Review

- 第一换元法(凑微分法)

$$\int f'(\varphi(x))\varphi'(x)\mathrm{d}x = \int f'(\varphi(x))\mathrm{d}\varphi(x) = f(\varphi(x)) + C$$

- 第二换元法

$$\int f'(u)\mathrm{d}u \overset{u=\varphi(x)}{=\!=\!=\!=} \int f'(\varphi(x))\varphi'(x)\mathrm{d}x$$

$$= g(x) + C = g(\varphi^{-1}(u)) + C$$

- 分部积分法 $\int u(x)\mathrm{d}v(x) = u(x)v(x) - \int v(x)\mathrm{d}u(x)$

- 联合积分法

§5.有理函数与三角有理函数的不定积分

1.有理函数 $\dfrac{p(x)}{q(x)}$ (p,q为多项式)

Thm.有理真分式 $\dfrac{p(x)}{q(x)}$ 可分解成最简分式之代数和:

(1)$q(x)$的一次k重因式$(x-a)^k$对应k项

$$\frac{A_1}{x-a},\frac{A_2}{(x-a)^2},\cdots,\frac{A_k}{(x-a)^k};$$

(2)$q(x)$的二次k重因式$((x+a)^2+b^2)^k$对应k项

$$\frac{B_1x+C_1}{(x+a)^2+b^2},\frac{B_2x+C_2}{((x+a)^2+b^2)^2},\cdots,\frac{B_kx+C_k}{((x+a)^2+b^2)^k}.$$

Remark. $\dfrac{Bx+C}{((x+a)^2+b^2)^k}$可以表示成形如$\dfrac{x+a}{((x+a)^2+b^2)^k}$, $\dfrac{1}{((x+a)^2+b^2)^k}$的简单分式的代数和.

- $\displaystyle\int\frac{\mathrm{d}x}{x-a}=\ln|x-a|+C,$

- $\displaystyle\int\frac{\mathrm{d}x}{(x-a)^k}=\frac{-1}{(k-1)(x-a)^{k-1}}+C,\quad (k>1)$

$$\int \frac{x+a}{(x+a)^2+b^2}\,\mathrm{d}x = \frac{1}{2}\ln((x+a)^2+b^2)+C,$$

$$\int \frac{x+a}{((x+a)^2+b^2)^k}\,\mathrm{d}x = \frac{-1}{2(k-1)((x+a)^2+b^2)^{k-1}},\quad (k>1)$$

$$J_k = \int \frac{1}{((x+a)^2+b^2)^k}\,\mathrm{d}x$$

$$J_1 = \frac{1}{b}\arctan\frac{x+a}{b}+C,$$

$$J_{k+1} = \frac{1}{2kb^2}\Big((x+a)((x+a)^2+b^2)^{-k}+(2k-1)J_k\Big).$$

Ex. $I=\int\frac{\mathrm{d}x}{1+x^2+x^4}$.

解：$1+x^2+x^4=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1)$.

$$\frac{1}{1+x^2+x^4}=\frac{ax+b}{x^2+x+1}+\frac{cx+d}{x^2-x+1}$$

$$\Rightarrow a=b=d=\frac{1}{2},c=-\frac{1}{2}.$$

$$\frac{1}{1+x^2+x^4}=\frac{1}{2}\frac{x+1}{x^2+x+1}-\frac{1}{2}\frac{x-1}{x^2-x+1}.$$

$$\int \frac{\mathrm{d}x}{1+x^2+x^4} = \frac{1}{2}\int \frac{x+1}{x^2+x+1}\mathrm{d}x - \frac{1}{2}\int \frac{x-1}{x^2-x+1}\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{x+1/2}{x^2+x+1}\mathrm{d}x + \frac{1}{4}\int \frac{\mathrm{d}x}{x^2+x+1}$$

$$-\frac{1}{2}\int \frac{x-1/2}{x^2-x+1}\mathrm{d}x + \frac{1}{4}\int \frac{\mathrm{d}x}{x^2-x+1}$$

$$= \frac{1}{4}\ln(x^2+x+1) + \frac{1}{2\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

$$-\frac{1}{4}\ln(x^2-x+1) + \frac{1}{2\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

2.三角有理式$R(\sin x, \cos x)$:$\sin x, \cos x$有限次四则运算

万能变换 $t = \tan\frac{x}{2}$, $x = 2\arctan t$, $\mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$,

