Week 8: Search Tree Algorithms

Tree Review

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Binary search trees …

- data structures designed for *O(log n)* search
- consist of nodes containing item (incl. key) and two links
- can be viewed as recursive data structure (subtrees)
- have overall ordering (data(Left) < root < data(Right))
- insert new nodes as leaves (or as root), delete from anywhere
- have structure determined by insertion order *(worst: O(n))*
- operations: insert, delete, search, …

Balanced Search Trees

Balanced BSTs

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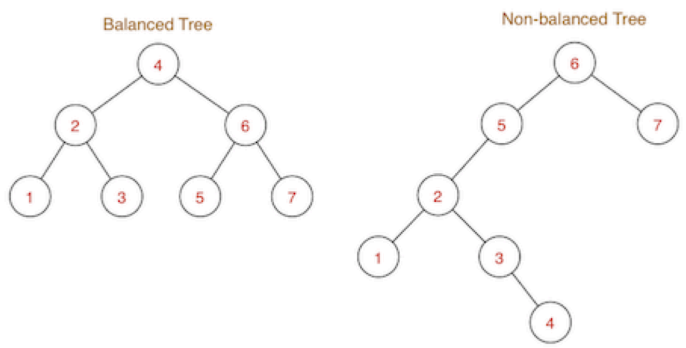
Reminder …

- Goal: build binary search trees which have
  - minimum height ⇒ minimum worst case search cost
- Best balance you can achieve for tree with *N* nodes:
  - tree height of log_2N ⇒ worst case search *O(log N)*



Three *strategies* to improving worst case search in BSTs:

- *randomise* — reduce chance of worst-case scenario occuring
- *amortise* — do more work at insertion to make search faster
- *optimise* — implement all operations with performance bounds

Randomised BST Insertion

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Effects of order of insertion on BST shape:

- best case (for at-leaf insertion): keys inserted in pre-order
  (median key first, then median of lower half, median of upper half, etc.)
- worst case: keys inserted in ascending/descending order
- average case: keys inserted in *random* order ⇒ $O(log_2 n)$

Tree ADT has no control over order that keys are supplied.

Can the algorithm itself introduce some *randomness*?

In the hope that this randomness helps to balance the tree …

... Randomised BST Insertion

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How can a computer pick a number at random?

- it cannot

Software can only produce *pseudo random numbers*.

- a pseudo random number may appear unpredictable
  - but is actually predictable
- ⇒ implementation may deviate from expected theoretical behaviour
  - more on this in week 10

... Randomised BST Insertion

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- Pseudo random numbers in C:

  ```
  rand() // generates random numbers in the range 0 .. RAND_MAX
  ```

  where the constant `RAND_MAX` is defined in `stdlib.h`
  (depends on the computer: on the CSE network, RAND_MAX = 2147483647)

To convert the return value of `rand()` to a number between 0 .. RANGE

- compute the remainder after division by RANGE+1

... Randomised BST Insertion

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Approach: normally do leaf insert, randomly do root insert.

```
insertRandom(tree,item)
|  Input  tree, item
|  Output tree with item randomly inserted
|
|  if tree is empty then
|     return new node containing item
|  end if
|  // p/q chance of doing root insert
```

```
|  if random number mod q < p then
|     return insertAtRoot(tree,item)
|  else
|     return insertAtLeaf(tree,item)
|  end if
```

E.g. 30% chance ⇒ choose $p=3$, $q=10$

... Randomised BST Insertion

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Cost analysis:

- similar to cost for inserting keys in random order: $O(\log_2 n)$
- does not rely on keys being supplied in random order

Approach can also be applied to deletion:

- standard method promotes inorder successor to root
- for the randomised method …
  - promote inorder successor from right subtree, OR
  - promote inorder predecessor from left subtree

Rebalancing Trees

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An approach to balanced trees:

- insert into leaves as for simple BST
- periodically, rebalance the tree

Question: how frequently/when/how to rebalance?

```
NewTreeInsert(tree,item):
|  Input  tree, item
|  Output tree with item randomly inserted
|
|  t=insertAtLeaf(tree,item)
|  if #nodes(t) mod k = 0 then
|     t=rebalance(t)
|  end if
|  return t
```

