Algorithms: COMP3121/9101

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11. INTRACTABILITY

Feasibility of Algorithms

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- We denote this by $A \in \mathbf{P}$.

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    - This can only change the constants involved in the expression $T(n) = O(n^k)$ but not the asymptotic bound.

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- In fact, every precise description without artificial redundancies will do.

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- For example, decision problem "integer x is not prime" is in **NP** because $A(x)$ is true just in case there exists integer y such that $B(x, y) =$"x is divisible by y" is true.
- Clearly, the problem "x is divisible by y" is decidable by an algorithm which runs in time polynomial in the length of x only.
- In fact, "integer x is not prime" is actually decidable in (deterministic) polynomial time, but this is a hard theorem to prove.

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- If each clause C_i involves exactly three variables we call such decision problem 3SAT.

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- However, so far, no one has been able to prove (or disprove) that this is indeed the case, despite a huge effort of very many very famous people!!
- Conjecture that NP is a strictly larger class of decision problems than P is known as "$P \neq NP$" hypothesis, and it is widely believed that it is one of the hardest open problems in the whole of Mathematics!!

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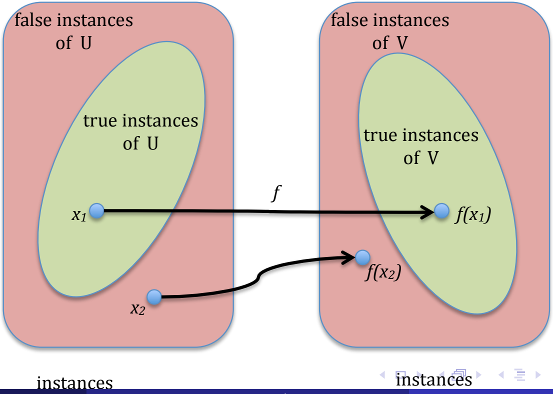

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- Clearly, (**??**) can be obtained from (**??**) using a simple polynomial time algorithm.

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- Unfortunately, this cannot be farthest from the truth!
- A vast number of practically important decision problems are NP complete!

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- Think of a mailman which has to deliver mail to several addresses and then return to the post office. Can he do it while traveling less than L kilometres in total?

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- In graph theoretic terms: Is it possible to color the vertices of a graph G with at most K colors so that no edge has both vertices of the same color.

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- As we will see, many other practically important problems are NP complete.

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- The Traveling Salesman Optimisation Problem is clearly NP hard:
- using a "black box" for solving it, we can solve the Traveling Salesman Decision problem:
- Given a weighted graph G and a number L we can determine if there is a cycle containing all vertices of the graph and whose length is at most L.
- We do that by solving the Traveling Salesman Optimisation Problem thus determining the length of the cycle of minimal possible length and comparing the length of such a cycle with L.
- Since all other NP problems are polynomial time reducible to the Traveling Salesman Decision problem (which is NP complete), then every other NP problem is solvable using a "black box" for the Traveling Salesman Optimisation Problem.

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- Thus, for a practical problem which appears to be hard, the strategy would be:
    - prove that the problem is indeed NP hard, to justify not trying solving the problem exactly;
    - look for an approximation algorithm which provides a feasible sub-optimal solution that it is not too bad.

Proving NP completeness

Warning: sometimes distinction between a problem in P and an NP complete problem can be subtle!

in P	**NP complete**
• Given a graph G and two vertices s and t, is there a path from s to t of length **at most** K?	• Given a graph G and two vertices s and t, is there a simple path from s to t of length **at least** K?
• Given a propositional formula in CNF form such that every clause has at most **two** propositional variables, does the formula have a satisfying assignment?	• Given a propositional formula in CNF form such that every clause has at most **three** propositional variables, does the formula have a satisfying assignment?
• Given a graph G, does G have a tour where every **edge** is traversed exactly once? (An *Euler tour*.)	• Given a graph G, does G have a tour where every **vertex** is visited exactly once? (A *Hamiltonian* cycle.)

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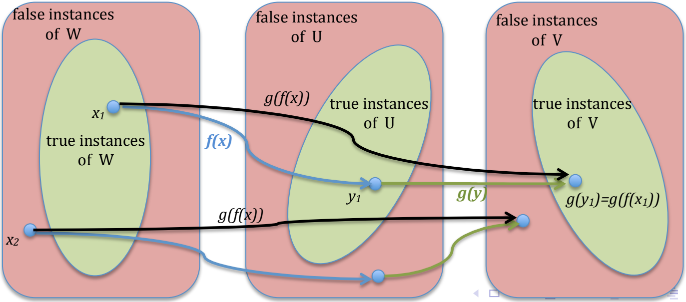

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  - Since $f(x)$ is the output of a polynomial time computable function, the length $|f(x)|$ of the output $f(x)$ can be at most a polynomial in $|x|$, i.e., for some polynomial (with positive coefficients) P we have $|f(x)| \leq P(|x|)$.

