# Algorithms: COMP3121/9101

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2. DIVIDE-AND-CONQUER

# A Puzzle

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- We have already seen a prototypical “serious” algorithm designed using such a method: the MERGE-SORT.
- We split the array into two, sort the two parts recursively and then merge the two sorted arrays.
- We now look at a closely related but more interesting problem of counting inversions in an array.

- Assume that you have $m$ users ranking the same set of $n$ movies. You want to determine for any two users $A$ and $B$ how similar their tastes are (for example, in order to make a recommender system).

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- Lets enumerate the movies on the ranking list of user $B$ by assigning to the top choice of user $B$ index 1, assign to his second choice index 2 and so on.

- For the $i^{th}$ movie on $B's$ list we can now look at the position (i.e., index) of that movie on $A's$ list, denoted by $a(i)$.

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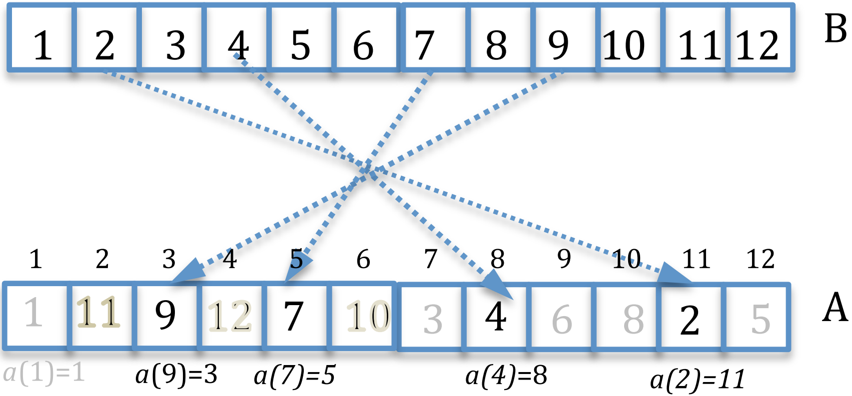

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- A good measure of how different these two users are, is the total number of *inversions*, i.e., total number of pairs of movies $i, j$ such that movie $i$ precedes movie $j$ on $B's$ list but movie $j$ is higher up on $A's$ list than the movie $i$.

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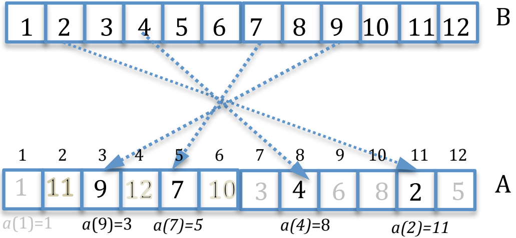

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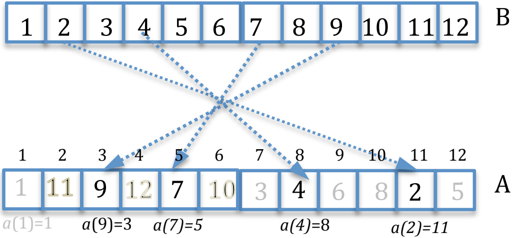


- For example 1 and 2 do not form an inversion because $a(1) < a(2)$ ($a(1) = 1$ and $a(2) = 11$ because $a(1)$ is on the first and $a(2)$ is on the $11^{th}$ place in $A$);

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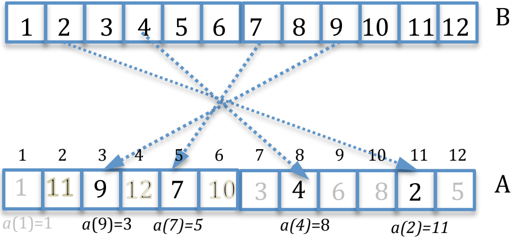


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- However, for example 4 and 7 do form an inversion because $a(7) < a(4)$ ($a(7) = 5$ because seven is on the fifth place in $A$ and $a(4) = 8$)

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- Clearly, since the total number of pairs is quadratic in $n$, we cannot afford to inspect all possible pairs.

