Algorithms
COMP3121/9101

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5. THE FAST FOURIER TRANSFORM
(not examinable material)

Our strategy to multiply polynomials fast:

- Given two polynomials of degree at most n,

$$P_A(x) = A_n x^n + \ldots + A_0; \qquad P_B(x) = B_n x^n + \ldots + B_0$$

1. convert them into value representation at $2n+1$ distinct points $x_0, x_1, \ldots, x_{2n}$:

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_{2n}, P_A(x_{2n}))\}$$
$$P_B(x) \leftrightarrow \{(x_0, P_B(x_0)), (x_1, P_B(x_1)), \ldots, (x_{2n}, P_B(x_{2n}))\}$$

2. multiply them point by point using $2n+1$ multiplications:

$$\Big\{(x_0, \underbrace{P_A(x_0)P_B(x_0)}_{P_C(x_0)}), \; (x_1, \underbrace{P_A(x_1)P_B(x_1)}_{P_C(x_1)}), \ldots, (x_{2n}, \underbrace{P_A(x_{2n})P_B(x_{2n})}_{P_C(x_{2n})})\Big\}$$

3. Convert such value representation of $P_C(x)$ to its coefficient form

$$P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \ldots + C_1 x + C_0;$$

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- So, we need $2n+1$ values of $P_A(x_i)$ and $P_B(x_i)$, $0 \leq i \leq 2n$.
- If we use $2n+1$ integers which are the smallest by their absolute value, i.e.,

  $$-n, -(n-1), \ldots, -1, 0, 1, \ldots, n-1, n$$

  among the values which we need to compute is
  $P_A(n) = A_0 + A_1 n + \ldots + A_{n-1} n^{n-1} + A_n n^n$
- We saw that the trouble is that, as the degree n of the polynomials $P_A(x)$ and $P_B(x)$ increases, the value of n^n increases very fast and causes rapid increase of the computational complexity of the algorithm for polynomial multiplication which we used in the generalised Karatsuba algorithm.
- **Key Question:** What values should we take for $x_0, \ldots, x_{2n}$ to avoid "explosion" of size when we evaluate x_i^n while computing $P_A(x_i) = A_0 + A_1 x + \ldots + A_n x_i^n$?

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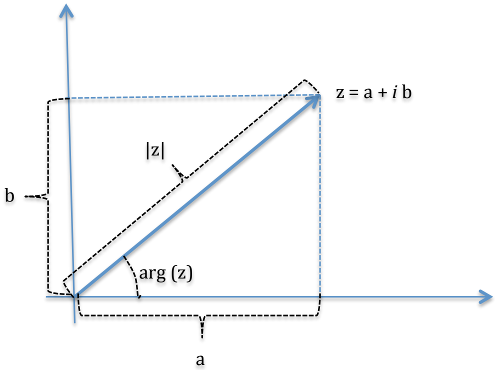
Complex numbers revisited

Complex numbers $z = a + ib$ can be represented using their *modulus* $|z| = \sqrt{a^2 + b^2}$ and their *argument*, $\arg(z)$, which is an angle taking values in $(-\pi, \pi]$ and satisfying:

$$z = |z|e^{i\arg(z)} = |z|(\cos\arg(z) + i\sin\arg(z)),$$

see figure below.



Complex numbers revisited

Recall that

$$z^n = \left(|z|e^{i\ \arg(z)}\right)^n = |z|^n e^{i\,n\arg(z)} = |z|^n(\cos(n\arg(z)) + i\sin(n\arg(z))),$$

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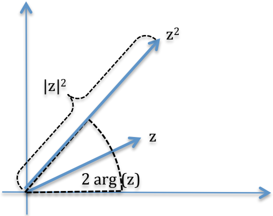
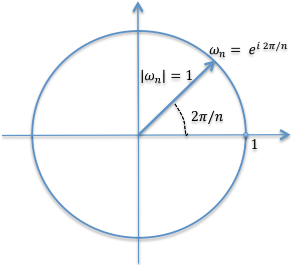
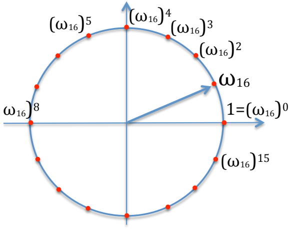


Complex roots of unity

- *Roots of unity of order n* are complex numbers which satisfy $z^n = 1$.
- If $z^n = |z|^n(\cos(n \arg(z)) + i \sin(n \arg(z))) = 1$ then

  $|z| = 1$ and $n \arg(z)$ is an integer multiple of 2π;
- Thus, $n \arg(z) = 2\pi\, k$, i.e., $\arg(z) = \dfrac{2\pi\, k}{n}$
- We denote $\omega_n = e^{i\, 2\pi/n}$; such ω_n is called a *primitive root of unity of order n.*





Roots of unity of order 16

- A root of unity ω of order n is *primitive* if all other roots of unity of the same order can be obtained as its powers ω^k.

