9101 Assignment 1
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Question 5

(a) $f(n) = log_2(n); \quad g(n) = \sqrt[10]{n}$

According to the fact, $f(n) \in O\big(g(n)\big), if\ and\ only\ if\ \lim_{n\to\infty}\frac{f(n)}{g(n)} < \infty$.

Here, by calculating,

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{log_2(n)}{n^{\frac{1}{10}}}$$

Because, $\lim_{n\to\infty} log_2(n) = \infty$, and $\lim_{n\to\infty} n^{\frac{1}{10}} = \infty$,

Then, using the $L'H\hat{o}pital's\ rule$,

$$\lim_{n\to\infty}\frac{log_2(n)}{n^{\frac{1}{10}}} = \lim_{n\to\infty}\frac{\frac{1}{nln2}}{\frac{1}{10n^{9/10}}} = \lim_{n\to\infty}\frac{10n^{9/10}}{nln2} = \lim_{n\to\infty}\frac{10n^{\frac{9}{10}}}{nln2} = \frac{10}{ln2}\cdot\lim_{n\to\infty}\frac{1}{n^{\frac{1}{10}}}$$

$$= \frac{10}{ln2}\cdot\frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} n^{\frac{1}{10}}} = \frac{10}{ln2}\cdot\frac{1}{\infty} = 0 < \infty$$

Therefore, $f(n) \in O\big(g(n)\big)$, that is $f(n) = O\big(g(n)\big)$

(b) $f(n) = n^n; \quad g(n) = 2^{nlog_2(n^2)}$

First, $g(n) = 2^{nlog_2(n^2)} = 2^{2nlog_2(n)} = 4^{nlog_2(n)}$

Then, $\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^n}{4^{nlog_2(n)}} = \lim_{n\to\infty}\left(\frac{n}{4^{log_2(n)}}\right)^n$,

Also because, $a^{log_b n} = n^{log_b a}$

Hence,

$$\operatorname{Lim}_{n\to\infty}\left(\frac{n}{4^{log_2(n)}}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{n}{n^{log_2 4}}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{n}{n^2}\right)^n = \lim_{n\to\infty}\left(\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\cdot ln(1/n)} = 0 < \infty$$

Therefore, $g(n)$ is an upper bound of $f(n)$, that is $f(n) = O\big(g(n)\big)$.

(c) $f(n) = n^{1+\sin(\pi n)}$; $g(n) = n$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{1+\sin(\pi n)}}{n} = \lim_{n\to\infty} \frac{n^{\sin(\pi n)}}{1}$, which is not convergent.

Here, because $n^{\sin(\pi n)}$ is a cyclically changing function.
Then,
(i) When n is an integer, e.g. {-1, 1, 2, 3, …}, $n^{\sin(\pi n)} = n^0 = 1$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{1+\sin(\pi n)}}{n} = 1$, and then, $f(n) = O\big(g(n)\big), or\ g(n) = O(f(n))$

(ii) When $n = 2k + \frac{1}{2}$,

case1: $n^{\sin(\pi n)} = n^{-1}, (e.g.\ n = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots)$, then $\lim_{n\to\infty} \frac{n^{1+\sin(\pi n)}}{n} = \frac{1}{n} = 0$, then $f(n) = O\big(g(n)\big)$

case2: $n^{\sin(\pi n)} = n^{1}, (e.g.\ n = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots)$, then $\lim_{n\to\infty} \frac{n^{1+\sin(\pi n)}}{n} = n = \infty$, then $g(n) = O(f(n))$

Based on (i) and (ii) above, neither $(n) = O\big(g(n)\big)\ nor\ g(n) = O(f(n))$