A visual (and mathematical) proof that neural nets can compute any function

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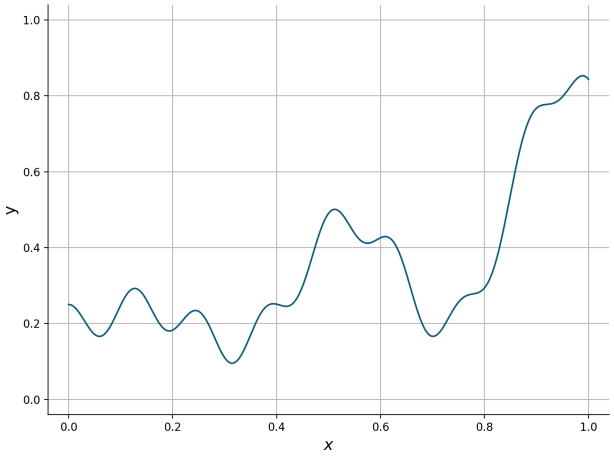
Presentations

- Using neural nets to recognize handwritten digits
- How the backpropagation algorithm works
- Improving the way neural networks learn
- A visual (and mathematical) proof that neural nets can compute any function
- Why are deep neural networks hard to train ?
- Deep learning

What it is about

An arbitrary function :



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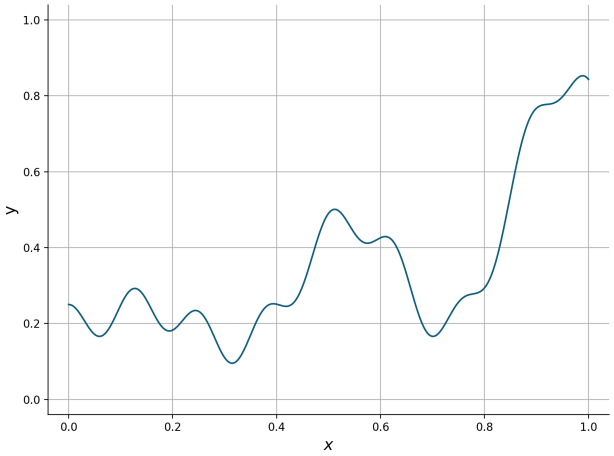

The MNIST dataset :

What it is about

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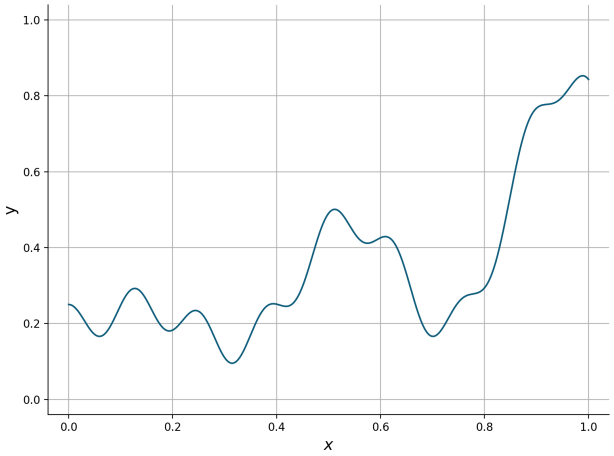

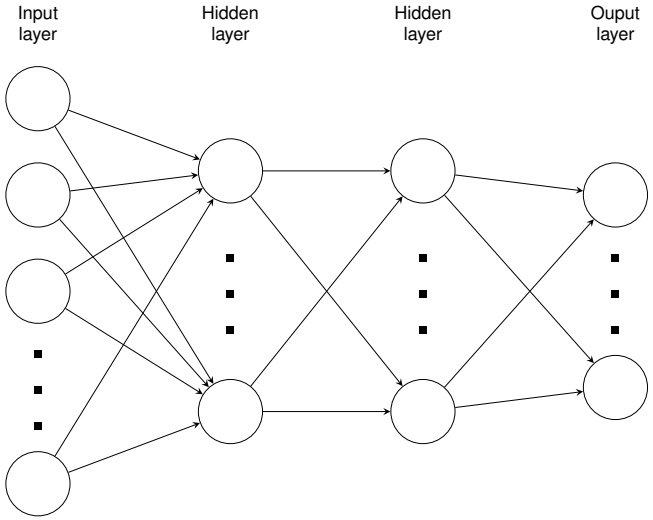

The MNIST dataset :

Caveat 1

"Neural nets can compute any function"

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"Neural nets can compute any function"

