

Financial Mathematics 32000

Lecture 2

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Pricing: problems and solutions

A description of a pricing problem and solution consists of:

- ▶ a **contract** (what are you pricing)
- ▶ a **model** of the **dynamics** (what are you assuming about the underlying risks)
- ▶ a **pricer** / solution **method** (how are you pricing it)

For example:

- ▶ Last Fall: contract = European, model = GBM, pricer = analytic
- ▶ Last week: contract \in {European, American, barrier}
model = GBM or binomial tree; pricer = binomial/trinomial tree

(**contract=lookback**, dynamics=GBM, pricer=tree):

(contract=general, **dynamics=local volatility**, pricer=tree)

(contract=European call/put, dynamics=GBM, **pricer=Taylor**)

Another path-dependent option

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A fixed-strike lookback with strike K , start date 0, and expiry T pays

$$\text{Call:} \quad \left(\max_{t \in [0, T] \cap \mathcal{T}} S_t - K \right)^+$$

$$\text{Put:} \quad \left(K - \min_{t \in [0, T] \cap \mathcal{T}} S_t \right)^+$$

where \mathcal{T} is some set of monitoring times.

Pricing a fixed-strike lookback call in a tree

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- Recall: the knockout option, also path-dependent, has price

$$C(t, S_t, \mathbf{1}_{\text{knockout prior to time } t})$$

Here the path-dependence is simple. The path-dependent state variable has only two states: 0/1 (live/dead).

And the “1” state does not need to be tracked in the tree.

- Lookback is more complicated than the barrier. At each node (t, S_t) in the tree, need to store a table of values associating each possible M_t with $C(t, S_t, M_t)$.

Example

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assumes r=0		running max		option price: fixed strike lookback call								
mu	-0.042	<div></div>		<div></div>		pu	0.15	K		100		
sigma	0.29					pm	0.66					
deltaX	0.25					pd	0.19					
deltaT	0.25											
		n=1	n=2		n=3		n=4		n=5			
		time 0	time 0.25		time 0.5		time 0.75		time 1.00			
j=4	271.83									<div>271.83 171.83</div>		
j=3	211.70						<div>211.70 120.60</div>			<div>211.70 111.70</div>		
j=2	164.87					<div>164.87 77.71</div>	<div>164.87 71.81</div>			<div>211.70 111.70 164.87 64.87</div>		
j=1	128.40			<div>128.40 42.47</div>	<div>128.40 38.40</div>	<div>164.87 64.87 128.40 33.80</div>	<div>164.87 64.87 128.40 28.40</div>			<div>164.87 64.87 128.40 28.40</div>		
j=0	100	<div>100.00 13.84</div>	<div>100.00 10.96</div>	<div>128.40 29.20 100.00 7.79</div>	<div>128.40 28.40 100.00 4.21</div>		<div>164.87 64.87 128.40 28.40 100.00 0.00</div>					
j=-1	77.88			<div>100.00 1.57</div>	<div>100.00 0.62</div>	<div>128.40 28.40 100.00 0.00</div>			<div>128.40 28.40 100.00 0.00</div>			
j=-2	60.65								<div>128.40 28.40 100.00 0.00</div>			
j=-3	47.24					<div>100.00 0.00</div>			<div>100.00 0.00</div>			
j=-4	36.79									<div>100.00 0.00</div>		

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Local volatility models

We want a model that is consistent with the observed non-constant implied vol skew. One approach: *local volatility* models specify the instantaneous volatility σ to be a function of (S_t, t) .

- ▶ In the continuous-time setting,

$$dS_t = rS_t dt + \sigma(S_t, t)S_t dW_t$$

where W is \mathbb{P} -BM.

- ▶ In the tree setting, we let σ depend on the node (S_t, t) .

Note that the local volatility σ is not the same thing as σ_{imp} .

Analytic computation of option prices is difficult in the diffusion setting, even for European calls/puts (exception: $\sigma(S_t, t) = \sigma(t)$). But by approximating the diffusion in a tree, the computations are easy.

