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1、

(a) $C(K) = e^{-rT} \int_0^{\infty} p(s,T)(S-K)^+ ds$

$\frac{\partial c}{\partial K}(K,T) = -e^{-rT} \int_0^{\infty} p(s,T) 1_{s>K} ds = -0.8 \times 0.5 = -0.4$

$\frac{\partial^2 c}{\partial K^2}(K,T) = e^{-rT} p(K,T) = 0.8 \times 0.15 = 0.12$

$C(99.5) \approx C(100) + (-0.5)C'(100) + \frac{1}{2}(-0.5)^2 C''(100)$

$= 8 - 0.5 \times (-0.4) + \frac{1}{2} \times 0.25 \times 0.12 = 8.215$

(b) $-C'(K) = e^{-rT} \int_0^{\infty} p(s,T) 1_{s>K} ds = 0.8 \times 0.5 = 0.4$

Thus, binary call price is 0.4

e^{-rT} − Binary call price = 0.8 − 0.4 = 0.4

Thus, price of a binary put is 0.4.

(c) $e^{rT} C''(100) = \frac{1}{0.8} \times 0.12 = 0.15$

probability = $(101-99) \times 0.15 = 0.3$

2.

(a) $$\frac{\partial c}{\partial t} + 0.75\frac{\partial^2 c}{\partial x^2} = 0$$

$$\frac{\partial c}{\partial t} \approx \frac{C(15.0, 0.1) - C(15, 0)}{0.1} = \frac{1.05 - C(15, 0)}{0.1}$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{C(15.5, 0.1) - 2C(15, 0.1) + C(14.5, 0.1)}{(0.5)^2}$$

$$= \frac{1.16 - 2\times 1.05 + 1.04}{0.5^2}$$

$$= 0.4$$

$$\frac{1.05 - C(15, 0)}{0.1} + 0.75\times 0.4 = 0.$$

$$\Rightarrow C(15, 0) = 1.08$$

(b) $$\frac{\partial c}{\partial t} = \frac{C(15, 0.2) - C(15, 0)}{0.2} = \frac{1.01 - C(15, 0)}{0.2}$$

$$\frac{1.01 - C(15, 0)}{0.2} + 0.75\times 0.4 = 0$$

$$\Rightarrow C(15, 0) = 1.07$$

(c)(a) $\dfrac{\partial c}{\partial t} \approx \dfrac{C(15.0,\ 0.1) - C(15,\ 0)}{0.1} = \dfrac{C(15,\ 0.1) - C(15,\ 0)}{0.1}$

$$\frac{\partial^2 c}{\partial x^2} = \frac{C(15.5,\ 0.1) - 2C(15,\ 0.1) + C(14.5,\ 0.1)}{(0.5)^2}$$

By $\dfrac{\partial c}{\partial t} + 0.75\dfrac{\partial^2 c}{\partial x^2} = 0$, we get

$$\frac{C(15,\ 0.1) - C(15,\ 0)}{0.1} + 0.75 \times \frac{C(15.5,\ 0.1) - 2C(15,\ 0.1) + C(14.5,\ 0.1)}{(0.5)^2} = 0$$

$\Rightarrow 10C(15,\ 0.1) - 10C(15,0) + 3C(15.5,\ 0.1) - 6C(15,\ 0.1) + 3C(14.5,\ 0.1) = 0.$

$\Rightarrow C(15,0) = 0.4C(15,0.1) + 0.3C(15.5,\ 0.1) + 0.3C(14.5,\ 0.1)$

Thus, we can guarantee that

$|C(15,\ 0)| \leq \max(|C(15,0.1)|,\ |C(15.5,\ 0.1)|,\ |C(14.5,\ 0.1)|)$

(c) (b) $\frac{\partial c}{\partial t} = \frac{C(15, 0.2) - C(15, 0)}{0.2}$

By $\frac{\partial c}{\partial t} + 0.75 \frac{\partial^2 C}{\partial x^2} = 0$, we get

$$\frac{C(15, 0.2) - C(15, 0)}{0.2}$$

$$+ 0.75 \times \frac{C(15.5, 0.1) - 2C(15, 0.1) + C(14.5, 0.1)}{(0.5)^2} = 0$$

$$\Rightarrow 5C(15, 0.2) - 5C(15, 0) + 3C(15.5, 0.1) - 6C(15, 0.1) + 3C(14.5, 0.1) = 0.$$

$$\Rightarrow C(15, 0) = C(15, 0.2) + 0.6C(15.5, 0.1) - 1.2C(15, 0.1) + 0.6C(14.5, 0.1)$$

We can not guarantee that.

I can give a counterexample.

