Homework

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3.3

3.3.13

(1)

设 $x = r\cos\theta, y = r\sin\theta$, 代入条件得:

$$r^2 = 2a^2\cos 2\theta, r^2 \geq a^2$$

解得

$$\cos 2\theta \geq \frac{1}{2}, -\frac{\pi}{6} \leq \theta \geq \frac{\pi}{6}, \frac{5\pi}{6} \leq \theta \leq \frac{7\pi}{6}$$

从而有对称性,

$$S = 4\int_0^{\frac{\pi}{6}} (\frac{1}{2}(2a^2\cos 2\theta - a^2))\,\mathrm{d}\theta = a^2(\sqrt{3} - \frac{\pi}{3})$$

3.3.14

(3)

注意到

$$\iint_D x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y = 4\iint_{D_1} x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y$$

其中 $D_1 = \{(x, y) \mid 0 \leq x + y \leq 1\}$

$$\iint_{D_1} x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_0^{1-y} x^2 + y^2 \,\mathrm{d}x = \int_0^1 \frac{1}{3}(1-y)^3 + y^2(1-y)\,\mathrm{d}y$$

$$= \int_0^1 \frac{1}{3} - y + 2y^2 - \frac{4}{3}y^3 \,\mathrm{d}y = \frac{1}{6}$$

从而

$$\iint_D x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y = 4\iint_{D_1} x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y = \frac{2}{3}$$

(4)

$$\iint_D x-y^2\,\mathrm{d}x\,\mathrm{d}y=\int_{\frac{1}{3}}^{2}\mathrm{d}y\int_{y^2-y-1}^{y^2+2y-2}x-y^2\,\mathrm{d}x=\frac{1}{2}\int_{\frac{1}{3}}^{2}(y-3)(3-y)\,\mathrm{d}y=-\frac{175}{54}$$

3.3.15

(1)

设

$$u=a_1x+b_1y+c_1,v=a_2x+b_2y+c_2,u^2+v^2=1$$
$$\frac{\partial(u,v)}{\partial(x,y)}=a_1b_2-a_2b_1$$
$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{a_1b_2-a_2b_1}$$

从而

$$\iint_D \mathrm{d}x\,\mathrm{d}y=\iint_{D'}\frac{1}{a_1b_2-a_2b_1}\,\mathrm{d}u\,\mathrm{d}v=\frac{\pi}{a_1b_2-a_2b_1}$$

3.3.17

注意到 x,y 轮换对称，所以

$$\iint_{x^2+y^2\le R^2}\frac{f(x)}{f(x)+f(y)}\,\mathrm{d}x\,\mathrm{d}y=\iint_{x^2+y^2\le R^2}\frac{f(y)}{f(x)+f(y)}\,\mathrm{d}x\,\mathrm{d}y=\frac{1}{2}\iint_{x^2+y^2\le R^2}\frac{f(x)+f(y)}{f(x)+f(y)}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{\pi R^2}{2}$$

从而原式为

$$\iint_{x^2+y^2\le R^2}\frac{af(x)+bf(y)}{f(x)+f(y)}\,\mathrm{d}x\,\mathrm{d}y=a\iint_{x^2+y^2\le R^2}\frac{f(x)}{f(x)+f(y)}\,\mathrm{d}x\,\mathrm{d}y+b\iint_{x^2+y^2\le R^2}\frac{f(y)}{f(x)+f(y)}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{(a+b)\pi R^2}{2}$$

3.3.18

$$F'(x)=\int_0^x(\frac{\partial\int_{t^2}^{x^2}f(t,s)\,\mathrm{d}s}{\partial x})\,\mathrm{d}t=\int_0^x 2xf(t,x^2)\,\mathrm{d}t=2x\int_0^x f(t,x^2)\,\mathrm{d}t$$

3.4.5

(1)

$$\iiint_V xy^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^1\int_0^x\int_0^{xy}xy^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^1\int_0^x xy^2\frac{1}{4}(xy)^4$$
$$=\frac{1}{4}\int_0^1\int_0^x x^5y^6\,\mathrm{d}x\,\mathrm{d}y=\frac{1}{4}\frac{1}{7}\int_0^1 x^5x^7=\frac{1}{4*7*13}=\frac{1}{364}$$

(4)

注意到 $\iiint_\Omega (x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0$ 从而

$$\iiint_\Omega (x+|y|+|z|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_\Omega (|y|+|z|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= 8\iiint_{x+y+z\le 1, x,y,z\ge 0} (|y|+|z|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 8\iiint_{\Omega'} (y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}y \int_0^{1-y-z} y+z\,\mathrm{d}x$$

$$= 8\int_0^1 \mathrm{d}z \int_0^{1-z} (y+z)(1-y-z)\,\mathrm{d}y = 8\int_0^1 \mathrm{d}z \int_0^{1-z} y+z-y^2-z^2-2yz\,\mathrm{d}y$$

$$= 8\int_0^1 (1-2z)\frac{1}{2}(1-z)^2 - \frac{1}{3}(1-z)^3 + (z-z^2)(1-z)\,\mathrm{d}z = \frac{2}{3}$$

