COMP6212 Coursework

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1 Derivatives Pricing

1.1 Black-Scholes model

Black-Scholes model is a pricing model and used to determine the price of call and put option on the expiration date. In the Black-Scholes model, K is the strike price, S is the value of the underlying asset, r is the risk-free interest rate, T is the time of maturity, t is the current time and $N(t)$ is the cumulative normal distribution.

1.1.1 Derivative

As $N(x)$ is the cumulative normal distribution, $N(x)$ in the Black-Scholes model could be denoted as equation 1.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)dy \tag{1}$$

As shown, $N(x)$ is from standard normal distribution with $\mu = 0, \sigma = 1$. Therefore, the derivative of $N(x)$ is the probability density distribution of standard normal distribution.

$$N'(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \tag{2}$$

1.1.2 Proof

As the derivative shown in equation 2,

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} \exp(-d_1^2/2) \tag{3}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} \exp(-d_2^2/2) \tag{4}$$

$N(d_1)$ and $N(d_2)$ could be denoted as above, in which,

$$d_1 = \frac{log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}} \tag{5}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)} \tag{6}$$

Therefore, $N'(d_2) = \frac{1}{\sqrt{2\pi}} \exp(-d_2^2/2) = \frac{1}{\sqrt{2\pi}} \exp(-(d_1 - \sigma\sqrt{(T-t)})^2/2)$
$= \frac{1}{\sqrt{2\pi}} \exp(-\frac{d_1^2}{2} + \sigma d_1\sqrt{(T-t)} - \frac{\sigma^2(T-t)}{2})$

Set up equation 7,

$$Equation = \log(SN'(d_1)) - \log(K\exp(-r(T-t))N'(d_2)) \tag{7}$$

Then, $Equation = \log(\frac{S}{\sqrt{2\pi}}\exp(-d_1^2/2)) - \log(\frac{K}{\sqrt{2\pi}}\exp(-r(T-t) - \frac{d_1^2}{2} + \sigma d_1\sqrt{(T-t)}) - \frac{\sigma^2(T-t)}{2}))$.

Next, substitute d_1 from $Equation$ with equation 5.

$$Equation = \log\frac{S}{K} + (r(T-t) - \sigma d_1\sqrt{(T-t)}) + \frac{\sigma^2(T-t)}{2})$$
$$= \log\frac{S}{K} + (r(T-t) - \sigma(\frac{log(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}})\sqrt{(T-t)}) + \frac{\sigma^2(T-t)}{2}) = 0$$

$Equation$ equals to 0, which means $\log(SN'(d_1)) = \log(K\exp(-r(T-t))N'(d_2))$.

Therefore, $SN'(d_1) = K\exp(-r(T-t))N'(d_2)$.

1.1.3 Derivatives

Since equation 5 and 6,

$\frac{\partial d_1}{\partial S} = \frac{1/S}{\sigma\sqrt{(T-t)}} = \frac{1}{\sigma S\sqrt{(T-t)}}$,

$\frac{\partial d_2}{\partial S} = \frac{\partial(d_1 - \sigma\sqrt{(T-t)})}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1/S}{\sigma\sqrt{(T-t)}} = \frac{1}{\sigma S\sqrt{(T-t)}}$.

1.1.4 2 proofs

$Proof 1$

As referred in the question, the solution for the call option price is as below.

$$c = SN(d_1) - K\exp(-r(T-t))N(d_2) \tag{8}$$

Then the derivative of equation 8 with respect to time could be represented as equation 9.

$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - K(r\exp(-r(T-t))N(d_2) + \exp(-r(T-t))N'(d_2)\frac{\partial d_2}{\partial t}) \tag{9}$$

As $SN'(d_1)$ is equal to $K\exp(-r(T-t))N'(d_2)$,

$$\begin{aligned}\frac{\partial c}{\partial t} &= SN'(d_1)\frac{\partial d_1}{\partial t} - Kr\exp(-r(T-t))N(d_2) - SN'(d_1)\frac{\partial d_2}{\partial t}\\ &= SN'(d_1)(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}) - Kr\exp(-r(T-t))N(d_2)\\ &= -Kr\exp(-r(T-t))N(d_2) + SN'(d_1)(\frac{\partial d_1}{\partial t} - (\frac{\partial d_1}{\partial t} + \sigma\frac{1}{2\sqrt{(T-t)}}))\\ &= -rK\exp(-r(T-t))N(d_2) - SN'(d_1)\sigma\frac{1}{2\sqrt{(T-t)}}\end{aligned}$$

$Proof 2$

From equation 8, the derivative of c with respect to S is as below.