Option pricing in the tree setting

Suppose we are given the local volatility function σ .

- ▶ Let $\Delta t = T/N$ (unless this fails to place important dates in the tree).
- ▶ To choose Δx , let σ_{avg} be some “representative” or “average” σ in the tree, and let σ_{max} be an upper bound on the σ in the tree. Two guidelines: to make local discretization error small, we want

$$\Delta x \approx \sigma_{avg} \sqrt{3\Delta t}$$

but for stability reasons, we want

$$\Delta x \geq \sigma_{max} \sqrt{\Delta t}$$

So we can let $\Delta x = \max(\sigma_{avg} \sqrt{3\Delta t}, \sigma_{max} \sqrt{\Delta t})$

Option pricing in the tree setting

- ▶ Then at each node, use the $\sigma(S_t, t)$ prevailing at that particular node to generate the ν and the probabilities for the branches out of that node. Same formulas as L1:

$$p_{u,d} = \frac{1}{2} \left[\frac{\sigma^2 \Delta t + \nu^2 (\Delta t)^2}{(\Delta x)^2} \pm \frac{\nu \Delta t}{\Delta x} \right], \quad p_m = 1 - \frac{\sigma^2 \Delta t + \nu^2 (\Delta t)^2}{(\Delta x)^2}$$

- ▶ Price options – including path-dependent and American-style options – as we did for GBM.

The only change is that σ (hence ν , p_u , p_m , p_d) vary across nodes.

What if σ is not given

Then *calibrate* it to the prices of listed options.

The general idea of calibration:

- ▶ Observe prices of liquidly traded assets (e.g. listed options)
- ▶ Choose the model's parameters (e.g. the σ function) in such a way that the model generates theoretical prices that match closely the observed prices.

Then one can apply that model, with the calibrated parameters, to

- ▶ Compute hedges and risk sensitivities
- ▶ Price illiquid options
- ▶ Price and hedge complex deals (e.g. exotic options, structured products)

Calibration of local volatility $\sigma(t)$

If σ is a non-random function of t and

$$dS_t = rS_t dt + \sigma(t)S_t dW_t$$

then $d \log S_t = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW_t$ so

$$\log S_T = \log S_0 + \left(r - \frac{\bar{\sigma}_T^2}{2}\right)T + \int_0^T \sigma(t) dW_t$$

$$\sim \text{Normal}\left(\log S_0 + \left(r - \frac{\bar{\sigma}_T^2}{2}\right)T, \bar{\sigma}_T^2 T\right) \text{ where } \bar{\sigma}_T := \sqrt{\frac{1}{T} \int_0^T \sigma^2(t) dt}$$

$\sigma_{imp} =$

(intuition: a nonrandomly weighted sum of indep normals is normal)

So time-0 call prices $C(K, T) = C^{BS}(\bar{\sigma}_T)$, thus $\sigma_{imp}(K, T) = \bar{\sigma}_T$.

Calibration: If given $C(K, T)$ at various expiries T , then obtain σ_{imp} and use boxed equation to (not uniquely) find $\sigma(t)$.

Calibration of local volatility $\sigma(S, t)$ in a trinomial tree

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It's harder when σ depends on (S, t) .

Here's the rough idea. Assume $r = 0$ and a tree with S -levels s_j with equal spacing $\Delta S = s_{j+1} - s_j$. Let $\Delta K := \Delta S$.

Given: $C_0(K, \tau)$, the time-0 price of a strike- K expiry- τ call, for all (K, τ) . Find at each node (s_j, t_n) : Local volatility σ which generates risk-neutral probabilities consistent with call prices (and stock prices).

