Algorithms
COMP3121/9101

School of Computer Science and Engineering
University of New South Wales

4. FAST LARGE INTEGER MULTIPLICATION - part A

Basics revisited: how do we multiply two numbers?

- The primary school algorithm:

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          X X X X  <- first input integer
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      X X X X       / O(n^2) elementary  multiplications
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- **Can we do it faster than in n^2 many steps??**

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$$= A_1B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0$$

- We have saved one multiplication, now we have only three: A_0B_0, A_1B_1 and $(A_1 + A_0)(B_1 + B_0)$.

$AB = A_1B_12^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0$

function $\textsc{Mult}(A, B)$
    if $|A| = |B| = 1$ **then return** AB
    else
        $A_1 \leftarrow \text{MoreSignificantPart}(A)$;
        $A_0 \leftarrow \text{LessSignificantPart}(A)$;
        $B_1 \leftarrow \text{MoreSignificantPart}(B)$;
        $B_0 \leftarrow \text{LessSignificantPart}(B)$;
        $U \leftarrow A_0 + A_1$;
        $V \leftarrow B_0 + B_1$;
        $X \leftarrow \textsc{Mult}(\mathrm{A_0}, \mathrm{B_0})$;
        $W \leftarrow \textsc{Mult}(\mathrm{A_1}, \mathrm{B_1})$;
        $Y \leftarrow \textsc{Mult}(\mathrm{U}, \mathrm{V})$;
        return $W\,2^n + (Y - X - W)\,2^{n/2} + X$
    end if
end function

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$$f(n) = c\,n = O(n^{\log_2 3 - \varepsilon}) \quad \text{for any} \quad 0 < \varepsilon < 0.5$$

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- Thus, the first case of the Master Theorem applies.
- Consequently,

$$T(n) = \Theta(n^{\log_2 3}) < \Theta(n^{1.585})$$

without going through the messy calculations!

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- So,

$$AB = A_2B_2\, 2^{4k} + (A_2B_1 + A_1B_2)2^{3k} + (A_2B_0 + A_1B_1 + A_0B_2)2^{2k} + \\ + (A_1B_0 + A_0B_1)2^{k} + A_0B_0$$

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$$AB = \underbrace{A_2B_2}_{C_4}\,2^{4k} + (\underbrace{A_2B_1 + A_1B_2}_{C_3})2^{3k} + (\underbrace{A_2B_0 + A_1B_1 + A_0B_2}_{C_2})2^{2k} +$$
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- Not clear at all how to get $C_0 - C_4$ with 5 multiplications only ...

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- We form the naturally corresponding polynomials:

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- Note that

$$A = A_2\,(2^k)^2 + A_1\,2^k + A_0 = P_A(2^k);$$
$$B = B_2\,(2^k)^2 + B_1\,2^k + B_0 = P_B(2^k).$$

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  with only 5 multiplications, we can then obtain the product of numbers A and B simply as

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- Thus, we compute

$$P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$$
$$P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$$

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- For $P_A(x) = A_2x^2 + A_1x + A_0$ we have

$$\begin{aligned}
P_A(-2) &= A_2(-2)^2 + A_1(-2) + A_0 = 4A_2 - 2A_1 + A_0\\
P_A(-1) &= A_2(-1)^2 + A_1(-1) + A_0 = A_2 - A_1 + A_0\\
P_A(0) &= A_2 0^2 + A_1 0 + A_0 = A_0\\
P_A(1) &= A_2 1^2 + A_1 1 + A_0 = A_2 + A_1 + A_0\\
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- Similarly, for $P_B(x) = B_2x^2 + B_1x + B_0$ we have

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\end{aligned}$$

- These evaluations involve only additions because $2A = A + A; \quad 4A = 2A + 2A.$

The Karatsuba trick: slicing into 3 pieces

- Having obtained $P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$ and
  $P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$
  we can now obtain $P_C(-2), P_C(-1), P_C(0), P_C(1), P_C(2)$ with only 5 multiplications of large numbers:

$$\begin{aligned} P_C(-2) &= P_A(-2)P_B(-2) \\ &= (A_0 - 2A_1 + 4A_2)(B_0 - 2B_1 + 4B_2) \end{aligned}$$

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The Karatsuba trick: slicing into 3 pieces

- Thus, if we represent the product $C(x) = P_A(x)P_B(x)$ in the coefficient form as $C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ we get