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - K\exp(-r(T-t))N'(d_2)\frac{\partial d_2}{\partial S} \tag{10}$$

As $SN'(d_1)$ is equal to $K\exp(-r(T-t))N'(d_2)$,

$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - SN'(d_1)\frac{\partial d_2}{\partial S} = N(d_1) + SN'(d_1)(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S})$.

As proved in 1.1.3, $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$.

Therefore, $\frac{\partial c}{\partial S} = N(d_1)$.

1.1.5 Solution of differential equation

Black-Scholes differential equation is given by equation 11

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc = 0 \tag{11}$$

If the expression for the call option price is the solution of Black-Scholes differential equation, then equation 8 should meet the equation 11. Substitute the left side of equation 11 with the derivatives obtained from 1.1.4 and $\frac{\partial^2 c}{\partial S^2} = N'(d_1)\frac{1}{S\sigma\sqrt{(T-t)}}$.

$$\begin{aligned}
Left\ side &= -rK\exp(-r(T-t))N(d_2) - SN'(d_1)\sigma\frac{1}{2\sqrt{(T-t)}} + rSN(d_1) + \tfrac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{(T-t)}} - rc \\
&= -rK\exp(-r(T-t))N(d_2) + rSN(d_1) - rc - SN'(d_1)\sigma\frac{1}{2\sqrt{(T-t)}} + \sigma SN'(d_1)\frac{1}{2\sqrt{(T-t)}} \\
&= r(-K\exp(-r(T-t))N(d_2) + SN(d_1) - c)
\end{aligned}$$

As in the call option price equation, $-K\exp(-r(T-t))N(d_2) + SN(d_1) = c$.

Therefore, $Left\ side = 0 = Right\ side$. The expression for the call option price is the solution of Black-Scholes differential equation.

1.2 Evaluation of the Model

In this section, we are going to evaluate the Black-Scholes model by comparing the price predicted using this model with the true price from the given data set.

For each call option, the price using Black-Scholes model is given by equation 8 with 5 and 6. Whereas, the prediction of a put option is given by equation 12.

$$p = -SN(-d_1) + K\exp(-r(T-t))N(-d_2) \tag{12}$$

As proved in [1], the value of a stock option, depends on σ but not on μ at all. The volatility, σ, also named as standard deviation, is a measure of uncertainty about the returns of the given stocks[1].

To estimate volatility from historical data, chapter 14.4 in [1] gives us an idea.

Define:
$n+1$: Number of observations
S_i: Stock price at end of ith interval, with $i = 0, 1, ..., n$
τ: Length of time interval in years
$\mu_i = ln(\frac{S_i}{S_{i-1}})$, $i = 1, 2, ..., n$
$\bar{\mu}$: The mean of μ_i
s: The standard deviation of the μ_i

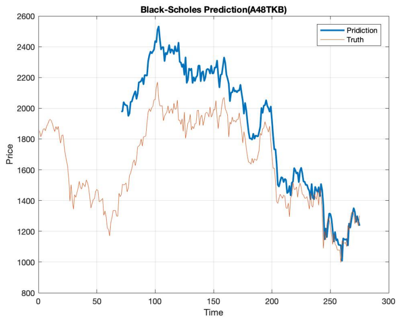
The algorithm is explained as below.

Firstly, denoting $\bar{\mu}$ using μ_i, $\bar{\mu} = \frac{1}{n}\sum_{i=1}^{n}\mu_i$
Secondly, computing standard deviation, $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\mu_i - \bar{\mu})^2} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\mu_i^2 - \frac{1}{n(n-1)}(\sum_{i=1}^{n}\mu_i)^2}$
As the standard deviation of μ_i is $\sigma\sqrt{\tau}$, s is an estimate of $\sigma\sqrt{\tau}$, $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$.