For a given continuous function

$$f : \mathbb{R}^n \to \mathbb{R}, x \mapsto f(x) \tag{1}$$

with $n \in \mathbb{N}$ one can always find for a desired accuracy $\varepsilon > 0$ a neural network with the output

$$g : \mathbb{R}^n \to \mathbb{R}, x \mapsto g(x) \tag{2}$$

that satisfies

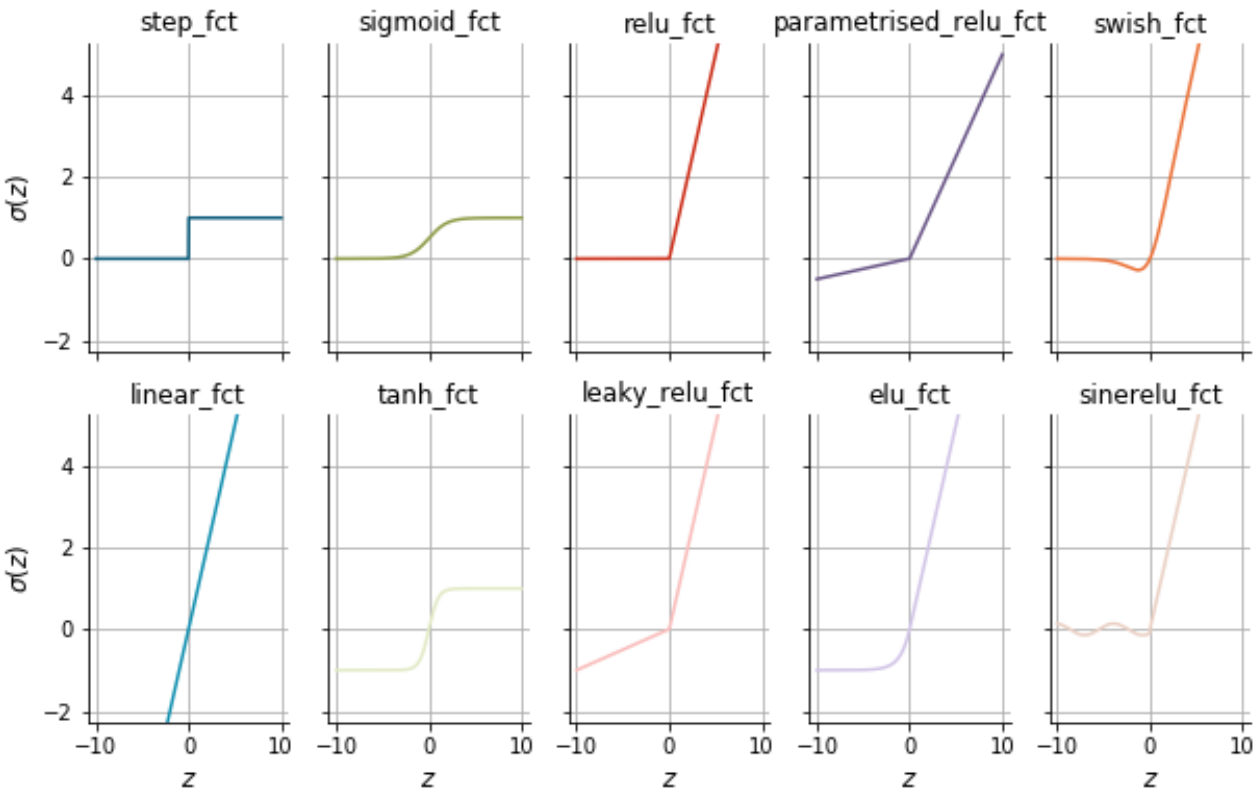
$$|f(x) - g(x)| < \varepsilon \tag{3}$$

for all $x \in \mathbb{R}^n$.

Caveat 2

A function f which can be approximated by a given continuous output of a neural network g upto a desired accuracy is an element of the class of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Activaltion functions



