Algorithms: COMP3121/9101

School of Computer Science and Engineering
University of New South Wales

10. LINEAR PROGRAMMING

Linear Programming problems - Example 1

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  - the price of all food per day is as low as possible.

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- Note that all constraints and the objective function are **linear**.

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- In order to win, you have to get at least 51% of each of the city, suburban and rural votes.
- You wish to win the election by cleverly making a promise that **appears** that it will blow as small hole in the budget as possible, i.e., that the total cost of your promises is as low as possible.

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- Such problems are MUCH harder to solve than the “plain” Linear Programming problems whose solutions can be real numbers.

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- Let the boldface $\mathbf{x}$ represent a (column) vector, $\mathbf{x} = \langle x_1 \ldots x_n \rangle^{\intercal}$.
- To get a more compact representation of linear programs we introduce a partial ordering on vectors $\mathbf{x} \in \mathbf{R}^n$ by $\mathbf{x} \leq \mathbf{y}$ if and only if the corresponding inequalities hold coordinate-wise, i.e., if and only if $x_j \leq y_j$ for all $1 \leq j \leq n$.

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  - maximize $\mathbf{c}^\intercal \mathbf{x}$
  - subject to the following two (matrix-vector) constraints:

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  $$\mathbf{x} \geq \mathbf{0}.$$
- Thus, to specify a Linear Programming optimisation problem we just have to provide a triplet $(A, \mathbf{b}, \mathbf{c})$;
- This is the usual form which is accepted by most standard LP solvers.

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- Equality constraints of the form $\sum_{i=1}^{n} a_{ij}x_i = b_j$ can be replaced by two inequalities: $\sum_{i=1}^{n} a_{ij}x_i \geq b_j$ and $\sum_{i=1}^{n} a_{ij}x_i \leq b_j$; thus, we can assume that all constraints are inequalities.

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- This poses no problem, because each occurrence of an unconstrained variable x_j can be replaced by the expression $x'_j - x^*_j$ where x'_j, x^*_j are new variables satisfying the constraints $x'_j \geq 0,\ x^*_j \geq 0$.

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- If $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector, we let $|\mathbf{x}| = (|x_1|, \ldots, |x_n|)$. Some problems are naturally translated into constraints of the form $|A\mathbf{x}| \leq \mathbf{b}$. This also poses no problem because we can replace such constraints with two linear constraints: $A\mathbf{x} \leq \mathbf{b}$ and $-A\mathbf{x} \leq \mathbf{b}$ because $|x| \leq y$ if and only if $x \leq y$ and $-x \leq y$.

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maximize $\quad z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \quad (3)$

subject to the constraints

$$x_1 + x_2 + 3x_3 \leq 30 \tag{4}$$

$$2x_1 + 2x_2 + 5x_3 \leq 24 \tag{5}$$

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- Standard Form: maximize $\mathbf{c}^\intercal\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
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- Since all variables are constrained to be non-negative, we are assured that

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- If we compare this with our objective (3) we see that if we choose y_1, y_2 and y_3 so that:

$$\begin{aligned}
y_1 + 2y_2 + 4y_3 &\geq 3 \\
y_1 + 2y_2 + y_3 &\geq 1 \\
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Combining this with (9) we get:

$$30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$$

Linear Programming - Standard Form

- Consequently, in order to find as tight upper bound for our objective $z(x_1, x_2, x_3)$ of the problem P:

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we have to look for y_1, y_2, y_3 which solve problem P^*:

$$\begin{aligned}
\text{minimise:} \quad & z^*(y_1,y_2,y_3) = 30y_1 + 24y_2 + 36y_3 & (10)\\
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  then $z^*(y_1, y_2, y_3) = 30y_1 + 24y_2 + 36y_3 \ge 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$ will be a tight upper bound for $z(x_1, x_2, x_3)$
- The new problem P^* is called the *dual problem* for the problem P.

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- So, at the first sight, looking for the multipliers y_1, y_2, y_3 did not help much, because it only reduced a maximisation problem to an equally hard minimisation problem.
- It is now useful to remember how we proved that the Ford - Fulkerson Max Flow algorithm in fact produces a **maximal flow**, by showing that it terminates only when we reach the capacity of a **minimal cut**.