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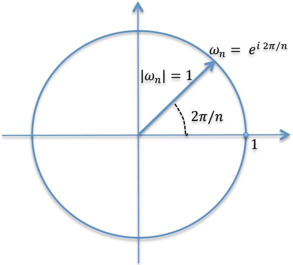

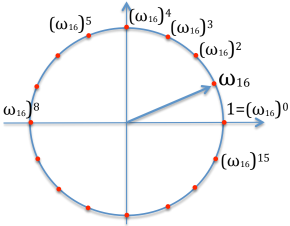

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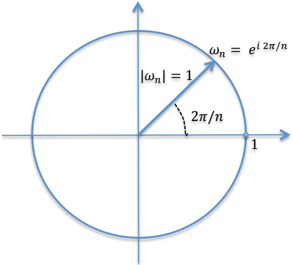

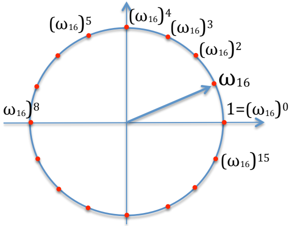

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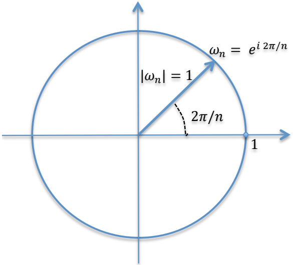

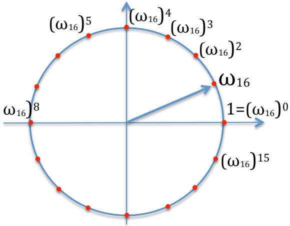

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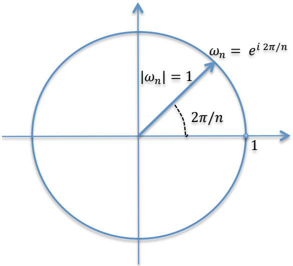

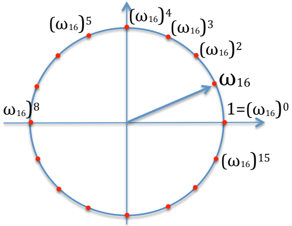

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- For $\omega_n = e^{i\,2\pi/n}$ and for all k such that $0 \leq k \leq n-1$,

$$((\omega_n)^k)^n = (\omega_n)^{n\,k} = ((\omega_n)^n)^k = 1^k = 1.$$

- Thus, $\omega_n^k = (\omega_n)^k$ is also a root of unity (and it can be shown that it is primitive just in case k is relatively prime with n).
- Since ω_n^k are roots of unity for $k = 0, 1, \ldots, n-1$ and there are at most n distinct roots of unity of order n (i.e., solutions to the equation $x^n - 1 = 0$) we conclude that every root of unity of order n must be of the form ω_n^k.
- For the product of any two roots of unity ω_n^k and ω_n^m of the same order we have $\omega_n^k \omega_n^m = \omega_n^{k+m}$.
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- If $k + m \geq n$ then $k + m = n + l$ for $l = (k+m) \bmod n$ and we have $\omega_n^k \omega_n^m = \omega_n^{k+m} = \omega_n^{n+l} = \omega_n^n \omega_n^l = 1 \cdot \omega_n^l = \omega_n^l$ where $0 \leq l < n$.
- Thus, the product of any two roots of unity of the same order is just another root of unity of the same order.

Complex roots of unity

- So in the set of all roots of unity of order n, i.e., $\{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$ we can multiply any two elements or raise an element to any power without going out of this set.
- Note that this is not true for addition, i.e., the sum of two roots of unity is NOT another root of unity!
- A most important property of the roots of unity is:

  The Cancelation Lemma: $\omega_{kn}^{km} = \omega_n^m$ for all integers k, m, n.

  Proof:

$$\omega_{kn}^{km} = (\omega_{kn})^{km} = (e^{i\frac{2\pi}{kn}})^{km} = e^{i\frac{2\pi km}{kn}} = e^{i\frac{2\pi m}{n}} = (e^{i\frac{2\pi}{n}})^m = \omega_n^m$$

- Thus, in particular, $(\omega_{2n}^k)^2 = \omega_{2n}^{2k} = (\omega_{2n}^2)^k = \omega_n^k$.
- So the squares of the roots of unity of order $2n$ are just the roots of unity of order n.