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\begin{aligned}
C_4(-2)^4 + C_3(-2)^3 + C_2(-2)^2 + C_1(-2) + C_0 &= P_C(-2) = P_A(-2)P_B(-2) \\
C_4(-1)^4 + C_3(-1)^3 + C_2(-1)^2 + C_1(-1) + C_0 &= P_C(-1) = P_A(-1)P_B(-1) \\
C_4 0^4 + C_3 0^3 + C_2 0^2 + C_1 \cdot 0 + C_0 &= P_C(0) = P_A(0)P_B(0) \\
C_4 1^4 + C_3 1^3 + C_2 1^2 + C_1 \cdot 1 + C_0 &= P_C(1) = P_A(1)P_B(1) \\
C_4 2^4 + C_3 2^3 + C_2 2^2 + C_1 \cdot 2 + C_0 &= P_C(2) = P_A(2)P_B(2).
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- Simplifying the left side we obtain

$$\begin{aligned}
16C_4 - 8C_3 + 4C_2 - 2C_1 + C_0 &= P_C(-2)\\
C_4 - C_3 + C_2 - C_1 + C_0 &= P_C(-1)\\
C_0 &= P_C(0)\\
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- Thus we have obtained $A \cdot B = P_A(2^k) P_B(2^k) = P_C(2^k)$ with only 5 multiplications!

The Karatsuba trick: slicing into 3 pieces

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- Thus we have obtained $A \cdot B = P_A(2^k) P_B(2^k) = P_C(2^k)$ with only 5 multiplications!Here is the complete algorithm:

function $\textsc{Mult}(A, B)$

obtain A_0, A_1, A_2 and B_0, B_1, B_2 such that $A = A_2\, 2^{2\,k} + A_1\, 2^k + A_0$; $\quad B = B_2\, 2^{2\,k} + B_1\, 2^k + B_0$;

form polynomials $P_A(x) = A_2 x^2 + A_1 x + A_0$; $\quad P_B(x) = B_2 x^2 + B_1 x + B_0$;

$$\begin{array}{ll}
P_A(-2) \leftarrow 4A_2 - 2A_1 + A_0 & P_B(-2) \leftarrow 4B_2 - 2B_1 + B_0 \\
P_A(-1) \leftarrow A_2 - A_1 + A_0 & P_B(-1) \leftarrow B_2 - B_1 + B_0 \\
P_A(0) \leftarrow A_0 & P_B(0) \leftarrow B_0 \\
P_A(1) \leftarrow A_2 + A_1 + A_0 & P_B(1) \leftarrow B_2 + B_1 + B_0 \\
P_A(2) \leftarrow 4A_2 + 2A_1 + A_0 & P_B(2) \leftarrow 4B_2 + 2B_1 + B_0
\end{array}$$

$$\begin{array}{ll}
P_C(-2) \leftarrow \textsc{Mult}(P_A(-2), P_B(-2)); & P_C(-1) \leftarrow \textsc{Mult}(P_A(-1), P_B(-1)); \\
P_C(0) \leftarrow \textsc{Mult}(P_A(0), P_B(0)); & \\
P_C(1) \leftarrow \textsc{Mult}(P_A(1), P_B(1)); & P_C(2) \leftarrow \textsc{Mult}(P_A(2), P_B(2))
\end{array}$$

$$C_0 \leftarrow P_C(0); \qquad C_1 \leftarrow \frac{P_C(-2)}{12} - \frac{2P_C(-1)}{3} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{12}$$

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form $P_C(x) = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0$; compute
$P_C(2^k) = C_4 2^{4k} + C_3 2^{3k} + C_2 2^{2k} + C_1 2^k + C_0$

return $P_C(2^k) = A \cdot B$.

end function

The Karatsuba trick: slicing into 3 pieces

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- Clearly, the first case of the MT applies and we get $T(n) = O(n^{\log_3 5}) < O(n^{1.47})$.

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- Recall that the original Karatsuba algorithm runs in time

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- Thus, we got a significantly faster algorithm.
- Then why not slice numbers A and B into even larger number of slices? Maybe we can get even faster algorithm?
- The answer is, in a sense, BOTH yes and no, so lets see what happens if we slice numbers into $p+1$ many (approximately) equal slices, where $p = 1, 2, 3, \ldots$....