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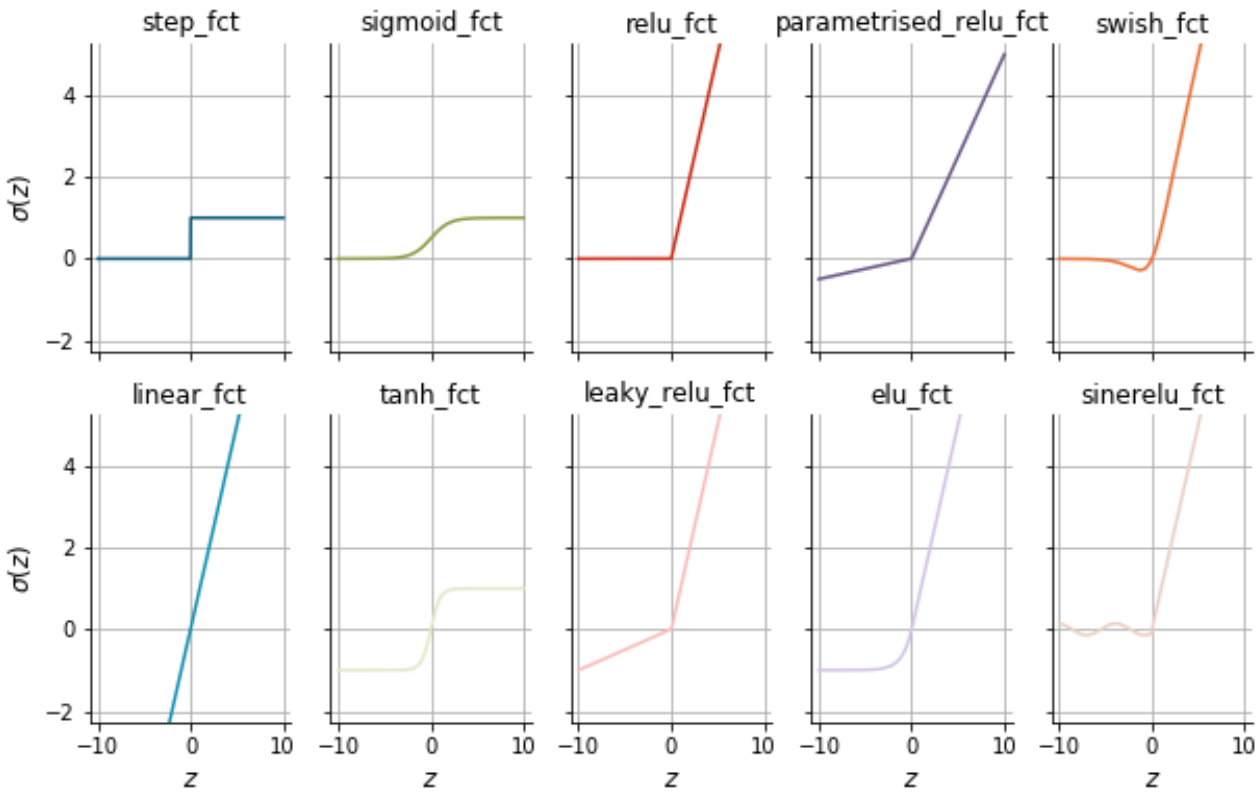
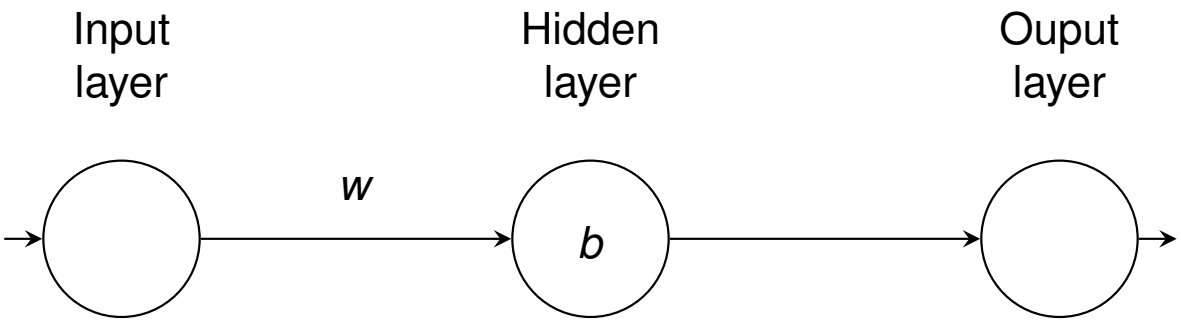