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The Discrete Fourier Transform

- Let $A = \langle A_0, A_1, \ldots, A_{n-1} \rangle$ be a sequence of n real or complex numbers.
- We can form the corresponding polynomial $P_A(x) = \sum_{j=0}^{n-1} A_j x^j$.
- We can evaluate it at all complex roots of unity of order n, i.e., we compute $P_A(\omega_n^k)$ for all $0 \leq k \leq n-1$.
- The sequence of values $\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \ldots, P_A(\omega_n^{n-1}) \rangle$, is called **the Discrete Fourier Transform (DFT)** of the sequence $A = \langle A_0, A_1, \ldots, A_{n-1} \rangle$.
- The value $P_A(\omega_n^k)$ is usually denoted by $\widehat{A}_k$ and the sequence of values $\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \ldots, P_A(\omega_n^{n-1}) \rangle$ is usually denoted by $\widehat{A} = \langle \widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_{n-1} \rangle$.
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New way for fast multiplication of polynomials

- If we multiply a polynomial

  $$P_A(x) = A_0 + \ldots + A_{n-1}x^{n-1}$$

  of degree $n-1$ with a polynomial

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  of degree $n-1+m-1 = m+n-2$ with $m+n-1$ coefficients.

- To uniquely determine such a polynomial $C(x)$ of degree $m+n-2$ we need $m+n-1$ many values.

- Thus, we will evaluate both $P_A(x)$ and $P_B(x)$ at all the roots of unity of order $n+m-1$ (instead of at $-(n-1), \ldots, -1, 0, 1, \ldots, m-1$ as we would in Karatsuba's method!)

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- Note that we defined the DFT of a sequence of length n as the values of the corresponding polynomial of degree $n-1$ at the n roots of unity of order n, i.e., ω_n^k $(0 \leq k \leq n-1)$.
- So the DFT of a sequence A is another sequence $\widehat{A}$ of exactly the same length; we do not have an operation which would evaluate a polynomial of degree $n-1$ at m roots of unity of order $m \neq n$.
- For that reason, since we need $n+m-1$ values of $P_C(x) = P_A(x)P_B(x)$, we pad A with $m-1$ zeros at the end, $(A_0, A_1, \ldots, A_{n-1}, \underbrace{0, \ldots, 0}_{m-1})$ to make it of length $m+n-1$, and similarly we pad B with $n-1$ zeros at the end, $(B_0, B_1, \ldots, B_{m-1}, \underbrace{0, \ldots, 0}_{n-1})$ to also obtain a sequence of length $n+m-1$.
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New way for fast multiplication of polynomials

- We can now compute the DFTs of the two (0 padded) sequences:

$$DFT(\langle A_0, A_1, \ldots, A_{n-1}, \underbrace{0, \ldots, 0}_{m-1} \rangle) = \langle \widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_{n+m-2} \rangle$$

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- We then use the inverse transformation for DFT, called IDFT, to recover the coefficients $\langle C_0, C_1, \ldots, C_{n+m-1} \rangle$ of the product polynomial $P_C(x)$ from the sequence $\langle \widehat{C}_0, \widehat{C}_1, \ldots, \widehat{C}_{n+m-1} \rangle$ of its values $C_k = P_C(\omega^k_{n+m-1})$ at the roots of unity of order $n + m - 1$.

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$$P_A(x) = A_0 + \ldots + A_{n-1}x^{n-1} + 0 \cdot x^n + \ldots + 0 \cdot x^{n+m-2};$$

$$P_B(x) = B_0 + \ldots + B_{m-1}x^{m-1} + 0 \cdot x^m + \ldots + 0 \cdot x^{n+m-2}$$

$\Downarrow$ DFT $\qquad\qquad$ $\Downarrow$ DFT

$$\{P_A(1), P_A(\omega_{n+m-1}), \ldots, P_A(\omega_{n+m-1}^{n+m-2})\}; \quad \{P_B(1), P_B(\omega_{n+m-1}), \ldots, P_B(\omega_{n+m-1}^{n+m-2})\}$$

$\Downarrow$ multiplication

$$\Big\{ \underbrace{P_A(1)P_B(1)}_{P_C(1)}, \quad \underbrace{P_A(\omega_{n+m-1})P_B(\omega_{n+m-1})}_{P_C(\omega_{n+m-1})}, \ldots, \underbrace{P_A(\omega_{n+m-1}^{n+m-2})P_B(\omega_{n+m-1}^{n+m-2})}_{P_C(\omega_{n+m-1}^{n+m-2})} \Big\}$$

$\Downarrow$ IDFT

$$P_C(x) = P_A(x) \cdot P_B(x) = \sum_{j=0}^{n+m-2} C_j x^j = \sum_{j=0}^{n+m-2} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i} \right)}_{C_j} x^j$$

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- Crucial fact: the values $P_A(\omega_n^k)$ for all k such that $0 \leq k < n$ can be computed in $\mathbf{O(n \log n)}$ time!
- Note that a direct evaluation of a polynomial of degree $n-1$ at n roots of unity of order n would take n^2 many multiplications, even if we precompute all powers $\omega_n^{k\,m}$, because we have to perform multiplications $A_m \cdot (\omega_n^k)^m$ for all $0 \leq m \leq n-1$ and all $0 \leq k \leq n-1$.
  - We can assume that n is a power of 2 - otherwise we can pad $P_A(x)$ with zero coefficients until its number of coefficients becomes equal to the nearest power of 2.
  - Exercise: Show that for every n which is not a power of two the smallest power of 2 larger or equal to n is smaller than $2n$.
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- **Problem of size n:**
  Evaluate a polynomial with n coefficients at n many roots of unity.

