# Week 7: Search Tree Data Structures

## Searching



An extremely common application in computing

- given a (large) collection of *items* and a *key* value
- find the item(s) in the collection containing that key
  - item = (key, $\text{val}_1$, $\text{val}_2$, …) (i.e. a structured data type)
  - key = value used to distinguish items (e.g. student ID)

Applications: Google, databases, .....

## ... Searching



Since searching is a very important/frequent operation,
many approaches have been developed to do it

Linear structures: arrays, linked lists, files

Arrays = random access. Lists, files = sequential access.

Cost of searching:

| | Array | List | File |
|---|---|---|---|
| Unsorted | O(n) (linear scan) | O(n) (linear scan) | O(n) (linear scan) |
| Sorted | O(log n) (binary search) | O(n) (linear scan) | O(log n) (*seek*, *seek*, …) |

- *O(n)* … linear scan (search technique of last resort)
- *O(log n)* … binary search, *search trees* (trees also have other uses)

Also (cf. COMP9021): hash tables (*O(1)*, but only under optimal conditions)

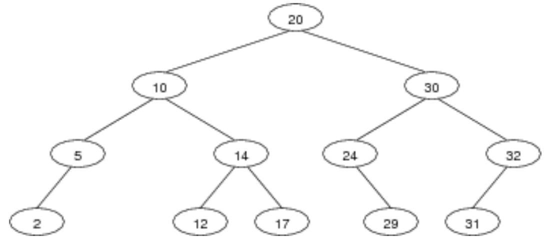
## ... Searching



Maintaining the order in sorted arrays and files is a costly operation.

*Search trees* are as efficient to search but more efficient to maintain.

Example: the following tree corresponds to the sorted array `[2,5,10,12,14,17,20,24,29,30,31,32]`:

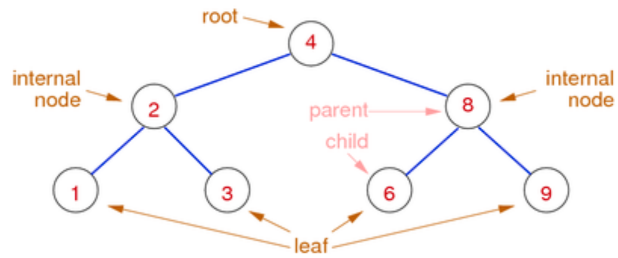



## Tree Data Structures



*Trees* are connected graphs

- consisting of nodes and edges (called *links*), with no cycles (no "up-links")
- each node contains a data value (or key+data)
- each node has links to ≤ *k* other child nodes (*k=2* below)

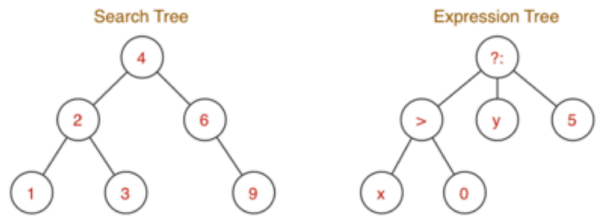



## ... Tree Data Structures



Trees are used in many contexts, e.g.

- representing hierarchical data structures (e.g. expressions)
- efficient searching (e.g. sets, symbol tables, …)

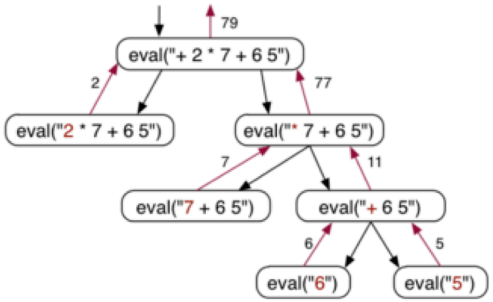


## ... Tree Data Structures



Trees can be used as a data structure, but also for *illustration*.