- The main idea is to tweak the MERGE-SORT algorithm, by extending it to recursively both sort an array $A$ **and** determine the number of inversions in $A$.

# Counting the number of inversions

- We split the array $A$ into two (approximately) equal parts $A_{top} = A[1 \ldots \lfloor n/2 \rfloor]$ and $A_{bottom} = A[\lfloor n/2 \rfloor + 1 \ldots n]$.

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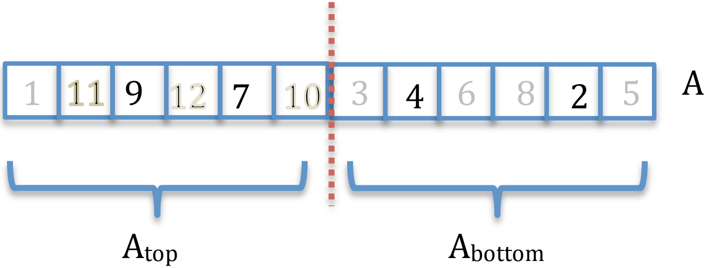
- Note that the total number of inversions in array $A$ is equal to the sum of the number of inversions $I(A_{top})$ in $A_{top}$ (such as 9 and 7) plus the number of inversions $I(A_{bottom})$ in $A_{bottom}$ (such as 4 and 2) plus the number of inversions $I(A_{top}, A_{bottom})$ across the two halves (such as 7 and 4).



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- We now recursively sort arrays $A_{top}$ and $A_{bottom}$ also obtaining in the process the number of inversions $I(A_{top})$ in the sub-array $A_{top}$ and the number of inversions $I(A_{bottom})$ in the sub-array $A_{bottom}$.

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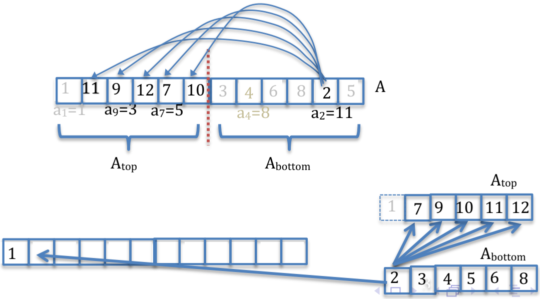
- When the next smallest element among all elements in both arrays is an element in $A_{bottom}$, such an element clearly is in an inversion with all the remaining elements in $A_{top}$ and we add the total number of elements remaining in $A_{top}$ to the current value of the number of inversions across $A_{top}$ and $A_{bottom}$.



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- **Next:** we study applications of divide and conquer to arithmetic of very large integers.

# Basics revisited: how do we add two numbers?

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- no asymptotically faster algorithm

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- "Simple" problems can actually turn out to be difficult!

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What we mean is that the product $AB$ can be calculated recursively by the following program:

**function** $\textsc{Mult}(A, B)$
  **if** $|A| = |B| = 1$ **then return** $AB$
  **else**
    $A_1 \leftarrow \text{MoreSignificantPart}(A);$
    $A_0 \leftarrow \text{LessSignificantPart}(A);$
    $B_1 \leftarrow \text{MoreSignificantPart}(B);$
    $B_0 \leftarrow \text{LessSignificantPart}(B);$
    $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0);$
    $Y \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_1);$
    $Z \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_0);$
    $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1);$
    **return** $W\,2^n + (Y + Z)\,2^{n/2} + X$
  **end if**
**end function**

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$$T\left(n\right) = 4\,T\left(\tfrac{n}{2}\right) + c\,n = 4\left(\left(\tfrac{n}{2}\right)^2 (c+1) - \tfrac{n}{2}\,c\right) + c\,n$$

