Real Analysis Notes

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This notes is referenced from the book ”Real Analysis: A Long-Form Mathematics Textbook” by Jay Cummings

1 Chapter 1: Reals

1.1 Basic Set Theory Definition

Definition 1.1.

- A set is an unordered collection of distinct objects, which are called elements.
- If x is an element of set S, x $\in$ S.
- Set-builder notation, S = {elements : conditions used to generate the elements}
- The set containing no elements is called the empty set and is denoted $\emptyset$
- A $\cap$ B = {x : x $\in$ A and x $\in$ B}
- A $\cup$ B = {x : x $\in$ A or x $\in$ B}
- A $\setminus$ B = {x : x $\in$ A and x $\notin$ B}
- A and B are disjoint if A $\cap$ B = $\emptyset$
- If A $\subseteq$ U for a universal set U (Typically U = $\mathbb{R}$), the the complement of A in U is A^c = U $\setminus$ A
- The Cartesian product A $\times$ B = {(a,b) : a $\in$ A and b $\in$ B}
- The power set of a set A is P(A) = {X: X $\subseteq$ A}
- $A_1, A_2, A_3, \ldots$ are all sets, then the union of all of them is
$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$
- $A_1, A_2, A_3, \ldots$ are all sets, then the intersection of all of them is
$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$
- You can also write the union and intersection of infinite sets.
$$\bigcup_{i=1}^{\infty} A_i \text{ and } \bigcap_{i=1}^{\infty} A_i$$
- The cardinality of a set A is the ”number” of elements in the set, and is denoted by $|A|$

1.2 Functions of sets

Definition 1.2.

- Given a pair of sets A and B, suppose that each element x ∈ A is associated to one element of B, which we denote f(x). Then f is said to be a function from A to B, denoted " f : A → B "
  - A is called the domain of f, and B is called the codomain of f. The set {f(x) : x ∈ A} is called the range of f.
- A function f : A → B is injective (or one-to-one) if f(a) = f(b) implies that a = b.
- A function f : A → B is surjective (or onto) if, for ever b ∈ B, there exists some a ∈ A such that f(a) = b.
- A function f : A → B is bijective if it is both injective and surjective.
- Note: We can use contrapositive to give the equivalent definition as well.
  - f is injective if a ≠ b implies $f(a) \neq f(b)$.
  - f is surjective if there does not exist a, b ∈ B for which $f(a) \neq$ b for all a ∈ A.

1.3 Number

Naturals
The natural number ℕ is the set {1, 2, 3, . . . }
Integers
The integers ℤ is the set {. . . , -3, -2, -1, 0, 1, 2, 3, . . . }, where each number is 1 away from the next.
Rationals
The next natural set of numbers to consider is the set of rational numbers:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$$

- They are nested: Between any two rational numbers is another one
- They are closed under addition, subtraction, multiplication and division:
  - If $p,\ q \in \mathbb{Q}$, then $p + q \in \mathbb{Q}$
  - If $p,\ q \in \mathbb{Q}$, then $p - q \in \mathbb{Q}$
  - If $p,\ q \in \mathbb{Q}$, then $p \cdot q \in \mathbb{Q}$
  - If $p,\ q \in \mathbb{Q}$, then $p \div q \in \mathbb{Q}$
- They are not algebraically closed: There exist polynomials with rational coefficients whose roots are not rational.

But rationals are not enough, there are numbers are cannot be represented by rational numbers.
Rationals
Numbers are that are not rational.

1.4 Fields and Ordered Fields

A field is a set that satisfies the classic multiplicative and additive properties.

Definition 1.3. A field is a nonempty set $\mathbb{F}$, along with two binary operations, addition $(+)$ and multiplication $(\cdot)$, satisfying the following axioms.

1. Commutative Law: If $a, b \in \mathbb{F}$, then $a + b = b + a$ and $a \cdot b = b \cdot a$
2. Distributive Law: If $a, b, c \in \mathbb{F}$, then $a \cdot (b + c) = a \cdot b + a \cdot c$
3. Associative Law. If $a, b, c \in \mathbb{F}$, then $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
4. Identity Law: There are special elements $0, 1 \in \mathbb{F}$, where $a + 0 = a$ and $a \cdot 1 = a$ for all $a \in \mathbb{F}$
5. Inverse Law: For each $a \in \mathbb{F}$, there is an element $-a in \mathbb{F}$ such that $a + (-a) = 0$. If $a \neq 0$, then there is also an element $a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$

2 Chapter 2: Cardinality

The number of elements in a set A is the cardinality of that set. In general, we say that two sets have the same size if there is a way to pair up the elements between the two sets. Equivalently, if there is a bijection between them, known as the bijection principle.