All of them are functions of the input vector $\boldsymbol{x}$ via

$$z = \boldsymbol{w} \cdot \boldsymbol{x} + b. \tag{4}$$

One hidden layer with one neuron



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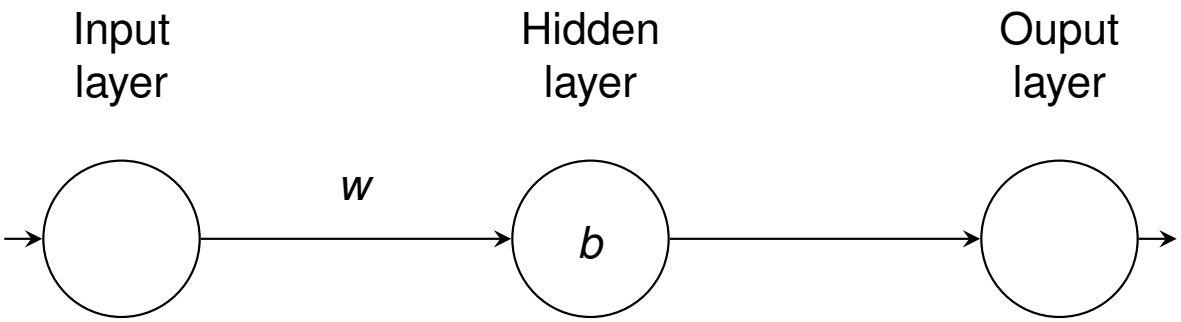

With the weight w, the bias b and the activation function

$$\sigma(z) = \frac{1}{1+e^{-z}} \qquad \Rightarrow \qquad \sigma_{wb}(x) = \frac{1}{1+e^{-(wx+b)}} \tag{5}$$

Behaviour of the sigmoid function

Sigmoid function :

$$\sigma(z) = \frac{1}{1+e^{-z}} \qquad \Rightarrow \qquad \sigma_{wb}(x) = \frac{1}{1+e^{-(wx+b)}} \tag{6}$$

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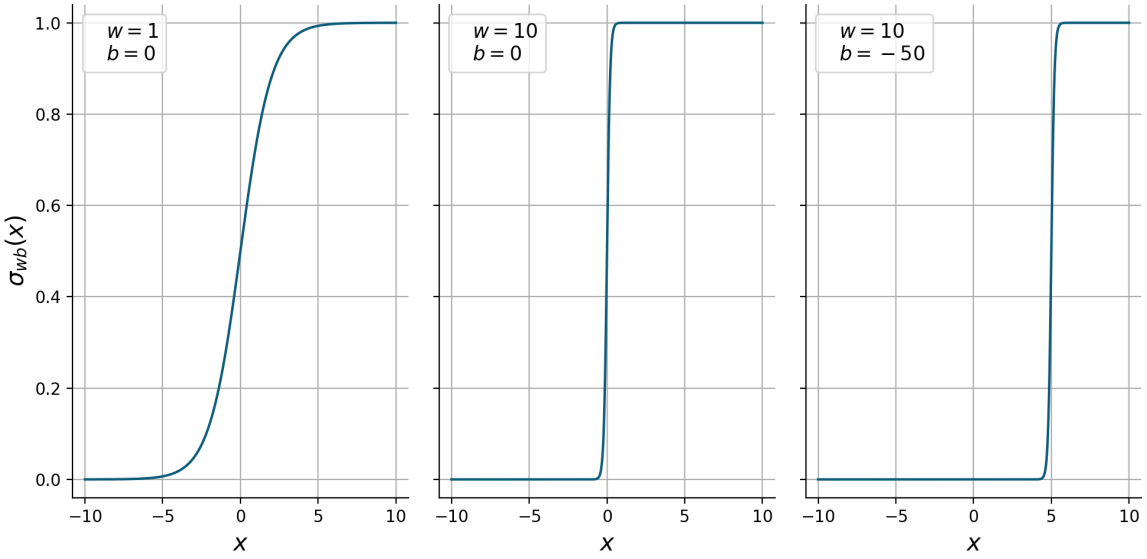
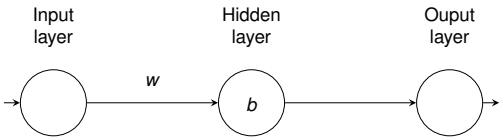
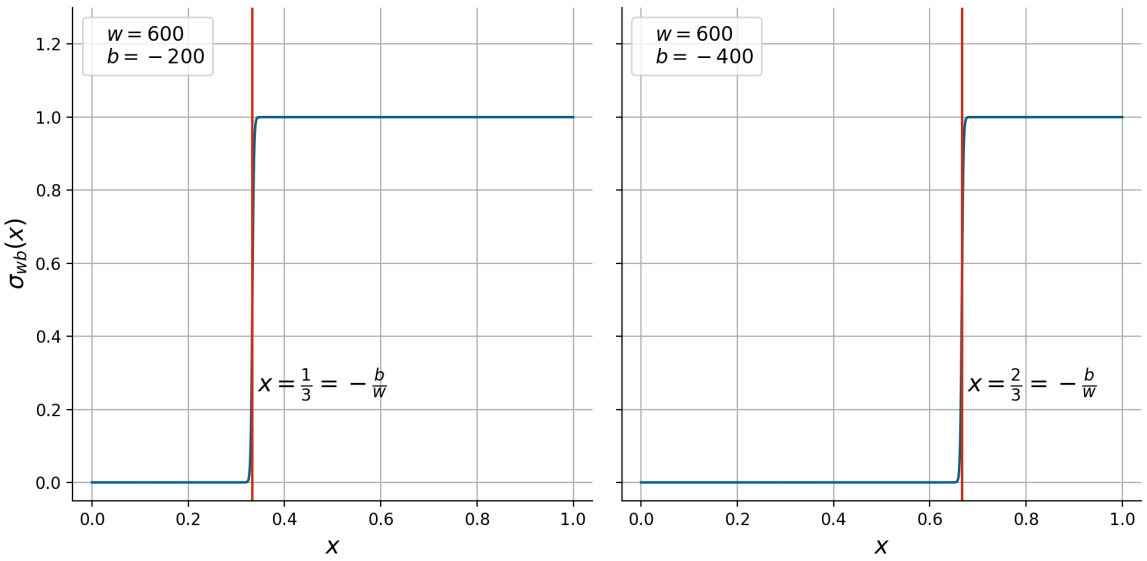




