Algorithms: COMP3121/9101

School of Computer Science and Engineering
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9. STRING MATCHING ALGORITHMS

String Matching algorithms

- Assume that you want to find out if a string $B = b_0 b_1 \ldots b_{m-1}$ appears as a (contiguous) substring of a much longer string $A = a_0 a_1 \ldots a_{n-1}$.

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- The “naive” string matching algorithm does not work well if B is much longer than what can fit in a single register; we need something cleverer.
- We now show how hashing can be combined with recursion to produce an efficient string matching algorithm.

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- Thus, we can identify each string with a sequence of integers by mapping each symbol s_i into a corresponding integer i:

$$\mathcal{A} = \{s_0, s_1, s_2, \ldots, s_{d-1}\} \longrightarrow \{0, 1, 2, \ldots, d-1\}$$

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- To any string $B = b_0 b_1 \ldots b_{m-1}$ we can now associate an integer whose digits in base d are integers corresponding to each symbol in B:

$$h(B) = h(b_0 b_1 b_2 \ldots b_m) = d^{m-1} b_0 + d^{m-2} b_1 + \ldots + d \cdot b_{m-2} + b_{m-1}$$

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- This can be done efficiently using the Horner's rule:

$$h(B) = b_{m-1} + d(b_{m-2} + d(b_{m-3} + d(b_{m-4} + \dots + d(b_1 + d \cdot b_0))) \dots)$$

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- Next we choose a large prime number p such that $(d+1)\,p$ still fits into a single register and define the hash value of B as $H(B) = h(B) \bmod p$.

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- For each contiguous substring $A_s = a_s a_{s+1} \ldots a_{s+m-1}$ of string A we also compute its hash value as

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- This is where recursion comes into play: we do not have compute the hash value $H(A_{s+1})$ of $A_{s+1} = a_{s+1} a_{s+2} \ldots a_{s+m}$ "from scratch", but we can compute it efficiently from the hash value $H(A_s)$ of $A_s = a_s a_{s+1} \ldots a_{s+m-1}$ as follows.

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Since

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by multiplying both sides by d we obtain

$$\begin{aligned}
&(d \cdot H(A_s)) \bmod p = \\
&= (d^m a_s + d^{m-1}a_{s+1} + \dots d \cdot a_{s+m-1}) \bmod p \\
&= (d^m a_s + (d^{m-1}a_{s+1} + \dots d^2 a_{s+m-2} + d\, a_{s+m-1} + a_{s+m}) \bmod p - a_{s+m}) \bmod p \\
&= (d^m\, a_s + H(A_{s+1}) - a_{s+m}) \bmod p
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- Thus, for every s except $s = 0$ the value of $H(A_s)$ can be computed in constant time independent of the length of the strings A and B.

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- However, as always when we use hashing, we cannot guarantee the worst case performance.
- So we now look for algorithms whose worst case performance can be guaranteed.

String matching finite automata

- A string matching finite automaton for a string S with k symbols has $k+1$ many states $0, 1, \ldots k$ which correspond to the number of characters matched thus far and a transition function $\delta(s, c)$ where s is a state and c is a character red at the moment.

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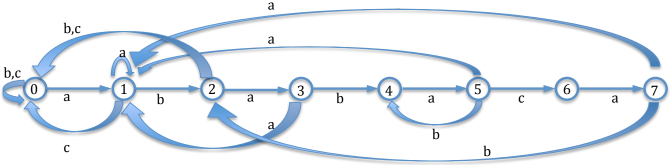
- We first look at the case when such $\delta(s, c)$ is given by a pre-constructed table.
- To make things easier to describe, we consider the string $S = ababaca$. The table defining $\delta(s, c)$ would then be

	input			
state	a	b	c	
0	**1**	0	0	a
1	1	**2**	0	b
2	**3**	0	0	a
3	1	**4**	0	b
4	**5**	0	0	a
5	1	4	**6**	c
6	**7**	0	0	a
7	1	2	0	



state transition diagram for string *ababaca*

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- Thus, if a happens to be $B[k+1]$, then $m = k+1$ and so $\delta(k, a) = k+1$ and $B_k a = B_{k+1}$.
- We do that by matching the string against itself: we can recursively compute a function $\pi(k)$ which for each k returns the largest integer m such that the prefix B_m of B is a proper suffix of B_k.

function Compute − Prefix − Function(B)

  $m \leftarrow \text{length}[B]$

  let $\pi[1..m]$ be a new array

  $\pi[1] = 0$

  $k = 0$

  for $q = 2$ to m **do**

    while $k > 0$ **and** $B[k+1] \neq B[q]$

      $k = \pi[k]$

    if $B[k+1] == B[q]$

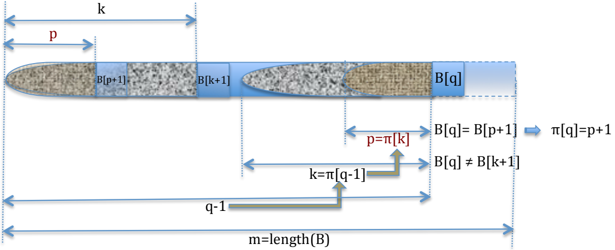
      $k = k + 1$

    $\pi[q] = k$

  end for

  return π

end function



Assume that length of B is m and that we have already found that $\pi[q-1] = k$; to compute $\pi[q]$ we check if $B[q] = B[k+1]$; if true then $\pi[q] = k+1$; if not true then we find $\pi[k] = p$; if now $B[q] = B[p+1]$ then $\pi[q] = p+1$.

The Knuth-Morris-Pratt algorithm

- We can now do our search for string B in a longer string A:

```
function KMP − Matcher(A, B)
    n ← length[A]
    m ← length[B]
    π = Compute − Prefix − Function(B)
    q = 0
    for i = 1 to n do
        while q > 0 and B[q + 1] ≠ A[i]
        q = π[q]
        if  B[q + 1] == A[i]
            q = q + 1
        if  q == m
            print pattern occurs with shift i − m
            q = π[q]
    end for
end function
```

Looking for imperfect matches

- Sometimes we are not interested in finding just the prefect matches, but also in matches that might have a few errors, such as a few insertions, deletions and replacements.

- So assume that we have a very long string $A = a_0a_1a_2a_3\ldots\ldots a_sa_{s+1}\ldots a_{s+m-1}\ldots\ldots a_{N-1}$, a shorter string $B = b_0b_1b_2\ldots b_{m-1}$ where $m << N$ and an integer $k << m$. We are interested in finding all matches for B in A which allow up to k many errors.

- Idea: split B into $k+1$ consecutive subsequences of (approximately) equal length. Then any match in A with at most k errors must contain a subsequence which is a perfect match for a subsequence of B. Thus, we look for all perfect matches for all of $k+1$ subsequences of B and for every hit we test by brute force if the remaining parts of B have sufficient number of matches in the appropriate parts of A.

PUZZLE!!

On a rectangular table there are 25 non-overlapping round coins of equal size placed in such a way that it is not possible to add another coin without overlapping any of the existing coins and without the coin falling off the table (for a coin to stay on the table its centre must be within the table). Show that it is possible to completely cover the table with 100 coins (of course with overlapping of coins).