- **Problem of size $n/2$:**
  Evaluate a polynomial with $n/2$ coefficients at $n/2$ many roots of unity.

- We reduced evaluation of our polynomial $P_A(x)$ with n coefficients at inputs $x = \omega_n^0,\ x = \omega_n^1,\ x = \omega_n^2, \ldots, x = \omega_n^{n-1}$ to evaluation of two polynomials $P_{A^{[0]}}(y)$ and $P_{A^{[1]}}(y)$ each with $n/2$ coefficients, at points $y = x^2$ for the same values of inputs x.

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- Note that by the Cancelation Lemma $\omega_n^{n/2} = \omega_{2n/2}^{n/2} = \omega_2 = -1$.
- Thus,

$$\omega_n^{k+n/2} = \omega_n^{n/2}\omega_n^k = \omega_2\omega_n^k = -\omega_n^k;$$

- We can now simplify evaluation of

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- Compare this with $P_A(\omega_n^m) = P_{A^{[0]}}(\omega_{n/2}^m) + \omega_n^m P_{A^{[1]}}(\omega_{n/2}^m)$ for $0 \leq m < n/2$.

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- Note that by the Cancelation Lemma $\omega_n^{n/2} = \omega_{2n/2}^{n/2} = \omega_2 = -1$.
- Thus,

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for $n/2 \leq k < n$ as follows:
let $k = n/2 + m$ where $0 \leq m < n/2$; then

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- We can now write a pseudo-code for our FFT algorithm:

FFT algorithm

```
function FFT($A$)
    $n \leftarrow \text{length}[A]$
    if $n = 1$ then return $A$
    else
        $A^{[0]} \leftarrow (A_0, A_2, \dots A_{n-2})$;
        $A^{[1]} \leftarrow (A_1, A_3, \dots A_{n-1})$;
        $y^{[0]} \leftarrow FFT(A^{[0]})$;
        $y^{[1]} \leftarrow FFT(A^{[1]})$;
        $\omega_n \leftarrow e^{i\frac{2\pi}{n}}$;
        $\omega \leftarrow 1$;                 % a variable to hold powers of $\omega_n$
        for $k = 0$ to $k = n/2 - 1$ do:      % $\underbrace{P_A(\omega_n^k)}_{y_k} = \underbrace{P_{A^{[0]}}(\omega_{n/2}^k)}_{y_k^{[0]}} + \omega_n^k \underbrace{P_{A^{[1]}}(\omega_{n/2}^k)}_{y_k^{[1]}}$
            $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$;
            $y_{n/2+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$      % $\underbrace{P_A(\omega_n^{n/2+k})}_{y_{n/2+k}} = \underbrace{P_{A^{[0]}}(\omega_{n/2}^k)}_{y_k^{[0]}} - \omega_n^k \underbrace{P_{A^{[1]}}(\omega_{n/2}^k)}_{y_k^{[1]}}$
            $\omega \leftarrow \omega \cdot \omega_n$;
        end for
        return $y$
    end if
end function
```

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- We have recursively reduced evaluation of a polynomial $P_A(x)$ with n coefficients at n roots of unity of order n to evaluations of two polynomials $P_{A^{[0]}}(y)$ and $P_{A^{[1]}}(y)$, each with $n/2$ coefficients, at $n/2$ many roots of unity of order $n/2$.
- Once we get these $n/2$ values of $P_{A^{[0]}}(y)$ and $P_{A^{[1]}}(y)$ we need $n/2$ additional multiplications to obtain the values of

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for all $0 \leq k < n/2$.

- Thus, we have reduced a problem of size n to two such problems of size $n/2$, plus a linear overhead.
- Consequently, our algorithm's run time satisfies the recurrence

$$T(n) = 2T(n/2) + c\,n$$

- The Master Theorem gives $T(n) = \Theta(n \log n)$.