Definition of s



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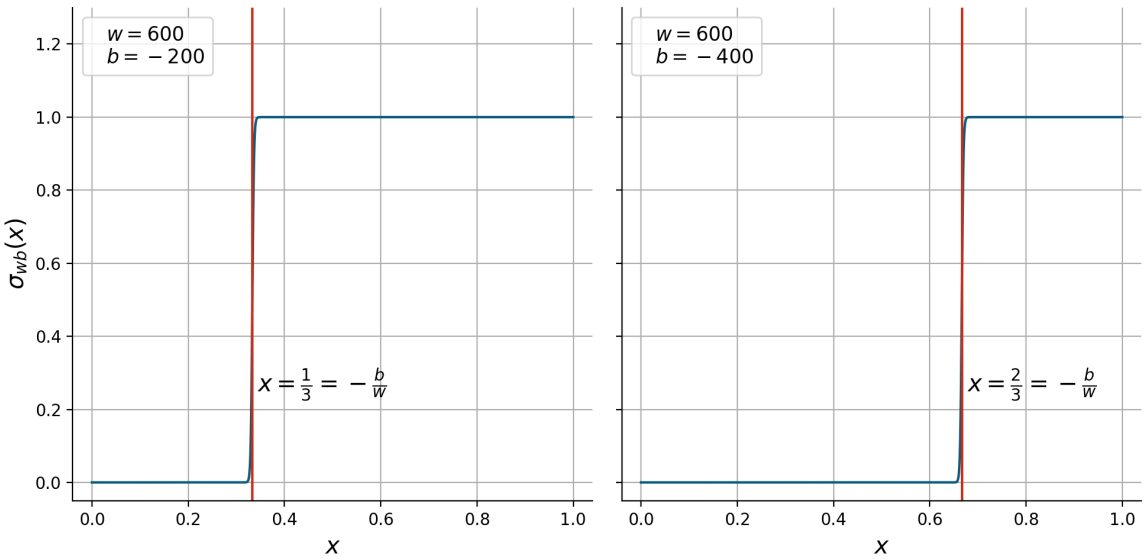

We define the parameter s as :

$$s := -\frac{b}{w} \tag{7}$$

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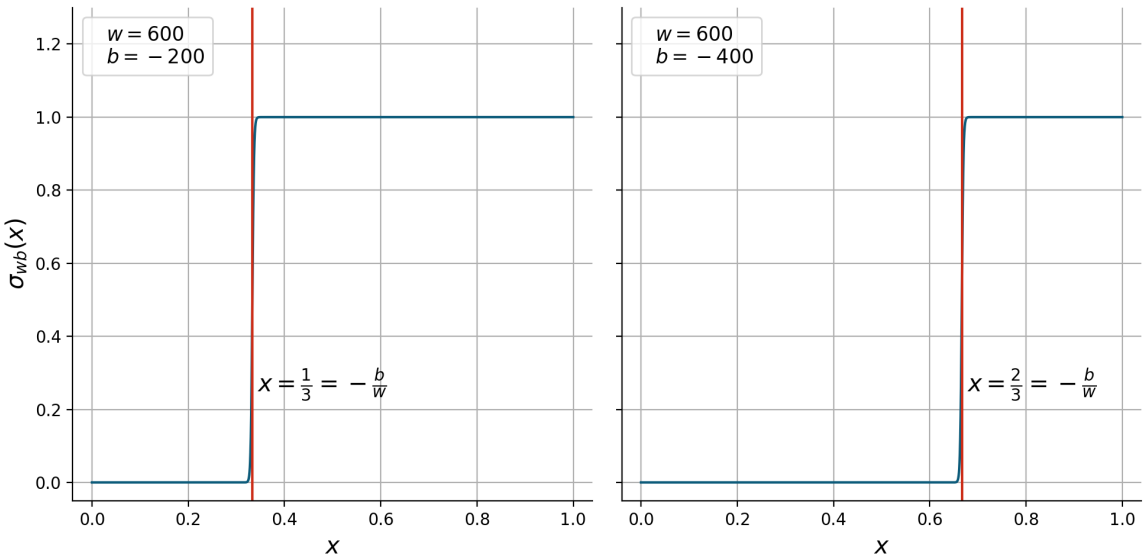

