^2+y^2)^2}=\frac{k^2}{(1+k^2)^2}$$ 与 k 有关. 故该极限不存在.

$\Rightarrow$ 在 $(0,0)$ 不可微.

4. (5)
$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{x-y}{x+y}\right)=\frac{2y}{(x+y)^2}\\ \frac{\partial z}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{x-y}{x+y}\right)=\frac{-2x}{(x+y)^2}\end{aligned}\right\}\Rightarrow dz=\frac{2y}{(x+y)^2}dx-\frac{2x}{(x+y)^2}dy$$

(8). $z=\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j=x^TAx$, $A=(1)_{n\times n}$ 是对称

$$\begin{aligned}z(x+\Delta x)-z(x)&=(x+\Delta x)^TA(x+\Delta x)-x^TAx\\ &=x^TAx+x^TA\Delta x+\Delta x^TAx+\Delta x^TA\Delta x-x^TAx\\ &=\left(x^TA\Delta x+\Delta x^TAx\right)+\Delta x^TA\Delta x\end{aligned}$$

由于 $\Delta x^TAx\in\mathbb{R}\Rightarrow \Delta x^TAx=(\Delta x^TAx)^T=x^TA^T\Delta x=x^TA\Delta x$

$\Rightarrow z(x+\Delta x)-z(x)=2x^TA\Delta x+\Delta x^TA\Delta x$

$\lim\limits_{\Delta x\to\vec{0}} \dfrac{\Delta x^T A \Delta x}{\|\Delta x\|} = \lim\limits_{\Delta x\to\vec{0}} \dfrac{\Delta x^T Q^T D Q \Delta x}{\|\Delta x\|}$. 这里 $D = QAQ^T$ 为对角矩阵. $D = diag\{\lambda_1 \cdots \lambda_n\}$. $\lambda_1 \cdots \lambda_n$ 为A的特征值. Q为单位正交矩阵. 设 $y = Q\Delta x$. 则 $\|y\|^2 = \langle y, y\rangle = \langle Q\Delta x, Q\Delta x\rangle = \langle Q^TQ\Delta x, \Delta x\rangle = \langle \Delta x, \Delta x\rangle = \|\Delta x\|^2$. 故 $\Delta x\to\vec{0}$ 时 $y\to\vec{0}$

$$\Rightarrow \lim_{\Delta x\to\vec{0}} \frac{\Delta x^T A\Delta x}{\|\Delta x\|} = \lim_{y\to\vec{0}} \frac{y^T D y}{\|y\|} = \lim_{y\to\vec{0}} \frac{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2}{\sqrt{y_1^2+\cdots+y_n^2}}$$

而 $0 \le \left|\dfrac{\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2}{\sqrt{y_1^2+\cdots+y_n^2}}\right| \le \dfrac{|\lambda_1| y_1^2}{\|y\|} + \cdots + \dfrac{|\lambda_n| y_n^2}{\|y\|} \le |\lambda_1||y_1| + \cdots + |\lambda_n||y_n| \to 0,\ (y_1, \cdots, y_n) \to (0, 0, \cdots, 0)$

$$\Rightarrow \lim_{\Delta x\to\vec{0}} \frac{\Delta x^T A \Delta x}{\|\Delta x\|} = 0$$

$$\Rightarrow z(x+\Delta x) - z(x) = 2x^T A \Delta x + o(\|\Delta x\|)$$

$\Rightarrow dz = 2x^T A\, dx$. 其中 $\begin{cases} x^T A = (x_1, x_2, \cdots, x_n)\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1\end{pmatrix} = (S, S, \cdots, S),\ S \triangleq \sum\limits_{i=1}^n x_i \\ dx = (dx_1\ dx_2 \cdots dx_n)^T \end{cases}$

$\Rightarrow dz = 2S\,dx_1 + 2S\,dx_2 + \cdots + 2S\,dx_n$, 其中 $S = \sum\limits_{i=1}^n x_i$

这里给出了更一般的情况，直接计算也是可以的

8. $\lim\limits_{(x,y)\to(0,0)} \sqrt[3]{xy} = 0 = f(0,0) \Rightarrow f(x,y)$ 在原点连续

$\left.\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0 \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0 \end{aligned}\right\} \Rightarrow f(x,y)$ 在 $(0,0)$ 偏导存在.