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$$T(n) = 4T\left(\frac{n}{2}\right) + c\,n \tag{3}$$

then

$$T(n) = n^2(c+1) - c\,n$$

**Proof:** By "fast" induction. We assume it is true for $n/2$:

$$T\left(\frac{n}{2}\right) = \left(\frac{n}{2}\right)^2 (c+1) - c\,\frac{n}{2}$$

and prove that it is also true for $n$:

$$\begin{aligned} T\left(n\right) &= 4\,T\left(\tfrac{n}{2}\right) + c\,n = 4\left(\left(\tfrac{n}{2}\right)^2 (c+1) - \tfrac{n}{2}\,c\right) + c\,n \\ &= n^2(c+1) - 2c\,n + c\,n = n^2(c+1) - c\,n \end{aligned}$$

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Remarkably, there is, but first some history:
In 1952, one of the most famous mathematicians of the $20^{th}$ century, Andrey Kolmogorov, conjectured that you cannot multiply in less than $\Omega(n^2)$ elementary operations. In 1960, Karatsuba, then a 23-year-old student, found an algorithm (later it was called "divide and conquer") that multiplies two n-digit numbers in $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.58\ldots})$ elementary steps, thus disproving the conjecture!! Kolmogorov was shocked!

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Take again our two input numbers $A$ and $B$, and split them into two halves:

$$A = A_1 2^{\frac{n}{2}} + A_0 \qquad \underbrace{XX \ldots X}_{\frac{n}{2}}\underbrace{XX \ldots X}_{\frac{n}{2}}$$

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$$AB = A_1 B_1 2^n + (A_1 B_0 + A_0 B_1) 2^{\frac{n}{2}} + A_0 B_0$$

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$$\begin{aligned} AB &= A_1B_1 2^n + (A_1B_0 + A_0B_1)2^{\frac{n}{2}} + A_0B_0 \\ &= A_1B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0 \end{aligned}$$

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- $AB$ can now be calculated as follows:

$$AB = A_1B_12^n + (A_1B_0 + A_0B_1)2^{\frac{n}{2}} + A_0B_0$$

$$= A_1B_12^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0$$

- So we have saved one multiplication at each recursion round!

- Thus, the algorithm will look like this:

**function** $\textsc{Mult}(A, B)$<br>
    **if** $|A| = |B| = 1$ **then return** $AB$<br>
    **else**<br>
        $A_1 \leftarrow \text{MoreSignificantPart}(A);$<br>
        $A_0 \leftarrow \text{LessSignificantPart}(A);$<br>
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        $U \leftarrow A_0 + A_1;$<br>
        $V \leftarrow B_0 + B_1;$<br>
        $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0);$<br>
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- How fast is this algorithm?

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$$\approx 3^{\log_2 n}T(1) + c\,n\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} = 3^{\log_2 n}T(1) + 2c\,n\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)$$

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Please review the basic properties of logarithms and the asymptotic notation from the review material (the first item at the class webpage under "class resources".)

# A Karatsuba style trick also works for matrices: Strassen's algorithm for faster matrix multiplication

- If we want to multiply two $n \times n$ matrices $P$ and $Q$, the product will be a matrix $R$ also of size $n \times n$. To obtain each of $n^2$ entries in $R$ we do $n$ multiplications, so matrix product by brute force is $\Theta(n^3)$.

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$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad Q = \begin{pmatrix} e & f \\ g & h \end{pmatrix}; \qquad R = \begin{pmatrix} r & s \\ t & u \end{pmatrix}.$$

- Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \tag{4}$$

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \tag{5}$$

- We obtain:
$$\begin{aligned} a\,e + b\,g &= r & \qquad a\,f + b\,h &= s \\ c\,e + d\,g &= t & \qquad c\,f + d\,h &= u \end{aligned}$$

- Prima facie, there are 8 matrix multiplications, each running in time $T\left(\frac{n}{2}\right)$ and 4 matrix additions, each running in time $O(n^2)$, so such a direct calculation would result in time complexity governed by the recurrence

$$T(n) = 8T\left(\frac{n}{2}\right) + c\,n^2$$

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- The first case of the Master Theorem gives $T(n) = \Theta(n^3)$, so nothing gained.