E.g. rebalance after every 20 insertions ⇒ choose $k=20$

Note: To do this efficiently we would need to change tree data structure and basic operations:

```
typedef struct Node {
   int  data;
   int  nnodes;      // #nodes in my tree
   Tree left, right; // subtrees
} Node;
```

... Rebalancing Trees

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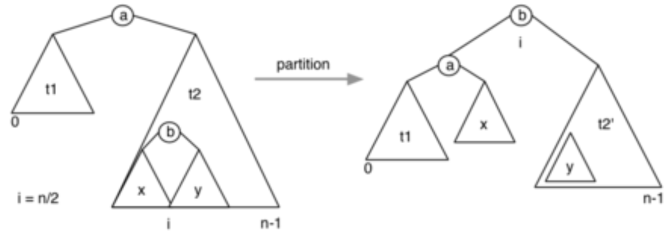
How to rebalance a BST? Move median item to root.



... Rebalancing Trees

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Implementation of rebalance:

```
rebalance(t):
|  Input  tree t with n nodes
|  Output t rebalanced
|
|  if n≥3 then
|  |  t=partition(t,⌊n/2⌋)          // put node with median key at root
|  |  left(t)=rebalance(left(t))  // then rebalance each subtree
|  |  right(t)=rebalance(right(t))
|  end if
|  return t
```

... Rebalancing Trees

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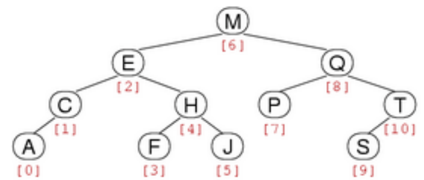
New operation on trees:

- **partition(tree,i)**: re-arrange tree so that element with index i becomes root



For tree with N nodes, indices are 0 .. $N-1$

... Rebalancing Trees

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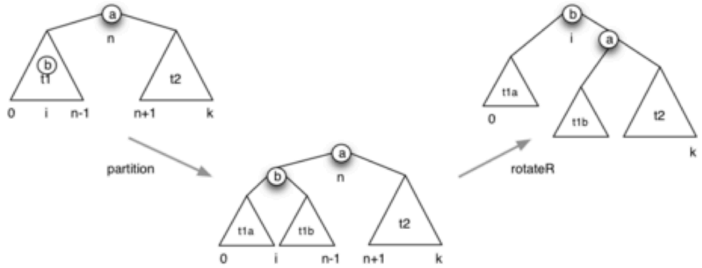
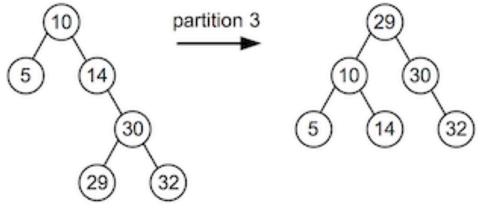
Partition: moves i^{th} node to root





... Rebalancing Trees

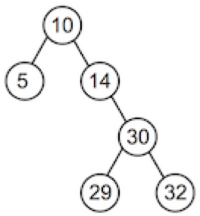
Implementation of partition operation:

```
partition(tree,i):
|  Input  tree with n nodes, index i
|  Output tree with item #i moved to the root
|
|  m=#nodes(left(tree))
|  if i < m then
|     left(tree)=partition(left(tree),i)
|     tree=rotateRight(tree)
|  else if i > m then
|     right(tree)=partition(right(tree),i-m-1)
|     tree=rotateLeft(tree)
|  end if
|  return tree
```

Note: size(tree) = n, size(left(tree)) = m, size(right(tree)) = n-m-1 (why -1?)

Exercise #1: Partition

Consider the tree `t`:



Show the result of `partition(t,3)`

... Rebalancing Trees

Analysis of rebalancing: visits every node ⇒ $O(N)$

Cost means not feasible to rebalance after each insertion.

When to rebalance? … Some possibilities:

- after every k insertions
- whenever "imbalance" exceeds threshold

Either way, we tolerate worse search performance for periods of time.