E.g. showing evaluation of a prefix arithmetic expression

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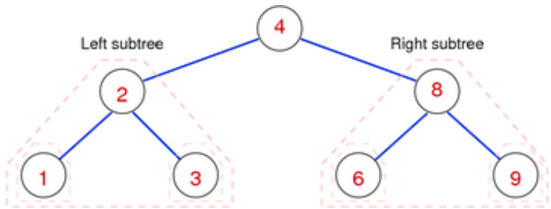
## ... Tree Data Structures



*Binary trees* (*k=2* children per node) can be defined recursively, as follows:

A *binary tree* is either

- empty (contains no nodes)
- consists of a *node*, with two *subtrees*
  - node contains a value
  - left and right subtrees are *binary trees*

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## ... Tree Data Structures



Other special kinds of tree

- *m-ary tree*: each internal node has exactly *m* children
- *Ordered tree*: all left values < root, all right values > root
- *Balanced tree*: has ≅minimal height for a given number of nodes
- *Degenerate tree*: has ≅maximal height for a given number of nodes

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# Search Trees



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# Binary Search Trees

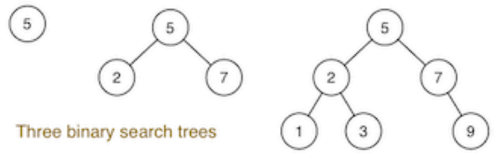


*Binary search trees* (or *BSTs*) have the characteristic properties

- each node is the root of 0, 1 or 2 subtrees
- all values in any left subtree are less than root
- all values in any right subtree are greater than root
- these properties applies over all nodes in the tree

*perfectly balanced trees* have the properties

- #nodes in left subtree = #nodes in right subtree
- this property applies over all nodes in the tree

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## ... Binary Search Trees



Operations on BSTs:

- *insert*(Tree,Item) … add new item to tree via key
- *delete*(Tree,Key) … remove item with specified key from tree
- *search*(Tree,Key) … find item containing key in tree
- plus, "bookkeeping" … new(), free(), show(), …

Notes:

- in general, nodes contain `Item`s; we just show `Item.key`
- keys are unique (not technically necessary)

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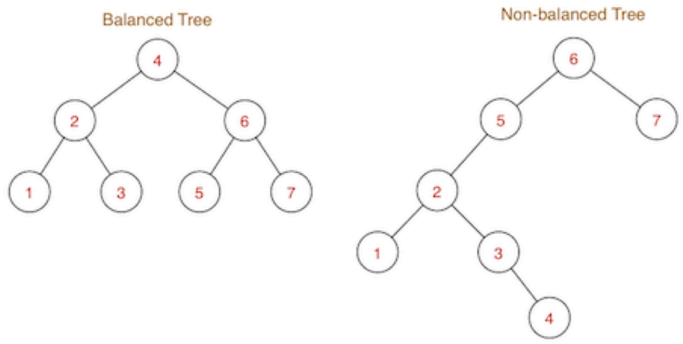
## ... Binary Search Trees



Examples of binary search trees:




Shape of tree is determined by order of insertion.

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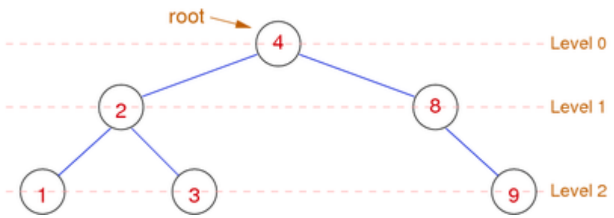
## ... Binary Search Trees

13/63

*Level* of node = path length from root to node

*Height* (or: *depth*) of tree = max path length from root to leaf




*Height-balanced tree*: ∀ nodes: height(left subtree) = height(right subtree)

Time complexity of tree algorithms is typically *O(height)*

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## Exercise #1: Insertion into BSTs

14/63

For each of the sequences below

- start from an initially empty binary search tree
- show tree resulting from inserting values in order given

(a) 4 2 6 5 1 7 3

(b) 6 5 2 3 4 7 1

(c) 1 2 3 4 5 6 7

Assume new values are always inserted as new leaf nodes

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(a) the balanced tree from 3 slides ago (height = 2)

(b) the non-balanced tree from 3 slides ago (height = 4)

(c) a fully degenerate tree of height 6

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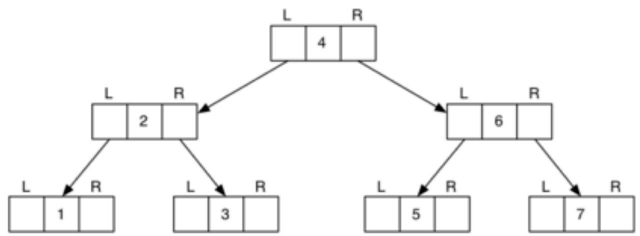
## Representing BSTs

16/63

Binary trees are typically represented by node structures

- containing a value, and pointers to child nodes

Most tree algorithms move *down* the tree.
If upward movement needed, add a pointer to parent.