Linear Programming - primal/dual problem forms

- The original, *primal* Linear Program P and its *dual* Linear Program can be easily described in the most general case:

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P: \quad & \text{maximize} & & z(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j, \\
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& & & x_1, \ldots, x_n \geq 0; \\
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or, in matrix form,

$$\begin{aligned} P: \quad & \text{maximize} \quad z(\mathbf{x}) = \mathbf{c}^{\intercal}\mathbf{x}, \ \text{subject to the constraints} \quad A\mathbf{x} \leq \mathbf{b} \ \text{ and } \ \mathbf{x} \geq 0; \\ P^*: \quad & \text{minimize} \quad z^*(\mathbf{y}) = \mathbf{b}^{\intercal}\mathbf{y}, \ \text{subject to the constraints} \quad A^{\intercal}\mathbf{y} \geq \mathbf{c} \ \text{ and } \ \mathbf{y} \geq 0. \end{aligned}$$

Weak Duality Theorem

- Recall that any vector $\mathbf{x}$ which satisfies the two constraints, $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^\intercal \mathbf{x}$ might be.

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- **Theorem** If $x = \langle x_1 \ldots x_n \rangle$ is any basic feasible solution for P and $y = \langle y_1 \ldots y_m \rangle$ is any basic feasible solution for P^*, then:

$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

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Proof: Since x and y are basic feasible solutions for P and P^* respectively, we can use the constraint inequalities, first from P^* and then from P to obtain

$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

Weak Duality Theorem

- Recall that any vector $\mathbf{x}$ which satisfies the two constraints, $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^{\intercal}\mathbf{x}$ might be.
- **Theorem** If $x = \langle x_1 \dots x_n \rangle$ is any basic feasible solution for P and $y = \langle y_1 \dots y_m \rangle$ is any basic feasible solution for P^*, then:

$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

Proof: Since x and y are basic feasible solutions for P and P^* respectively, we can use the constraint inequalities, first from P^* and then from P to obtain

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- Thus, the value of (the objective of P^* for) any feasible solution of P^* is an upper bound for the set of all values of (the objective of P for) all feasible solutions of P, and

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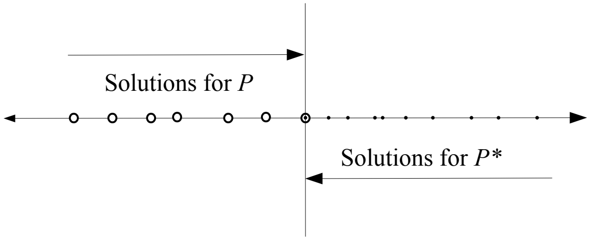
- every feasible solution of P is a lower bound for the set of feasible solutions for P^*.

Weak Duality Theorem



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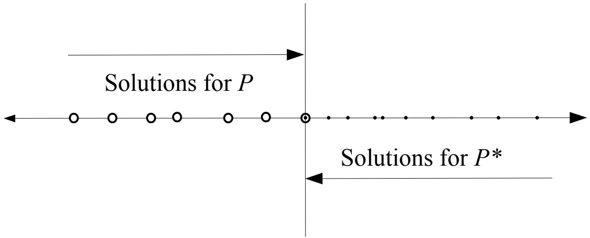
Weak Duality Theorem



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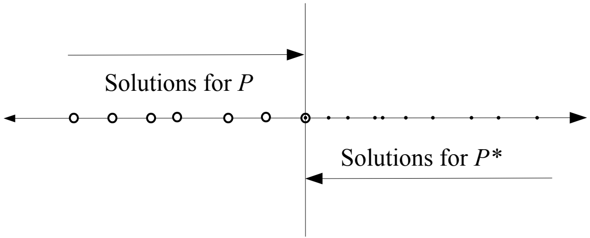
- If we use a search procedure to find an optimal solution for P we know when to stop: when such a value is also a feasible solution for P^*.

Weak Duality Theorem



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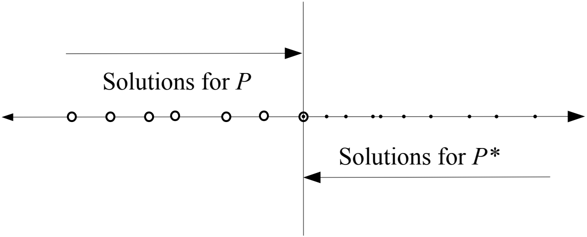
- This is why the most commonly used LP solving method, the SIMPLEX method, produces optimal solution for P, because it stops at a value of the primal objective which is also a value of the dual objective.

Weak Duality Theorem



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- This is why the most commonly used LP solving method, the SIMPLEX method, produces optimal solution for P, because it stops at a value of the primal objective which is also a value of the dual objective.
- See the Lecture Notes for the details and an example of how the SIMPLEX algorithm runs.

PUZZLE!!

Five sisters are alone in their house. Sharon is reading a book, Jennifer is playing chess, Cathrine is cooking and Ana is doing laundry. What is Helen, the fifth sister, doing?