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Matrix representation of polynomial evaluation

- Evaluation of a polynomial $P_A(x) = A_0 + A_1 x + \ldots + A_{n-1}x^{n-1}$ at roots of unity ω_n^k of order n can be represented in the matrix form as follows:

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \ldots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} = \begin{pmatrix} \widehat{A}_0 \\ \widehat{A}_1 \\ \widehat{A}_2 \\ \vdots \\ \widehat{A}_{n-1} \end{pmatrix}$$

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Another remarkable feature of the roots of unity:

- To obtain the inverse of the above matrix, all we have to do is just change the signs of the exponents and divide everything by n:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

To see this, note that if we compute the product

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

the (i, j) entry in the product matrix is equal to a product of i^{th} row and j^{th} column:

$$\begin{pmatrix} 1 & \omega_n^i \; \omega_n^{2\cdot i} & \dots & \omega_n^{i\cdot(n-1)} \end{pmatrix} \begin{pmatrix} 1 \\ \omega_n^{-j} \\ \omega_n^{-2j} \\ \vdots \\ \omega_n^{-(n-1)j} \end{pmatrix} = \sum_{k=0}^{n-1} \omega_n^{ik} \omega_n^{-jk} = \sum_{k=0}^{n-1} \omega_n^{(i-j)k}$$

We now have two possibilities:

1. $i = j$: then
$$\sum_{k=0}^{n-1} \omega_n^{(i-j)k} = \sum_{k=0}^{n-1} \omega_n^0 = \sum_{k=0}^{n-1} 1 = n;$$

2. $i \neq j$: then $\sum_{k=0}^{n-1} \omega_n^{(i-j)k}$ represents a sum of a geometric progression with the ratio ω_n^{i-j} and thus
$$\sum_{k=0}^{n-1} \omega_n^{(i-j)k} = \frac{1-\omega_n^{(i-j)n}}{1-\omega_n^{i-j}} = \frac{1-(\omega_n^n)^{i-j}}{1-\omega_n^{i-j}} = \frac{1-1}{1-\omega_n^{i-j}} = 0$$

So,

$$\begin{pmatrix} 1 & \omega_n^i & \omega_n^{2\cdot i} & \ldots & \omega_n^{i\cdot(n-1)} \end{pmatrix} \begin{pmatrix} 1 \\ \omega_n^{-j} \\ \omega_n^{-2j} \\ \vdots \\ \omega_n^{-(n-1)j} \end{pmatrix} = \sum_{k=0}^{n-1} \omega_n^{(i-j)k} = \begin{cases} n & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{4}$$

We now have two possibilities:

1. $i = j$: then
$$\sum_{k=0}^{n-1} \omega_n^{(i-j)k} = \sum_{k=0}^{n-1} \omega_n^0 = \sum_{k=0}^{n-1} 1 = n;$$

2. $i \neq j$: then $\sum_{k=0}^{n-1} \omega_n^{(i-j)k}$ represents a sum of a geometric progression with the ratio ω_n^{i-j} and thus
$$\sum_{k=0}^{n-1} \omega_n^{(i-j)k} = \frac{1-\omega_n^{(i-j)n}}{1-\omega_n^{i-j}} = \frac{1-(\omega_n^n)^{i-j}}{1-\omega_n^{i-j}} = \frac{1-1}{1-\omega_n^{i-j}} = 0$$

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So we get:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

$$= \begin{pmatrix} n & 0 & 0 & \dots & 0 \\ 0 & n & 0 & \dots & 0 \\ 0 & 0 & n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix} = n \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

- We now have

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} =$$

$$= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}$$

- This means that to covert from the values

$$\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1}) \rangle$$

which we denoted by $\langle \widehat{A}_0, \widehat{A}_1, \widehat{A}_2, \dots, \widehat{A}_{n-1} \rangle$ back to the coefficient form

$$P_A(x) = A_0 + A_1 x + A_2 x^2 + A_{n-1} x^{n-1}$$

we can use **the same** FFT algorithm with the only change that:

1. the root of unity ω_n is replaced by $\overline{\omega_n} = e^{-i\frac{2\pi}{n}}$,
2. the resulting output values are divided by n.

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$$= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}$$

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Inverse Fast Fourier Transform (IFFT):

```
function IFFT*($\widehat{A}$)
    $n \leftarrow \text{length}(\widehat{A})$
    if $n = 1$ then return $\widehat{A}$
    else
        $\widehat{A}^{[0]} \leftarrow (\widehat{A}_0, \widehat{A}_2, \dots \widehat{A}_{n-2})$;
        $\widehat{A}^{[1]} \leftarrow (\widehat{A}_1, \widehat{A}_3, \dots \widehat{A}_{n-1})$;
        $y^{[0]} \leftarrow IFFT^*(\widehat{A}^{[0]})$;
        $y^{[1]} \leftarrow IFFT^*(\widehat{A}^{[1]})$;
        $\omega_n \leftarrow e^{-i\frac{2\pi}{n}}$;            ⇐   different from FFT
        $\omega \leftarrow 1$;
        for $k = 0$ to $k = n/2 - 1$ do;
            $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$;
            $y_{n/2+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$
            $\omega \leftarrow \omega \cdot \omega_n$;
        end for
        return $y$;
    end if
end function
```

```
function IFFT($\widehat{A}$)            ⇐   different from FFT
    return $IFFT^*(\widehat{A})/\text{length}(\widehat{A})$
end function
```

Important note:

Computer science books take the forward DFT operation to be the evaluation of the corresponding polynomial at all roots of unity $\omega_n^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$ and the InverseDFT to be the evaluation of the polynomial at the complex conjugates of the roots of unity, i.e., at $\omega_n^{-k} = \cos\frac{2\pi k}{n} - i\sin\frac{2\pi k}{n}$.