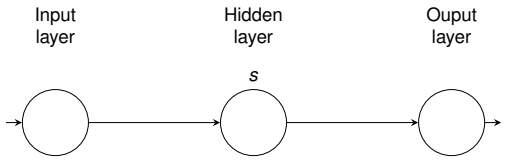

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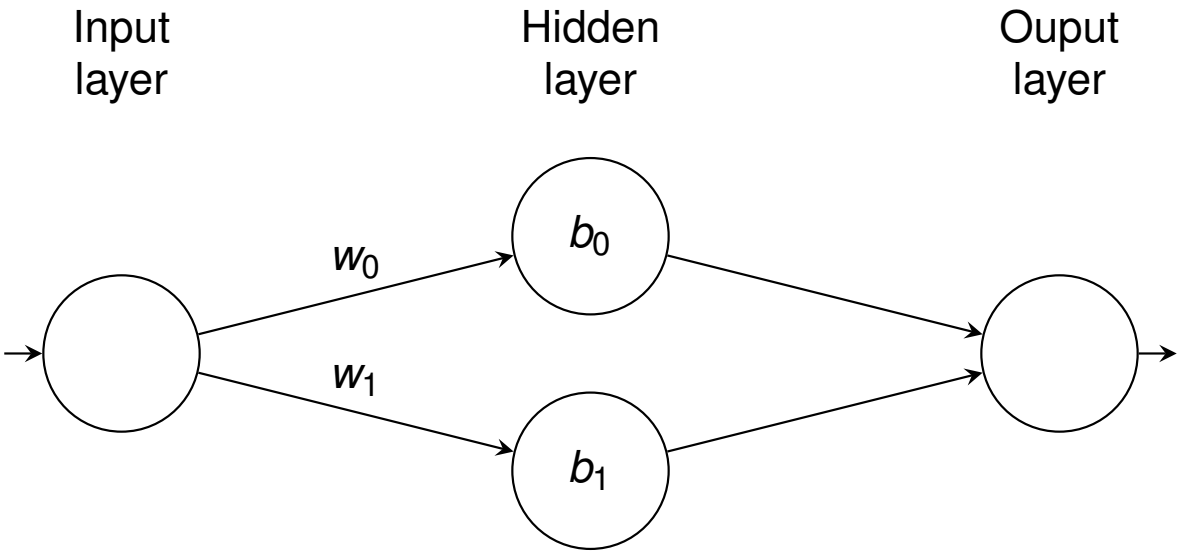
One hidden layer with two neurons



Sum of the two sigmoid functions

Sum of two sigmoid functions :

$$\sigma_{w_0 b_0}(x) + \sigma_{w_1 b_1}(x) = \frac{1}{1 + e^{-(w_0 x + b_0)}} + \frac{1}{1 + e^{-(w_1 x + b_1)}} \tag{8}$$