2.1 The bijection principle

- Two sets have the same size if and only if there is a bijection between them.

2.2 Counting Infinities

Let S and be sets. Then,

- $|S| = |T|$ if and only if there is a bijection from S to T.
- $|S| \leq |T|$ if and only if there is an injection from S to T.
  - There can be elements in T not mapped by S.
- $|S| \geq |T|$ if and only if there is a surjection from S to T.
  - There can be multiple elements mapping S to T.

Specific sets

Ex 1: There are the same number of natural numbers as there are natural numbers larger than 1 (that is, $|\mathbb{N}| = |\{2, 3, 4, \dots\}|$). Let

$$f : \mathbb{N} \to \{2, 3, 4, \dots\}, n \longmapsto n + 1$$

a bijection.* Two sets can have the same size even though one is a proper subset of the other.
Ex 2: There are the same umber of natural numbers as even natural numbers (that is, $|\mathbb{N}| = |2\mathbb{N}|$). What's the bijection that shows this? Let

$$f : \mathbb{N} \to 2\mathbb{N}, n \longmapsto 2n$$

Two sets can have the same size even though one is a proper subset of the other and the larger one even ave infinitely many more elements than the smaller one.

Theorem 2.1. ($|\mathbb{Z}| = |\mathbb{Q}|$). There are the same number of integers as rational numbers.

Proof. We need to show a bijection. Note that if we can show that there are the same number of positive integers as positive real rational numbers, then by simply adding a minus sign to everything, we will get a full bijection. So we will focus on just the positive case. This is done through a winding bijection where a number maps to the next number. $1 \to 2/1 \to 1/2 \to 1/3 \to 2/2 \to 3/1 \to \dots$. We can hit every pair. However, there is a small problem. Each rational number is hit more than once. The rational number p/q are the same for (p,q), (2p,2q), (3p,3q), ... To fix this, we can just skip over the rational number also considered. Clearly we will not run out of rational numbers, so this does indeed pair up everything. so

$$f(n) = \text{ the } n^{th} \text{ new rational number you reach.}$$

□

Even though there are infinitely many rational numbers between two consecutive integers, the two sets still have the same size.

Theorem 2.2. ($|\mathbb{R}| > |\mathbb{N}|$). There are more real numbers than natural numbers. The cardinality of $\mathbb{R}$ is strictly larger than the cardinality of $\mathbb{N}$. This implies that some infinities are bigger than others.

Proof. Since $\mathbb{N} \subseteq \mathbb{R}$, clearly $|\mathbb{N}| \leq |\mathbb{R}|$. To show that they are not equal, we must prove that there is n bijection between $\mathbb{R}$ and $\mathbb{N}$. Again use the pairing idea and prove by contradiction. In fact, we will prove the stronger statement that there are more real numbers in (0,1) than there are natural numbers. (This would of course would prove the larger statement since then we could say $|\mathbb{R}| \geq |(0,1)| > |\mathbb{N}|$. Assume for a contradiction that there does exist some way to pair up the naturals with the reals in (0,1). Writing the reals in decimal notation, assume the pairing is this:

$$1 \leftrightarrow 0 \cdot a_{11}a_{12}a_{13}a_{14}\ldots$$

$$2 \leftrightarrow 0 \cdot a_{21}a_{22}a_{23}a_{24}\ldots$$

$$3 \leftrightarrow 0 \cdot a_{31}a_{32}a_{33}a_{34}\ldots$$

$$\vdots$$

So the left of the arrows is every natural number, and on the right of the arrows is every number in the interval (0,1), and they are paired up in some way. (Each a_{ij} is some digit, from 0 to 9). Now focus on the diagonals $a_{11}, a_{22}, \ldots$. We let a new real number be different than the first number in its 1st position, different than the second number in its 2nd positive and so on. The number will have decimal expansion:

$$b = 0.b_1b_2b_3b_4\ldots, \text{where } b_i \neq a_{ii} \text{ for all } i.$$

Then $b \in (0,1)$, but b is not found in the list! We know b is not found paired with 1 since it is different at a_{11} and so forth. In general, we know that b is not paired with k because that number are different in the k^{th} positive ($b_k \neq a_{kk}$). So this real number b is not found anywhere, and thus reach a contradiction. □

*There are different sizes of infinity, and $|\mathbb{N}|, |\mathbb{Z}|$ and $|\mathbb{Q}|$ are all a smaller infinity than $|\mathbb{R}|$. The smallest infinity is $|\mathbb{N}|$, and since these are the counting numbers this infinity is called the countable infinity.