Does it solve the problem? … Not completely ⇒ Solution: real balanced trees (later)

Splay Trees

Splay Trees

A kind of "self-balancing" tree …

Splay tree insertion modifies insertion-at-root method:

- by considering *parent-child-granchild* (three level analysis)
- by performing double-rotations based on p-c-g orientation

The idea: appropriate double-rotations improve tree balance.

... Splay Trees

Splay tree implementations also do *rotation-in-search*:

- by performing double-rotations also when searching

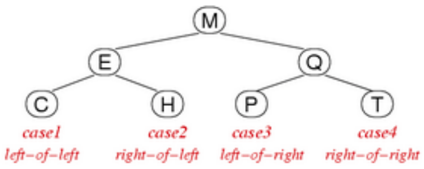
The idea: provides similar effect to periodic rebalance.

⇒ improves balance but makes search more expensive

... Splay Trees

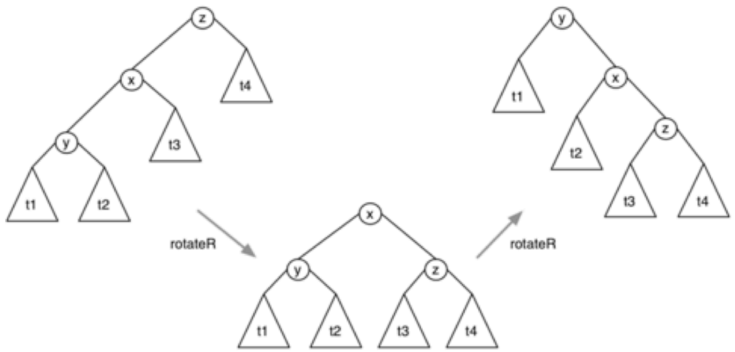
Cases for splay tree double-rotations:

- case 1: grandchild is left-child of left-child $\Rightarrow$ double right rotation from top
- case 2: grandchild is right-child of left-child
- case 3: grandchild is left-child of right-child
- case 4: grandchild is right-child of right-child $\Rightarrow$ double left rotation from top



... Splay Trees

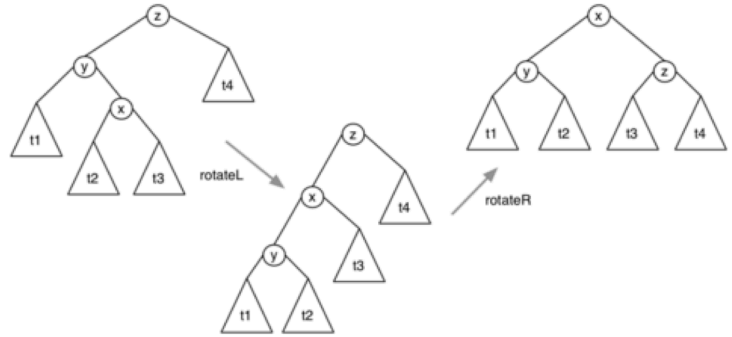
Double-rotation case for left-child of left-child ("zig-zig"):



Note: both rotations at the root (unlike insertion-at-root)

... Splay Trees

Double-rotation case for right-child of left-child ("zig-zag"):



Note: rotate subtree first (like insertion-at-root)