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## ... Representing BSTs

17/63

Typical data structures for trees …

```
// a Tree is represented by a pointer to its root node
typedef struct Node *Tree;

// a Node contains its data, plus left and right subtrees
typedef struct Node {
   int  data;
   Tree left, right;
} Node;

// some macros that we will use frequently
#define data(tree)  ((tree)->data)
#define left(tree)  ((tree)->left)
#define right(tree) ((tree)->right)
```

We ignore items ⇒ `data` in `Node` is just a key

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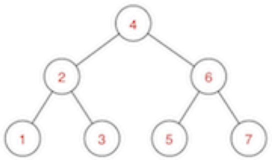
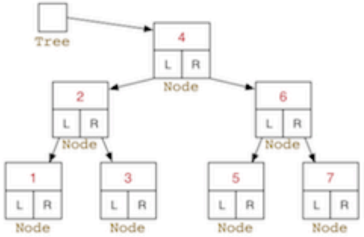
## ... Representing BSTs

18/63

Abstract data vs concrete data …




## Tree Algorithms



## Searching in BSTs

Most tree algorithms are best described recursively

```
TreeSearch(tree,item):
|  Input  tree, item
|  Output true if item found in tree, false otherwise
|
|  if tree is empty then
|     return false
|  else if item < data(tree) then
|     return TreeSearch(left(tree),item)
|  else if item > data(tree) then
|     return TreeSearch(right(tree),item)
|  else          // found
|    return true
|  end if
```



## Insertion into BSTs

Insert an item into appropriate subtree

```
insertAtLeaf(tree,item):
|  Input  tree, item
|  Output tree with item inserted
|
|  if tree is empty then
|     return new node containing item
|  else if item < data(tree) then
|     return insertAtLeaf(left(tree),item)
|  else if item > data(tree) then
|     return insertAtLeaf(right(tree),item)
|  else
|     return tree    // avoid duplicates
|  end if
```



## Tree Traversal

Iteration (traversal) on …

- `Lists` … visit each value, from first to last
- `Graphs` … visit each vertex, order determined by DFS/BFS/…

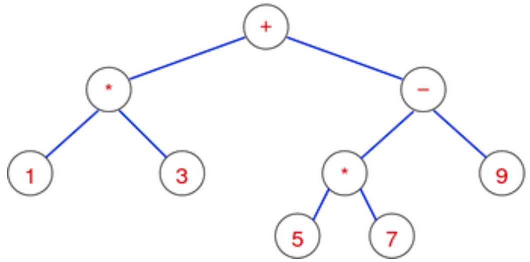
For binary `Trees`, several well-defined visiting orders exist:

- *preorder* (NLR) … visit root, then left subtree, then right subtree
- *inorder* (LNR) … visit left subtree, then root, then right subtree
- *postorder* (LRN) … visit left subtree, then right subtree, then root
- *level-order* … visit root, then all its children, then all their children



## ... Tree Traversal

Consider "visiting" an expression tree like:

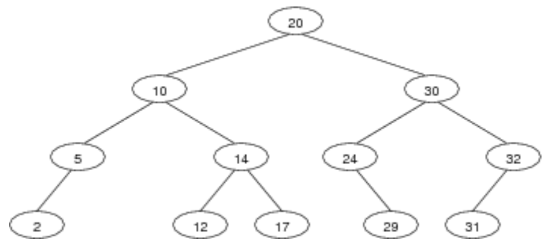



NLR: + * 1 3 - * 5 7 9 (prefix-order: useful for building tree)
LNR: 1 * 3 + 5 * 7 - 9 (infix-order: "natural" order)
LRN: 1 3 * 5 7 * 9 - + (postfix-order: useful for evaluation)
Level: + * - 1 3 * 9 5 7 (level-order: useful for printing tree)



## Exercise #2: Tree Traversal

Show NLR, LNR, LRN traversals for the tree




NLR (preorder): 20 10 5 2 14 12 17 30 24 29 32 31

LNR (inorder): 2 5 10 12 14 17 20 24 29 30 31 32

LRN (postorder): 2 5 12 17 14 10 29 24 31 32 30 20

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### Exercise #3: Non-recursive traversals

26/63

Write a non-recursive *preorder* traversal algorithm.