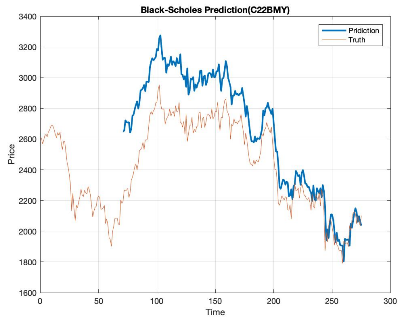
In this question, we want to evaluate the price from $T/4+1$ to T and the volatility is estimated using a slide window from $t - T/4$ to t. Randomly choose three stocks from call option data sets and three stocks from put option data sets. Figure 1 shows the predicted prices of three random stocks in call option with the true prices. And figure 3 shows the predicted and true value of stocks in put option.

To see whether systematic differences exists, I plot the difference between prediction and truth. Figure 2 shows obvious tendency in three stocks. With time goes by, the difference between prediction and truth is close to zero. Therefore, systematic difference exists at the beginning of prediction and would be vanishing with sufficient period of time. The circumstance in the put option figure 4 is more complex.

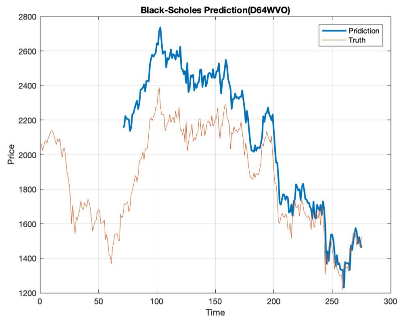
There are much more fluctuation. However, we can see the similar tendency from three plots, up, down and finally close to zero. Therefore, the systematic differences exist.



(a) Stock A48TKB; Strike price: 5600

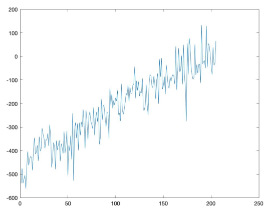


(b) Stock C22BMY; Strike price: 4800

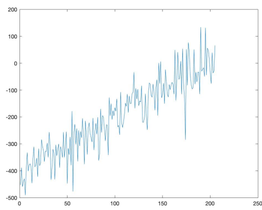


(c) Stock D64WVO; Strike price: 5375

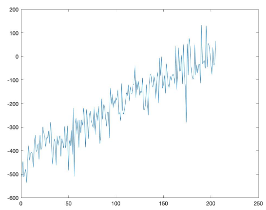
Figure 1: The predicted price and truth of three random stocks in call option.



(a) Stock A48TKB



(b) Stock C22BMY



(c) Stock D64WVO

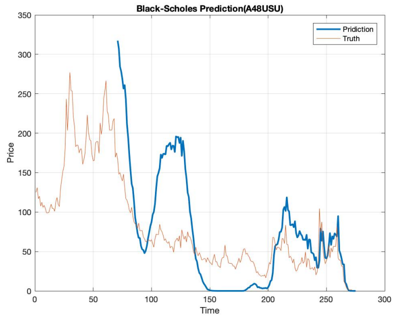
Figure 2: The difference between prediction and truth of these three stocks against time.

1.3 Implied volatility

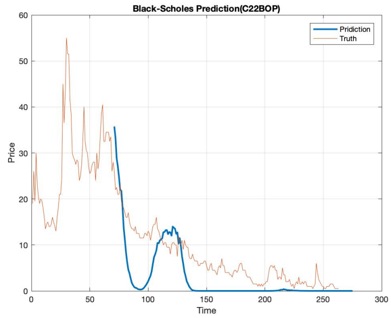
Implied volatility can be explained that how far the traders expect the stock to move. It cannot be directly observed or calculated with a well defined function. A reasonable way to find the value of the implied volatility is to do an iterative search. The iteration function is given by equation 13, in which P is the price and $v(\sigma_n)$ is the derivative of volatility σ_n.

$$\sigma_{n+1} = \sigma_n - \frac{BS(\sigma_n) - P}{v(\sigma_n)} \tag{13}$$

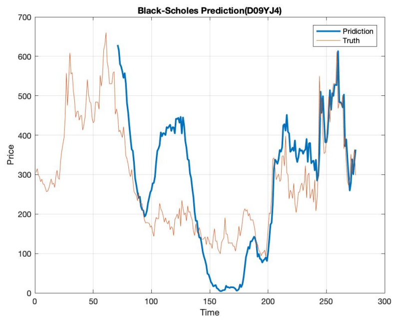
In this coursework, to compute the implied volatility, I am going to use a closed form estimate of implied volatility given by Brenner and Subrahmanyam (1988), $\sigma = \sqrt{\frac{2\pi}{T}} \cdot \frac{c}{S}$, as an initial input, σ_0. To see