Definition 2.1. If S is an infinite set, the S is countable if $|S| = |\mathbb{N}|$. Otherwise S is uncountable.

- Notice that A being a countable set means that there is a bijection $f: \mathbb{N} \to A$ where f(1), f(2), f(3), … is a listing of all elements in A. Therefore an infinite set A being countable is equivalent to being able to write $A = \{a_1, a_2, a_3, a_4, \ldots\}$

Theorem 2.3. *(Sizes of infinity).* There are different sizes of infinity, with countable infinity being the smallest. Moreover, $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are countable while $\mathbb{R}$ is uncountable.

2.3 Undecidable Statements

We know that $|\mathbb{N}| < |\mathbb{R}|$. Is there any infinity between these two? An astounding fact is that, within the axioms of set theory (called ZFC), whether or not there exists such an infinity can neither be proven nor disproven. Its truth value is undecidable, no proof can possibly exist. There are statements in math which are impossible to prove and also impossible to disprove (but we are able to prove that they are unprovable).

Here is the continuum hypothesis, formally stated.

Definition 2.2. *The continuum hypothesis.* There is no set whose cardinality is strictly between that of the naturals and reals. That is, there does not exist a set S for which

$$|\mathbb{N}| < |S| < |\mathbb{R}|$$

But how does one prove that a statement is undecidable? To say something is undecidable means that, within the axioms of our particular theory, the statement can neither be proven nor disproven. And so, since all proofs ultimately can be traced back and deduced from just the axioms, whether something is decidable relies fundamentally on the chosen axioms. That said, we are not picking axioms at random; the fundamental axioms that nearly all of mathematics is built on give undecidable statements.
Ex: As a theorem, it was shown that the parallel postulate is undecidable. The approach to show this are both of these:

- The five axioms for the planar geometry of Euclid's Elements are consistent (they don't self-contradict).
- These five axioms are also consistent: Axioms 1-4 from planar geometry, along with an axiom that says "If l is a straight line and P is a point not on the line l, then there are <u>at least two</u> lines that pass through P and are parallel to l." where (it was changed originally from <u>at most one</u> to <u>at least two</u>).

If you can show that the assumption that the parallel postulate is true leads to no contradictions (which means that you are unable to disprove it, based on the first four axioms), and the assumption that the parallel postulate is false leads to no contradictions (which means you can't prove it, based on the first four axioms), then it must be the case that the parallel postulate is neither provable nor disprovable. Its truth value is completely independent on your axioms.
The continuum hypothesis was settled in this same way. As mentioned, ZFC set theory is the most fundamental collection of axioms in mathematics, which nearly all of math is based upon. And Kurt Gödel gave a consistent structure satisfying the axioms of ZFC set theory in which the continuum hypothesis was assumed to be true, while Paul Cohen gave a consistent structure satisfying the axioms of ZFC set theory in which the continuum hypothesis was assumed to be false. Combined, this proves that the continuum hypothesis is neither provable nor disprovable (in ZFC). That is, the continuum hypothesis (which asks what is presumably a basic question about the infinite) is undecidable.

2.4 Number of Infinities

We know of two distinct infinities, but just how many infinities are there? The first theorem in this direction is the power set theorem. The power set P(A) of a set A is the set of all subsets of A.

Theorem 2.4. ($|A| < |P(A)|$). If a set and $P(A)$ is the power set of A, then $|A| < |P(A)|$.

Proof. Assume for a contradiction that $|A| \geq |P(A)|$. That is, assume that there is a surjection f from A to $P(A)$. Since f is a surjection, for every $T \subseteq A$, there is some element $t \in A$ where $f(t) = T$. To reach our contradiction, we will construct a set $B \subset A$ which is not hit.
For each a there is one special property about the set $f(a)$ that we are going to care about. Is $a \in f(a)$ or is $a \notin f(a)$? In general, consider the set of all elements a such that $a \notin f(a)$, and we call this set B.

$$B = \{a \in A \colon a \notin f(a)\}$$

If we show that there is no b where $f(b) = B$, then we are done. We will have discovered an element of P(A) that was not hit by f, a contradiction.
<u>Claim.</u> There is no $b \in A$ such that $f(b) = B$
<u>Proof of claim.</u> Assume for a contradiction that there does exist some $b \in A$ such that $f(b) = B$. Note by the definition of B that:

$$b \in B \text{ if and only if } b \notin f(b)$$

But since we assumed that $f(b) = B$, this is equivalent to:

$$b \in f(b) \text{ if and only if } b \notin f(b)$$

which is clearly a contradiction □

There are infinitely many infinities!

