Financial Mathematics 32000

Lecture 5

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2023 April 19

UNIT 3: Monte Carlo

Generating random variables

Variance reduction: Antithetic variates

Monte Carlo estimate

Let Y be a discounted payoff. Example: $Y = e^{-\int_0^T r_t \mathrm{d}t}(S_T - K)^+$.

Want to calculate the time-0 price $C = \mathbb{E}Y$

Generate $Y_1, Y_2, \ldots$ independently and identically distributed as Y.

(*How?*)

irrespective of independence

Then the random variable

$\mathbb{E}(Y_1 + \cdots + Y_M)$

$$\hat{C}_M := \frac{Y_1 + Y_2 + \cdots + Y_M}{M}$$

$= M\mathbb{E}Y_1$

is the Monte Carlo estimate of C. Note that

"unbiased" $$\mathbb{E}\hat{C}_M = C$$

By the strong law of large numbers, with probability 1 we have

$$\hat{C}_M \to C \text{ as } M \to \infty.$$

How fast does convergence occur?

because $Y_1, \ldots, Y_m$ are independent

Let $\sigma^2 := \mathrm{Var}(Y)$. (Here σ does not denote volatility.) Then

$$\mathrm{Var}(\hat{C}_M) = \frac{1}{M^2}\mathrm{Var}(Y_1 + Y_2 + \cdots + Y_M) = \frac{1}{M^2}(M\sigma^2) = \frac{\sigma^2}{M}$$

By the Central Limit Theorem, we have convergence in distribution

$$\frac{\hat{C}_M - \mathbb{E}\hat{C}_M}{\sqrt{\mathrm{Var}\ \hat{C}_M}} \xrightarrow{d} N(0,1) \qquad \text{hence} \qquad \frac{\hat{C}_M - C}{\sigma/\sqrt{M}} \xrightarrow{d} N(0,1)$$

as $M \to \infty$. Conclusion still holds using sample stdev in place of σ:

$$\frac{\hat{C}_M - C}{\hat{\sigma}_M/\sqrt{M}} \xrightarrow{d} N(0,1)$$

because $\hat{\sigma}_M^2 \to \sigma^2$ where

$$\hat{\sigma}_M^2 := \frac{1}{M-1}\sum_{m=1}^{M}(Y_m - \hat{C}_M)^2.$$

is the "sample variance" and $\hat{\sigma}_M$ is the "sample standard deviation."

Confidence intervals

So an asymptotic (for large M) confidence interval of $100(1-p)\%$ is

$$\left(\hat{C}_M - \mathcal{N}^{-1}\big(1-\frac{p}{2}\big)\frac{\hat{\sigma}_M}{\sqrt{M}},\ \hat{C}_M + \mathcal{N}^{-1}\big(1-\frac{p}{2}\big)\frac{\hat{\sigma}_M}{\sqrt{M}}\right)$$

where $\mathcal{N}$ is the standard Normal cdf.

- $\mathcal{N}^{-1}\big(1-\frac{p}{2}\big)$ tells us: a radius of *how many standard deviations* of a normal distribution contains $100(1-p)\%$ of the probability?
- The $\hat{\sigma}_M/\sqrt{M}$ is called the *standard error*.

  It gives us the estimated standard deviation of $\hat{C}_M$.
- Example: Let $p = 0.05$. Then $\mathcal{N}^{-1}\big(1-\frac{p}{2}\big) \approx 1.96$ and

  $$\left(\hat{C}_M - 1.96\frac{\hat{\sigma}_M}{\sqrt{M}},\ \hat{C}_M + 1.96\frac{\hat{\sigma}_M}{\sqrt{M}}\right)$$

  is an asymptotic 95% confidence interval for C.

Confidence intervals

- If $\hat{\sigma} = 20$ and we run $M = 10000$ simulations, then a 95% confidence interval has radius

$$1.96 \times \frac{20}{\sqrt{10000}} = 0.40$$

  To reduce this to 0.04, we need to take $M = 1$ million.

- So we will want to use *variance reduction* techniques, which reformulate the problem to keep σ small, or which carefully introduce dependence in the simulations to keep $\operatorname{Var} \hat{C}_M$ small.