Generalizing Karatsuba's algorithm

The general case - slicing the input numbers A, B into $p+1$ many slices

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- Slice A, B into $p+1$ pieces each:

$$\begin{aligned} A &= A_p 2^{kp} + A_{p-1}2^{k(p-1)} + \cdots + A_0 \\ B &= B_p 2^{kp} + B_{p-1}2^{k(p-1)} + \cdots + B_0 \end{aligned}$$

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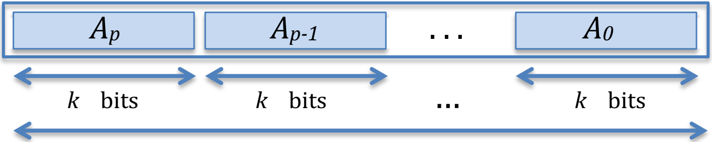

A divided into $p+1$ slices each slice k bits = $(p+1)\,k$ bits in total

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- We form the naturally corresponding polynomials:

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  $A = P_A(2^k);\;\; B = P_B(2^k);\;\; AB = P_A(2^k)P_B(2^k) = (P_A(x) \cdot P_B(x))\,|_{x=2^k}$

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$$P_C(x) = P_A(x) \cdot P_B(x);$$

  - then we evaluate $P_C(2^k)$.

- Note that $P_C(x) = P_A(x) \cdot P_B(x)$ is of degree $2p$:

$$P_C(x) = \sum_{j=0}^{2p} C_j x^j$$

Generalizing Karatsuba's algorithm

- Example:

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&(A_3x^3 + A_2x^2 + A_1x + A_0)(B_3x^3 + B_2x^2 + B_1x + B_0) = \\
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$$P_A(x) \cdot P_B(x) = \sum_{j=0}^{2p}\left(\sum_{i+k=j} A_iB_k\right) x^j = \sum_{j=0}^{2p} C_j\, x^j$$

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- We need to find the coefficients $C_j = \sum_{i+k=j} A_iB_k$ without performing $(p+1)^2$ many multiplications necessary to get all products of the form A_iB_k.

A VERY IMPORTANT DIGRESSION:

If you have two sequences $\vec{A} = (A_0, A_1, \ldots, A_{p-1}, A_p)$ and $\vec{B} = (B_0, B_1, \ldots, B_{m-1}, B_m)$, and if you form the two corresponding polynomials

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then the sequence $\vec{C} = (C_0, C_1, \ldots, C_{p+m})$ of the coefficients of the product polynomial, with these coefficients given by

$$C_j = \sum_{i+k=j} A_i B_k, \quad \text{for} \quad 0 \leq j \leq p+m,$$

is **extremely important** and is called the **LINEAR CONVOLUTION** of sequences $\vec{A}$ and $\vec{B}$ and is denoted by $\vec{C} = \vec{A} \star \vec{B}$.

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- Convolutions are bread-and-butter of signal processing, and for that reason it is **extremely important** to find fast ways of multiplying two polynomials of possibly very large degrees.
- In signal processing these degrees can be greater than 1000.
- This is the main reason for us to study methods of fast computation of convolutions (aside of finding products of large integers, which is what we are doing at the moment).

Coefficient vs value representation of polynomials

- Every polynomial $P_A(x)$ of degree p is uniquely determined by its values at any $p+1$ distinct input values $x_0, x_1, \ldots, x_p$:

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_p, P_A(x_p))\}$$

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- For $P_A(x) = A_p x^p + A_{p-1}x^{p-1} + \ldots + A_0$, these values can be obtained via a matrix multiplication:

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^p \\ 1 & x_1 & x_1^2 & \ldots & x_1^p \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_p & x_p^2 & \ldots & x_p^p \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_p \end{pmatrix} = \begin{pmatrix} P_A(x_0) \\ P_A(x_1) \\ \vdots \\ P_A(x_p) \end{pmatrix}. \tag{1}$$

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- It can be shown that if x_i are all distinct then this matrix is invertible.
- Such a matrix is called *the Vandermonde matrix*.

Coefficient vs value representation of polynomials - ctd.

- Thus, if all x_i are all distinct, given any values $P_A(x_0), P_A(x_1), \ldots, P_A(x_p)$ the coefficients $A_0, A_1, \ldots, A_p$ of the polynomial $P_A(x)$ are uniquely determined:

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  2. a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_p, P_A(x_p))\}$$

Coefficient vs value representation of polynomials- ctd.

- If we fix the inputs $x_0, x_1, \ldots, x_p$ then commuting between a representation of a polynomial $P_A(x)$ via its coefficients and a representation via its values at these points is done via the following two matrix multiplications, with matrices made up from **constants**:

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- Thus, for fixed input values $x_0, \ldots, x_p$ this switch between the two kinds of representations is done in **linear time**!

Our strategy to multiply polynomials fast:

1. Given two polynomials of degree at most p,

$$P_A(x) = A_p x^p + \ldots + A_0; \qquad P_B(x) = B_p x^p + \ldots + B_0$$

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- **Note:** since the product of the two polynomials will be of degree $2p$ we need the values of $P_A(x)$ and $P_B(x)$ at $2p+1$ points, rather than just $p+1$ points!

