COMP3121/9101/3821/9801
Lecture Notes
More on Dynamic Programming (DP)

LiC: Aleks Ignjatovic

THE UNIVERSITY OF
NEW SOUTH WALES

School of Computer Science and Engineering
The University of New South Wales
Sydney 2052, Australia

1 Turtle Tower

You are given n turtles, and for each turtle you are given its weight and its strength. The strength of a turtle is the maximal weight you can put on it without cracking its shell. Find the largest possible number of turtles which you can stack one on top of the other without cracking any turtle. (Hint: for each turtle consider the sum of its weight and its strength.)

Solution

This problem illustrates well several important steps involved in applying DP.

Let the turtles be denoted by $T_1, \dots, T_n$ where the turtles are in an arbitrary order, and let the weight of turtle T_i be $W(T_i)$ and its strength $S(T_i)$. We will say that a tower of turtles is *legitimate* if the strength of each turtle in the tower is larger or equal to the weight of all turtles placed on top of it.

Generalizing the problem. DP involves building a solution of a problem recursively from solutions of its subproblems. Thus, instead of solving just the problem itself, we solve a whole set of its subproblems. For example, in this problem we might want to solve all of the following subproblems for all $1 \leq j \leq n$:

Subproblem P(j): Find the largest possible number of turtles from the set $\{T_1, \dots, T_j\}$ which you can stack one on top of the other without cracking any turtle.

We might now try to build a solution to $P(j)$ by extending an optimal solution to a problem $\{P(i) : 1 \leq i < j\}$ if possible. However, a longest chain built by choosing from turtles $\{T_i : 1 \leq i \leq j\}$ and which includes T_j might contain turtle T_j NOT as the last turtle in the tower but somewhere in the middle. Thus, such chain is not an extension of an optimal chain for any of the sub problems $\{P(i) : 1 \leq i < j\}$ because solutions to such problems are built from turtles with indices strictly smaller than i.

Choosing the right ordering. Thus, we have to find a proper set of subproblems together with an appropriate ordering on such subproblems which allows a recursive construction and which does not miss all optimal solutions, i.e., so that at least one optimal solution can be obtained by such a recursion. In this particular problem finding such ordering is tricky: one should order turtles according to the sum of their weight and their strength, in an increasing way.

Claim: If there exists a legitimate tower of height k, then there must also exist a tower of height k which is non-decreasing with respect to weight plus strength of a turtle, i.e. if there is a legitimate tower of turtles of height m there must be a tower $\langle t_1, \dots, t_m \rangle$ of the same height such that

$$i < j \Rightarrow W(t_i) + S(t_i) \leq W(t_j) + S(t_j). \tag{1.1}$$

To prove this claim we will show that in fact any legitimate tower of turtles can be reordered into another legitimate tower of turtles which is non-decreasing with respect to the weight plus strength of turtles. Thus, let $\langle t_1, \ldots, t_m \rangle$ be any legitimate tower; it is enough to show that if two consecutive turtles t_i and t_{i+1} in that tower satisfy

$$W(t_{i+1}) + S(t_{i+1}) < W(t_i) + S(t_i), \tag{1.2}$$

then these two turtles t_i and t_{i+1} can be swapped and still obtain a legitimate tower. If we prove this, we can then use the BubbleSort algorithm to get the whole tower into a non-decreasing order of weight plus strength of each turtle.

So the original tower τ and the new tower τ^* obtained by swapping t_i and t_{i+1} look like this:

$$\begin{aligned} \tau =& \langle t_1, \ldots t_{i-1}, t_i, t_{i+1}, t_{i+2} \ldots, t_m \rangle \\ \tau^* =& \langle t_1, \ldots t_{i-1}, t_{i+1}, t_i, t_{i+2}, \ldots, t_m \rangle \end{aligned}$$