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  - In total, the computation of $g(f(x)$ terminates in at most $P(|x|) + Q(P(|x|))$ many steps, which is a polynomial bound in $|x|$.

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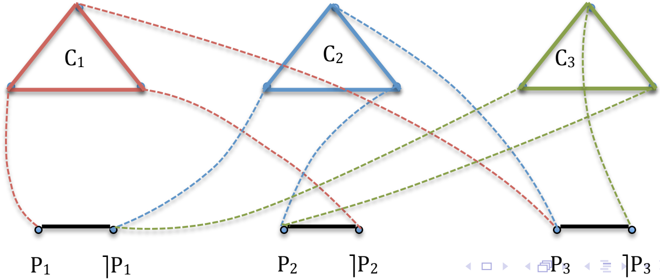

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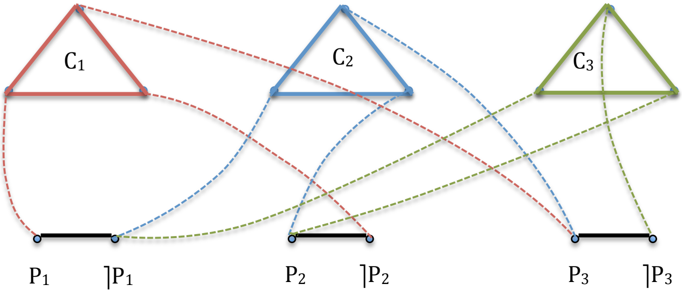

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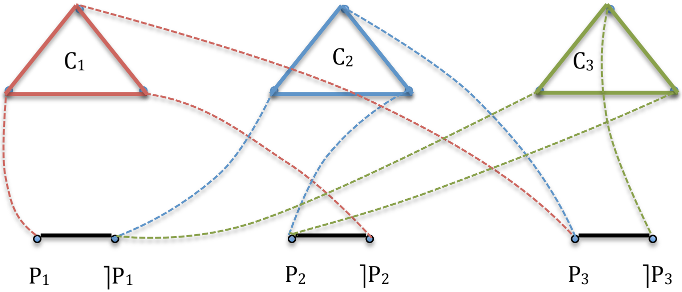

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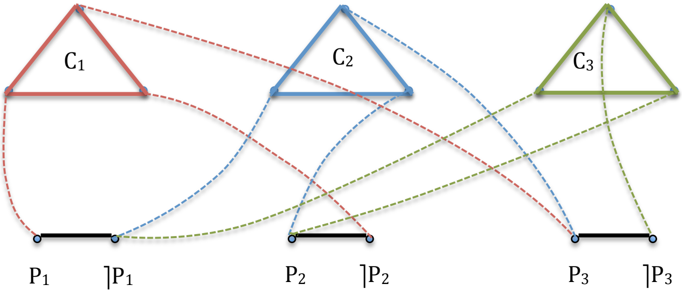

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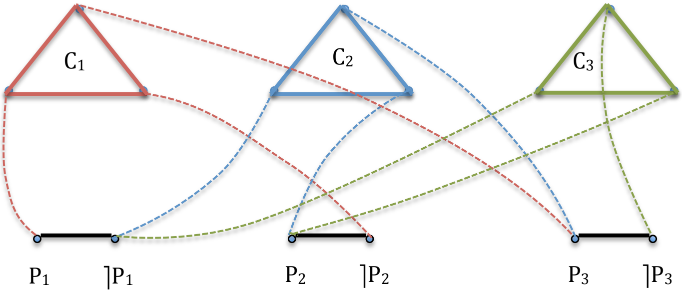

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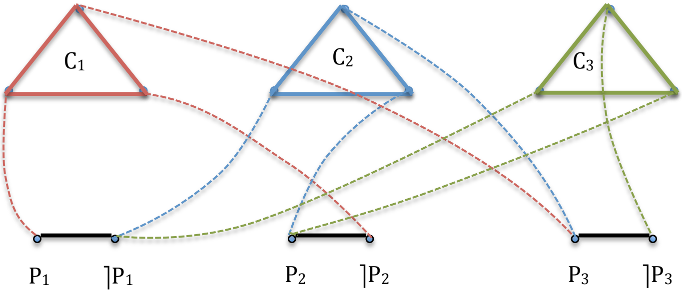