Assume that you have a stack ADT available.

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```
showBSTreePreorder(t):
|  Input tree t
|
|  push t onto new stack S
|  while stack is not empty do
|  |  t=pop(S)
|  |  print data(t)
|  |  if right(t) is not empty then
|  |     push right(t) onto S
|  |  end if
|  |  if left(t) is not empty then
|  |     push left(t) onto S
|  |  end if
|  end while
```

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## Joining Two Trees

28/63

An auxiliary tree operation …

Tree operations so far have involved just one tree.

An operation on two trees: `t = joinTrees(t1,t2)`

- Pre-conditions:
  - takes two BSTs; returns a single BST
  - max(key($t_1$)) < min(key($t_2$))
- Post-conditions:
  - result is a BST (i.e. fully ordered)
  - containing all items from $t_1$ and $t_2$

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### ... Joining Two Trees

29/63

Method for performing tree-join:

- find the min node in the right subtree ($t_2$)
- replace min node by its right subtree
- elevate min node to be new root of both trees

Advantage: doesn't increase height of tree significantly

$x \le \text{height}(t) \le x+1$, where $x = \max(\text{height}(t_1),\text{height}(t_2))$

Variation: choose deeper subtree; take root from there.

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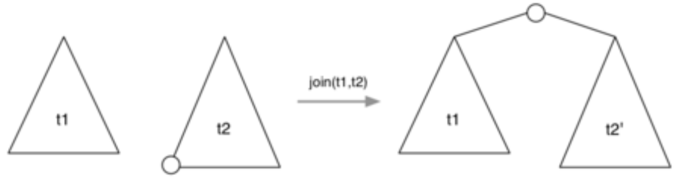
### ... Joining Two Trees

30/63

Joining two trees:




Note: t2' may be less deep than t2

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### ... Joining Two Trees

31/63

Implementation of tree-join