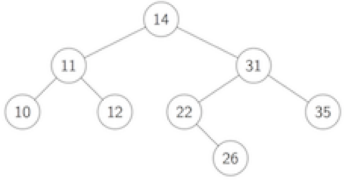
... Splay Trees

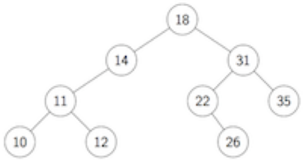
Algorithm for splay tree insertion:

```
insertSplay(tree,item):
|  Input  tree, item
|  Output tree with item splay-inserted
|
|  if tree is empty then return new node containing item
|  else if item=data(tree) then return tree
|  else if item<data(tree) then
|  |  if left(tree) is empty then
|  |     left(tree)=new node containing item
|  |  else if item<data(left(tree)) then
|  |        // Case 1: left-child of left-child "zig-zig"
|  |     left(left(tree))=insertSplay(left(left(tree)),item)
|  |     tree=rotateRight(tree)
|  |  else if item>data(left(tree)) then
|  |        // Case 2: right-child of left-child "zig-zag"
|  |     right(left(tree))=insertSplay(right(left(tree)),item)
|  |     left(tree)=rotateLeft(left(tree))
|  |  end if
|  |  return rotateRight(tree)
|  else     // item>data(tree)
|  |  if right(tree) is empty then
|  |     right(tree)=new node containing item
|  |  else if item<data(right(tree)) then
|  |        // Case 3: left-child of right-child "zag-zig"
|  |     left(right(tree))=insertSplay(left(right(tree)),item)
|  |     right(tree)=rotateRight(right(tree))
|  |  else if item>data(right(tree)) then
|  |        // Case 4: right-child of right-child "zag-zag"
|  |     right(right(tree))=insertSplay(right(right(tree)),item)
|  |     tree=rotateLeft(tree)
|  |  end if
|  |  return rotateLeft(tree)
|  end if
```

Exercise #2: Splay Trees

Insert `18` into this splay tree:





... Splay Trees

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Searching in splay trees:

```
searchSplay(tree,item):
|  Input  tree, item
|  Output address of item if found in tree
|         NULL otherwise
|
|  if tree=NULL then
|     return NULL
|  else
|  |  tree=splay(tree,item)
|  |  if data(tree)=item then
|  |     return tree
|  |  else
|  |     return NULL
|  |  end if
|  end if
```

where `splay()` is similar to `insertSplay()`,
except that it doesn't add a node … simply moves `item` to root if found, or nearest node if not found

Exercise #3: Splay Trees

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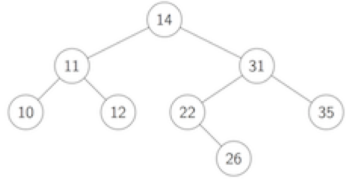
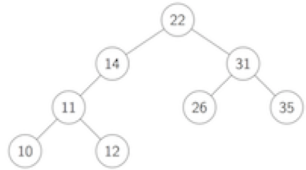
If we search for `22` in the splay tree



… how does this affect the tree?



... Splay Trees

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Why take into account both child and grandchild?

- moves accessed node to the root
- *moves every ancestor of accessed node roughly halfway to the root*

⇒ better amortized cost than insert-at-root

... Splay Trees

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Analysis of splay tree performance:

- assume that we "splay" for both insert and search
- consider: m insert+search operations, n nodes
- *Theorem*. Total number of comparisons: average $O((n+m)\cdot log(n+m))$

Gives good overall (amortized) cost.

- insert cost not significantly different to insert-at-root
- search cost increases, but …
    - improves balance on each search
    - moves frequently accessed nodes closer to root

But … still has worst-case search cost $O(n)$

Real Balanced Trees

Better Balanced Binary Search Trees

So far, we have seen …

- randomised trees … make poor performance unlikely
- occasional rebalance … fix balance periodically
- splay trees … reasonable amortized performance
- but both types still have *O(n)* worst case

Ideally, we want both average/worst case to be *O(log n)*

- AVL trees … fix imbalances as soon as they occur
- 2-3-4 trees … use varying-sized nodes to assist balance
- red-black trees … isomorphic to 2-3-4, but binary nodes

AVL Trees

AVL Trees

Invented by Georgy Adelson-Velsky and Evgenii Landis

Approach:

- insertion (at leaves) may cause imbalance
- repair balance as soon as we notice imbalance
- repairs done locally, not by overall tree restructure

A tree is unbalanced when: abs(height(left)-height(right)) > 1

This can be repaired by at most two rotations:

- if left subtree too deep …
  - if data inserted in left-right grandchild ⇒ left-rotate left subtree
  - rotate right
- if right subtree too deep …
  - if data inserted in right-left grandchild ⇒ right-rotate right subtree
  - rotate left

Problem: determining height/depth of subtrees may be expensive.

