



Review

- $\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$

- $a < b < c$, 则 $f \in R[a, c] \Leftrightarrow f \in R[a, b] \& f \in R[b, c].$

此时 $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$

- $f(x) \leq g(x) \Rightarrow \int_a^b f(x) dx \leq \int_a^b g(x) dx.$

- $f \in R[a, b] \Rightarrow |f| \in R[a, b],$ 且 $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$

- $f, g \in R[a, b] \Rightarrow fg \in R[a, b].$



- $f \in R[a, b], f \geq 0 \Rightarrow \sqrt{f} \in R[a, b].$

- $f \in R[a, b], |f| \geq \lambda > 0 \Rightarrow \frac{1}{f} \in R[a, b].$

- Cauchy不等式

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx.$$

- 积分第一中值定理

$f \in C[a, b], g \in R[a, b], g$ 不变号, 则 $\exists \xi \in [a, b], s.t.$

$$\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx.$$



§ 3. 微积分基本定理—Newton-Leibniz公式

Def. 称 F 为 f 在区间 I 上的一个原函数,若

$$F'(x) = f(x), \quad \forall x \in I.$$

Remark. 求原函数与求导互为逆运算.

Def. f 在区间 I 上有原函数,称 f 在 I 上的原函数全体为 f 在 I 上的不定积分,记为 $\int f(x)dx$.

Remark. F 和 G 都为 f 在区间 I 上的原函数,则

$$(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0.$$

$$F(x) - G(x) \equiv \text{const.}$$



Remark.

(1) F 为 f 在区间 I 上的原函数, 则 $\int f(x)dx = F(x) + C$,

C 为任意常数.

$$(2) \int [\alpha f(x) + \beta g(x)] dx = \alpha \int f(x) dx + \beta \int g(x) dx,$$

$$(3) \frac{d}{dx} \int f(x) dx = f(x), \int f'(x) dx = f(x) + C.$$

$$\int \cos x dx = \sin x + C,$$

$$\int \sin x dx = -\cos x + C,$$

$$\int \sec^2 x dx = \tan x + C,$$

$$\int \csc^2 x dx = -\cot x + C,$$

$$\int \sec x \tan x dx = \sec x + C,$$

$$\int \csc x \cot x dx = -\csc x + C,$$

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$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1), \quad \int \frac{1}{x} dx = \ln|x| + C,$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1),$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C,$$

$$\int \frac{dx}{1+x^2} = \arctan x + C = -\operatorname{arccot} x + C,$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C.$$



$$\begin{aligned}\text{Ex. } \int \frac{2x^2}{1+x^2} dx &= \int \left(2 - \frac{2}{1+x^2} \right) dx \\ &= 2 \int dx - 2 \int \frac{1}{1+x^2} dx = 2x - 2 \arctan x + C.\end{aligned}$$

$$\begin{aligned}\text{Ex. } \int \frac{1}{\cos^2 x \cdot \sin^2 x} dx &= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \cdot \sin^2 x} dx \\ &= \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} = \tan x - \cot x + C.\end{aligned}$$

$$\text{Ex. } \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \int \left(\frac{1}{a-x} + \frac{1}{a+x} \right) dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C.$$



Thm.(微积分基本定理)

$f \in R[a, b], F(x) = \int_a^x f(t)dt \ (a \leq x \leq b)$, 则

(1) $F \in C[a, b]$;

(2) 若 f 在 $x_0 \in [a, b]$ 连续, 则 F 在 x_0 可导, 且 $F'(x_0) = f(x_0)$;

(3) 若 $f \in C[a, b]$, 则 F 是 f 在 $[a, b]$ 上的一个原函数. 若 G 为 f 的任一个原函数, 则

$$\int_a^b f(t)dt = G(b) - G(a) \triangleq G(x) \Big|_a^b. \text{ (Newton-Leibniz)}$$

Proof. (1) $f \in R[a, b]$, 则 $\exists M > 0, s.t. |f(x)| \leq M \ (a \leq x \leq b)$.

$$|F(x) - F(x_0)| = \left| \int_{x_0}^x f(t)dt \right| \leq \left| \int_{x_0}^x |f(t)|dt \right| \leq M |x - x_0|.$$



(2) f 在 $x_0 \in [a, b]$ 连续, 则 $\forall \varepsilon > 0, \exists \delta > 0, s.t.$

$$|f(x) - f(x_0)| < \varepsilon, \quad \forall |x - x_0| < \delta, x \in [a, b].$$

于是, $\forall 0 < |x - x_0| < \delta, x \in [a, b]$, 有

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t) dt - f(x_0) \right| \\ &= \left| \frac{1}{x - x_0} \int_{x_0}^x [f(t) - f(x_0)] dt \right| \\ &\leq \frac{1}{|x - x_0|} \left| \int_{x_0}^x |f(t) - f(x_0)| dt \right| \leq \varepsilon. \end{aligned}$$



(3) $f \in C[a, b]$, 由(2)知, F 是 f 在 $[a, b]$ 上的一个原函数.