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

$$\int R(\sin x, \cos x)\mathrm{d}x = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2\mathrm{d}t}{1+t^2}.$$

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Ex. $I=\int\frac{1+\sin x}{1+\cos x}\mathrm{d}x.$

解法一：令$t=\tan\frac{x}{2}$，则 $x=2\arctan t,\ \mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t,$

$$\int\frac{1+\sin x}{1+\cos x}\mathrm{d}x=\int\frac{1+\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t=\int\frac{(1+t)^2}{2}\cdot\frac{2}{1+t^2}\mathrm{d}t$$

$$=\int\left(1+\frac{2t}{1+t^2}\right)\mathrm{d}t=t+\ln(1+t^2)+C=\tan\frac{x}{2}-2\ln\left|\cos\frac{x}{2}\right|+C.$$

解法二：$I=\int\frac{1}{2\cos^2\frac{x}{2}}\mathrm{d}x+\int\frac{\sin x}{1+\cos x}\mathrm{d}x$

$$=\tan\frac{x}{2}-\ln(1+\cos x)+C.$$

Remark.万能变换不一定简单.

Ex. $\int \frac{\tan x}{a^2\cos^2 x + b^2\sin^2 x}\mathrm{d}x$

$$= \int \frac{\tan x}{a^2 + b^2\tan^2 x}\cdot\frac{\mathrm{d}x}{\cos^2 x} = \int \frac{\tan x}{a^2 + b^2\tan^2 x}\mathrm{d}\tan x$$

$$= \frac{1}{2}\int \frac{\mathrm{d}\tan^2 x}{a^2 + b^2\tan^2 x} = \frac{1}{2b^2}\ln(a^2 + b^2\tan^2 x) + C.$$

3.可化为有理式的简单无理式

$$1)\ \int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)\mathrm{d}x \quad (ad-bc\neq 0)$$

令 $t=\sqrt[n]{\frac{ax+b}{cx+d}}$, 则 $x=\frac{b-dt^n}{ct^n-a}$, $\mathrm{d}x=\frac{ad-bc}{(ct^n-a)^2}nt^{n-1}\mathrm{d}t$,

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)\mathrm{d}x=\int R\left(\frac{b-dt^n}{ct^n-a}, t\right)\frac{ad-bc}{(ct^n-a)^2}nt^{n-1}\mathrm{d}t.$$

Ex. $I=\int \frac{\mathrm{d}x}{\sqrt[3]{(x-1)(x+1)^2}}$

解: 令 $t=\sqrt[3]{\frac{x+1}{x-1}}$, 则 $x=\frac{t^3+1}{t^3-1}$, $\mathrm{d}x=\frac{-6t^2}{(t^3-1)^2}\mathrm{d}t$.

$$I=\int \sqrt[3]{\frac{x+1}{x-1}}\frac{\mathrm{d}x}{x+1}=\int \frac{-3}{t^3-1}\mathrm{d}t=\int\left(\frac{-1}{t-1}+\frac{t+2}{t^2+t+1}\right)\mathrm{d}t$$

$$=-\ln|t-1|+\frac{1}{2}\ln(t^2+t+1)+\sqrt{3}\arctan\frac{2t+1}{\sqrt{3}}+C$$

$$=\frac{1}{2}\ln|x-1|-\frac{3}{2}\ln\left|\sqrt[3]{\frac{x+1}{x-1}}-1\right|+\sqrt{3}\arctan\frac{2}{\sqrt{3}}\left(\sqrt[3]{\frac{x+1}{x-1}}-\frac{1}{2}\right)+C.$$

$$2)\int R(x,\sqrt{ax^2+bx+c})\mathrm{d}x,(a\neq 0)$$

- $\int R(x,\sqrt{(x+p)^2+q^2})\mathrm{d}x$ 令 $x+p=q\tan t,|t|<\dfrac{\pi}{2}$
- $\int R(x,\sqrt{(x+p)^2-q^2})\mathrm{d}x$ 令 $x+p=q\sec t,t\in[0,\dfrac{\pi}{2})\cup(\dfrac{\pi}{2},\pi]$
- $\int R(x,\sqrt{q^2-(x+p)^2})\mathrm{d}x$ 令 $x+p=q\sin t,|t|<\dfrac{\pi}{2}$

Euler第一替换：若$a>0$,令$\sqrt{ax^2+bx+c}=\pm\sqrt{a}x+t.$

此时,$x=\dfrac{t^2-c}{b\mp 2\sqrt{a}t}$, $\sqrt{ax^2+bx+c}=\dfrac{\sqrt{a}(t^2-c)}{-2\sqrt{a}t\pm b}+t.$

Euler第二替换：若$c>0$，令$\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$.