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$$P_A(x) = A_p x^p + \ldots + A_0; \qquad P_B(x) = B_p x^p + \ldots + B_0$$

convert them into value representation at $2p+1$ distinct points $x_0, x_1, \ldots, x_{2p}$:

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   - **Note:** since the product of the two polynomials will be of degree $2p$ we need the values of $P_A(x)$ and $P_B(x)$ at $2p+1$ points, rather than just $p+1$ points!

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3. Convert such value representation of $P_C(x) = P_A(x)P_B(x)$ back to coefficient form

$$P_C(x) = C_{2p}x^{2p} + C_{2p-1}x^{2p-1} + \ldots + C_1 x + C_0;$$

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Fast multiplication of polynomials - ctd.

- We now perform $2p+1$ **multiplications of large numbers** to obtain

$$P_A(-p)P_B(-p),\ \dots\ ,\ P_A(-1)P_B(-1),\quad P_A(0)P_B(0),\quad P_A(1)P_B(1),\ \dots\ ,\ P_A(p)P_B(p)$$

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Fast multiplication of polynomials - ctd.

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- Thus the coefficients C_i can be obtained in linear time.
- So here is the algorithm we have just described:

function $\textsc{Mult}(A, B)$
  if $|A| = |B| < p+1$ **then return** AB
  else
    obtain $p+1$ slices $A_0, A_1, \ldots, A_p$ and $B_0, B_1, \ldots, B_p$ such that

$$A = A_p 2^{p\,k} + A_{p-1} 2^{(p-1)\,k} + \ldots + A_0$$

$$B = B_p 2^{p\,k} + B_{p-1} 2^{(p-1)\,k} + \ldots + B_0$$

    form polynomials

$$P_A(x) = A_p x^p + A_{p-1} x^{(p-1)} + \ldots + A_0$$

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    for $m = -p$ to $m = p$ **do**
      compute $P_A(m)$ and $P_B(m)$;
      $P_C(m) \leftarrow \textsc{Mult}(P_A(m) P_B(m))$
    end for
    compute $C_0,\ C_1, \ldots\ C_{2p}$ via

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    form $P_C(x) = C_{2p} x^{2p} + \ldots + C_0$ and compute $P_C(2^k)$
    return $P_C(2^k) = A \cdot B$
  end if
end function

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- it is easy to see that the values of the two polynomials we are multiplying have at most $k+s$ bits where s is a constant which depends on p but does NOT depend on k:

$$P_A(m) = A_p m^p + A_{p-1} m^{p-1} + \cdots + A_0 : \quad -p \leq m \leq p.$$

This is because each A_i is smaller than 2^k because each A_k has k bits; thus

$$|P_A(m)| < p^p(p+1) \times 2^k \quad \Rightarrow \quad \log_2 |P_A(m)| < \log_2(p^p(p+1)) + k = s + k$$

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- Since s is constant, its impact can be neglected.

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- so we get:

$$T(n) = \Theta\left(n^{\log_b a}\right) = \Theta\left(n^{\log_{p+1}(2p+1)}\right)$$

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- Thus, we would have to slice the input numbers into $2^{10} = 1024$ pieces!!

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- The moral is: **In practice, asymptotic estimates are useless if the size of the constants hidden by the O-notation are not estimated and found to be reasonably small!!!**

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- Every mobile phone performs thousands of FFT runs each second, for example to compress your speech signal or to compress images taken by your camera, to mention just a few uses of the FFT.

PUZZLE!

The warden meets with 23 new prisoners when they arrive. He tells them, "You may meet today and plan a strategy. But after today, you will be in isolated cells and will have no communication with one another. In the prison there is a switch room, which contains two light switches labeled A and B, each of which can be in either the on or the off position. I am not telling you their present positions. The switches are not connected to anything. After today, from time to time whenever I feel so inclined, I will select one prisoner at random and escort him to the switch room. This prisoner will select one of the two switches and reverse its position. He must move one, but only one of the switches. He can't move both but he can't move none either. Then he will be led back to his cell. No one else will enter the switch room until I lead the next prisoner there, and he'll be instructed to do the same thing. I'm going to choose prisoners at random. I may choose the same guy three times in a row, or I may jump around and come back. But, given enough time, everyone would eventually visit the switch room many times. At any time anyone of you may declare to me: "We have all visited the switch room. If it is true, then you will all be set free. If it is false, and somebody has not yet visited the switch room, you will be fed to the alligators."

What is the strategy the prisoners can devise to gain their freedom?