Corollary 2.4.1. *(There exist infinitely many infinities).* There exist infinitely man distinct infinite cardinalities.

Proof. By Theorem 2.4, the following is a chain of distinct infinite cardinalities

$$|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < |P(P(P(\mathbb{N})))| < |P(P(P(P(\mathbb{N}))))| < \dots$$

□

We can identify infinitely many infinities. When we say there are infinitely many infinities, which infinity is it? BY the above there are at least countably many infinities. But is there an uncountable number of infinities? Similar to the continuum hypothesis, there is not even an answer to this question. In some sense, the "number" of infinities is more than any infinity. //
Assume for a contradiction that A is a set which contains a set of each infinite size (a set of size $|\mathbb{N}|$, a set of size $|\mathbb{R}|$, and all the rest), and let B be the union of all of the sets in A. The contradiction comes when we consider P(B). By Theorem 2.4,

$$|B| < |P(B)|$$

Here is the contradiction: The cardinality $|P(B)|$ can not possibly be in A. And here's why: If it were equal to $|b|$ for some $b \in A$, then since $b \subseteq B$ we would know that $|b| \leq |B|$. But combining this with the one above,

$$|b| \leq |B| < |P(B)| = |b|$$

which is impossible. So, there are more infinities than we can imagine. There are more than infinity of them, for any of the infinities. However, every set should have a cardinality. It turns out that there was another contradiction in the above, when referring to A as "the set of all infinities". Such a collection is actually not a set.

3 Chapter 3: Sequences

Definition 3.1. A sequence of real numbers is a function $a: \mathbb{N} \to \mathbb{R}$

3.1 Bounded Sequences

Definition 3.2. A sequence (a_n) is bounded if the range $\{a_n : n \in \mathbb{N}\}$ is bounded. There exists a lower bound $L \in \mathbb{R}$ and an upper bound $U \in \mathbb{R}$ where

$$L \leq a_n \leq U$$

for all n.

Proposition 3.1. *(Bounded $\Leftrightarrow |a_n| \leq C$).* A sequence (a_n) is bounded if and only if there exists some $C \in \mathbb{R}$ for which $|a_n| \leq C$ for all n.

Proof. Recall again that boundedness means that there exists a lower bound $L \in \mathbb{R}$ and an upper bound $U \in \mathbb{R}$ where $L \leq a_n \leq U$ for al n. We prove each direction.
$\Leftarrow$ Assume that there exists a $C \in \mathbb{R}$ where

$$|a_n| \leq C$$

Then

$$-C \leq a_n \leq C$$

And so by setting $L = -C$ and $U = C$ we have shown that

$$L \leq a_n \leq U$$

which means that a_n is bounded. This concludes the backward direction.
$\Rightarrow$ If a_n is bounded, then there exists such an L and U. Let $C = max\{|L|, U\}$. This implies that $C \geq U$ and (since $C \geq |L|$, that) $-C \leq -|L|$. Thus, for all n we have

$$-C \leq -|L| \leq L \leq a_n \leq U \leq C$$

So we see that

$$-C \leq a_n \leq C$$

which is the same as $|a_n| \leq C$. □

3.2 Convergent Sequences

Definition 3.3. A sequence (a_n) converges to $a \in \mathbb{R}$ if for all $\epsilon > 0$ there exists some N such that $|a_n - a| < \epsilon$ for all $n > N$. When this happens, a is called the limit of (a_n).

Notation The following are notationally equivalent:

- The sequences (a_n) converges to a
- $a_n \to a$ as $n \to \infty$
- $a_n \to a$
- $\lim_{n\to\infty} a_n$

Outline: To show that $a_n \to a$, begin with preliminary work.

1. Scratch work: Start with $|a_n - a| < \epsilon$ and unravel to solve for n. This tells you which N to pick for step 3.
   Now for the actual proof.
2. Let $\epsilon > 0$
3. Let N be the final value of n you got in your scratch work, and let $n > N$.
4. Redo scratch work (without ϵ's), but at the end use N to show that $|a_n - a| < \epsilon$.

Definition 3.4. Let $\epsilon > 0$. The ϵ-neighborhood of a point a is the interval

$$(a - \epsilon, a + \epsilon)$$

With this, we can rephrase the definition of convergence like so:

Definition 3.5. A sequence (a_n) converges to $a \in \mathbb{R}$ if for al $\epsilon > 0$ there exists some N such that a_n is in the ϵ-neighborhood of a for all $n > N$.

3.3 Divergent Sequences