此时，$x=\dfrac{-b\pm2\sqrt{c}t}{a-t^2}$，$\sqrt{ax^2+bx+c}=\dfrac{-b\pm2\sqrt{c}t}{a-t^2}t\pm\sqrt{c}$.

Euler第三替换：若ax^2+bx+c有相异实根α,β，令

$$\sqrt{ax^2+bx+c}=(x-\alpha)t,$$

此时，$a(x-\alpha)(x-\beta)=(x-\alpha)^2t^2$，　$x=\dfrac{a\beta-\alpha t^2}{a-t^2}$.

$$\text{Ex.}\, I=\int\frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}}$$

解: 令$\sqrt{x^2\pm a^2}=x+t$, 则

$$x=\frac{-t^2\pm a^2}{2t},\ \mathrm{d}x=\frac{t^2\pm a^2}{-2t^2}\mathrm{d}t,\ \sqrt{x^2\pm a^2}=\frac{t^2\pm a^2}{2t},$$

$$\int\frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}}=\int\frac{\mathrm{d}t}{-t}=-\ln|t|+C$$

$$=-\ln\left|\sqrt{x^2\pm a^2}-x\right|+C=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C.\ \square$$

Ex. $I=\int \frac{\mathrm{d}x}{x^2\sqrt{4x^2-3x-1}}$

解法一: $4x^2-3x-1=(4x+1)(x-1)$, 令

$$\sqrt{(4x+1)(x-1)}=(4x+1)t,$$

则 $x=\frac{t^2+1}{1-4t^2}, \mathrm{d}x=\frac{10t\mathrm{d}t}{(1-4t^2)^2}, \sqrt{4x^2-3x-1}=\frac{5t}{1-4t^2},$

$$I=\int \frac{2(1-4t^2)\mathrm{d}t}{(t^2+1)^2}=-8\int \frac{\mathrm{d}t}{t^2+1}+10\int \frac{\mathrm{d}t}{(t^2+1)^2}=-8I_1+10I_2,$$

$$\arctan t + C = I_1 = \frac{t}{t^2+1} + \int \frac{2t^2\mathrm{d}t}{(t^2+1)^2} = \frac{t}{t^2+1} + 2\arctan t - 2I_2,$$

$$I_2 = \frac{1}{2}\left(\frac{t}{t^2+1} + \arctan t\right) + C,$$

$$I = -8I_1 + 10I_2 = -3\arctan t + \frac{5t}{t^2+1} + C$$

$$= -3\arctan\frac{\sqrt{4x^2-3x-1}}{4x+1} + \frac{5\sqrt{4x^2-3x-1}}{x} + C.$$

解法二: 令$\sqrt{4x^2-3x-1}=-2x+t$, 则 $x=\dfrac{1+t^2}{4t-3}$,

$$\mathrm{d}x=\frac{2(2t^2-3t-2)}{(4t-3)^2}\mathrm{d}t,\ \sqrt{4x^2-3x-1}=\frac{2t^2-3t-2}{4t-3},$$

$$I=2\int\frac{(4t-3)\mathrm{d}t}{(1+t^2)^2}=4\int\frac{\mathrm{d}(t^2+1)}{(1+t^2)^2}-6\int\frac{\mathrm{d}t}{(1+t^2)^2}$$

$$=-\frac{4}{1+t^2}-\frac{3t}{t^2+1}-3\arctan t+C$$

$$=\frac{\sqrt{4x^2-3x-1}-2x}{x}-3\arctan(2x+\sqrt{4x^2-3x-1})+C.\square$$

Remark.不同解法得出的结果可能形式不同.

Remark.初等函数的原函数不一定是初等函数,如

$$e^{x^2}, \sin x^2, \cos x^2, \frac{\sin x}{x}, \frac{\cos x}{x},$$

$$\frac{1}{\ln x}, \sqrt{1-a^2\sin^2 x}\,(0<a<1).$$

作业：习题5.5

No.1(2,6),2(4,7),

3(3,6),4(6,8,9)