(a) Stock A48USU; Strike price: 6400

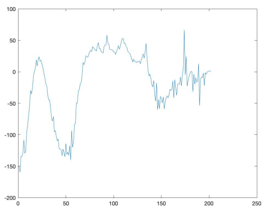


(b) Stock C22BOP; Strike price: 4800

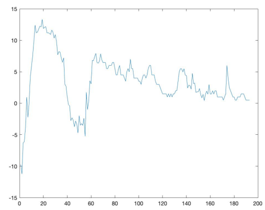


(c) Stock D64WYJ4; Strike price: 7200

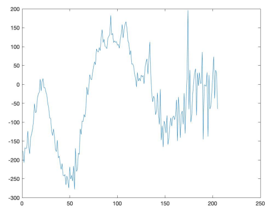
Figure 3: The predicted price and truth of three random stocks in put option



(a) Stock A48USU



(b) Stock C22BOP



(c) Stock D64WYJ4

Figure 4: The difference between prediction and truth of these three stocks against time.

the relationship between implied volatility(IV) and historical volatility(HV) computed from the data, I plot a scatter plot(Figure 5), in which y axis stands for IV and x axis stands for HV. A higher value of implied volatility means people are willing to pay more money to the option contract as they expect a bigger movement in the stock[2].

A volatility smile is a "smile-like" pattern. When the implied volatility is plotted against strike prices, the slopes of the line are upward on the both sides. To obtain a better performance, I chose a subset of the overall call option data set. The "smile" is shown in figure 6. From this "smile" figure, we can see that the minimum implied volatility is around 0.1 with the strike price at the middle of the selected subset. Besides, when the strike price is away from the middle value, no matter too low or too high, the value of the implied volatility would be much bigger.

1.4 Black-Scholes vs Binomial lattice method

According to the results from previous chapters, we can find that it is easy to implement a Black-Scholes model to price an option as only we need to do is to input the accurate parameters, such as strike price,

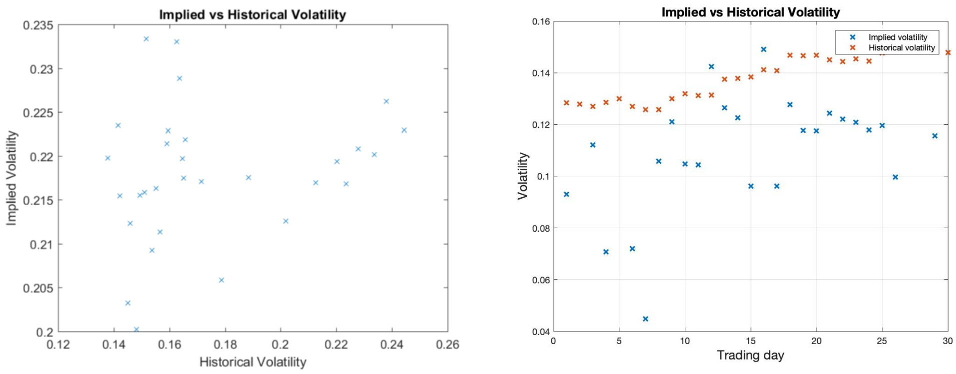


Figure 5: Left: implied against historical volatility. Randomly choose one option from the overall data set and compute its HV and IV. Right: a more reasonable and clear way to present HV and IV as they are changing with time going by.

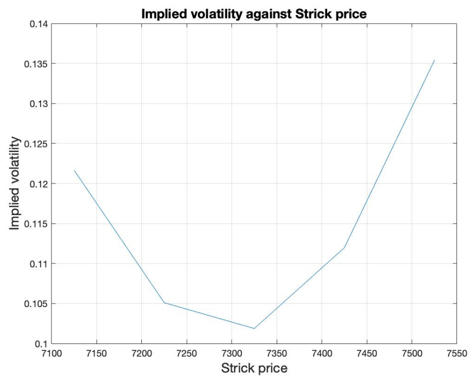


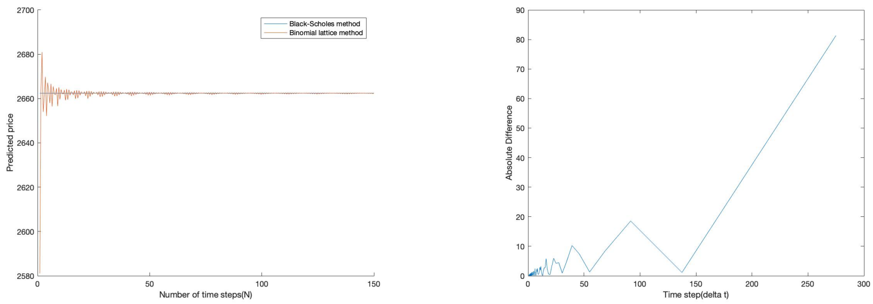
Figure 6: Volatility smile. Implied volatility against strike price.