```
joinTrees(t1,t2):
|  Input  trees t1,t2
|  Output t1 and t2 joined together
|
|  if t1 is empty then return t2
|  else if t2 is empty then return t1
|  else
|  |  curr=t2, parent=NULL
|  |  while left(curr) is not empty do     // find min element in t2
|  |     parent=curr
|  |     curr=left(curr)
|  |  end while
|  |  if parent≠NULL then
|  |     left(parent)=right(curr)  // unlink min element from parent
|  |     right(curr)=t2
|  |  end if
|  |  left(curr)=t1
|  |  return curr                  // curr is new root
|  end if
```

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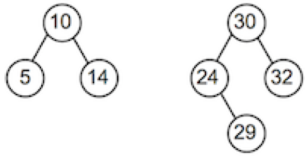
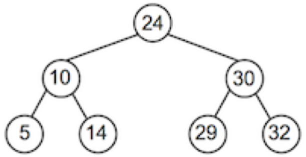
### Exercise #4: Joining Two Trees

32/63

Join the trees






## Deletion from BSTs

Insertion into a binary search tree is easy.

Deletion from a binary search tree is harder.

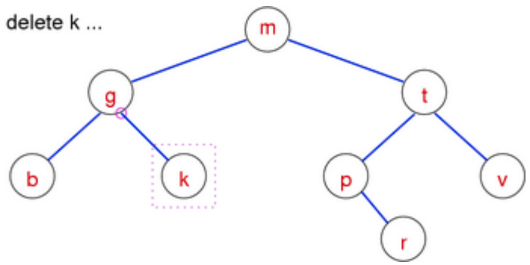
Four cases to consider …

- empty tree … new tree is also empty
- zero subtrees … unlink node from parent
- one subtree … replace by child
- two subtrees … replace by successor, join two subtrees



### ... Deletion from BSTs

Case 2: item to be deleted is a leaf (zero subtrees)

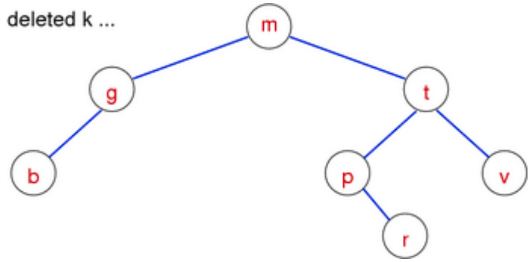



Just delete the item



### ... Deletion from BSTs

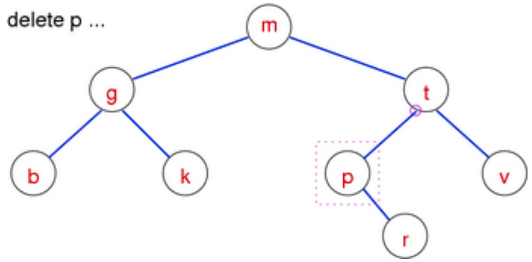
Case 2: item to be deleted is a leaf (zero subtrees)






### ... Deletion from BSTs

Case 3: item to be deleted has one subtree

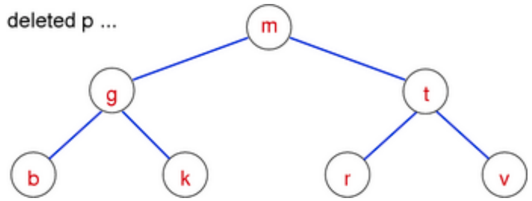



Replace the item by its only subtree



### ... Deletion from BSTs

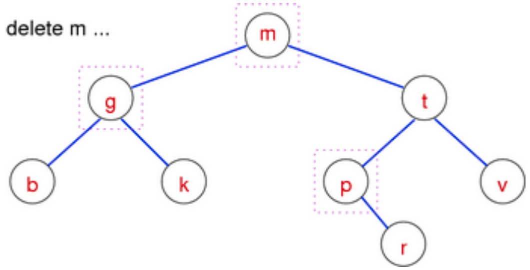
Case 3: item to be deleted has one subtree






### ... Deletion from BSTs

Case 4: item to be deleted has two subtrees

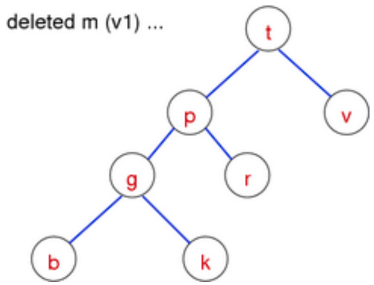



Version 1: right child becomes new root, attach left subtree to min element of right subtree

## ... Deletion from BSTs



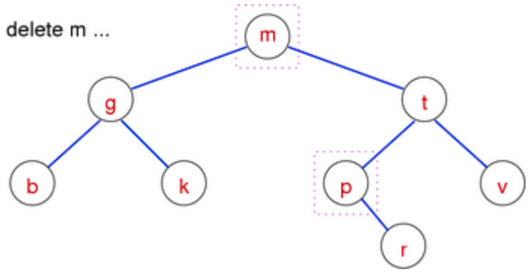
Case 4: item to be deleted has two subtrees




## ... Deletion from BSTs



Case 4: item to be deleted has two subtrees

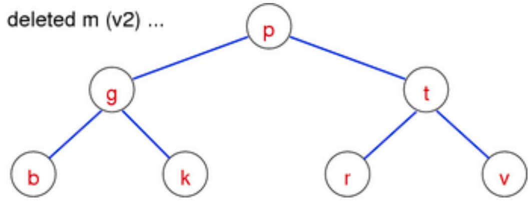



Version 2: *join* left and right subtree

## ... Deletion from BSTs



Case 4: item to be deleted has two subtrees




## ... Deletion from BSTs



Pseudocode (version 2 for case 4)

