Homework

王俊琪

March 20, 2022

1.6

1.6.4

Proof. Suppose that $p = u^2 - x^2, q = u^2 - y^2, r = u^2 - z^2$

$$f(p, q, r) = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial p}(2u\frac{\partial u}{\partial x} - 2x) + \frac{\partial f}{\partial q}((2u\frac{\partial u}{\partial x}) + \frac{\partial f}{\partial r}((2u\frac{\partial u}{\partial x})$$

$$x\frac{\partial f}{\partial p} = (\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q} + \frac{\partial f}{\partial r})(u\frac{\partial u}{\partial x})$$

$$\frac{1}{x}\frac{\partial u}{\partial x} = \frac{1}{u}\frac{\frac{\partial f}{\partial p}}{(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q} + \frac{\partial f}{\partial r})}$$

$$\sum \frac{1}{x}\frac{\partial u}{\partial x} = \frac{1}{u}\frac{(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q} + \frac{\partial f}{\partial r})}{(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q} + \frac{\partial f}{\partial r})} = \frac{1}{u}$$

□

1.6.5

Notice that

$$u = \frac{x + y}{2}, v = \frac{x - y}{2}, z = \frac{(x + y)^2(x - y)^2}{16}$$

therefore

$$f(x, y, z) = z - \frac{1}{16}x^4 - \frac{1}{16}y^4 + \frac{1}{8}x^2y^2 = 0$$

$$\frac{\partial f}{\partial z} = 1 \neq 0$$

therefore

$$\begin{cases} 1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \\ 0 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \end{cases}$$

$$\left\{\begin{array}{r}1=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}\\ 0=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\end{array}\right.$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=\frac{1}{2},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial y}=\frac{1}{2}$$

$$\frac{\partial z}{\partial x}=2uv^2\frac{\partial u}{\partial x}+2vu^2\frac{\partial v}{\partial x}=uv(u+v)$$

$$\frac{\partial z}{\partial y}=uv(v-u)$$

1.6.7

$$\left\{\begin{array}{r}2x\frac{\mathrm{d}x}{\mathrm{d}z}+2y\frac{\mathrm{d}y}{\mathrm{d}z}=z\\ \frac{\mathrm{d}x}{\mathrm{d}z}+\frac{\mathrm{d}y}{\mathrm{d}z}+1=0\end{array}\right.$$

We get

$$\left\{\begin{array}{r}\frac{\mathrm{d}x}{\mathrm{d}z}=0\\ \frac{\mathrm{d}y}{\mathrm{d}z}=-1\end{array}\right.$$

$$\left\{\begin{array}{r}2(\frac{\mathrm{d}x}{\mathrm{d}z})^2+2x\frac{\mathrm{d}^2x}{\mathrm{d}z^2}+2(\frac{\mathrm{d}y}{\mathrm{d}z})^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}z^2}\\ \frac{\mathrm{d}^2y}{\mathrm{d}z^2}+\frac{\mathrm{d}^2x}{\mathrm{d}z^2}=0\end{array}\right.$$

We get

$$\left\{\begin{array}{r}\frac{\mathrm{d}^2x}{\mathrm{d}z^2}=-\frac{1}{4}\\ \frac{\mathrm{d}^2y}{\mathrm{d}z^2}=\frac{1}{4}\end{array}\right.$$

1.6.10

(1)

$$u=e^{2x}\cos 2y$$

$$v=e^{2x}\sin 2y$$

$$Jh(1,0)=\begin{bmatrix}2e^{2x}\cos 2y & -2e^{2x}\sin 2y\\ 2e^{2x}\sin 2y & -2e^{2x}\cos 2y\end{bmatrix}=\begin{bmatrix}2e^2 & 0\\ 0 & 2e^2\end{bmatrix}$$

$DJh(1,0)=(4e^4)\neq 0$
从而：$f\circ g$ 是可逆的

1.7

1.7.1

(5)

we have

$$\left\{\begin{array}{r}x=u_0\cos v_0\\ y=u_0\sin v_0\\ z=av_0\end{array}\right.$$

$$\overrightarrow{n} = \begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \cos v_0 & \sin v_0 & 0 \\ -u_0 \sin v_0 & u_0 \cos v_0 & a \end{bmatrix} = (6a \sin v_0, -a \cos v_0, u_0)$$