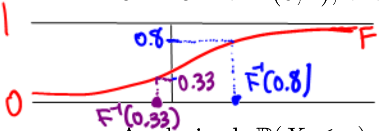
UNIT 3: Monte Carlo

Generating random variables

Variance reduction: Antithetic variates

The inverse CDF method

- Assume the existence of a pseudo-random number generator whose output can be treated as if it is IID uniform on $(0, 1)$. Python: `numpy.random.Generator` has method `random()`
- To generate a random variable X having a CDF F, generate $U \sim \text{Uniform}(0, 1)$, then apply F^{-1}, the inverse CDF, to produce



$$X := F^{-1}(U)$$

As desired, $\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x)$.
(If F not invertible, letting $F^{-1}(u) := \min\{x : F(x) \geq u\}$ works.)
Intuition: randomly choose the *percentile* between 0% and 100%, uniformly. The inverse CDF finds the corresponding value of X.

Generating normal random variables

- Python: `numpy.random.Generator` has method `normal()`

But what if you need to build your own?

- Could do $\mathcal{N}^{-1}(U)$, if an implementation of the inverse of the normal CDF $\mathcal{N}$ is available. Excel: `NORMSINV(RAND())`

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- Box-Muller method: If (X, Y) are independent Normal$(0, 1)$, then $R := X^2 + Y^2$ has CDF $\mathbb{P}(R \leq r) = 1 - e^{-r/2}$. Given R, the point (X, Y) is uniformly distributed on the circle of radius $\sqrt{R}$. So generate *pairs* of independent normals by drawing U_1 and U_2 IID from a Uniform$(0, 1)$ distribution, and taking

$$R := -2\log(U_1)$$

$$(X, Y) := (\sqrt{R}\cos(2\pi U_2), \sqrt{R}\sin(2\pi U_2))$$

Rainfall follows a bivariate standard normal distribution. 50% of the rain falls within radius 1 around center, how much falls within radius 2?

Python

```
[1]  import numpy
     rng = numpy.random.default_rng(seed=0)
```

```
[2]  rng.random(size=5)
```

```
array([0.63696169, 0.26978671, 0.04097352, 0.01652764, 0.81327024])
```

```
[3]  rng.random(size=5)
```

```
array([0.91275558, 0.60663578, 0.72949656, 0.54362499, 0.93507242])
```

```
[4]  rng.normal(size=5)
```

```
array([-0.62327446,  0.04132598, -2.32503077, -0.21879166, -1.24591095])
```

```
[5]  rng.normal(size=(2,5))
```

```
array([[-0.73226735, -0.54425898, -0.31630016,  0.41163054,  1.04251337],
       [-0.12853466,  1.36646347, -0.66519467,  0.35151007,  0.90347018]])
```

Simple dynamics

If Y is a known function of random variables with distributions that you can readily simulate, then it is easy to generate $Y_1, Y_2, \ldots$

Example: Consider a call paying $(S_T - K)^+$ where S is GBM.

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

Then

$$Y = e^{-rT}(S_T - K)^+ = e^{-rT}(S_0 e^{(r-\sigma^2/2)T+\sigma W_T} - K)^+$$

where $W_T \sim N(0, T)$. So let

$$Y_m := e^{-rT}(S_T - K)^+ = e^{-rT}(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(m)}} - K)^+$$

where the $Z^{(m)}$ are IID standard normal: $N(0, 1)$.

But sometimes we need to simulate entire path

- But what about more complicated dynamics, such as

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(S_t, t)S_t\mathrm{d}W_t$$

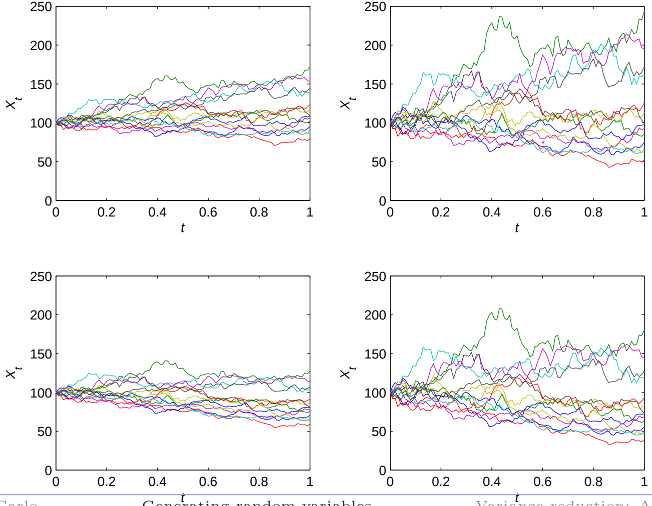
- Or what if σ or r follows a process driven by a second BM.
- What about more complicated contracts, such as an Asian option or a barrier option.