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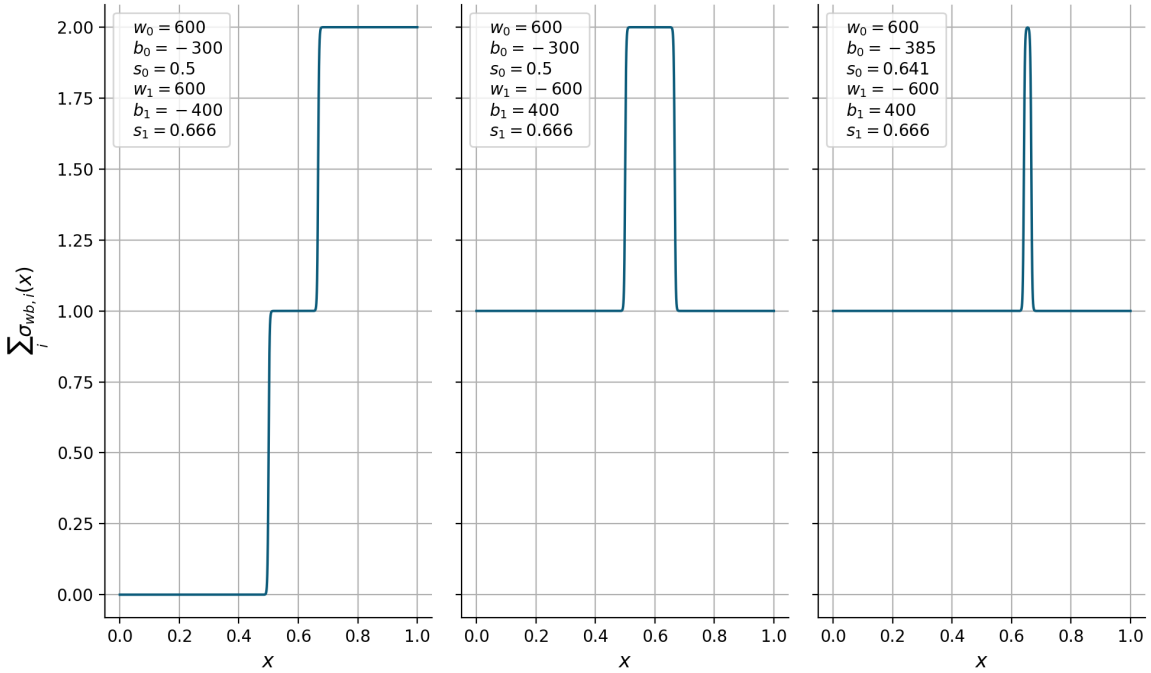
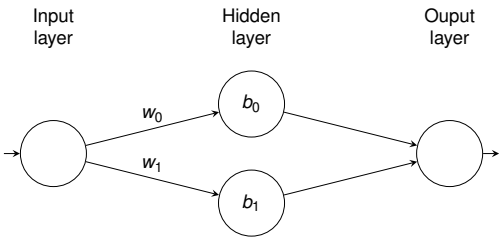
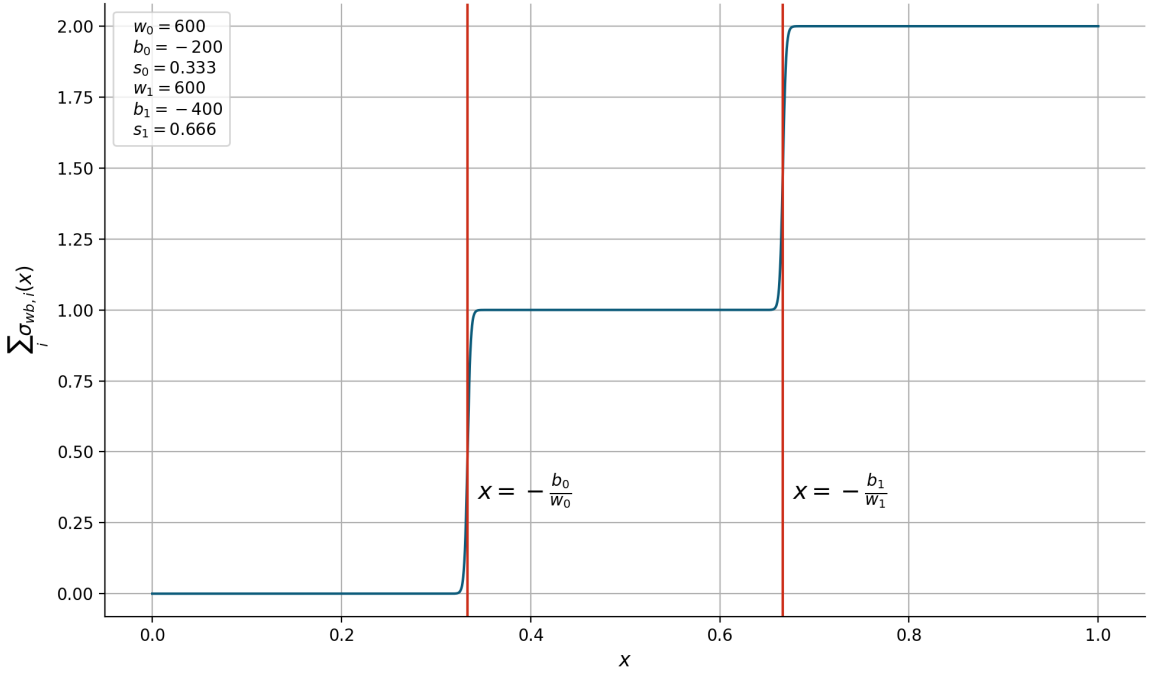
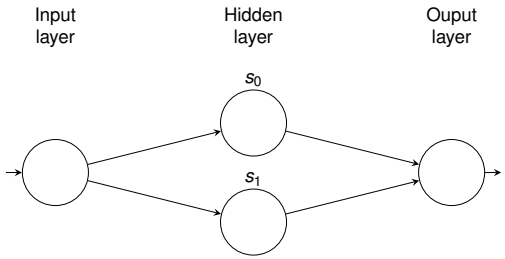