When $C(15, 0.2) = 0.9$, $C(15.5, 0.1) = 1$, $C(15, 0.1) = 0.6$, $C(14.5, 0.1) = 0.4$

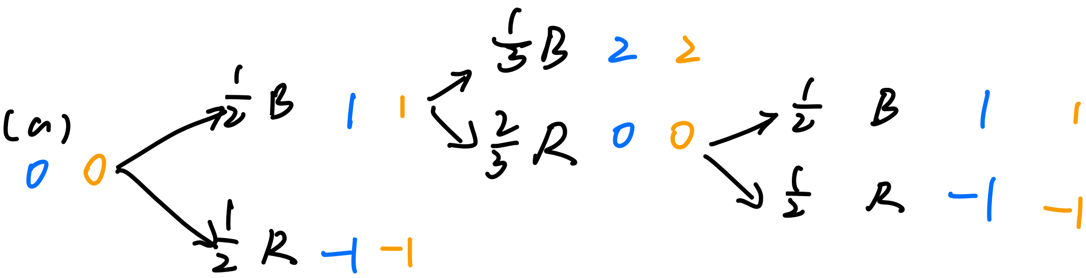
$C(15, 0) = 1.02$

$> \max(|C(15, 0.1)|, |C(15.5, 0.1)|, |C(14.5, 0.1)|, |C(15, 0.2)|)$

The stability property does not hold.

4.

(a)

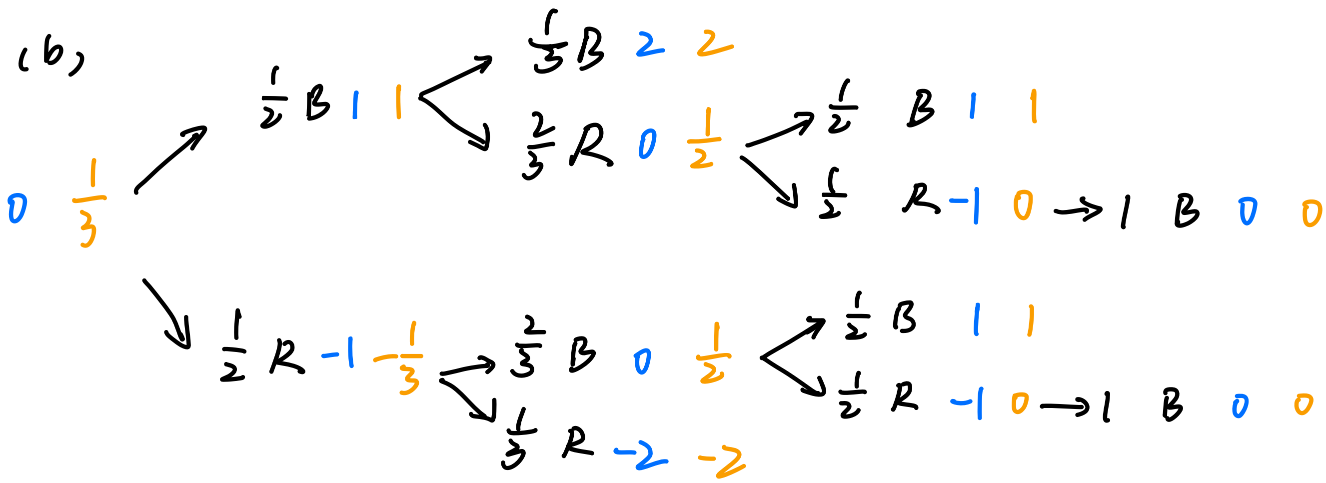


The payoffs in blue and optimized values in orange.

My expected optimized profit is 0 if I play.

Thus, we can either play or not, because the expected is the same, i.e. 0.

(b)



The payoffs in blue and optimized values in orange.

If I play, the expected optimized profit is $\frac{1}{3}$.

I should play.

5. (a)

$$\varphi_{a,b}(w) = \mathbb{E}\left[e^{iw(aR_1+bR_2)}\right]$$

$$= \mathbb{E}\left[e^{iawR_1+ibwR_2}\right]$$

$$= F(aw, bw)$$

(b) $\frac{X_T}{Y_T} > K \iff \log X_T - \log Y_T > \log K.$

$$\iff R_1 - R_2 > \log K.$$

$$\mathbb{E}\left[\mathbb{1}_{R_1-R_2>\log K}\right] = \mathbb{P}(R_1-R_2 > \log K)$$

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{\varphi_{R_1-R_2}(z)}{iz} e^{-iz\log K}\right] dz.$$

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{F(z,-z)}{iz} e^{-iz\log K}\right] dz.$$