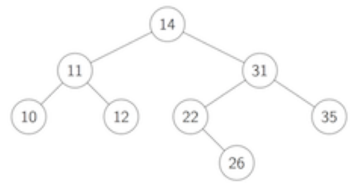
… AVL Trees

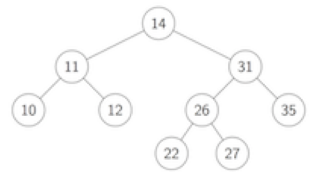
Implementation of AVL insertion

```
insertAVL(tree,item):
|  Input  tree, item
|  Output tree with item AVL-inserted
|
|  if tree is empty then
|     return new node containing item
|  else if item=data(tree) then
|     return tree
|  else
|  |  if item<data(tree) then
|  |     left(tree)=insertAVL(left(tree),item)
|  |  else if item>data(tree) then
|  |     right(tree)=insertAVL(right(tree),item)
|  |  end if
|  |  if height(left(tree))-height(right(tree)) > 1 then
|  |     if item>data(left(tree)) then
|  |        left(tree)=rotateLeft(left(tree))
|  |     end if
|  |     tree=rotateRight(tree)
|  |  else if height(right(tree))-height(left(tree)) > 1 then
|  |     if item<data(right(tree)) then
|  |        right(tree)=rotateRight(right(tree))
|  |     end if
|  |     tree=rotateLeft(tree)
|  |  end if
|  |  return tree
|  end if
```


Exercise #4: AVL Trees

Insert 27 into the AVL tree





What would happen if you now insert 28?

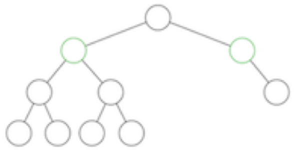
You may like the animation at www.cs.usfca.edu/~galles/visualization/AVLtree.html

… AVL Trees

Analysis of AVL trees:

- trees are *height*-balanced; subtree depths differ by +/-1

- average/worst-case search performance of *O(log n)*
- *require* extra data to be stored in each node ("height")
- may not be *weight*-balanced; subtree sizes may differ

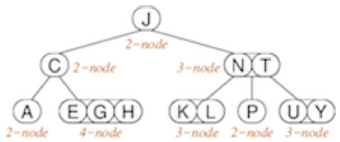


2-3-4 Trees

2-3-4 Trees

2-3-4 trees have three kinds of nodes

- 2-nodes, with two children (same as normal BSTs)
- 3-nodes, two values and three children
- 4-nodes, three values and four children



... 2-3-4 Trees

2-3-4 trees are ordered similarly to BSTs

In a *balanced 2-3-4 tree*:

- all leaves are at same distance from the root

2-3-4 trees grow "upwards" by splitting 4-nodes.

... 2-3-4 Trees

Possible 2-3-4 tree data structure:

```
typedef struct node {
   int          order;     // 2, 3 or 4
   int          data[3];   // items in node
   struct node *child[4];  // links to subtrees
} node;
```

... 2-3-4 Trees

Searching in 2-3-4 trees:

```
Search(tree,item):
|  Input  tree, item
|  Output address of item if found in 2-3-4 tree
|         NULL otherwise
|
|  if tree is empty then
|     return NULL
|  else
|  |  i=0
|  |  while i<tree.order-1 and item>tree.data[i] do
|  |     i=i+1    // find relevant slot in data[]
|  |  end while
|  |  if item=tree.data[i] then    // item found
|  |     return address of tree.data[i]
|  |  else       // keep looking in relevant subtree
|  |     return Search(tree.child[i],item)
|  |  end if
|  end if
```

... 2-3-4 Trees

2-3-4 tree searching cost analysis:

- as for other trees, worst case determined by height *h*
- 2-3-4 trees are always balanced ⇒ height is *O(log n)*
- worst case for height: all nodes are 2-nodes
  same case as for balanced BSTs, i.e. $h \cong \log_2 n$
- best case for height: all nodes are 4-nodes
  balanced tree with branching factor 4, i.e. $h \cong \log_4 n$

Insertion into 2-3-4 Trees

Starting with the root node:

repeat

- if current node is full (i.e. contains 3 items)
  - split into two 2-nodes
  - promote middle element to parent
    - if no parent ⇒ middle element becomes the new root 2-node