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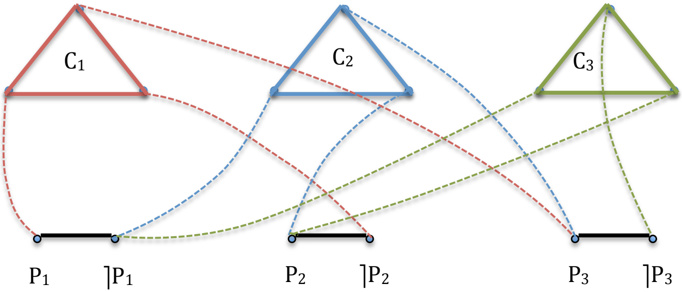

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  1. Each triangle must have at least two vertices chosen;
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- This is in total $2M + N$ points; thus each triangle must have *exactly* two vertices chosen and each segment must have *exactly* one of its ends chosen.

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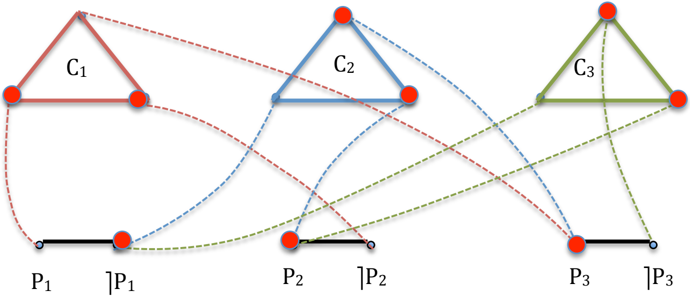

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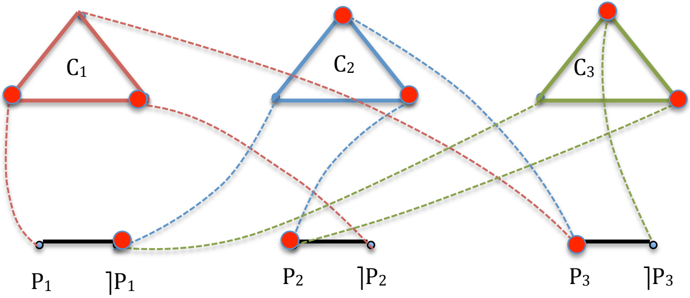

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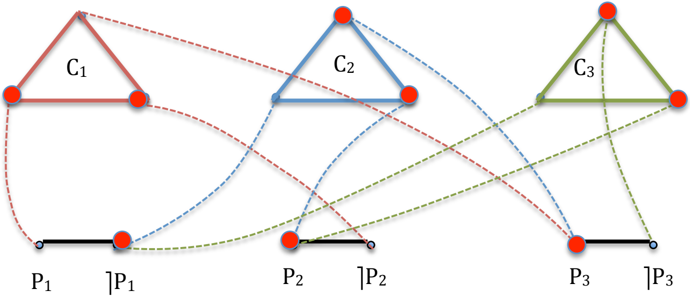

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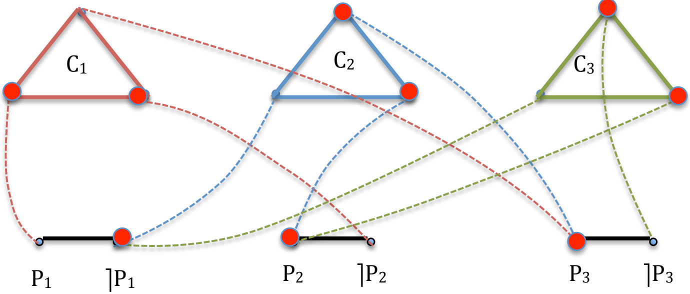

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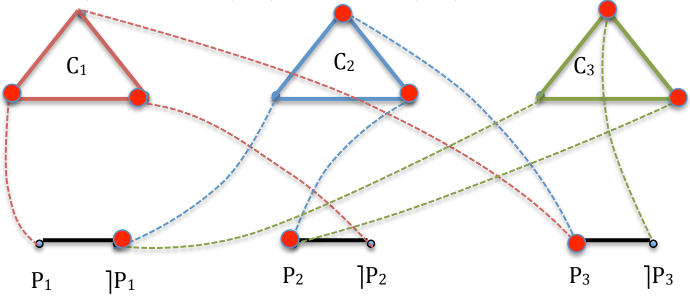