However, Electrical engineering books do it just opposite, the direct DFT evaluates the polynomial at ω_n^{-k} and the InverseDFT at ω_n^k!

While for the purpose of multiplying polynomials both choices are equally good, the choice made by the electrical engineers is much better for all other purposes. We will explain this in the Advanced Algorithms 4121 when we do the JPEG.

We did here only multiplication of polynomials, and did not apply it to multiplication of large integers. This is possible to do but one has to be careful because roots of unity are represented by floating point numbers so you have to show that if you do FFT with sufficient precision you can round off the results and obtain correct integer values, but all of this is tricky.

Earlier results along this line produced algorithms for multiplication of large integers which operate in time $n\log n\log(\log n)$ but very recently David Harvey of the School of Mathematics at UNSW came up with an algorithm to multiply large integers which runs in time $n\log n$.

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Back to fast multiplication of polynomials

$P_A(x) = A_0 + A_1 x + \ldots + A_{n-1} x^{n-1}$ $\qquad$ $P_B(x) = B_0 + B_1 x + \ldots + B_{n-1} x^{n-1}$

$\Downarrow$ DFT $O(n \log n)$ $\qquad$ $\Downarrow$ DFT $O(n \log n)$

$$\{P_A(1), P_A(\omega_{2n-1}), P_A(\omega_{2n-1}^2), \ldots, P_A(\omega_{2n-1}^{2n-2})\}; \quad \{P_B(1), P_B(\omega_{2n-1}), P_B(\omega_{2n-1}^2), \ldots, P_B(\omega_{2n-1}^{2n-2})\}$$

$\Downarrow$ multiplication $O(n)$

$$\{P_A(1)P_B(1), \quad P_A(\omega_{2n-1})P_B(\omega_{2n-1}), \ldots, P_A(\omega_{2n-1}^{2n-2})P_B(\omega_{2n-1}^{2n-2})\}$$

$\Downarrow$ IDFT $O(n \log n)$

$$P_C(x) = \sum_{j=0}^{2n-2} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i} \right)}_{C_j} x^j = \sum_{j=0}^{2n-2} C_j x^j = P_A(x) \cdot P_B(x)$$

Thus, the product $P_C(x) = P_A(x)\, P_B(x)$ of two polynomials $P_A(x)$ and $P_B(x)$ can be computed in time $O(n \log n)$.

Back to fast multiplication of polynomials

$$P_A(x) = A_0 + A_1 x + \ldots + A_{n-1}x^{n-1} \qquad\qquad P_B(x) = B_0 + B_1 x + \ldots + B_{n-1}x^{n-1}$$

$$\Downarrow \text{ DFT} \quad \color{darkred}{O(n \log n)} \qquad\qquad\qquad \Downarrow \text{ DFT} \quad \color{darkred}{O(n \log n)}$$

$$\{P_A(1), P_A(\omega_{2n-1}), P_A(\omega_{2n-1}^2), \ldots, P_A(\omega_{2n-1}^{2n-2})\}; \quad \{P_B(1), P_B(\omega_{2n-1}), P_B(\omega_{2n-1}^2), \ldots, P_B(\omega_{2n-1}^{2n-2})\}$$

$$\Downarrow \text{ multiplication} \quad \color{darkred}{O(n)}$$

$$\{P_A(1)P_B(1), \quad P_A(\omega_{2n-1})P_B(\omega_{2n-1}), \ldots, P_A(\omega_{2n-1}^{2n-2})P_B(\omega_{2n-1}^{2n-2})\}$$

$$\Downarrow \text{IDFT} \quad \color{darkred}{O(n \log n)}$$

$$P_C(x) = \sum_{j=0}^{2n-2} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i}\right)}_{C_j} x^j = \sum_{j=0}^{2n-2} C_j x^j = P_A(x) \cdot P_B(x)$$

Thus, the product $P_C(x) = P_A(x)\, P_B(x)$ of two polynomials $P_A(x)$ and $P_B(x)$ can be computed in time $\color{darkred}{O(n \log n)}$.