- go back to parent node
- if current node is a leaf
    - insert Item in this node, order++
- if current node is not a leaf
    - go to child where Item belongs

until Item inserted

... Insertion into 2-3-4 Trees

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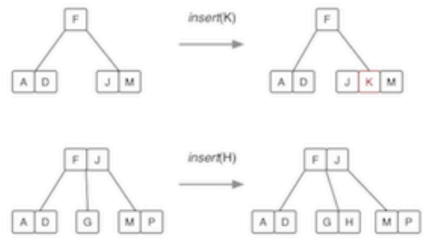
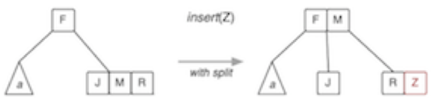
Insertion into a 2-node or 3-node:



Insertion into a 4-node (requires a split):



... Insertion into 2-3-4 Trees

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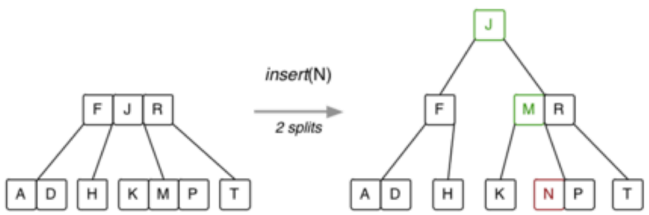
Splitting the root:



... Insertion into 2-3-4 Trees

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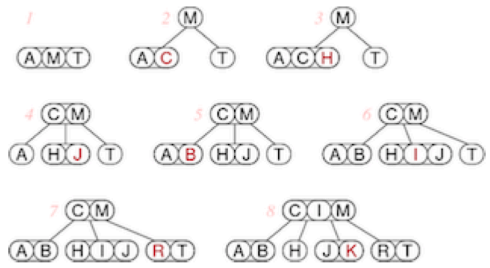
Building a 2-3-4 tree … 7 insertions:



Exercise #5: Insertion into 2-3-4 Tree

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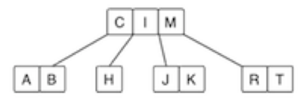
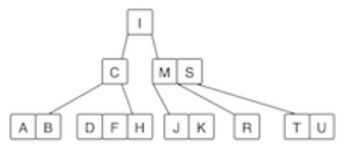
Show what happens when D, S, F, U are inserted into this tree:





... Insertion into 2-3-4 Trees

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Insertion algorithm:

```
insert(tree,item):
|  Input  2-3-4 tree, item
|  Output tree with item inserted
|
|  node=root(tree), parent=NULL
|  repeat
|  |  if node.order=4 then
|  |  |  promote = node.data[1]      // middle value
|  |  |  nodeL   = new node containing node.data[0]
|  |  |  nodeR   = new node containing node.data[2]
|  |  |  if parent=NULL then
|  |  |     make new 2-node root with promote,nodeL,nodeR
|  |  |  else
|  |  |     insert promote,nodeL,nodeR into parent
|  |  |     increment parent.order
|  |  |  end if
|  |  |  node=parent
|  |  end if
|  |  if node is a leaf then
|  |     insert item into node
```

```
|  |     increment node.order
|  |  else
|  |  |  parent=node
|  |  |  if item<node.data[0] then
|  |  |     node=node.child[0]
|  |  |  else if item<node.data[1] then
|  |  |     node=node.child[1]
|  |  |  else
|  |  |     node=node.child[2]
|  |  |  end if
|  |  end if
|  until item inserted
```

... Insertion into 2-3-4 Trees

Variations on 2-3-4 trees …

Variation #1: why stop at 4? why not 2-3-4-5 trees? or *M*-way trees?

- allow nodes to hold up to *M-1* items, and at least *M/2*
- if each node is a disk-page, then we have a *B-tree* (databases)
- for B-trees, depending on `Item` size, *M > 100/200/400*

Variation #2: don't have "variable-sized" nodes

- use standard BST nodes, augmented with one extra piece of data
- implement similar strategy as 2-3-4 trees → red-black trees.

Red-Black Trees

Red-Black Trees

Red-black trees are a representation of 2-3-4 trees using BST nodes.