We may need to simulate the whole path.

$d\log X_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$

Geometric Brownian motion: $\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$

Let $X_0 = 100$. Trajectories for $\mu = -0.15, +0.15$ and $\sigma = 0.20, 0.40$:



Simulating the path of a state variable: Euler method

Suppose X satisfies

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$

Divide the time interval $[0, T]$ into N parts: $\Delta t = T/N$, $t_n = n\Delta t$.

Define the mth simulated path by initializing $X_0^{(m)} = X_0$,

and given X_{t_n}, obtain $X_{t_{n+1}}$ by

$$X_{t_{n+1}}^{(m)} = X_{t_n}^{(m)} + a(X_{t_n}^{(m)}, t_n)\Delta t + b(X_{t_n}^{(m)}, t_n)Z_n^{(m)}\sqrt{\Delta t}$$

where $Z_n^{(m)}$ are IID standard normal. Evaluate the discounted payoff for the mth path. Take the average across all paths $m = 1, \ldots, M$.

This extends directly to multidimensional state vectors X and multidimensional standard Brownian motion W.

Euler method convergence

If S is GBM, and if you need to generate path, apply Euler to log S, not S. When S needed, exponentiate the log S.

With some assumptions on a and b, the Euler method has *weak* order of convergence 1 for general f, meaning

$$|\mathbb{E}f(X_T) - \mathbb{E}f(X_T^{(m)})| = O(\Delta t)$$

(Estimating $\mathbb{E}f(X_T^{(m)})$ produces additional error, not included here.)

Error analysis: Let $a_t = a(X_t)$ and $b_t = b(X_t)$. First subinterval:

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$$\begin{aligned} X_{t_1} &= X_0 + \int_0^{t_1} a_t \mathrm{d}t + \int_0^{t_1} b_t \mathrm{d}W_t \\ &= X_0 + \int_0^{t_1} \left(a_0 + \int_0^t a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\mathrm{d}s + \int_0^t b\frac{\partial a}{\partial x}\mathrm{d}W_s \right)\mathrm{d}t \\ &\quad + \int_0^{t_1} \left(b_0 + \int_0^t a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\mathrm{d}s + \int_0^t b\frac{\partial b}{\partial x}\mathrm{d}W_s \right)\mathrm{d}W_t \end{aligned}$$

The Euler scheme keeps three terms to generate $X_{t_1}^{(m)}$

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Crude analysis of discretization error of Euler scheme

- Intuition for weak error: $|\mathbb{E}(X_T - X_T^{(m)})|$ is $O(\Delta t)$, because there are N time steps, and $\mathbb{E}$ of error is $O(\Delta t)^2$ at each step (the only nonzero-$\mathbb{E}$ error term in previous equation is the $\mathrm{d}s\mathrm{d}t$ term).
- Euler scheme has strong order of convergence 1/2, meaning that

$$\mathbb{E}|X_T - X_T^{(m)}| = O(\Delta t)^{1/2} \qquad \text{as } \Delta t \to 0$$

Intuition for strong error: Biggest ignored term is $\mathrm{d}W_s \mathrm{d}W_t$.
Variance of error at one time step $= O(\mathrm{Var}(\Delta W)^2) = O(\Delta t)^2$.
Variance of total error $X_T - X_T^{(m)}$ after N time steps $= O(\Delta t)$.
Standard deviation of $X_T - X_T^{(m)}$ after N time steps $= O(\Delta t)^{1/2}$.
This suggests that strong order of convergence is 1/2.

