Algorithms Tutorial 5
Solutions

1. In the country of Pipelistan there are several oil wells, several oil refineries and many distribution hubs connected by oil pipelines. To visualise Pipelistan's oil infrastructure, just imagine a directed graph with k source vertices (the oil wells), m sinks (refineries) and n vertices which are distribution hubs linked with one way pipelines. You are given the graph and the capacity $C(i, j)$ of each pipeline going from a vertex i to a vertex j. You want to instal the smallest possible number of flow meters on some of these pipelines so that the total throughput of oil from all the wells to all refineries can be computed exactly from the readings of all of these meters. Each meter shows the direction of the flow and the quantity of flow per minute. Design an efficient algorithm for deciding on which pipelines to place the flow meters.

2. Given an undirected graph with vertices numbered $1, 2, \ldots, n$, partition the vertices into two disjoint subsets such that vertex 1 and n are in different subsets, and the number of edges with both ends in the same subset is maximised.

3. Assume that you are given a network flow graph with a source s, a sink t and two other distinct vertices u and v. Design an algorithm which returns a smallest capacity cut among all cuts for which the vertex u is in the same side of the cut as the source s and vertex v is in the same side as the sink t and which runs in polynomial time.

4. Assume that you are given a network flow graph with a source s, a sink t and two other distinct vertices u and v. Design an algorithm which returns a smallest capacity cut among all cuts for which vertices u and v are in the same side of the cut.

5. You know that $n + 2$ spies S, $s_1, s_2, \ldots, s_n$ and T are communicating through certain number of communication channels; in fact, for each i and each j you

know if there is a channel through which spy s_i can send a secret message to spy s_j or if there is no such a channel (i.e., you know what the graph looks like, with spies as vertices and communication channels as directed edges).

(a) Your task is to design an algorithm which finds the fewest number of channels which you need to compromise (for example, by placing a listening device on that channel) so that spy S cannot send a message to spy T through a sequence of intermediary spies without the message being passed through at least one compromised channel.

(b) Assume now that you cannot compromise channels because they are encrypted, so the only thing you can do is bribe some of the spies. Design an algorithm which finds the smallest number of spies which you need to bribe so that S cannot send a message to T without the message going through at least one of the bribed spies as an intermediary.

6. There are N cities (labelled $1, 2, \ldots, N$), connected by M bidirectional roads. Each road connects two different cities. A pair of cities may be connected by multiple roads. A well known criminal is currently in city 1 and wishes to get to city N via road. To catch the criminal, the police have decided to block the minimum number of roads possible to make it impossible to get from city 1 to city N. However, some roads are major roads. In order to avoid disruption, the police cannot close any major roads. Find the minimum number of roads to block to prevent the criminal from going from city 1 to city N, or output -1 if the police cannot stop the criminal.

7. An $n \times n$ grid is an undirected graph consisting of n rows, each row containing n vertices, with vertices connected with edges to all of their immediate neighbours (2 at all of the 4 corners, 3 at all of the 4 sides and 4 in the interior of the grid). Vertices with exactly 4 neighbours are called the internal vertices; vertices with exactly 3 neighbours are called the side vertices. The escape problem is, given $m \leq 4(n-1)$ many internal vertices in the grid and m side vertices on the 4 sides of the grid, connect each of m many distinct internal vertices with distinct m side vertices by non intersecting paths or return impossible when there is no such a solution. Note that any of m internal vertices can be connected to any of m side vertices.

8. A band of M criminals has infiltrated a secure building, which is structured as an $N \times N$ square grid of rooms, each of which has a door on all of its sides. Thus: from an internal room, we can move to any of the four neighbouring

rooms, from a room on the side of the building, we can move to three other rooms or leave the building, and from a corner room, we can move to two other rooms or leave the building. The criminals were able to shut down the building's security system before entering, but during their nefarious activities, the security system became operational again, so they decided to abort the mission and attempt to escape. The building has a sensor in each room, which becomes active when an intruder is detected, but only triggers the alarm if it is activated again. Thus, the criminals may be able to escape if they can all reach the outside of the building without any two of them passing through a same room. Given the M different rooms which the criminals occupy when the security system is reactivated, determine whether all M criminals can escape without triggering the alarm.

9. Several families are coming to a birthday celebration at a restaurant. You have arranged that v many tables will serve only vegetarian dishes, p many tables will not serve pork and r many remaining tables will serve food with pork. You know that V many families are all vegetarians, P_1 many families do not eat pork but do not mind eating vegetarian dishes, P_2 many families do not eat pork but hate vegetarian dishes. Also R_1 many families have no dietary restrictions and would also not mind eating vegetarian dishes or food without pork, R_2 many families have no dietary restrictions but hate vegetarian dishes but can eat food without pork. Finally, S many families are from Serbia and cannot imagine not eating pork. You are also given the number of family members in each family and the number of seats at each table. Your conundrum is to place the guests at the tables so that their food preferences are respected and no two members from the same family sit at the same table. In case the problem has no solutions your algorithm should output statement "no solution".

10. Today was just a regular day for everyone in Krypton until a news flashed that a meteor is going to destroy Krypton in X days. Krypton has N cities, some of which are connected by bidirectional roads. You are given a road map of Krypton; for every two cities C_i and C_j which are connected by a (direct) road from C_i straight to C_j you are given the value $t(i, j)$ which is the number of days to travel from city C_i to city C_j. (You can of course also go from a city C_m to city C_k without a direct road from C_m to C_k by going through a sequence of intermediate cities connected by direct roads.) In each city C_i the Krypton Government built q_i pods to carry inhabitants in case of any calamity, which will transport them to Earth. City C_i has population p_i. As soon as the people hear this news they try to save themselves by acquiring these pods either at

their own city or in other city before the meteor destroys everything. Note that a pod can carry only one person. Find the largest number of invaders the Earth will have to deal with. (20 pts)

11. You have been told of the wonder and beauty of a very famous painting. It is painted in the hypermodern style, and so it is simply an N by N grid of squares, with each square coloured either black or white. You have never seen this picture for yourself, but have been told some details of it by a friend. Your friend has told you the value of N and the number of white squares in each row and each column. Additionally, your friend has also been kind enough to tell you the specific colour of some squares: some squares are black, some are white, and the rest she simply could not remember. The more details she tells you, the more amazing this painting becomes but you begin to wonder that perhaps it's simply too good to be true. Thus, you wish to design a polynomial time algorithm that determines whether or not such a painting can exist.