# Homework

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## 4.7

### 4.7.3

**(5)**

$$\oiint_{S+}\mathbf{A}\,\mathrm{d}\mathbf{S}=\oiint_{S+}\frac{x\,\mathrm{d}y\wedge\mathrm{d}z+y\,\mathrm{d}z\wedge\mathrm{d}x+z\,\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0$$

且在除$(0,0)$上的点均连续。

考虑曲面

$$S_0: x^2+y^2+z^2=\epsilon^2$$

$$\oiint_{S+}\mathbf{A}\,\mathrm{d}\mathbf{S}+\oiint_{S_0-}\mathbf{A}\,\mathrm{d}\mathbf{S}=\oiint_{S_1+}\mathbf{A}\,\mathrm{d}\mathbf{S}=\iiint_{\Omega}\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0$$

$$\oiint_{S_0+}\mathbf{A}\,\mathrm{d}\mathbf{S}=\oiint_{S_0+}\frac{x\,\mathrm{d}y\wedge\mathrm{d}z+y\,\mathrm{d}z\wedge\mathrm{d}x+z\,\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{\frac{3}{2}}}=3\iint_{y^2+z^2\le\epsilon^2}\frac{\sqrt{\epsilon^2-y^2-z^2}\,\mathrm{d}y\,\mathrm{d}z}{\epsilon^3}$$

$$-3\iint_{y^2+z^2\le\epsilon^2}\frac{\sqrt{\epsilon^2-y^2-z^2}(-\,\mathrm{d}y\,\mathrm{d}z)}{\epsilon^3}=6\iint_{y^2+z^2\le\epsilon^2}\frac{\sqrt{\epsilon^2-y^2-z^2}\,\mathrm{d}y\,\mathrm{d}z}{\epsilon^3}=6\int_0^{2\pi}\mathrm{d}\theta\int_0^{\epsilon}r\sqrt{\epsilon^2-r^2}\,\mathrm{d}r=4\pi$$

从而

$$\oiint_{S+}\mathbf{A}\,\mathrm{d}\mathbf{S}=4\pi$$

### 4.7.5

**(1)**

由Stokes公式

$$\oint_{L+}y\,\mathrm{d}x+z\,\mathrm{d}y+x\,\mathrm{d}z=\iint_{\Omega}(-1-1-1)\mathbf{n}^o\,\mathrm{d}S$$

又

$$\mathbf{n}^o=(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3})$$

从而

$$\oint_{L+} y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z = \iint_{\Omega} (-1-1-1)\mathbf{n}^o\,\mathrm{d}S = -\sqrt{3}\iint_{\Omega}\mathrm{d}S = \sqrt{3}\pi R^2$$

## 4.7.6

**(2)**

$$\frac{y}{(x+z)^2+y^2}\,\mathrm{d}x + \frac{-(x+z)}{(x+z)^2+y^2}\,\mathrm{d}y + \frac{y}{(x+z)^2+y^2}\,\mathrm{d}z$$

$$X = \frac{y}{(x+z)^2+y^2}, Y = \frac{-(x+z)}{(x+z)^2+y^2}, Z = \frac{y}{(x+z)^2+y^2}$$

$$rot\mathbf{V} = (\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z})\mathbf{i} + (\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x})\mathbf{j} + (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\mathbf{k} = \mathbf{0}$$

从而，存在$u$,使得

$$\mathrm{d}u = X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z$$

注意到

$$\int \frac{y}{(x+z)^2+y^2}\,\mathrm{d}x = \arctan\frac{x+z}{y} + C(y,z)$$

$$\int \frac{-(x+z)}{(x+z)^2}\,\mathrm{d}y = \arctan\frac{x+z}{y} + C(z,x)$$

$$\int \frac{y}{(x+z)^2+y^2}\,\mathrm{d}z = \arctan\frac{x+z}{y} + C(x,y)$$

从而

$$u = \arctan\frac{x+z}{y} + C$$

## 4.7.7

**(1)**

注意到

$$\mathbf{V} = (y+z, z+x, x+y)$$

为$R^3$上的连续可微向量场

$$rot\mathbf{V} = (\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z})\mathbf{i} + (\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x})\mathbf{j} + (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\mathbf{k} = \mathbf{0}$$

从而该曲线积分与路径无关。

$$\int_{(0,0,0)^{1,2,1}} (y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\,\mathrm{d}z = 0 + \int_0^2 1\,\mathrm{d}y + \int_0^1 3\,\mathrm{d}z = 5$$

# 5.2

## 5.2.3

**(1)**

由根值判敛法,

$$\lim_{n\to+\infty}\sqrt[n]{\frac{2^n}{\sqrt{n^n}}}=\lim_{n\to+\infty}\frac{2}{\sqrt{n}}=0<1$$

从而收敛

**(2)**

由根值判敛法,

$$\lim_{n\to+\infty}\sqrt[n]{\frac{1}{3^n}(1+\frac{1}{n})^{n^2}}=\lim_{n\to+\infty}\frac{1}{3}(1+\frac{1}{n})^n=\frac{e}{3}<1$$

从而收敛

**(3)**

记

$$u_n=\frac{1}{n^p(\ln n)^q(\ln\ln n)^r}$$

(1)当$p>1$时，取$v_n=\frac{1}{n^{\frac{1+p}{2}}},\forall q\in\mathbf{R}$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n^{\frac{1-p}{2}}}{(\ln n)^q(\ln\ln n)^r}=+\infty$$