所求切平面方程为

$$a \sin v_0 (x - u_0 \cos v_0) - a \cos v_0 (y - u_0 \sin v_0) + u_0 (z - a v_0) = 0$$

所求法线方程为

$$\frac{x - u_0 \cos v_0}{a \sin v_0} = \frac{y - u_0 \sin v_0}{-a \cos v_0} = \frac{z - a v_0}{u_0}$$

(6)

we have

$$\begin{cases} x = 1 + 2 = 3 \\ y = 1 + 2^2 = 5 \\ z = 1 + 2^3 = 9 \end{cases}$$

$$\overrightarrow{n} = \begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2u & 3u^2 \\ 1 & 2v & 3v^2 \end{bmatrix} = (6uv^2 - 6vu^2, 3u^2 - 3v^2, 2v - 2u) = (12, -9, 2)$$

所求切平面方程为

$$12(x - 3) - 9(y - 5) + 2(z - 9) = 0$$

所求法线方程为

$$\frac{x - 3}{12} = \frac{y - 5}{-9} = \frac{z - 9}{2}$$

1.7.2

考虑球面上任意一个点 $P(x_0, y_0, z_0)$
过 P 的法向量为 $(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2})$
由于过 P 点的法线与坐标轴正方向成等角
从而有

$$\frac{2x_0}{a^2} = \frac{2y_0}{b^2} = \frac{2z_0}{c^2}$$

解得

$$P = (\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}})$$

或者

$$P = (-\frac{a}{\sqrt{a^2 + b^2 + c^2}}, -\frac{b}{\sqrt{a^2 + b^2 + c^2}}, -\frac{c}{\sqrt{a^2 + b^2 + c^2}})$$

1.7.3

考虑曲面上一点 $P(x_0, y_0, z_0)$ 法向量为 $(2x_0, 4y_0, 6z_0)$ 从而过这个点的切平面方程为

$$x_0(x_{x0}) + 2y_0(y - y_0) + 3z_0(z_{z0}) = 0$$

又由于 $x_0^2 + 2y_0^2 + 3z_0^2 = 21$ 从而切平面方程为

$$x_0 x + 2y_0 y + 3z_0 z = 21$$

由于两平面平行，

$$\frac{x_0}{1} = \frac{2y_0}{4} = \frac{3z_0}{6}$$

解得

$$x_0 = 1, y_0 = 2, z_0 = 2$$

或

$$x_0 = -1, y_0 = -2, z_0 = -2$$

从而切平面方程为

$$x + 4y + 6z = \pm 21$$

1.7.5

切线方程为

$$\begin{cases} 2(x-1) - 4(y+2) + 2(z-1) = 0 \\ (x-1) + (y+2) + (z-1) = 0 \end{cases}$$

切方向为 $(2, -4, 2) \times (1, 1, 1) = (-6, 0, 6)$
从而, 切线方程为

$$\frac{x-1}{-1} = \frac{y+2}{0} = \frac{z-1}{1}$$

法平面方程为

$$x - z = 0$$

1.7.6

Proof. 考虑螺旋线

$$\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$$

考虑其上一点 $x_0 = (a\cos t_0, a\sin t_0, bt_0)$ 切向量为 $\overrightarrow{n} = (-a\sin t_0, a\cos t_0, b)$ 切线方程为

$$\frac{x - a\cos t_0}{-a\sin t_0} = \frac{y - a\sin t_0}{a\cos t_0} = \frac{z - bt_0}{b}$$

考虑 z 轴的方向向量 $\overrightarrow{b} = (0, 0, 1)$
则所成的角的余弦值为 $\frac{\overrightarrow{n} \cdot \overrightarrow{b}}{|\overrightarrow{n}||\overrightarrow{b}|} = \frac{b}{\sqrt{a^2+b^2}}$
从而与 z 轴形成定角 □

1.8

1.8.2

(2)

$Jf(0,0) = (0,0)$

$$H(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

从而，$z = \dfrac{\cos x}{\cos y}$ 在点 $(0,0)$ 处的二阶 Taylor 多项式为 $1 + \frac{1}{2}(x^2 + y^2)$

(3)

$Jf(0,0) = (0,1)$

$$H(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

从而，$z = e^{-x}$ 在点 $(0,0)$ 处的二阶 Taylor 多项式为 $y - \frac{1}{2}(2xy + y^2)$