# Strassen's algorithm for faster matrix multiplication

- However, we can instead evaluate:

$$A = a\,(f-h); \qquad B = (a+b)\,h; \qquad C = (c+d)\,e \qquad D = d\,(g-e);$$
$$E = (a+d)\,(e+h); \quad F = (b-d)(g+h); \quad H = (a-c)\,(e+f).$$

- We now obtain

$$\begin{aligned}
E + D - B + F &= (a\,e + d\,e + a\,h + d\,h) + (d\,g - d\,e) - (a\,h + b\,h) + (b\,g - d\,g + b\,h - d\,h) \\
&= a\,e + b\,g = r; \\
A + B &= (a\,f - a\,h) + (a\,h + b\,h) = a\,f + b\,h = s; \\
C + D &= (c\,e + d\,e) + (d\,g - d\,e) = c\,e + d\,g = t; \\
E + A - C - H &= (a\,e + d\,e + a\,h + d\,h) + (a\,f - a\,h) - (c\,e + d\,e) - (a\,e - c\,e + a\,f - c\,f) \\
&= c\,f + d\,h = u.
\end{aligned}$$

- We have obtained all 4 components of $C$ using only 7 matrix multiplications and 18 matrix additions/subtractions.
- Thus, the run time of such recursive algorithm satisfies $T(n) = 7T(n/2) + O(n^2)$ and the Master Theorem yields $T(n) = \Theta(n^{\log_2 7}) = O(n^{2.808})$.
- In practice, this algorithm beats the ordinary matrix multiplication for $n > 32$.

# Next time:

1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??

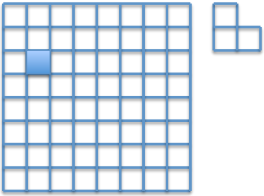
# Next time:

1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??
2. Can we avoid messy computations like:

$$
\begin{aligned}
T(n) &= 3T\left(\frac{n}{2}\right) + cn = 3\left(3T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right) + cn = 3^2T\left(\frac{n}{2^2}\right) + c\frac{3n}{2} + cn \\
&= 3^2\left(3T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right) + c\frac{3n}{2} + cn = 3^3T\left(\frac{n}{2^3}\right) + c\frac{3^2n}{2^2} + c\frac{3n}{2} + cn \\
&= 3^3T\left(\frac{n}{2^3}\right) + cn\left(\frac{3^2}{2^2} + \frac{3}{2} + 1\right) = \\
&= 3^3\left(3T\left(\frac{n}{2^4}\right) + c\frac{n}{2^3}\right) + cn\left(\frac{3^2}{2^2} + \frac{3}{2} + 1\right) = \\
&= 3^4T\left(\frac{n}{2^4}\right) + cn\left(\frac{3^3}{2^3} + \frac{3^2}{2^2} + \frac{3}{2} + 1\right) = \\
&\ldots \\
&= 3^{\lfloor \log_2 n \rfloor}T\left(\frac{n}{\lfloor 2^{\log_2 n} \rfloor}\right) + cn\left(\left(\frac{3}{2}\right)^{\lfloor \log_2 n \rfloor - 1} + \ldots + \frac{3^2}{2^2} + \frac{3}{2} + 1\right) \\
&\approx 3^{\log_2 n}T(1) + cn\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} \\
&= 3^{\log_2 n}T(1) + 2cn\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)
\end{aligned}
$$

# PUZZLE!

You are given a $2^n \times 2^n$ board with one of its cells missing (i.e., the board has a hole); the position of the missing cell can be arbitrary. You are also given a supply of "dominoes" each containing 3 such squares; see the figure:



Your task is to design an algorithm which covers the entire board with such "dominoes" except for the hole.

Hint: Do a divide-and-conquer recursion!


**That's All, Folks!!**