- each node needs one extra value to encode link type
- but we no longer have to deal with different kinds of nodes

Link types:

- *red* links … combine nodes to represent 3- and 4-nodes
- *black* links … analogous to "ordinary" BST links (child links)

Advantages:

- standard BST search procedure works unmodified
- get benefits of 2-3-4 tree self-balancing (although deeper)

Red-Black Trees

Definition of a *red-black tree*

- a BST in which each node is marked red or black
- no two red nodes appear consecutively on any path
- a red node corresponds to a 2-3-4 sibling of its parent
- a black node corresponds to a 2-3-4 child of its parent

Balanced red-black tree

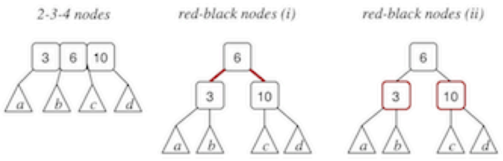
- all paths from root to leaf have same number of black nodes

Insertion algorithm: avoids worst case *O(n)* behaviour

Search algorithm: standard BST search

... Red-Black Trees

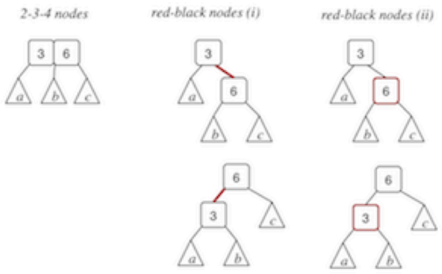
Representing 4-nodes in red-black trees:



Some texts colour the links rather than the nodes.

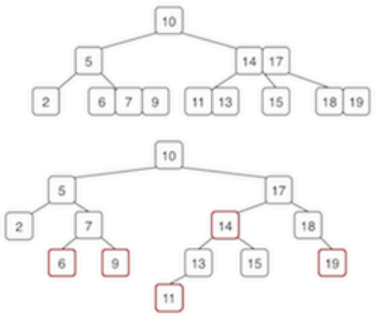
... Red-Black Trees

Representing 3-nodes in red-black trees (two possibilities):



... Red-Black Trees

Equivalent trees (one 2-3-4, one red-black):



... Red-Black Trees

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Red-black tree implementation:

```
typedef enum {RED,BLACK} Colr;
typedef struct node *RBTree;
typedef struct node {
   int    data;   // actual data
   Colr   colour;  // relationship to parent
   RBTree left;   // left subtree
   RBTree right;  // right subtree
} node;

#define colour(tree) ((tree)->colour)
#define isRed(tree)  ((tree) != NULL && (tree)->colour == RED)
```

RED = node is part of the same 2-3-4 node as its parent (sibling)

BLACK = node is a child of the 2-3-4 node containing the parent

... Red-Black Trees

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New nodes are always red:

```
RBTree newNode(Item it) {
   RBTree new = malloc(sizeof(Node));
   assert(new != NULL);
   data(new) = it;
   colour(new) = RED;
   left(new) = right(new) = NULL;
   return new;
}
```

... Red-Black Trees

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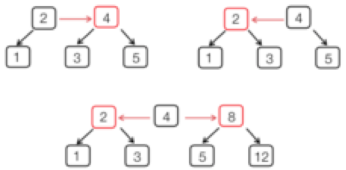
`Node.colour` allows us to distinguish links

- black = parent node is a "real"parent
- red = parent node is a 2-3-4 neighbour



... Red-Black Trees

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Search method is standard BST search:

```
SearchRedBlack(tree,item):
|  Input  tree, item
|  Output true if item found in red-black tree
|         false otherwise
|
|  if tree is empty then
|     return false
|  else if item<data(tree) then
|     return SearchRedBlack(left(tree),item)
|  else if item>data(tree) then
|     return SearchRedBlack(right(tree),item)
|  else          // found
|    return true
|  end if
```

Red-Black Tree Insertion

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Insertion is more complex than for standard BSTs

- need to recall direction of last branch (L or R)
- need to recall whether parent link is red or black
- splitting/promoting implemented by rotateLeft/rotateRight
- several cases to consider depending on colour/direction combinations