expiration time, interest rate and so on. However, the shortcoming of this model is lacking flexibility. It is not flexible enough because we cannot change anything in the period of time, T, from the beginning to the expiration time. In this case, a much more flexible way is Binomial lattice model. In this model, the period of time is broke down to a number of time steps, δt, i.e. $\delta t = \frac{T}{N}$, in which N is the number of time steps. In each time step, there are two possible moving directions of current stock price, up and down, and the value of arrived point is calculated by the current price S, the time step δt, and volatility σ. So any adjustment to the current parameters, like current stock prices, interest rates, etc, could be put into the calculation at any time.

Different from Black-Scholes model only with final outputs, Binomial lattice model shows all the possible paths of the stock prices. Therefore, when the option with American style rather than European style, using Binomial lattice method would be a good choice. However, the problem of this model is how to choose a suitable time step. With a sufficient number of time steps, the results of two kinds of models would be similar[4]. Figure 7 compares the difference between Black-Scholes and Binomial lattice methods. To well illustrate, in this figure, I plot the predicted prices given by two methods and their difference against the number of time steps(N) and time step(δt) respectively. According to $\delta t = \frac{T}{N}$, we can know that δt is determined by N, i.e. a bigger N value corresponds a smaller δt.

1.5 American call and put option

In this part of code, the model begins to work backward. The first line counts the number of time steps. The value of "tau" is from 1 to N and records the step in which the computation is working. For example, when "tau" is equal 1, it means the backtrack process is in the first step(from terminal layer



(a) Predicted price (b) Absolute difference

Figure 7: Left: x axis represents the number of time steps(N), and y axis is the predicted price calculated by B-S and B-L methods. With the number of time steps, N, increasing, the result from the B-L model is close to the result from B-S one. Right: x axis stands for time step, δt, and y axis is the absolute difference of the prediction given by two models. With δt decreasing, the value of absolute difference is close to zero.

to the previous layer). When "tau" is equal 2, the computation moves from second layer, counting from the terminal layer, to the third layer. The second line is to choose values from two adjacent points in current layer to output a new value in the previous layer. In this "for" loop, there are two lines which are used to compute the value in the previous layer. The first line is to back calculate based on the current values. In a maths way, it can be explained as $f = exp(-r\delta t)(pf_{i+1,j+1} + (1-p)f_{i,j+1})$, in which $j+1$ is the number of current layer and i, $i+1$ stand for two adjacent points with current values. The second line is to compare the calculated value with the value of difference between strike price and current price and then keep the higher value as the option value in the current layer. In the maths way, it can be denoteda as $f_{i,j} = \max(K - S_{i,j}, exp(-r\delta t)(pf_{i+1,j+1} + (1-p)f_{i,j+1}))$. After this two "for" loops, the final value is the option value. This code is for put option. If price a call option, only need to substitute "PVals(i)=max(hold, SVals(i)-K);" to "PVals(i)=max(hold, K-SVals(i));".

2 Paper by Hutchinson

Paper [5] trains a neural network type model, RBF model, to approximate the results generating by Black-Scholes model and performs well, although the RBF model is without complex financial parameters. In this sector, I am going to compare these two models.