设 G 是 f 在 $[a, b]$ 上的任一原函数, 则

$$(G(x) - F(x))' = G'(x) - F'(x) = f(x) - f(x) \equiv 0, x \in [a, b].$$

因此, \exists 常数 C , s.t. $G(x) \equiv F(x) + C$, 从而

$$\int_a^x f(t) dt = F(x) = F(x) - F(a) = G(x) - G(a), \forall x \in [a, b].$$

特别地, $\int_a^b f(t) dt = G(b) - G(a).$ \square

Question. $f \in R[a, b]$, 是否有 $\left(\int_a^x f(t) dt\right)' = f(x), \forall x \in [a, b]$?



Ex. 求 $\int e^{|x|} dx$.

解: $e^{|x|}$ 在 \mathbb{R} 上连续, 因而 $\forall x \in \mathbb{R}, F(x) = \int_0^x e^{|t|} dt$ 有定义, 且

$$F'(x) = e^{|x|} = \begin{cases} e^x, & x \geq 0 \\ e^{-x}, & x < 0 \end{cases}, \quad F(x) = \begin{cases} e^x + C_1, & x \geq 0 \\ -e^{-x} + C_2, & x < 0 \end{cases}.$$

$F(x)$ 在 $x = 0$ 处连续, 则 $1 + C_1 = -1 + C_2$,

$$\int e^{|x|} dx = F(x) + C = C + \begin{cases} e^x, & x \geq 0 \\ -e^{-x} + 2, & x < 0 \end{cases}. \quad \square$$



Thm. f 连续, u, v 可导, $G(x) = \int_{v(x)}^{u(x)} f(t) dt$, 则

$$G'(x) = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x).$$

解: 令 $F(u) = \int_a^u f(t) dt$, 则 $F'(u) = f(u)$.

$$\begin{aligned} G(x) &= \int_a^{u(x)} f(t) dt - \int_a^{v(x)} f(t) dt \\ &= F(u(x)) - F(v(x)) \end{aligned}$$

$$\begin{aligned} G'(x) &= F'(u(x)) \cdot u'(x) - F'(v(x)) \cdot v'(x) \\ &= f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x). \square \end{aligned}$$



Ex. f 连续, $F(x) = \int_a^x (x-t)f(t)dt$, 求 $F''(x)$.

解: $F(x) = x \int_a^x f(t)dt - \int_a^x tf(t)dt,$

$$F'(x) = \int_a^x f(t)dt + xf(x) - xf(x) = \int_a^x f(t)dt, \quad F''(x) = f(x). \square$$

Ex. $\lim_{x \rightarrow +\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt} = \lim_{x \rightarrow +\infty} \frac{2e^{x^2} \int_0^x e^{t^2} dt}{e^{2x^2}}$

$$= \lim_{x \rightarrow +\infty} \frac{2 \int_0^x e^{t^2} dt}{e^{x^2}} = \lim_{x \rightarrow +\infty} \frac{2e^{x^2}}{2xe^{x^2}} = 0. \square$$



Ex. k, n 为非负整数, $a \in \mathbb{R}$, 则

$$\int_a^{a+2\pi} \sin kt \, dt = 0;$$

$$\int_a^{a+2\pi} \cos kt \, dt = \begin{cases} 0, & k \neq 0 \\ 2\pi & k = 0 \end{cases};$$

$$\int_a^{a+2\pi} \sin kt \cdot \cos nt \, dt = 0;$$

$$\int_a^{a+2\pi} \sin kt \cdot \sin nt \, dt = \begin{cases} 0, & k \neq n \\ \pi, & k = n \neq 0 \end{cases};$$

$$\int_a^{a+2\pi} \cos kt \cdot \cos nt \, dt = \begin{cases} 0, & k \neq n \\ \pi, & k = n \end{cases}.$$



Ex. $\sigma_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$, 求 $\lim_{n \rightarrow +\infty} \sigma_n$.

解: $\sigma_n = \frac{1}{n} \left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}} \right]$ 是函数 $\frac{1}{1+x}$ 在 $[0,1]$ 上

关于 n 等分分割的一个 Riemann 和. 而 $\frac{1}{1+x}$ 在 $[0,1]$ 上可积,

$\ln(1+x)$ 是 $\frac{1}{1+x}$ 的一个原函数, 故

$$\lim_{n \rightarrow +\infty} \sigma_n = \int_0^1 \frac{1}{1+x} dx = \ln(1+x) \Big|_0^1 = \ln 2. \square$$



Ex. $\sigma_n = \frac{1}{n+1} + \frac{1}{n+3} + \cdots + \frac{1}{n+(2n+1)}$, 求 $\lim_{n \rightarrow +\infty} \sigma_n$.