... Red-Black Tree Insertion

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High-level description of insertion algorithm:

```
insertRB(tree,item,inRight):
|  Input  tree, item, inRight indicating direction of last branch
|  Output tree with it inserted
|
|  if tree is empty then
|     return newNode(item)
|  else if item=data(tree) then
```

```
|     return tree
|   end if
|   if left(tree) and right(tree) both are RED then
|      split 4-node in a red-black tree
|   end if
|   recursive insert a la BST, re-arrange links/colours after insert
|   return modified tree

insertRedBlack(tree,item):
|   Input  red-black tree, item
|   Output tree with item inserted
|
|   tree=insertRB(tree,item,false)
|   colour(tree)=BLACK
|   return tree
```

... Red-Black Tree Insertion

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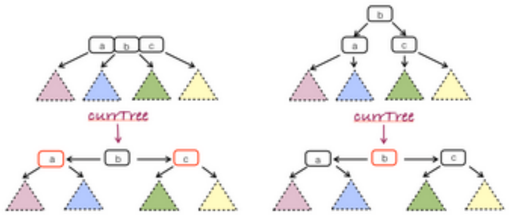
Splitting a 4-node, in a red-black tree:



Algorithm:

```
colour(currentTree)=RED
colour(left(currentTree))=BLACK
colour(right(currentTree))=BLACK
```

... Red-Black Tree Insertion

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Simple recursive insert (a la BST):

Algorithm:

```
if item<data(tree) then
   left(tree)=insertRB(left(tree),item,false)
   re-arrange links/colours after insert
else        // item larger than data in root
   right(tree)=insertRB(right(tree),item,true)
   re-arrange links/colours after insert
end if
```

Not affected by colour of `tree` node.

... Red-Black Tree Insertion

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Re-arrange links/colours after insert:

Step 1 — "normalise" direction of successive red links

Algorithm:

```
if inRight and both currentTree and left(currentTree) are red then
   currentTree=rotateRight(currentTree)
end if
```

Symmetrically,

- if not inRight and both currentTree and right(currentTree) are red
  ⇒ left rotate currentTree

... Red-Black Tree Insertion

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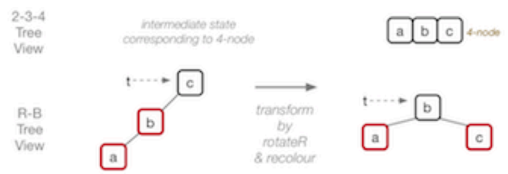
Re-arrange links/colours after insert:

Step 2 — two successive red links = newly-created 4-node



Algorithm:

```
if left(currentTree) and left(left(currentTree)) are red then
   currentTree=rotateRight(currentTree)
   colour(currentTree)=BLACK
   colour(right(currentTree))=RED
end if
```

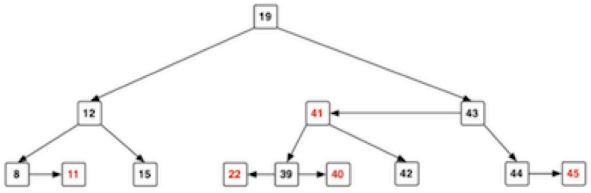
Symmetrically,

- if both right(currentTree) and right(right(currentTree)) are red
  ⇒ left rotate currentTree, then re-colour currentTree and left(currentTree)

... Red-Black Tree Insertion

Example of insertion, starting from empty tree:

```
22, 12, 8, 15, 11, 19, 43, 44, 45, 42, 41, 40, 39
```



Red-black Tree Performance

Cost analysis for red-black trees:

- tree is well-balanced; worst case search is $O(log_2\ n)$
- insertion affects nodes down one path; max #rotations is $2{\cdot}h$
  (where h is the height of the tree)

Only disadvantage is complexity of insertion/deletion code.

Note: red-black trees were popularised by Sedgewick.

Summary

- Randomised at-leaf/at-root insertion
- Tree operations
  - tree partition
  - joining trees
- Self-adjusting trees
  - Splay trees
  - AVL trees
  - 2-3-4 trees
  - Red-black trees

- Suggested reading:
  - Sedgewick, Ch. 12.9
  - Sedgewick, Ch. 13.1-13.4