而此时$\frac{1+p}{2}>1$,从而$\sum v_n$收敛
故由比率判敛法，$\sum u_n$收敛
(2)当$p<1$时，取$v_n=\frac{1}{n^{\frac{1+p}{2}}},\forall q\in\mathbf{R}$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n^{\frac{1-p}{2}}}{(\ln n)^q(\ln\ln n)^r}=+\infty$$

而此时$\frac{1+p}{2}<1$,从而$\sum v_n$发散
故由比率判敛法，$\sum u_n$发散
(3)当$p=1$时

$$u_n=\frac{1}{n(\ln n)^q(\ln\ln n)^r}$$

$\sum u_n$与积分

$$\int_3^{+\infty}\frac{1}{x(\ln x)^q(\ln\ln x)}\,\mathrm{d}x$$

同敛散

$$\int_3^{+\infty}\frac{1}{x(\ln x)^q(\ln\ln x)}\,\mathrm{d}x=\int_{\ln 3}^{+\infty}\frac{1}{e^t t^q(\ln t)^r}\,\mathrm{d}e^t=\int_{\ln 3}^{+\infty}\frac{1}{t^q(\ln t)^r}\,\mathrm{d}t$$

同理，
（1）当$q>1$时，级数收敛
（2）当$q<1$时，级数发散
（3）当$q=1$时，
　（i）当$r>1$时，级数收敛
　（ii）当$r\le 1$时，级数发散

**(4)**

设$u_n=\frac{1}{\sqrt[3]{n+1}}\ln\frac{n+2}{n}$ 由Taylor

$$u_n=\frac{1}{\sqrt[3]{n+1}}\ln\frac{n+2}{n}\sim\frac{2}{n\sqrt[3]{n+1}}\sim\frac{2}{n^{\frac{4}{3}}}$$

从而级数$\sum u_n$收敛

**(5)**

设$u_n=(\sin(\frac{\pi}{4}+\frac{1}{n}))^n$
由根值判敛法

$$\lim_{n\to+\infty}\sqrt[n]{u_n}=\lim_{n\to+\infty}\sin(\frac{\pi}{4}+\frac{1}{n})=\frac{\sqrt{2}}{2}$$

从而级数$\sum u_n$收敛

**(6)**

由比率判敛法
设$u_n=\frac{\ln(n!)}{n!}$

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to+\infty}\frac{\ln((n+1)!)}{(n+1)!}\frac{n!}{\ln n!}=\lim_{n\to+\infty}\frac{\ln n!+\ln(n+1)}{(n+1)\ln n!}$$

$$=\lim_{n\to+\infty}\frac{1}{n+1}+\frac{\ln(n+1)}{(n+1)\ln n!}=0$$

从而级数$\sum u_n$收敛

**(7)**

设$u_n=e^{-\frac{n^2+1}{n+1}}$ 由根值判敛法

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}e^{-\frac{n^2+1}{n^2+n}}=\frac{1}{e}<1$$

从而级数$\sum u_n$收敛

**(8)**

设$u_n = \frac{(\ln^n n)n!}{n^n}$ 由比率判敛法

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}(\frac{n}{n+1})^n\frac{\ln^{n+1}(n+1)}{\ln^n n} = \frac{1}{e}\lim_{n\to\infty}\frac{\ln^{n+1}(n+1)}{\ln^n n} = +\infty$$

从而级数$\sum u_n$发散

**(9)**

设$u_n = \frac{3n-1}{2^n+2^{-n}}$ 由比率判敛法

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{3n+2}{3n-1}\frac{2^n+2^{-n}}{2^{n+1}+2^{-(n+1)}} = \lim_{n\to\infty}\frac{3n+2}{3n-1}\frac{2^{2n+1}+2}{2^{2n+2}+1} = \frac{1}{2}$$

从而级数$\sum u_n$收敛

**(10)**

设$u_n = \frac{1}{1+a^n}$
（1）$a \le 1$时
级数$\sum u_n$发散
（2）$a > 1$时
由比率判敛法

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1+a^n}{1+a^{n+1}} = \lim_{n\to\infty}\frac{1}{a}+\frac{1-\frac{1}{a}}{1+a^{n+1}} = \frac{1}{a} < 1$$

从而级数$\sum u_n$收敛

## 5.2.5

由于数列$\{nu_n\}$有界，从而存在$M > 0$

$$nu_n < M, u_n < \frac{M}{n}$$

又已知$\sum \frac{M}{n^2}$收敛
根据比较判敛法

$$\frac{u_n}{n} < \frac{M}{n^2}$$

从而$\sum \frac{u_n}{n}$收敛

## 5.4.8

**(2)**

注意到

$$\lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) = \lim_{n\to\infty} n(\frac{1}{n+1}\frac{(n+1)^p(q+n+1)}{n^p} - 1)$$
$$= \lim_{n\to\infty} \frac{n}{n+1}(\frac{n+1}{n})^p q + n((\frac{n+1}{n})^p - 1) = q + \lim_{n\to\infty} n((\frac{n+1}{n})^p - 1)$$
$$= q + n((\frac{n+1}{n})^p - 1)\frac{(n+1)^p - n^p}{n^{p-1}} = q + p$$

从而，当$q+p\leq 1$时，原级数收敛

当$q+p>1$时，原级数发散