$\dfrac{\partial f}{\partial l}(0,0) = \lim\limits_{t\to 0^+} \dfrac{f(at, bt) - f(0,0)}{\sqrt{a^2+b^2}\,t} = \lim\limits_{t\to 0^+} \dfrac{\sqrt[3]{ab}}{\sqrt{a^2+b^2}\,t^{\frac{1}{3}}} = +\infty$ 不存在.

9. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$: $\lim\limits_{\substack{x\to 0 \\ y = kx^3 (k\ne 0)}} \dfrac{x^3}{y} = \dfrac{1}{k}$ 与k有关 $\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ 不存在. 故 $f(x,y)$ 在原点处不连续.

设 $\vec{l^0} = (\cos\alpha, \sin\alpha)$. 则沿任意方向 l^0 $(0\le\alpha<2\pi)$ 的方向导数为:

$\left.\begin{aligned} &\text{当}\alpha\ne 0,\pi.\ \frac{\partial f}{\partial l^0}(0,0) = \lim_{t\to 0^+} \frac{f(t\cos\alpha, t\sin\alpha) - f(0,0)}{t} = \lim_{t\to 0^+} \frac{t\cos^3\alpha}{\sin\alpha} = 0 \\ &\text{当}\alpha=0\text{或}\pi\ \frac{\partial f}{\partial l^0}(0,0) = \lim_{t\to 0^+} \frac{f(t\cos\alpha, 0) - f(0,0)}{t} = \lim_{t\to 0^+} \frac{0-0}{t} = 0 \end{aligned}\right\} \Rightarrow \dfrac{\partial f}{\partial l^0}(0,0) = 0,\ \forall 0\le\alpha<2\pi$

$\Rightarrow f(x,y)$ 沿任意方向的方向导数存在

要注意讨论alpha为0或者\pi的情况

13. $\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x - y - z \in C(R^3) \\ \frac{\partial u}{\partial y} &= 2y - x + z \in C(R^3) \\ \frac{\partial u}{\partial z} &= 2z - x + y \in C(R^3) \end{aligned}\right\} \Rightarrow u(x,y,z)$ 在 R^3 上可微.

故 $\max\left(\dfrac{\partial u}{\partial l^0}(P)\right) = \|grad\,u(P)\| = \left.\sqrt{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial z}\right)^2}\right|_P = \sqrt{0^2+2^2+2^2} = 2\sqrt{2}$ 为最大值

方向 $\vec{l^0}$ 与 $grad\,u(P)$ 同向: $\vec{l^0} = \dfrac{grad\,u(P)}{\|grad\,u(P)\|} = \dfrac{(0,2,2)}{2\sqrt{2}} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ 时取得最值

$\dfrac{\partial u}{\partial l^0}(P) = grad\,u(P)\cdot\vec{l^0}$, 故 $\dfrac{\partial u}{\partial l^0}(P) = 0 \Leftrightarrow grad\,u(P)\cdot l^0 = 0 \Leftrightarrow \vec{l^0} = \pm\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ 时方向导数为0

形如 (a, b, −b) 的方向即可

15. (2) 记 $\sqrt{x^2+y^2+z^2}=r$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{-x}{r^3}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) = \frac{-r^3 + x\cdot\frac{\partial}{\partial x}(r^3)}{r^6} = \frac{-r^3+3x^2 r}{r^6} = \frac{-r^2+3x^2}{r^5}$$

同理 $\frac{\partial^2 u}{\partial y^2} = \frac{-r^2+3y^2}{r^5}$, $\frac{\partial^2 u}{\partial z^2} = \frac{-r^2+3z^2}{r^5}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