法一: $\lim_{n \rightarrow +\infty} \sigma_n = \lim_{n \rightarrow +\infty} \frac{1}{2} \sum_{k=1}^n \frac{2}{n} \cdot \frac{1}{1 + \frac{2k-1}{n}} + \lim_{n \rightarrow +\infty} \frac{1}{n + (2n+1)}$

$$= \frac{1}{2} \int_0^2 \frac{1}{1+x} dx + 0 = \frac{1}{2} \ln(1+x) \Big|_0^2 = \frac{1}{2} \ln 3. \square$$

法二: $\lim_{n \rightarrow +\infty} \sigma_n = \lim_{n \rightarrow +\infty} \sum_{k=1}^n \frac{1}{n} \cdot \frac{1}{1 + \frac{2(k-1/2)}{n}} + \lim_{n \rightarrow +\infty} \frac{1}{n + (2n+1)}$

$$= \int_0^1 \frac{1}{1+2x} dx + 0 = \frac{1}{2} \ln(1+2x) \Big|_0^1 = \frac{1}{2} \ln 3. \square$$



Ex. $\int_0^x e^t dt = xe^{\theta(x)x}$, $\theta(x) \in (0,1)$. 求证: $\lim_{x \rightarrow +\infty} \theta(x) = 1$.

Proof. $xe^{\theta(x)x} = \int_0^x e^t dt = e^t \Big|_{t=0}^x = e^x - 1$.

$$\lim_{x \rightarrow +\infty} \theta(x) = \lim_{x \rightarrow +\infty} \frac{\ln(e^x - 1) - \ln x}{x}$$

$$= \lim_{x \rightarrow +\infty} \left(\frac{e^x}{e^x - 1} - \frac{1}{x} \right)$$

$$= 1. \square$$



Ex. f 可导, $f'(0) \neq 0$, $\int_0^x f(t)dt = f(\xi(x))x$, $\xi(x) \in (0, x)$, 求 $\lim_{x \rightarrow 0} \frac{\xi(x)}{x}$.

解: 令 $F(x) = \int_0^x f(t)dt$, 则 $F'(x) = f(x)$, $F''(x) = f'(x)$,

$$F(x) = F(0) + F'(0)x + \frac{1}{2!} F''(0)x^2 + o(x^2), x \rightarrow 0.$$

于是 $f(\xi(x))x = f(0)x + \frac{1}{2} f'(0)x^2 + o(x^2),$

$$\lim_{x \rightarrow 0} \frac{f(\xi(x)) - f(0)}{x} = \lim_{x \rightarrow 0} \left(\frac{1}{2} f'(0) + o(1) \right) = \frac{1}{2} f'(0),$$

$$\lim_{x \rightarrow 0} \frac{\xi(x)}{x} = \frac{\lim_{x \rightarrow 0} \frac{f(\xi(x)) - f(0)}{x}}{\lim_{x \rightarrow 0} \frac{f(\xi(x)) - f(0)}{\xi(x)}} = \frac{\frac{1}{2} f'(0)}{f'(0)} = \frac{1}{2}. \square$$



Ex. $f \in C^2[-1,1]$, $f(0) = 0$, 则 $\exists \xi \in [-1,1]$, s.t. $f''(\xi) = 3 \int_{-1}^1 f(x) dx$.

证法一: $f(0) = 0$, 则 $f(x) = f'(0)x + \frac{f''(\xi_x)}{2}x^2$, ξ_x 介于 0 与 x 之间.

$f \in C^2[-1,1]$, 记 $M = \max_{-1 \leq x \leq 1} f''(x)$, $m = \min_{-1 \leq x \leq 1} f''(x)$, 则

$$f'(0)x + \frac{m}{2}x^2 \leq f(x) \leq f'(0)x + \frac{M}{2}x^2, x \in [-1,1].$$

$$\frac{m}{3} = \frac{m}{2} \int_{-1}^1 x^2 dx \leq \int_{-1}^1 f(x) dx \leq \frac{M}{2} \int_{-1}^1 x^2 dx = \frac{M}{2} \cdot \frac{1}{3} x^3 \Big|_{-1}^1 = \frac{M}{3}.$$

$f \in C^2[-1,1]$, 由介值定理, $\exists \xi \in [-1,1]$, s.t. $f''(\xi) = 3 \int_{-1}^1 f(x) dx$.

证法二: $F(x) = \int_0^x f(t) dt$, 带 Lagrange 余项的 2 阶 Taylor 公式. \square



**作业：习题5.3 No.4,5,7,
12(6),13(1),14,15**