```
TreeDelete(t,item):
|  Input  tree t, item
|  Output t with item deleted
|
|  if t is not empty then          // nothing to do if tree is empty
|  |  if item < data(t) then       // delete item in left subtree
|  |     left(t)=TreeDelete(left(t),item)
|  |  else if item > data(t) then  // delete item in right subtree
|  |     right(t)=TreeDelete(right(t),item)
|  |  else                         // node 't' must be deleted
|  |  |  if left(t) and right(t) are empty then
|  |  |     new=empty tree                    // 0 children
|  |  |  else if left(t) is empty then
|  |  |     new=right(t)                      // 1 child
|  |  |  else if right(t) is empty then
|  |  |     new=left(t)                       // 1 child
|  |  |  else
|  |  |     new=joinTrees(left(t),right(t))   // 2 children
|  |  |  end if
|  |  |  free memory allocated for t
|  |  |  t=new
|  |  end if
|  end if
|  return t
```

# Balanced Binary Search Trees

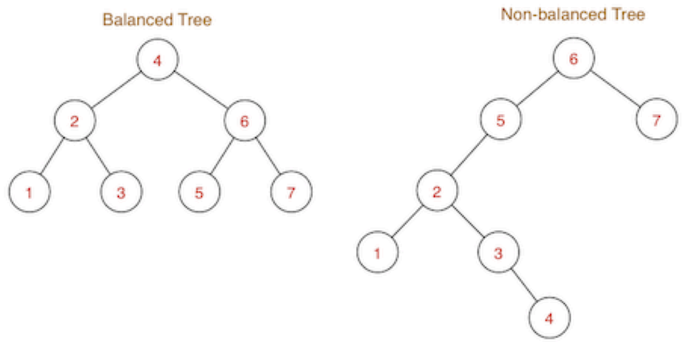


Goal: build binary search trees which have

- minimum height ⇒ minimum worst case search cost

Best balance you can achieve for tree with $N$ nodes:

- abs(#nodes(LeftSubtree) - #nodes(RightSubtree)) ≤ 1, for every node
- height of $log_2 N$ ⇒ worst case search $O(log\ N)$






## Operations for Rebalancing

To assist with rebalancing, we consider new operations:

Left rotation

- move right child to root; rearrange links to retain order

Right rotation

- move left child to root; rearrange links to retain order

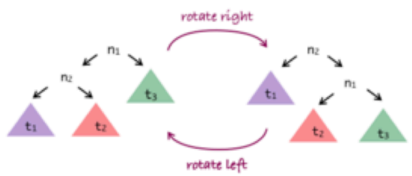
Insertion at root

- each new item is added as the new root node



## Tree Rotation

In tree below: $t_1 < n_2 < t_2 < n_1 < t_3$






### ... Tree Rotation

Method for rotating tree T right:

- $N_1$ is current root; $N_2$ is root of $N_1$'s left subtree
- $N_1$ gets new left subtree, which is $N_2$'s right subtree
- $N_1$ becomes root of $N_2$'s new right subtree
- $N_2$ becomes new root

Left rotation: swap left/right in the above.

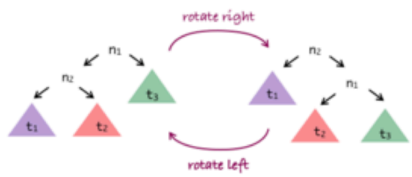
Cost of tree rotation: *O(1)*



### ... Tree Rotation

Algorithm for right rotation:

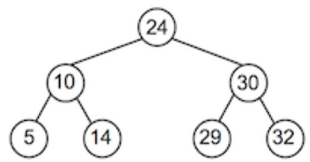
```
rotateRight(n1):
|  Input  tree n1
|  Output n1 rotated to the right
|
|  if n1 is empty or left(n1) is empty then
|     return n1
|  end if
|  n2=left(n1)
|  left(n1)=right(n2)
|  right(n2)=n1
|  return n2
```



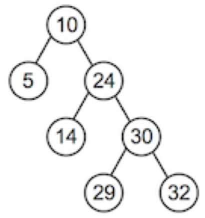



### Exercise #5: Tree Rotation

Consider the tree `t`:



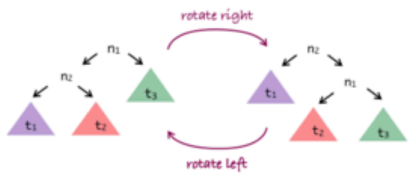

Show the result of `rotateRight(t)`



## Exercise #6: Tree Rotation



Write the algorithm for left rotation




