Homework

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1.9

1.9.1

(3)

先求驻点

$$\begin{cases} \dfrac{\partial u}{\partial x} = \cos x - \cos(x+y+z) = 0 \\ \dfrac{\partial u}{\partial y} = \cos y - \cos(x+y+z) = 0 \\ \dfrac{\partial u}{\partial z} = \cos z - \cos(x+y+z) = 0 \end{cases}$$

解得

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

或者

$$\begin{cases} x = \frac{\pi}{2} \\ y = \frac{\pi}{2} \\ z = \frac{\pi}{2} \end{cases}$$

或者

$$\begin{cases} x = \pi \\ y = \pi \\ z = \pi \end{cases}$$

考虑任意一个点处的 $Hesse$ 矩阵

$$H(x,y,z) = \begin{bmatrix} -\sin x + \sin(x+y+z) & \sin(x+y+z) & \sin(x+y+z) \\ \sin(x+y+z) & -\sin y + \sin(x+y+z) & \sin(x+y+z) \\ \sin(x+y+z) & \sin(x+y+z) & -\sin z + \sin(x+y+z) \end{bmatrix}$$

$$H(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \text{负定}$$

则函数在 $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ 处取极大值 4
且可以直接证明，u 在 $(0,0,0)(\pi,\pi,\pi)$ 处取最小值 0

1.9.2

$$\begin{cases} \frac{\partial z}{\partial x} = 4x + 2z\frac{\partial z}{\partial x} + 8z + 8x\frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = 0 \\ \frac{\partial z}{\partial y} = 4y + 2z\frac{\partial z}{\partial y} + 8x\frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} = 0 \end{cases}$$

解得

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{-4x-8z}{8x+2z-1} = 0 \\ \frac{\partial z}{\partial y} = \frac{-4y}{8x+2z-1} = 0 \end{cases}$$

解得

$$\begin{cases} x = -2z \\ y = 0 \end{cases}$$

代回原方程，得 $-7z^2 - z + 8 = 0$
解得 $z = 1$ 或 $z = -\frac{8}{7}$
注意到

$$\frac{\partial^2 z}{\partial x^2} = \frac{(-4 - 8\frac{\partial z}{\partial x})(8x + 2z - 1) - (-4x - 8z)(8 + 2\frac{\partial z}{\partial x})}{(8x + 2z - 1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-4(8x + 2z - 1) - (-4y)(2\frac{\partial z}{\partial y})}{(8x + 2z - 1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-8\frac{\partial z}{\partial y}(8x + 2z - 1) - (-4x - 8z)(2\frac{\partial z}{\partial y})}{(8x + 2z - 1)^2}$$

从而

$$H(\frac{16}{7}, 0) = \begin{bmatrix} -\frac{4}{15} & 0 \\ 0 & -\frac{4}{15} \end{bmatrix} \text{负定}$$

有极大值 $-\frac{8}{7}$

$$H(-2, 0) = \begin{bmatrix} \frac{4}{15} & 0 \\ 0 & \frac{4}{15} \end{bmatrix} \text{正定}$$

有极小值 1

1.9.7

(3)

$$\begin{cases} u = x^2 + y^2 + z^2 \\ \frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1 \end{cases}$$

考虑 $x^2 + y^2 + z^2 + \lambda(\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} - 1)$ 我们有

$$\begin{cases} \frac{\partial f}{\partial x} = 2x + \frac{2\lambda}{16}x = 0 \\ \frac{\partial f}{\partial y} = 2y + \frac{2\lambda}{9}y = 0 \\ \frac{\partial f}{\partial z} = 2z + \frac{2\lambda}{4}z = 0 \\ x^2 + y^2 + z^2 = 0 \end{cases}$$

从而，x, y, z 中有两个为 0，有一个不为 0, 验证可得
从而 u 有极大值 16, 有极小值 4

1.9.8

注意到，当 $x^2+y^2+z^2<4$ 时，

$$u=(x-y)^2+(y-z)^2\geq 0$$

$$/\begin{cases} \frac{\partial u}{\partial x}=2x-2y=0 \\ \frac{\partial u}{\partial y}=4y-2x-2z=0 \\ \frac{\partial u}{\partial z}=2z-2y=0 \end{cases}$$

解得 $x=y=z$，此时 u 有极小值 0，无极大值
当 $x^2+y^2+z^2=4$ 时，考虑 $u-\lambda(x^2+y^2+z^2-4)$

$$\begin{cases} \frac{\partial f}{\partial x}=2x-2y-2\lambda x=0 \\ \frac{\partial f}{\partial y}=4y-2x-2z-2\lambda y=0 \\ \frac{\partial f}{\partial z}=2z-2y-2\lambda z=0 \\ x^2+y^2+z^2=4 \end{cases}$$