Milstein scheme

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Milstein scheme: Don't ignore the term

$$\int_0^{t_1} \int_0^t b(X_s) \frac{\partial b}{\partial x}(X_s) \mathrm{d}W_s \mathrm{d}W_t.$$

Approximate it as

$$\begin{aligned} b(X_0) \frac{\partial b}{\partial x}(X_0) \int_0^{t_1} \int_0^t \mathrm{d}W_s \mathrm{d}W_t &= b(X_0) \frac{\partial b}{\partial x}(X_0) \int_0^{t_1} W_t \mathrm{d}W_t \\ &= \frac{1}{2} b(X_0) \frac{\partial b}{\partial x}(X_0)(W_{t_1}^2 - t_1) \end{aligned}$$

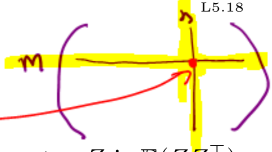
So Milstein is

$$X_{t_{n+1}}^{(m)} = X_{t_n}^{(m)} + a(X_{t_n}^{(m)})\Delta t + b(X_{t_n}^{(m)}) Z_n^{(m)} \sqrt{\Delta t} + \frac{1}{2} b \frac{\partial b}{\partial x}(X_{t_n}^{(m)})([Z_n^{(m)}]^2 - 1)\Delta t$$

Milstein has strong order of convergence 1, and weak order 1.

Weak convergence (important in option pricing): same order as Euler.

Covariance and correlation matrices



$\mathrm{Cov}(Z^{[m]}, Z^{[n]})$

Recall that the covariance matrix of a zero-mean vector Z is $\mathbb{E}(ZZ^\top)$.

Let M be a real symmetric matrix. The following are equivalent:

- M is a covariance matrix of some vector.
- M is *positive semi-definite*, which means that $x^\top Mx \geq 0$ for all real vectors x.
- The eigenvalues of M are all nonnegative.
- The principal minors of M are all nonnegative. (Principal minors = the determinants of the matrices formed by crossing out any rows and corresponding columns of M).
- M has a *Cholesky decomposition* $LL^\top = M$ for some real lower-triangular matrix L with diagonal entries ≥ 0.

Covariance and correlation matrices

Moreover, the following are equivalent:

- M is a correlation matrix
- M is a covariance matrix and its diagonal elements are all 1.

Moreover, if M is a 3×3 matrix, the following are equivalent

- M is a correlation matrix.
- M is symmetric, its entries $\in [-1, 1]$, its diagonal entries $= 1$, and $\det M \geq 0$.

Interview question

Suppose $\operatorname{Corr}(X,Y) = \operatorname{Corr}(X,Z) = \operatorname{Corr}(Y,Z) = \rho$.

What are the possible values of ρ?

$$0 \le \det\begin{bmatrix}1&\rho&\rho\\ \rho&1&\rho\\ \rho&\rho&1\end{bmatrix} = 1\begin{vmatrix}1&\rho\\ \rho&1\end{vmatrix} - \rho\begin{vmatrix}\rho&\rho\\ \rho&1\end{vmatrix} + \rho\begin{vmatrix}\rho&1\\ \rho&\rho\end{vmatrix}$$

$$= 1(1-\rho^2) - \rho(\rho-\rho^2) + \rho(\rho^2-\rho)$$

$$= 2\rho^3 - 3\rho^2 + 1$$

$$= (\rho-1)(2\rho^2-\rho-1)$$

$$= (\rho-1)(2\rho+1)(\rho-1)$$

$$= (\rho-1)^2(2\rho+1)$$

$$\begin{array}{r|rrrr} 1 & 2 & -3 & 0 & 1 \\ & & 2 & -1 & -1 \\ \hline & 2 & -1 & -1 & 0 \end{array}$$

Answer: $-\frac{1}{2} \le \rho \le 1$

Generating correlated Brownian motions

To get a D-dimensional vector $\underset{D\times 1}{\bar{W}}$ of BM with correlation matrix $\underset{D\times D}{H}$, find a matrix $L \in \mathbb{R}^{D\times D}$ such that $\underset{D\times D}{L}\underset{D\times D}{L^\top} = \underset{D\times D}{H}$, and let $\underset{D\times 1}{\bar{W}} = \underset{D\times D}{L}\underset{D\times 1}{W}$, where W is standard BM in $\mathbb{R}^D$.