- The probability of reaching node (s_j, t_n) equals $(\Delta K)\text{Fly}_0(s_j, t_n)$ where $\text{Fly}_0(K, \tau)$ is the time-0 price of a *butterfly*:

$$\text{Fly}_0(K, \tau) := \frac{C_0(K - \Delta K, \tau) - 2C_0(K, \tau) + C_0(K + \Delta K, \tau)}{(\Delta K)^2}$$

Calibration of local volatility in a trinomial tree

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- The probability of reaching node (s_j, t_n) and then going up equals $(\Delta t / \Delta K) \text{Cal}_0(s_j, t_n)$, where $\text{Cal}_0(K, \tau)$ is the time-0 price of a *calendar spread*:

$$\text{Cal}_0(K, \tau) := \frac{C_0(K, \tau + \Delta t) - C_0(K, \tau)}{\Delta t}$$

Hence the conditional up-probability from node (s_j, t_n) is the ratio

$$p_u = \frac{\mathbb{P}(\text{reach } (s_j, t_n) \text{ then up})}{\mathbb{P}(\text{reach } (s_j, t_n))} = \frac{(\Delta t / \Delta K) \text{Cal}_0}{(\Delta K) \text{Fly}_0}$$

Calibration of local volatility in a trinomial tree

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Comments:

- ▶ We have derived a discrete version of the “Dupire equation.”
We’ll return to this formula in the PDE context.
- ▶ If you try to implement this formula directly, the calibrated σ will be sensitive to noise in the option price observations.
In practice some regularization/smoothing procedure is advisable.
- ▶ Calibration is much easier in the case where σ depends only on t instead of jointly on (S, t) .

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Taylor approximation

Postpone error analysis until we do finite difference methods.

A key tool in that error analysis will be Taylor approximation:

If f has $n + 1$ continuous derivatives in a neighborhood of x_0 then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \\ + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + O(x - x_0)^{n+1}$$

as $x \rightarrow x_0$.

We may use Taylor approximation for two different purposes:

- ▶ To approximate prices and sensitivities – example on next page.
- ▶ To analyze the error in tree or finite difference calculation of prices and sensitivities

Interview question

No calculators allowed. Your interviewer says to you:

Spot is 100. No dividends. What's the price of a European-style 1-year at-the-money-forward (ATMF) vanilla option with 20% implied volatility?

(ATMF at time 0 means $K = F_0$. Recall $F_0 = S_0 e^{rT}$ if no divs.)

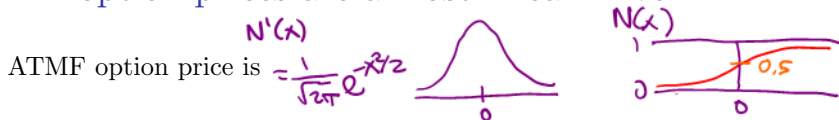
Answer: Black-Scholes call price is

$$S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_{1,2} := d_{+,-} := \frac{\log(S_0 e^{rT} / K)}{\sigma \sqrt{T}} \pm \frac{\sigma \sqrt{T}}{2}$$

ATMF option prices are almost linear in vol



$$S_0 N(d_1) - K e^{-rT} N(d_2) = S_0 (N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2))$$

For small $|x|$,

$$N(x) = N(0) + N'(0)x + \frac{1}{2}N''(0)x^2 + O(x^3) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x + 0 + O(x^3).$$

$$x_0=0$$

So option price is approximately

$$S_0 \left(\frac{1}{2} + \frac{\sigma\sqrt{T}/2}{\sqrt{2\pi}} - \frac{1}{2} + \frac{\sigma\sqrt{T}/2}{\sqrt{2\pi}} \right) = \frac{S_0\sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.4 \times S_0\sigma\sqrt{T}.$$

Your answer: 8 dollars

(True answer: 7.97 dollars)

Follow-up question

Same assumptions. What's the delta of the ATMF call?

Differentiating the approximate option price $S_0\sigma\sqrt{T}/\sqrt{2\pi}$ with respect to S_0 , we have a delta of

$$r=0 \quad \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.08$$

Or do we? I thought the delta of an ATMF call should be close to 0.5.

$$C(S, K)$$

$$\frac{\partial C(S, S)}{\partial S} \Big|_{S=S_0} \approx 0.08 \quad \text{but this isn't ATM delta}$$

$$\frac{\partial C(S, K)}{\partial S} \Big|_{S=K=S_0} \approx 0.5$$