2.1 RBF model

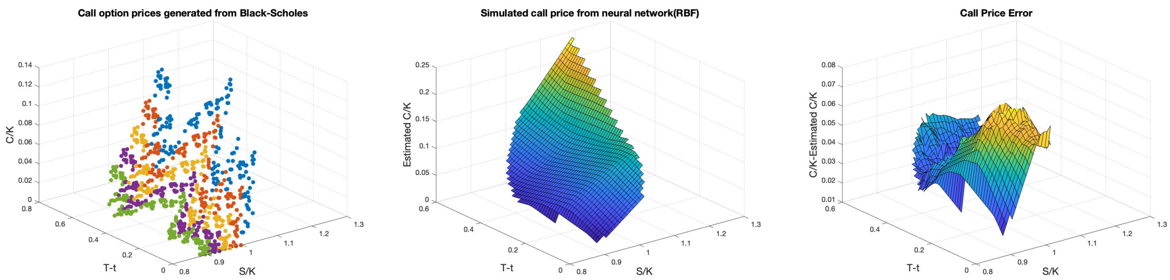
The form of RBF model is given by equation 14, in which data vector $\boldsymbol{x} = [S/X, (T-t)]^T$ and nonlinear function $\phi_j(\boldsymbol{x}) = [(\boldsymbol{x} - \boldsymbol{m}_j)^T \boldsymbol{\Sigma}_j (\boldsymbol{x} - \boldsymbol{m}_j) + b_j]^{1/2}$.

$$c = \sum_{j=1}^{J} \lambda_j \phi_j(\boldsymbol{x}) + \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{x} + w_0 \tag{14}$$

In this part, we ignore bias terms, b_j and set up $J = 4$. J is the number of components of a Gaussian mixture model. To train the RBF model, firstly we need to use a MATLAB function *fitgmdist* to fit a Gaussian mixture model with J components. In each Gaussian model, extract mean vector $\boldsymbol{m}_j$ and covariance matrix $\boldsymbol{\Sigma}_j$. And then compute the corresponding nonlinear distance ϕ_j. For computing convenience in the computer, denote the model in a matrix form. Firstly, construct a design matrix. Then $design\ matrix = [\phi_1(\boldsymbol{x}), \phi_2(\boldsymbol{x}), \phi_3(\boldsymbol{x}), \phi_4(\boldsymbol{x}), \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{1}]$, and each $\phi_j(\boldsymbol{x})$ has as many rows as the number of data points given by X. After constructing the model, regarding the value given by the B-S model as truth, optimize by implementing least squares method to estimate the weights of the RBF model, including $\lambda_1, \lambda_2, \lambda_3, \lambda_4, w_1, w_2, w_0$.

2.2 Comparison between two models

To compare two models, firstly randomly select several call options to construct a dataset with the predicted call option prices given by Black-Scholes model. Next, split the dataset into a train set and a test set, with a proportion 6:4. Then train a RBF model on the train set. Finally, compare the values of the prediction computed by two models in the test set. The results are presented in the figure 8. In



(a) Price from B-S model (b) Price from RBF model (c) Difference between two models

Figure 8: Call option prices normalized by strike prices against normalized stock prices and time to expiration. Left: call option prices computed using Black-Scholes model. Middle: call option prices computed by neural network model. Right: Difference between the estimated values from two models.

figure 8c, the range of z axis is from 0 to 0.08, close to zero. Therefore, this non-parametric model learns well as its outputs are very close to the outputs from the Black-Scholes model.

Lastly, to compare the "hedge ratio" computed by two models. The "hedge ratio" can be denoted as $\Delta = \frac{\partial c}{\partial S}$. The difference between the results from two models is given by figure 9. The Delta error is from

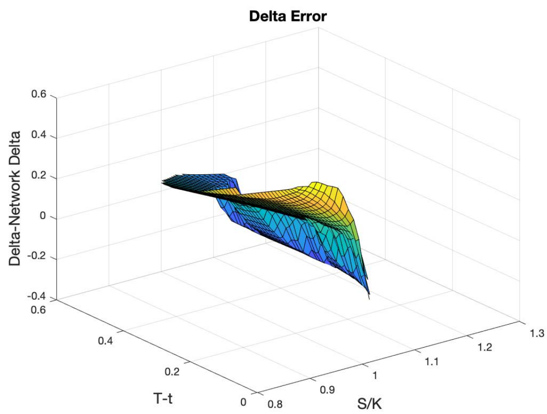


Figure 9: Delta error against normalized stock prices and time to expiration. Delta error is calculated by Delta given by B-S model subtracting Delta given by RBF model.

-0.4 to 0.6 and many points concentrate on the zero point. But from this figure, we can see that there are relatively more fluctuations, which means $\hat{\Delta}$ computed by RBF is not always close to Δ given by B-S model. It is not extremely stable and with a not tiny variance. So compared with the performance of predicted option prices, the model is not very well in hedge ratio.

References

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