So the process $\mathrm{d}X_t = a(X_t,t)\mathrm{d}t + b(X_t,t)\mathrm{d}\bar{W}_t$ becomes

$$\underset{J\times 1}{\mathrm{d}X_t} = \underset{J\times 1}{a(X_t,t)}\mathrm{d}t + \underset{J\times D}{b(X_t,t)}\underset{D\times D}{L}\ \underset{D\times 1}{\mathrm{d}W_t}$$

Simulation by Euler method:

$$\underset{J\times 1}{X^{(m)}_{t_{n+1}}} = \underset{J\times 1}{X^{(m)}_{t_n}} + \underset{J\times 1}{a(X^{(m)}_{t_n},t_n)}\Delta t + \underset{J\times D}{b(X^{(m)}_{t_n},t_n)}\underset{D\times D}{L}\ \underset{D\times 1}{Z^{(m)}_n}\sqrt{\Delta t}$$

where $Z^{(m)}_n$ are IID standard normal in $\mathbb{R}^D$. How to find L?

- `numpy.random.Generator.multivariate_normal` generates LZ
- `numpy.linalg.cholesky` returns L given H

Simulating correlated Brownian motions

Example: if we want $\bar{W} = \binom{W^{[1]}}{W^{[2]}}$ with $\mathrm{corr}(\Delta W^{[1]}, \Delta W^{[2]}) = \rho$, then

$$H = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

and Cholesky finds $LL^\top = H$ where

$$L = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \Delta B^{[1]} \\ \Delta B^{[2]} \end{pmatrix} = \begin{pmatrix} \Delta B^{[1]} \\ \rho \Delta B^{[1]} + \sqrt{1-\rho^2}\, \Delta B^{[2]} \end{pmatrix}$$

If Cholesky routine unavailable, can solve for L by traversing [the upper or lower triangular part of] H entry-by-entry. Each entry gives rise to an equation involving elements of L and only one unknown.

Generating correlated Brownian motions

$J \times D$

Sometimes it is not necessary to simulate the entire path.

- Suppose $X_T - X_0$ is known to be multivariate normal with mean μT and covariance matrix HT.
- Suppose the option payoff depends only on X_T.

Then no need to divide $[0, T]$ into N steps. No need for Euler.

- Just generate

$$X_T^{(m)} := X_0 + \mu T + L\sqrt{T} Z^{(m)}$$

$D \times 1$ $D \times 1$ $D \times 1$ $D \times D$ $D \times 1$

where $LL^\top = H$ and the $Z^{(m)}$ are IID standard normal in $\mathbb{R}^D$.

UNIT 3: Monte Carlo

Generating random variables

Variance reduction: Antithetic variates

Variance reduction

Recall the radius of a $100(1-p)\%$ confidence interval is

$$\mathcal{N}^{-1}\left(1-\frac{p}{2}\right)\sqrt{\mathrm{Var}(\hat{C})}$$

Variance reduction techniques try to construct alternative estimators with smaller variance. We will examine four:

- Antithetic variates: $\hat{C}_M^{\mathrm{av}}$
- Control variates: $\hat{C}_M^{\mathrm{cv}}$
- Importance sampling: $\hat{C}_M^{\mathrm{is}}$
- Conditional Monte Carlo: $\hat{C}_M^{\mathrm{cmc}}$

Antithetic variates

Let Y be discounted payoff. So

$$C = \mathbb{E}Y.$$

Ordinary MC: $\hat{C} = \frac{1}{M}(Y_1 + \cdots + Y_M)$, where each Y_m is IID as Y.

Example: Call on stock under GBM:

$$Y_m = e^{-rT}(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z_m} - K)^+$$

where we generate the Z_m to be IID $N(0,1)$ for $m = 1, \ldots, M$.

Antithetic Variates: If you draw a realization from the qth percentile, then you should *also draw one from the* $(100-q)$*th percentile.*

Symmetry of normal $\Rightarrow$ for each realization of Z, rerun also with $-Z$.