The only turtle which has more weight on its back in the new tower τ^* than it had in the (legitimate) tower τ is turtle t_i; thus, it is enough to show that the weight on its back is smaller than its strength, i.e., that

$$\sum_{j=1}^{i-1} W(t_j) + W(t_{i+1}) < S(t_i) \tag{1.3}$$

Since the original tower was legitimate, we have

$$\sum_{j=1}^{i-1} W(t_j) + W(t_i) \leq S(t_{i+1})$$

Adding $W(t_{i+1})$ to both sides of this inequality we get

$$\sum_{j=1}^{i-1} W(t_j) + W(t_i) + W(t_{i+1}) \leq W(t_{i+1}) + S(t_{i+1}) \tag{1.4}$$

However, we assumed that $W(t_{i+1}) + S(t_{i+1}) < W(t_i) + S(t_i)$; see (1.2), thus by combining this with (1.4) we get

$$\sum_{j=1}^{i-1} W(t_j) + W(t_i) + W(t_{i+1}) \leq W(t_{i+1}) + S(t_{i+1}) < W(t_i) + S(t_i). \tag{1.5}$$

We can now cancel out $W(t_i)$ which appears on both sides of

$$\sum_{j=1}^{i-1} W(t_j) + W(t_i) + W(t_{i+1}) < W(t_i) + S(t_i), \tag{1.6}$$

and obtain

$$\sum_{j=1}^{i-1} W(t_j) + W(t_{i+1}) < S(t_i), \tag{1.7}$$

which is precisely what we had to prove; see (1.3).

The above claim has the following important consequence: we can restrict ourselves to non-decreasing towers and still obtain an optimal solution.

Thus, we order turtles according to the sum of their strength plus their weight, i.e., assume that for all $i, j \leq n$

$$i < j \Rightarrow W(T_i) + S(T_i) \leq W(T_j) + S(T_j), \tag{1.8}$$

and proceed by recursion along such an ordering.

Let us now first try to solve the problem using the trick which we have used before: for every $i \leq n$ we try to look for the tallest non-decreasing tower of turtles which ends with turtle T_i.

However, there is a problem: let $\langle t_1, \ldots t_m, T_i\rangle$ be the tallest legitimate tower ending with T_i and with $\{t_1, \ldots t_{m-1}, t_m\} \subseteq \{T_1, \ldots, T_{i-1}\}$. Unfortunately, the tower $\langle t_1, \ldots, t_{m-1}, t_m\rangle$ might not be the tallest possible tower ending with turtle t_m; there might be a legitimate tower with at least $m+1$ turtles $\langle t_1^*, \ldots, t_m^*, t_m\rangle$ which ends with t_m, but such tower could be too heavy to put on top of T_i, i.e., tower $\langle t_1^*, \ldots, t_m^*, t_m, T_i\rangle$ might crack turtle T_i. Thus, an optimal tower for T_i might not be obtained by extending an optimal tower for T_j for some $j < i$. We need to generalise our problem more carefully to have *the optimal substructure*, i.e., to be able to build new optimal solutions from the previous ones.

To guarantee that optimal towers can be obtained from the previous optimal towers, for every $i \leq n$ we build a sequence of **lightest possible towers of every possible height**, built from turtles $\langle T_1, \ldots T_{i-1}, T_i\rangle$.

Thus, we solve the following generalisations of our problem, for every $j \leq n$:

Subproblem P(j): For each $k \leq j$ for which there exists a tower of turtles of length k built from turtles $\{T_1, \ldots, T_j\}$ (not necessarily containing T_j), find the lightest one.

Now the recursion works: assume that we have solved subproblem $P(i-1)$. To obtain the lightest tower θ_k^i of any length k built from turtles $\{T_1, \ldots, T_i\}$ we look at the lightest tower θ_k^{i-1} of length k and the lightest tower θ_{k-1}^{i-1} of length $k-1$, both built from turtles $T_1, T_2, \ldots, T_{i-1}$. If the tower obtained by extending θ_{k-1}^{i-1} with T_i is both legitimate and lighter than θ_k^{i-1} we let θ_k^i be such a tower; otherwise we let $\theta_k^i = \theta_k^{i-1}$. This produces an optimal solution because if it is possible at all to place a tower of length l on top of a turtle, then it is certainly possible to place the lightest tower of length l on top of this turtle. Also, if the highest tower built from turtles $T_1, T_2, \ldots, T_{i-1}$ is of height m and T_i can extend it, we get the first possible legitimate tower of height $m+1$ built from turtles $T_1, T_2, \ldots, T_{i-1}, T_i$.