Computing the convolution $C = A * B$

$$A = \langle A_0, A_1, \ldots, A_{n-1} \rangle \qquad\qquad B = \langle B_0, B_1, \ldots, B_{n-1} \rangle$$

$$\Downarrow \quad O(n) \qquad\qquad \Downarrow \quad O(n)$$

$$P_A(x) = A_0 + A_1 x + \ldots + A_{n-1} x^{n-1} \qquad P_B(x) = B_0 + B_1 x + \ldots + B_{n-1} x^{n-1}$$

$$\Downarrow \text{ DFT} \quad O(n \log n) \qquad\qquad \Downarrow \text{ DFT} \quad O(n \log n)$$

$$\{P_A(1), P_A(\omega_{2n-1}), P_A(\omega_{2n-1}^2), \ldots, P_A(\omega_{2n-1}^{2n-2})\}; \quad \{P_B(1), P_B(\omega_{2n-1}), P_B(\omega_{2n-1}^2), \ldots, P_B(\omega_{2n-1}^{2n-2})\}$$

$$\Downarrow \text{ multiplication} \quad O(n)$$

$$\{P_A(1)P_B(1), \quad P_A(\omega_{2n-1})P_B(\omega_{2n-1}), \ldots, P_A(\omega_{2n-1}^{2n-2})P_B(\omega_{2n-1}^{2n-2})\}$$

$$\Downarrow \text{ IDFT} \quad O(n \log n)$$

$$P_C(x) = \sum_{j=0}^{2n-2} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i} \right)}_{C_j} x^j$$

$$\Downarrow$$

$$C = \left\langle \sum_{i=0}^{j} A_i B_{j-i} \right\rangle_{j=0}^{j=2n-2}$$

Convolution $C = A * B$ of sequences A and B is computed in time $O(n \log n)$.

Computing the convolution $C = A * B$

$$A = \langle A_0, A_1, \ldots, A_{n-1} \rangle \qquad\qquad B = \langle B_0, B_1, \ldots, B_{n-1} \rangle$$

$$\Downarrow \quad {\color{red} O(n)} \qquad\qquad\qquad \Downarrow \quad {\color{red} O(n)}$$

$$P_A(x) = A_0 + A_1 x + \ldots + A_{n-1} x^{n-1} \qquad P_B(x) = B_0 + B_1 x + \ldots + B_{n-1} x^{n-1}$$

$$\Downarrow \ \mathrm{DFT} \quad {\color{red} O(n \log n)} \qquad\qquad\qquad \Downarrow \ \mathrm{DFT} \quad {\color{red} O(n \log n)}$$

$$\{P_A(1), P_A(\omega_{2n-1}), P_A(\omega_{2n-1}^2), \ldots, P_A(\omega_{2n-1}^{2n-2})\}; \quad \{P_B(1), P_B(\omega_{2n-1}), P_B(\omega_{2n-1}^2), \ldots, P_B(\omega_{2n-1}^{2n-2})\}$$

$$\Downarrow \ \text{multiplication} \quad {\color{red} O(n)}$$

$$\{P_A(1)P_B(1), \quad P_A(\omega_{2n-1})P_B(\omega_{2n-1}), \ldots, P_A(\omega_{2n-1}^{2n-2})P_B(\omega_{2n-1}^{2n-2})\}$$

$$\Downarrow \mathrm{IDFT} \quad {\color{red} O(n \log n)}$$

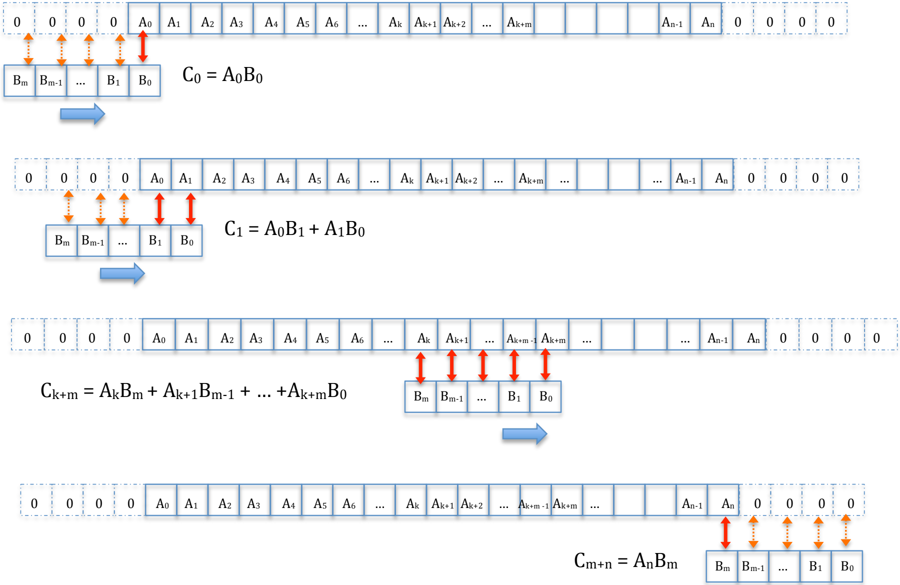
$$P_C(x) = \sum_{j=0}^{2n-2} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i} \right)}_{C_j} x^j$$

$$\Downarrow$$

$$C = \left\langle \sum_{i=0}^{j} A_i B_{j-i} \right\rangle_{j=0}^{j=2n-2}$$

Convolution $C = A * B$ of sequences A and B is computed in time ${\color{red} O(n \log n)}$.