3.4.6

$$\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \frac{\cos z}{(1-z)^2}\,\mathrm{d}z = \int_0^1 \frac{\cos z}{(1-z)^2}\,\mathrm{d}z \int_z^1 \mathrm{d}y \int_y^1 \mathrm{d}x = \int_0^1 \frac{\cos z}{(1-z)^2}(\frac{1}{2}(1-z)^2)\,\mathrm{d}z$$

$$= \frac{1}{2}\int_0^1 \cos z\,\mathrm{d}z = -\sin 1$$

3.4.7

(2)

设

$$x = r\cos\theta, y = r\sin\theta\sin\varphi, z = r\sin\theta\cos\varphi$$

则

$$\tan^2\theta \le 1, r \le R, 0 \le \theta \le \frac{\pi}{4}, \frac{3\pi}{4} \le \theta \le \pi$$

$$\iiint_\Omega x^2+y^2+z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{2\pi} \mathrm{d}\varphi(\int_0^{\frac{\pi}{4}} \sin\theta\,\mathrm{d}\theta + \int_{\frac{3\pi}{4}}^{\pi} \sin\theta\,\mathrm{d}\theta) \int_0^R r^4\,\mathrm{d}r$$

$$= \frac{1}{5}\int_0^{2\pi} \mathrm{d}\varphi(\int_0^{\frac{\pi}{4}} \sin\theta\,\mathrm{d}\theta + \int_{\frac{3\pi}{4}}^{\pi} \sin\theta\,\mathrm{d}\theta) R^5 = \frac{1}{5}2\pi(2-\sqrt{2})R^5$$

(5)

设

$$x = r\sin\theta\sin\varphi, y = r\sin\theta\cos\varphi, z = r\cos\theta$$

则

$$0 \le r \le 2, r^2 - 4r\cos\theta \le 0, r \le 4\cos\theta, \cos\theta \ge 0, 0 \le \theta, \varphi \le \frac{\pi}{2}$$

$$\iiint_\Omega xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{\frac{\pi}{3}} \mathrm{d}\theta \int_0^2 r^5 \sin^3\theta\cos\theta\sin\varphi\cos\varphi\,\mathrm{d}r$$

$$
+\int_0^{\frac{\pi}{2}} \mathrm{d}\varphi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{4\cos\theta} r^5 \sin^3\theta\cos\theta\sin\varphi\cos\varphi\,\mathrm{d}r
$$

$$
= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin\varphi\,\mathrm{d}\varphi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{6}(4\cos\theta)^6\sin^3\theta\cos\theta\,\mathrm{d}\theta
$$

$$
+\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin\varphi\,\mathrm{d}\varphi \int_0^{\frac{\pi}{3}} \frac{32}{3}\sin^3\theta\cos\theta\,\mathrm{d}\theta = \frac{3}{4}+\frac{4}{15}=\frac{91}{60}
$$

3.4.8

(1)

设

$$
x = ar\sin\theta\cos\varphi, y = br\sin\theta\sin\varphi, z = cr\cos\theta, r\le 1, 0\le\theta\le\pi, 0\le\varphi\le 2\pi
$$

$$
\iiint_\Omega \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^1 abcr^2\sqrt{1-r^2}\,\mathrm{d}r
$$

$$
= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\cos^2 z\sin^2 z\,\mathrm{d}z = 4\pi\int_0^{\frac{\pi}{2}} abc\cos^2 z\sin^2 z\,\mathrm{d}z = \frac{\pi abc}{4}
$$

(2)

$$
\iiint_\Omega x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^3\mathrm{d}z\int_{\frac{z}{2}}^{z}\mathrm{d}x\int_{\sqrt{z}}^{2\sqrt{z}} x^2\,\mathrm{d}y = \int_0^3\mathrm{d}z\int_{\frac{z}{2}}^{z} x^2\sqrt{z}\,\mathrm{d}x = \int_0^3\frac{1}{3}z^3\sqrt{z}\,\mathrm{d}z
$$

$$
= \frac{21\sqrt{3}}{4}
$$

3.4.9

(7)

设

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
$$

从而

$$
\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$

设

$$
x_1 = \rho\sin\theta\cos\varphi, y_1 = \rho\sin\theta\sin\varphi, z = \rho\cos\theta
$$

$$
\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |\frac{\det(A^{-1})D(x_1,y_1,z_1)}{D(\rho,\theta,\varphi)}| = \frac{\rho^2\sin\theta}{|\det(A)|}\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta
$$

从而

$$S = \frac{4}{3\det(A)}\pi r^3$$

3.4.11

$$\lim_{r\to 0^+}\frac{1}{r^3}\iiint_{x^2+y^2+z^2\le R^2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \lim_{r\to 0^+}\frac{1}{r^3}f(x_0,y_0,z_0)\iiint_{x^2+y^2+z^2\le R^2}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{4\pi}{3}\lim_{r\to 0^+} f(x_0,y_0,z_0), x_0^2+y_0^2+z_0^2 \le r^2 \to 0^+$$

由于 $f(x,y,z)$ 连续，则

$$\lim_{r\to 0^+}\frac{1}{r^3}\iiint_{x^2+y^2+z^2\le R^2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{4\pi}{3}f(0,0,0)$$