MA-202 NAC

تاریخ:

NUMERICAL ANALYSIS AND COMPUTATION

LEC, 01

Pre-requisite:

Basic knowledge about Multivariable Calculus includes:-

1- Limit: The function is defined at $x = c$ is

$$\lim_{x \to c} f(x) = L \quad \text{(any no.)}$$

Examples: (i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x \to 1} \frac{x^2+2}{x+1} = \frac{3}{2}$

(iii) $\lim_{x \to -1} \frac{x^2-1}{x+1} = -2$

2- Continuity: A function f is said to continuous at a no. 'c' iff three conditions are satisfied ⇒

(i) $f(c)$ is defined

(ii) $\lim_{x \to c} f(x)$ exists OR $L.H.L = R.H.L. = f(c)$

(iii) $\lim_{x \to c} f(x) = f(c)$

No break in graph of f

Take

$f(x) = x^{4/3}$ is continuous in $[-1, 3]$

$f'(x) = \frac{4}{3} x^{\frac{1}{3}}$ is continuous in $[-1, 3]$

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$f''(x) = \frac{4}{9}x^{-2/3}$ is not continuous in $[-1,3]$

break in graph at $x=0$.

3- The Derivative or Slope :-

(i) Basic Derivative Rules:

{Power rule, product rule, quotient rule, exponential rule, log rule}

(ii) Derivatives of (a) Trigonometric functions

(b) Inverse Trig. functions

(c) Hyperbolic Functions

(d) Inverse Hyp. functions

$$\text{Slope} = \left.\frac{dy}{dx}\right|_{x=c} = f(x_0)$$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$ b/w two points

$$f'(x) = \frac{y - y_0}{x_1 - x_0}$$

Taylor's series

$$f(x+h) = f(x) + h f'(x) + \frac{h^2 f''(x)}{2!} + \cdots$$

Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2 f''(0)}{2!} + \cdots$$

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4- Techniques of Integration

(i) Basic Rules of integration:

{Power Rule, Product Rule, Exponential Rule, Log Rule}

(ii) Integral Formulas of (a) Trig. Functions

(b) Inverse Trig. Fnc.

(c) Hyperbolic Fnc.

(d) Inverse Hyp. Fnc.

(iii) Concept of Definite Integral:

$\int_a^b f(x)dx$ has a definite value $\phi(b)-\phi(a)$, so it is called Definite Integral.

$\Rightarrow$ $y=f(x)$ shows area under curve from $x=a$ to $x=b$.

a) $\int_a^b f(x)dx = \phi(b)-\phi(a)$

b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

c) $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, $a<c<b$.

Motivation:

It is widely used for forecasting and predicting in the Field of machine learning. Most mathematical models are based on the solutions of ODE, PDE, or Integral Eqs.

It investigates and provides accurate solutions to real life problems from the field of Science, Engg., Biology, Astrophysics and Finance.

When we come across the problems which cannot be solved directly. A numerical solution can be obtained For problems, where an analytical solution does not exist.

Numerical Method only uses evaluation of standard functions and the operations $(+, -, \times, \div)$.

The process of finding a solution is to reduce the original problem to a repetition of the same step or series of steps so that the computations become automatic. Such process is called a Numerical Method. The derivation and analysis of such method is called Numerical Analysis.

Error Analysis

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As Numerical solutions are not exact solutions. There is a chance to occur error in numerical solution.

Error:- The difference between true value and approximate value is said to be error.

Absolute error: Let $\hat{x}$ is approximation of x, then absolute error is defined as

$$E_x = |x - \hat{x}|$$

Relative error: Relative error is defined as

$$R_x = \frac{|x - \hat{x}|}{|x|}, \quad x \neq 0$$

Examples:

1- Let $x = 3.141592$, $\hat{x} = 3.14$

$$E_x = |x - \hat{x}|$$

$$= |3.141592 - 3.14|$$

$$= 0.001592$$

and $R_x = \dfrac{|x - \hat{x}|}{|x|} = \dfrac{0.001592}{3.141592}$

$$= 0.000507$$

2- Take $x_1 = 1000000$, $\hat{x}_1 = 999996$

$$E_{x_1} = |x_1 - \hat{x}_1|$$

$$= |1000000 - 999996| = 4$$

and $R_{x_1} = \dfrac{|x_1 - \hat{x}_1|}{|x_1|} = \dfrac{4}{1000000} = 0.000004$

3- Let $p = 0.000012$, $\hat{p} = 0.000009$

$$E_p = |p - \hat{p}|$$

$$= |0.000012 - 0.000009|$$

$$= 0.000003$$

$$R_p = \frac{|p - \hat{p}|}{|p|} = \frac{0.000003}{0.000012} = 0.25$$

SOURCES OF ERRORS

There are many sources of errors, but we will discuss three major/main sources of error:

(i) Gross errors

(ii) Rounding off errors

(iii) Truncation errors

ⓐ **Gross Errors:** The gross errors are either caused by human mistakes or by the computer.

How occur?

(i) Misreading, Misquoting the figures or quantities. Read 1027791 instead of 1027991 etc.

(ii) Use of inaccurate Formula.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \quad x = \frac{b \pm \sqrt{b^2-4ac}}{2a}$$

Correct ↑ incorrect ↑

(iii) Use of inaccurate data: sampling, measuring etc.

Such types of errors can be avoided by enough care.

(b) **Rounding off Errors:** The error introduced by rounding off numbers to a limited number of decimal places is called the round off error.

For example

Distance between two cities A and B is 215.2967KM. Rounded to whole number is 215KM.

error = 0.2967KM.

$\pi = 3.14159265359$

Round-off Rules

a) If the first discarded digit is less than 5, then the previous digit is unchanged.

56.43 rounded to 1DP 56.4

b) If the first discarded digit is greater than 5, then previous digit is increased by 1

56.46 rounded to 1DP 56.5

c) If the discarded digit is exactly 5, then the previous digit is

1- unchanged if it is even
2- increased by 1 if it is odd.

56.45 → 56.4

56.75 → 56.8

The most commonly used rule for rounding off is

"If the discarded digit ≥ 5, so we add 1 to last retained digit."

If a number is correct to n decimal places, it has a rounding error

$$|e| \leq \frac{1}{2} \cdot 10^{-n}$$

Truncation Error

Truncation is the replacement of one series by another with fewer terms. The error arising from this approximation is called the truncation error.

Several types of series expansions commonly occur, some are as follows:

a) Binomial expansion
b) Maclaurin's series
c) Taylor's series
d) Infinite geometric progression
etc

a) Binomial expansion:

$$\left(1+\frac{1}{x}\right)^x = 1+(x)\left(\frac{1}{x}\right)+\frac{x(x-1)}{2!}\left(\frac{1}{x}\right)^2+\frac{x(x-1)(x-2)}{3!}\left(\frac{1}{x}\right)^3 + \cdots$$

$$= 1+1+\frac{1}{2!}\cdot\frac{x-1}{x}+\frac{1}{3!}\cdot\frac{(x-1)}{x}\cdot\frac{x-2}{x}+\cdots$$

$$= 2+\frac{1}{2!}\left(1-\frac{1}{x}\right)+\frac{1}{3!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)+\cdots$$

Using $\lim_{x\to\infty}$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left[2+\frac{1}{2!}\left(1-\frac{1}{x}\right)+\frac{1}{3!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)+\cdots\right]$$

$$= 2+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

$$= 2+0.5+0.1667+\cdots$$

Hence we have the expansion in the form

$$\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^x = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots$$

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