Visualizing Convolution $C = A * B$



$0 \quad 0 \quad 0 \quad 0 \quad A_0 \quad A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6 \quad \ldots \quad A_k \quad A_{k+1} \quad A_{k+2} \quad \ldots \quad A_{k+m} \quad A_{n-1} \quad A_n \quad 0 \quad 0 \quad 0 \quad 0$

$B_m \quad B_{m-1} \quad \ldots \quad B_1 \quad B_0$

$C_0 = A_0B_0$

$0 \quad 0 \quad 0 \quad 0 \quad A_0 \quad A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6 \quad \ldots \quad A_k \quad A_{k+1} \quad A_{k+2} \quad \ldots \quad A_{k+m} \quad \ldots \quad \ldots \quad A_{n-1} \quad A_n \quad 0 \quad 0 \quad 0 \quad 0$

$B_m \quad B_{m-1} \quad \ldots \quad B_1 \quad B_0$

$C_1 = A_0B_1 + A_1B_0$

$0 \quad 0 \quad 0 \quad 0 \quad A_0 \quad A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6 \quad \ldots \quad A_k \quad A_{k+1} \quad \ldots \quad A_{k+m-1} \quad A_{k+m} \quad \ldots \quad \ldots \quad A_{n-1} \quad A_n \quad 0 \quad 0 \quad 0 \quad 0$

$C_{k+m} = A_kB_m + A_{k+1}B_{m-1} + \ldots + A_{k+m}B_0$

$B_m \quad B_{m-1} \quad \ldots \quad B_1 \quad B_0$

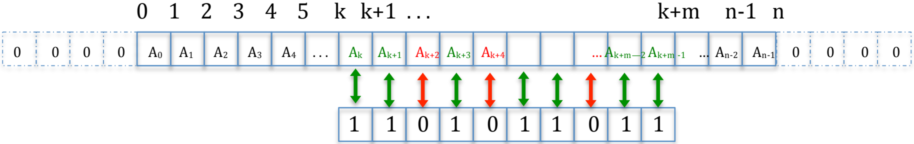
$0 \quad 0 \quad 0 \quad 0 \quad A_0 \quad A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6 \quad \ldots \quad A_k \quad A_{k+1} \quad A_{k+2} \quad \ldots \quad A_{k+m-1} \quad A_{k+m} \quad \ldots \quad A_{n-1} \quad A_n \quad 0 \quad 0 \quad 0 \quad 0$

$C_{m+n} = A_nB_m$

$B_m \quad B_{m-1} \quad \ldots \quad B_1 \quad B_0$

An Exercise

- Assume you are given a map of a straight sea shore of length $100n$ meters as a sequence on $100n$ numbers such that A_i is the number of fish between i^{th} meter of the shore and $(i+1)^{th}$ meter, $0 \leq i \leq 100n - 1$. You also have a net of length n meters but unfortunately it has holes in it. Such a net is described as a sequence N of n ones and zeros, where 0's denote where the holes are. If you throw such a net starting at meter k and ending at meter $k+n$, then you will catch only the fish in one meter stretches of the shore where the corresponding bit of the net is 1; see the figure.


0 1 2 3 4 5 k k+1 . . .
k+m n-1 n
0 0 0 0 A0 A1 A2 A3 A4 . . . Ak Ak+1 Ak+2 Ak+3 Ak+4 ... Ak+m—2 Ak+m -1 ... An-2 An-1 0 0 0 0
1 1 0 1 0 1 1 0 1 1

$$C = A_k + A_{k+1} + 0 + A_{k+2} + 0 + A_{k+4} + \ldots + 0 + A_{k+m-2} + A_{k+m-1}$$

Find the spot where you should place the left end of your net in order to catch the largest possible number of fish using an algorithm which runs in time $O(n \log n)$.

Hint: Let N' be the net sequence N in the reverse order; Compute $A * B'$ and look for the peak of that sequence.

PUZZLE!!

- On a circular highway there are n petrol stations, unevenly spaced, each containing a different quantity of petrol. It is known that the total quantity of petrol on all stations is enough to go around the highway once, and that the tank of your car can hold enough fuel to make a trip around the highway. Prove that there always exists a station among all of the stations on the highway, such that if you take it as a starting point and take the fuel from that station, you can continue to make a complete round trip around the highway, never emptying your tank before reaching the next station to refuel.