(3) ① $\left.\begin{array}{l} \frac{\partial u}{\partial x} = e^x\cos y \\ \frac{\partial v}{\partial y} = e^x \cos y \end{array}\right\} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\left.\begin{array}{l} \frac{\partial u}{\partial y} = -e^x \sin y \\ \frac{\partial v}{\partial x} = e^x \sin y \end{array}\right\} \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

② $\left.\begin{array}{l} \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^x\cos y) = e^x \cos y \\ \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-e^x \sin y) = -e^x\cos y \end{array}\right\} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

由于u为初等函数，故 $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y \partial x}$ 均为连续函数

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

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习题1.5

4. $$\frac{dz}{dx} = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}$$

$$= \left(\ln(u-v) + \frac{u}{u-v}\right)(-e^{-x}) + \left(\frac{-u}{u-v}\right)\frac{1}{x}$$

$$= -\left(\ln(u-v) + \frac{u}{u-v}\right)e^{-x} - \frac{u}{x(u-v)}$$

$$= -\left[\ln(e^{-x} - \ln x) + \frac{e^{-x}}{e^{-x} - \ln x}\right]e^{-x} - \frac{e^{-x}}{x(e^{-x} - \ln x)}$$

5. $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta\right] + \left[\left(\frac{\partial u}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

8. $$\begin{cases} w = x+y+z \\ u = x \\ v = x+y \end{cases} \quad w(u,v),\ z(x,y)$$

$$\Rightarrow z(x,y) = w(u,v) - x - y = w\big(u(x,y), v(x,y)\big) - x - y$$

$$\Rightarrow \begin{cases} \dfrac{\partial z}{\partial x} = \left(\dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial v}\dfrac{\partial v}{\partial x}\right) - 1 = \dfrac{\partial w}{\partial u} + \dfrac{\partial w}{\partial v} - 1 \\ \dfrac{\partial z}{\partial y} = \left(\dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial w}{\partial v}\dfrac{\partial v}{\partial y}\right) - 1 = \dfrac{\partial w}{\partial v} - 1 \end{cases}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} - 1\right) = \left(\frac{\partial^2 w}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v\partial u}\cdot\frac{\partial v}{\partial x}\right) + \left(\frac{\partial^2 w}{\partial u\partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v\partial u} + \frac{\partial^2 w}{\partial u\partial v} + \frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v} - 1\right) = \frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x} = \frac{\partial^2 w}{\partial u\partial v} + \frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v} - 1\right) = \frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y} = \frac{\partial^2 w}{\partial v^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v\partial u} - \frac{\partial^2 w}{\partial u\partial v} + \frac{\partial w}{\partial u}$$

故 $$\frac{\partial^2 w}{\partial u^2} + \underbrace{\frac{\partial^2 w}{\partial v\partial u} - \frac{\partial^2 w}{\partial u\partial v}}_{=0} + \frac{\partial w}{\partial u} = 0$$

注意在光滑性足够的
情况下混合偏导相等

典型错误

1.4 节
2（3）题

$$(3)\ f(x,y)=\begin{cases}\dfrac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0;\end{cases}$$

先计算偏导数，再用定义证明是否可微

$$(3)\ \lim_{\substack{(x,y)\to(0,0)\\ y=kx}}\frac{\frac{x^2y^2}{(x^2+y^2)^{3/2}}-0}{(x^2+y^2)^{1/2}}=\lim_{\substack{(x,y)\to(0,0)\\ y=kx}}\frac{k^2x^4}{(1+k^2)^2x^4}=\frac{k^2}{(1+k^2)^2}$$ 不是定值. 故不可微.

4（8）题
要注意形如$x_i x_j (i\neq j)$的有两项

). 初等函数. 显然可微.

$$\therefore\ dz=\sum_{i=1}^{n}\frac{\partial z}{\partial x_i}dx_i=\cancel{x_i+\sum_{j=1}^{n}x_j}\ \sum_{i=1}^{n}\left(x_i+\sum_{j=1}^{n}x_j\right)dx_i$$