由于 x,y,z 不全为 0，从而

$$\begin{vmatrix} 2-2\lambda & -2 & 0 \\ -2 & 4-2\lambda & -2 \\ 0 & -2 & 2-2\lambda \end{vmatrix}=0$$

解得 $\lambda=0$或者$\lambda=3$
对应的 $x=y=z=\frac{2}{\sqrt{3}}$
或者 $y=\frac{2\sqrt{2}}{\sqrt{3}}=-2x=-2z$ 此时有极小值 0，有极大值 12

1.9.9

设一顶点坐标为 (x,y,z)，考虑以其为一顶点的内切长方体，体积为 $8xyz$
考虑 $f=8xyz-\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1)$

$$\begin{cases} \frac{\partial f}{\partial x}=8yz-\frac{2\lambda x}{a^2}=0 \\ \frac{\partial f}{\partial y}=8xz-\frac{2\lambda y}{b^2}=0 \\ \frac{\partial f}{\partial z}=8xy-\frac{2\lambda z}{c^2}=0 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \end{cases}$$

解得 $\lambda^2=\frac{a^2b^2c^2}{12}$
此时 $8xyz$ 有极大值 $\frac{8abc}{3\sqrt{3}}$

1.9.10

(1)

设下底为 y 上底为 x，高为 h
不妨设梯形面积为 1, 则有 $xh+yh=2$

考虑 $u=x+2\sqrt{h^2+\frac{(y-x)^2}{4}}-\lambda(xh+yh-2)$

$$\begin{cases} 1+\frac{2x-2y}{2\sqrt{4h^2+x^2+y^2-2xy}}-\lambda h=0 \\ \frac{2y-2x}{2\sqrt{4h^2+x^2+y^2-2xy}}-\lambda h=0 \\ \frac{8h}{2\sqrt{4h^2+x^2+y^2-2xy}}-\lambda(x+y)=0 \end{cases}$$

解得 $\lambda=\frac{1}{2h}$

$$x=\frac{2\sqrt{3}}{3}h, y=\frac{4\sqrt{3}}{3}h$$

时，目标函数取极小值。这时腰长为 $\frac{2\sqrt{3}}{3}h$
这个时候上底，下底和腰的长度之比为 $2:1:1$

2.2

2.2.1

(1)

$$\lim_{a\to 0}\int_{-1}^{1}\sqrt{x^2+a^2}\,\mathrm{d}x$$

注意到：

$$\int\sqrt{x^2+a^2}\,\mathrm{d}x=x\sqrt{x^2+a^2}-\int x\frac{2x}{2\sqrt{x^2+a^2}}\,\mathrm{d}x+C=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}\,\mathrm{d}x+\int\frac{a^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x+C$$

$$=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}\,\mathrm{d}x+a^2\ln(\sqrt{a^2+x^2}+x)+C(\text{C 为常数})$$

从而

$$\int_{-1}^{1}\sqrt{x^2+a^2}\,\mathrm{d}x=a^2\ln\left(\frac{\sqrt{a+1}+1}{|a|}\right)+\sqrt{a^2+1}$$

从而

$$\lim_{a\to 0}\int_{-1}^{1}\sqrt{x^2+a^2}\,\mathrm{d}x=\lim_{a\to 0}a^2\ln\left(\frac{\sqrt{a+1}+1}{|a|}\right)+\sqrt{a^2+1}=1$$

2.2.2

(4)

$$F^{'}(t)=\int_0^t f(x+t,x-t)\,\mathrm{d}x=\int_0^t\frac{\partial f(x+t,x-t)}{\partial t}+f(2t,0)=\int_0^t f_1^{'}(x+t,x-t)-f_2^{'}(x+t,x-t)$$
$$+f(2t,0)$$

2.2.4

$$\frac{\partial u}{\partial t}=\frac{1}{2}(a\varphi^{'}(x+at)-\varphi^{'}(x-at))+\frac{1}{2a}(a\phi(x+at)+a\phi(x-at))$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{1}{2}(a^2\varphi^{''}(x+at)+a^2\varphi^{''}(x-at))+\frac{a}{2}(\phi^{'}(x+at)-\phi^{'}(x-at))$$

$$\frac{\partial u}{\partial x}=\frac{1}{2}(\varphi^{'}(x+at)+\varphi^{'}(x-at))+\frac{1}{2a}(\phi(x+at)-\phi(x-at))$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{2}(\varphi^{''}(x+at)+\varphi^{''}(x-at))+\frac{1}{2a}(\phi^{'}(x+at)-\phi^{'}(x-at))$$

从而有

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$