The solution to our problem is now obtained from the solution to $P(n)$ by picking the longest tower obtained by solving $P(n)$.

Exercise: Use a similar reasoning to solve the following problem: Given a sequence of n numbers find the longest increasing subsequence **in nlog n steps**.

2 Minimizing Total Variation of a Sequence

The total variation $V(s)$ of a sequence of numbers $s = \langle x_1, x_2, \ldots, x_k \rangle$ is defined as follows: if s has only one element, i.e., if $k = 1$, then $V(s) = 0$; if $k > 1$ then

$$V(s) = \sum_{i=1}^{k-1} |x_{i+1} - x_i|.$$

Assume that you are given a sequence of n numbers $a_1, a_2, \ldots a_n$. Split such a sequence into two subsequences (preserving the ordering of elements) such that the sum of total variations of the two sequences is as small as possible, i.e., find subsequences σ_1, σ_2 such that

$$\begin{aligned} \sigma_1 &= \langle x_{i_1}, \ldots x_{i_k} \rangle; & \qquad i_1 < i_2 < \ldots < i_k \\ \sigma_2 &= \langle x_{j_1}, \ldots x_{j_{n-k}} \rangle; & \qquad j_1 < j_2 < \ldots < j_{n-k} \end{aligned}$$

$$\{i_1, i_2, \ldots, i_k\} \cup \{j_1, j_2, \ldots, j_{n-k}\} = \{1, 2, 3, \ldots, n\}$$

and such that $V(\sigma_1) + V(\sigma_2)$ is as small as possible.

Solution: This is another tricky one. One might be tempted to reason as follows. Let us try to solve recursively the following subproblems for all $m \leq n$:

Subproblem P(j): Split the subsequence $s = \langle a_1, a_2, \ldots, a_m \rangle$ into two sequences such that the sum of their total variations is minimal.

Assume that we have an optimal solution for the sequence $s = \langle a_1, a_2, \ldots, a_m \rangle$ and that a such solution results in two subsequences, $s_1 = \langle x_1, x_2, \ldots, x_k \rangle$ whose last element is x_k for some $k < m$ and a sequence $s_2 = \langle y_1, y_2, \ldots, y_{m-k} \rangle$ whose last element is $y_{m-k} = a_m$; see Figure 2.1 (top).

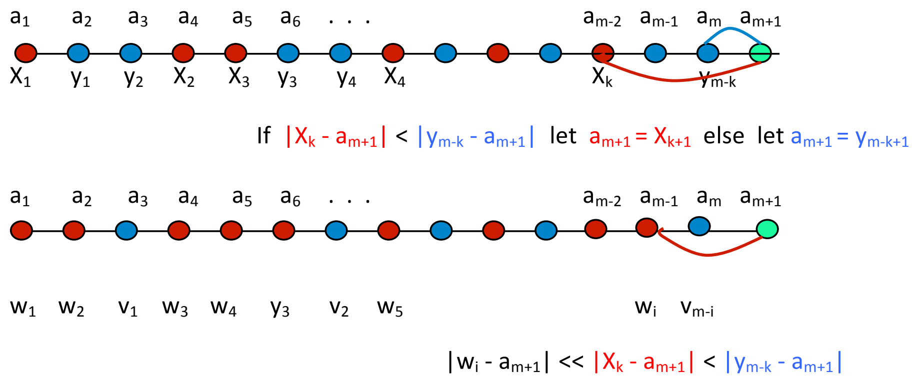

Figure 2.1:

To obtain an optimal solution for the sequence $s = \langle a_1, a_2, \ldots, a_m, a_{m+1} \rangle$ we look at the values of $|x_k - a_{m+1}|$ and $|y_{m-k} - a_{m+1}|$; if $|x_k - a_{m+1}| < |y_{m-k} - a_{m+1}|$ then we add a_{m+1} to s_1 by setting $x_{k+1} = a_{m+1}$; otherwise we add a_{m+1} to s_2 by setting $y_{m-k+1} = a_{m+1}$.