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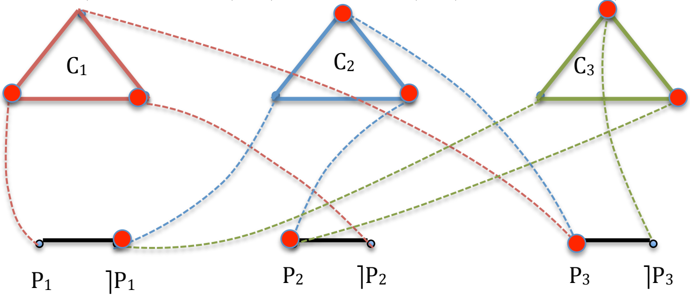

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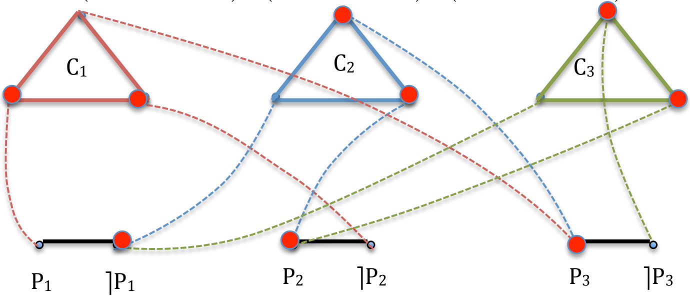

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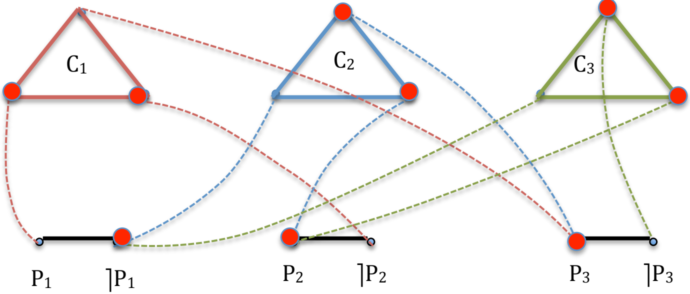

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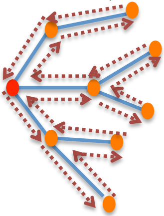
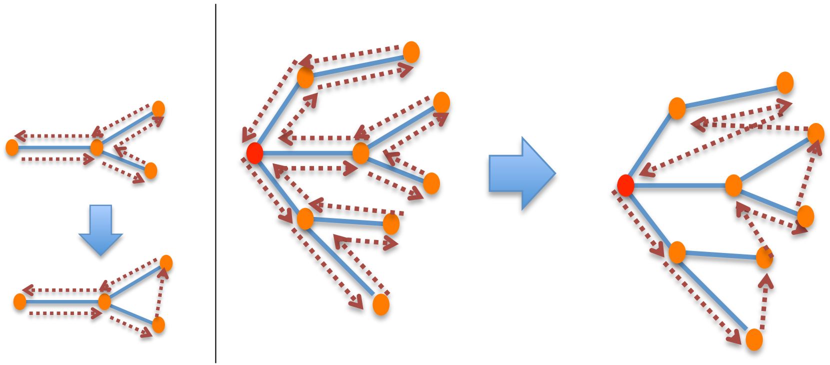
- If we do depth first traverse of the tree, we will travel in total a distance of $2w(T) \leq 2opt$.



Dealing with NP hard optimisation problems



- We now take shortcuts to avoid visiting vertices more than once; because of the triangle inequality, this operation does not increase the length of the tour.

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- On the other hand, the most general Traveling Salesman Problem does not allow any approximate solution at all: if $P \neq NP$, then for no $K > 0$ there can be a polynomial time algorithm which for every instance produces a tour which is at most K times longer than the optimal tour of minimal possible length, no matter how large K is chosen!

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- To prove this, we show that if there existed $K > 0$ and a polynomial time algorithm producing a tour which is at most K times longer than the optimal tour, then we could obtain an algorithm which solves in polynomial time the Hamiltonian Cycle Problem, i.e., which for every graph G determines if G contains a cycle visiting all vertices exactly once, which is impossible because this problem is known to be NP complete.

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- In this case the optimal tour through G^* has length of at least $(K \cdot n) + (n-1) \cdot 1 > K \cdot n$.
- Thus, our approximation algorithm can return a tour of length at most $K \cdot n$ if an only if it actually returns a tour of length of size n, which happens just in case G has a Hamiltonian cycle.
- This is impossible, because this would be a polynomial time decision procedure for determining in G has a Hamiltonian cycle.