```
rotateLeft(n2):
|  Input  tree n2
|  Output n2 rotated to the left
|
|  if n2 is empty or right(n2) is empty then
|     return n2
|  end if
|  n1=right(n2)
|  right(n2)=left(n1)
|  left(n1)=n2
|  return n1
```

# Insertion at Root



Previous description of BSTs inserted at leaves.

Different approach: insert new item at root.

Potential disadvantages:

- large-scale rearrangement of tree for each insert

Potential advantages:

- recently-inserted items are close to root
- low cost if recent items more likely to be searched

## ... Insertion at Root

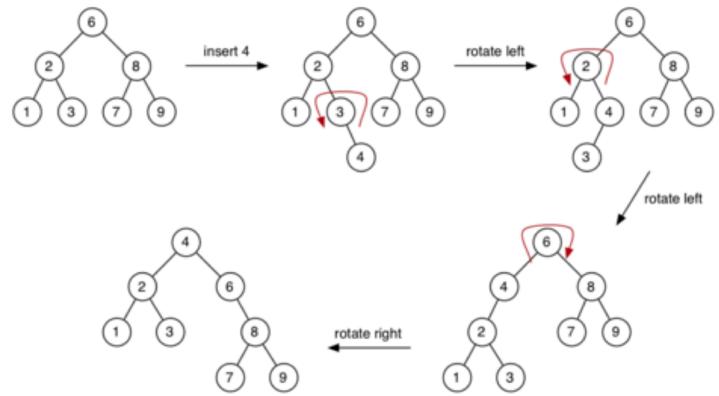


Method for inserting at root:

- base case:
  - tree is empty; make new node and make it root
- recursive case:
  - insert new node as root of appropriate subtree
  - lift new node to root by rotation

## ... Insertion at Root

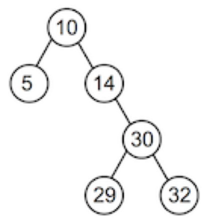




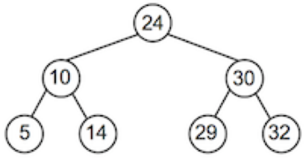

## Exercise #7: Insertion at Root



Consider the tree `t`:



Show the result of `insertAtRoot(t,24)`




## ... Insertion at Root

58/63

Analysis of insertion-at-root:

- same complexity as for insertion-at-leaf: *O(height)*
- tendency to be balanced, but no balance guarantee
- benefit comes in searching
  - for some applications, search favours recently-added items
  - insertion-at-root ensures these are close to root
- could even consider "move to root when found"
  - effectively provides "self-tuning" search tree

  ⇒ Real-balanced trees (week 8)

# Application of BSTs: Sets

59/63

Trees provide efficient search.

Sets require efficient search

- to find where to insert/delete
- to test for set membership

Logical to implement a set ADT via `BSTree`

## ... Application of BSTs: Sets

60/63

Assuming we have `Tree` implementation

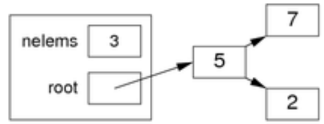
- which precludes duplicate key values
- which implements insertion, search, deletion

then `Set` implementation is

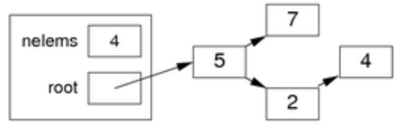
- `SetInsert(Set,Item)` ≡ `TreeInsert(Tree,Item)`
- `SetDelete(Set,Item)` ≡ `TreeDelete(Tree,Item.Key)`
- `SetMember(Set,Item)` ≡ `TreeSearch(Tree,Item.Key)`

## ... Application of BSTs: Sets

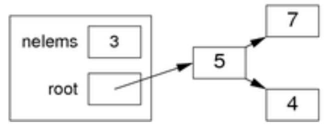
61/63




After SetInsert(s,4):




After SetDelete(s,2):




## ... Application of BSTs: Sets

62/63

Concrete representation:

```
#include "BSTree.h"

typedef struct SetRep {
   int   nelems;
   Tree  root;
} SetRep;

typedef SetRep *Set;

Set newSet() {
   Set S = malloc(sizeof(SetRep));
   assert(S != NULL);
   S->nelems = 0;
   S->root = newTree();
   return S;
}
```

# Summary

63/63

- Binary search tree (BST) data structure
- Tree traversal
- Basic BST operation: insertion, join, deletion, rotation

- Suggested reading:

- Sedgewick, Ch. 12.5-12.6, 12.8

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