Unfortunately this does not necessarily produce an optimal solution for the sequence $s = \langle a_1, a_2, \ldots, a_m, a_{m+1} \rangle$ for the following reason. There might be another, suboptimal way to split the sequence $s = \langle a_1, a_2, \ldots, a_m \rangle$ into two sequences $\langle w_1, w_2, \ldots, w_i \rangle$ and $\langle v_1, v_2, \ldots, v_{m-i} \rangle$, i.e., such that

$$\sum_{u=1}^{i} |w_{u+1} - w_u| + \sum_{u=1}^{m-i} |v_{u+1} - v_u| > \sum_{u=1}^{k} |x_{u+1} - x_u| + \sum_{u=1}^{m-i} |y_{u+1} - y_u|$$

but such that $|w_i - a_{m+1}|$ is much smaller than $|x_k - a_{m+1}|$ and $|y_{m-k} - a_{m+1}|$, so that

$$\sum_{u=1}^{i} |w_{u+1} - w_u| + \sum_{u=1}^{m-i} |v_{u+1} - v_u| + |w_i - a_{m+1}| <$$
$$\sum_{u=1}^{k} |x_{u+1} - x_u| + \sum_{u=1}^{m-i} |y_{u+1} - y_u| + \min(|x_k - a_{m+1}|, |y_{m-k} - a_{m+1}|);$$

see the bottom of Figure 2.1.

To solve this problem we generalize it to the following "two dimensional" problems for all $i < j \leq n$:

Subproblem P(i,j): Split the subsequence $s = \langle a_1, a_2, \ldots, a_j \rangle$ into two sequences such that one ends with a_i and the other with a_j such that the sum of their total variations is minimal.

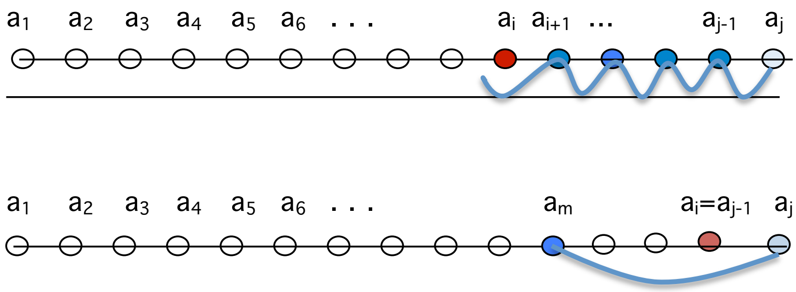

Figure 2.2:

To obtain a solution for the subproblem $P(i, j)$ from the previously obtained solutions we consider two cases.

1. If $i < j-1$ (see Figure 2.2 top) then we simply extend the optimal solution for $P(i, j-1)$ by adding a_j to the subsequence ending with a_{j-1}, because this is the only option, since the other sequence must have ended with a_i.

2. If $i = j-1$ (see Figure 2.2 bottom) then we consider optimal solutions for all subproblems $P(m, j-1)$ for all $m < j-1$ extending the subsequence ending with a_m and choosing m which results in the smallest total variation of such obtained sequences. We now compare such minimal value with a solution in which one sequence is the entire sequence $\langle a_1, a_2, \ldots, a_{j-1} \rangle$ and the other sequence just the singleton $\langle a_j \rangle$ and pick the one which has the smaller total variance as the solution for the problem $P(i, j)$.

Exercise: Reduce the above solution to a "single dimension". (Hint: simply consider only problems of the form $P(i-1, i)$.