Antithetic variates

In the same example, for each $m = 1, \ldots, M$, let

$$\begin{aligned}
\tilde{Z}_m &:= -Z_m \\
\tilde{Y}_m &:= e^{-rT}\left(S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}\tilde{Z}_m} - K\right)^+ \\
Y_m^{\text{av}} &:= \frac{Y_m + \tilde{Y}_m}{2} \\
\hat{C}_M^{\text{av}} &:= \frac{1}{M}\left(Y_1^{\text{av}} + Y_2^{\text{av}} + \cdots + Y_M^{\text{av}}\right)
\end{aligned}$$

This is the antithetic-variate Monte Carlo estimate.

Its expectation is

$$\mathbb{E}\hat{C}_M^{\text{av}} = \frac{1}{M}\sum_m \mathbb{E}Y_m^{\text{av}} = \mathbb{E}\left(\frac{Y + \tilde{Y}}{2}\right) = \frac{\mathbb{E}Y + \mathbb{E}\tilde{Y}}{2} = \mathbb{E}Y = C$$

so $\hat{C}_M^{\text{av}}$ is an unbiased estimate of C.

Antithetic variates: variance analysis

Variance of AV estimate is

$$\begin{aligned}\mathrm{Var}(\hat{C}_M^{\mathrm{av}}) &= \mathrm{Var}\left(\frac{1}{M}\sum_m Y_m^{\mathrm{av}}\right) = \frac{1}{M^2} M \mathrm{Var}(Y^{\mathrm{av}}) \\ &= \frac{1}{M}\mathrm{Var}\left(\frac{Y+\tilde{Y}}{2}\right) = \frac{1}{M}\frac{\mathrm{Var}Y + 2\mathrm{Cov}(Y,\tilde{Y}) + \mathrm{Var}(\tilde{Y})}{4} \\ &= \frac{1}{M}\left(\frac{1}{2}\mathrm{Var}Y + \frac{1}{2}\mathrm{Cov}(Y,\tilde{Y})\right)\end{aligned}$$

Compare to ordinary MC:

$$\mathrm{Var}(\hat{C}) = \frac{1}{M}\mathrm{Var}(Y).$$

Note that $\mathrm{Cov}(Y,\tilde{Y}) \le \sqrt{\mathrm{Var}(Y)\mathrm{Var}(\tilde{Y})} = \mathrm{Var}(Y)$

hence $\mathrm{Var}(\hat{C}_M^{\mathrm{av}}) \le \mathrm{Var}(\hat{C})$.

But maybe this overstates the benefit of AV.

Antithetic variates: variance analysis

Maybe better to compare

$$\mathrm{Var}(\hat{C}_M^{\mathrm{av}}) = \frac{1}{M}\left(\frac{1}{2}\mathrm{Var}Y + \frac{1}{2}\mathrm{Cov}(Y,\tilde{Y})\right)$$

against

$$\mathrm{Var}(\hat{C}_{2M}) = \frac{1}{2M}\mathrm{Var}(Y).$$

So $\mathrm{Var}(\hat{C}_M^{\mathrm{av}})$ is smaller iff $\mathrm{Cov}(Y,\tilde{Y}) < 0$.

This often holds, but for some payoffs, it doesn't.

Antithetic variates: variance analysis

Because

$$\mathrm{Var}(\hat{C}_M^{\mathrm{av}}) = \frac{1}{M}\mathrm{Var}(Y^{\mathrm{av}}),$$

an asymptotic $100(1-p)\%$ confidence interval has endpoints

$$\hat{C}_M^{\mathrm{av}} \pm \mathcal{N}^{-1}\left(1 - \frac{p}{2}\right)\frac{\hat{\sigma}_M^{\mathrm{av}}}{\sqrt{M}}$$

where

$$\hat{\sigma}_M^{\mathrm{av}} := \sqrt{\frac{1}{M-1}\sum_m (Y_m^{\mathrm{av}} - \hat{C}_M^{\mathrm{av}})^2}.$$

Note that $\hat{\sigma}_M^{\mathrm{av}}$ is the sample standard deviation of the *pair averages*, not of the individual realizations.

Dividing it by $\sqrt{M}$ gives the standard error $\hat{\sigma}_M^{\mathrm{av}}/\sqrt{M}$, an estimate of the standard deviation of $\hat{C}_M^{\mathrm{av}} = \frac{1}{M}\sum_m Y_m^{\mathrm{av}}$.