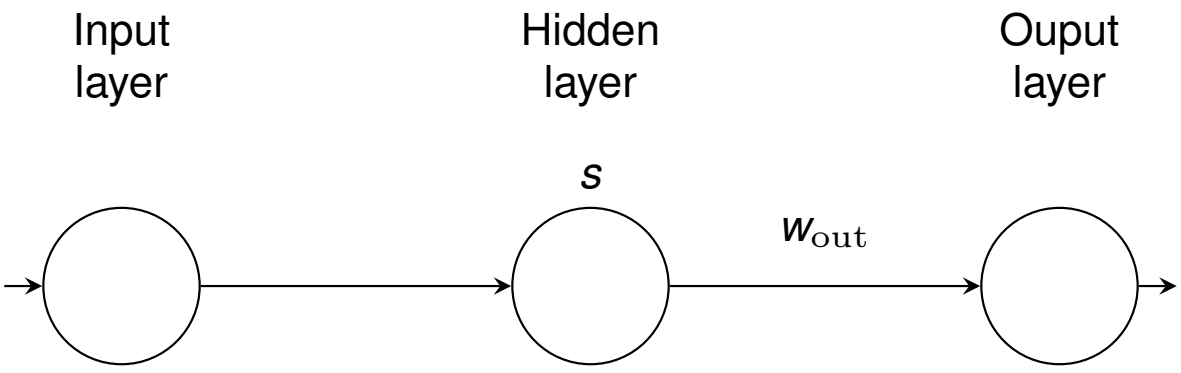


Identify s for one hidden layer with two neurons





One hidden layer with one neuron and weighted output



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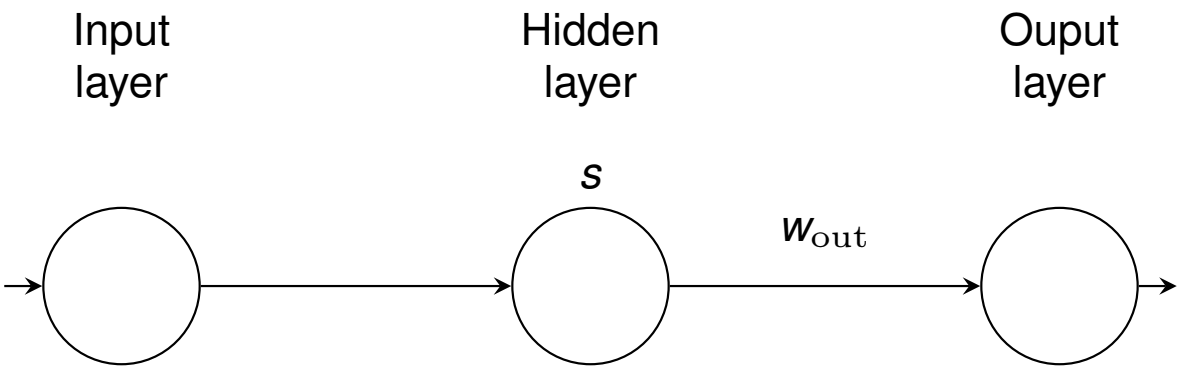

With the outputs or activations a of the neurons from the hidden layer we get for the activation of the neuron in the output layer

$$A_{w,\mathrm{out}}(x) := \sum_i w_{\mathrm{out},i} a_i(x) \tag{9}$$

by assuming it to be a linear neuron and neglecting the its bias b_{out}.

Weighted output of one hidden layer with one neuron

The activation of the neuron in the output layer by assuming it to be a linear neuron and neglecting the its bias b_{out}.

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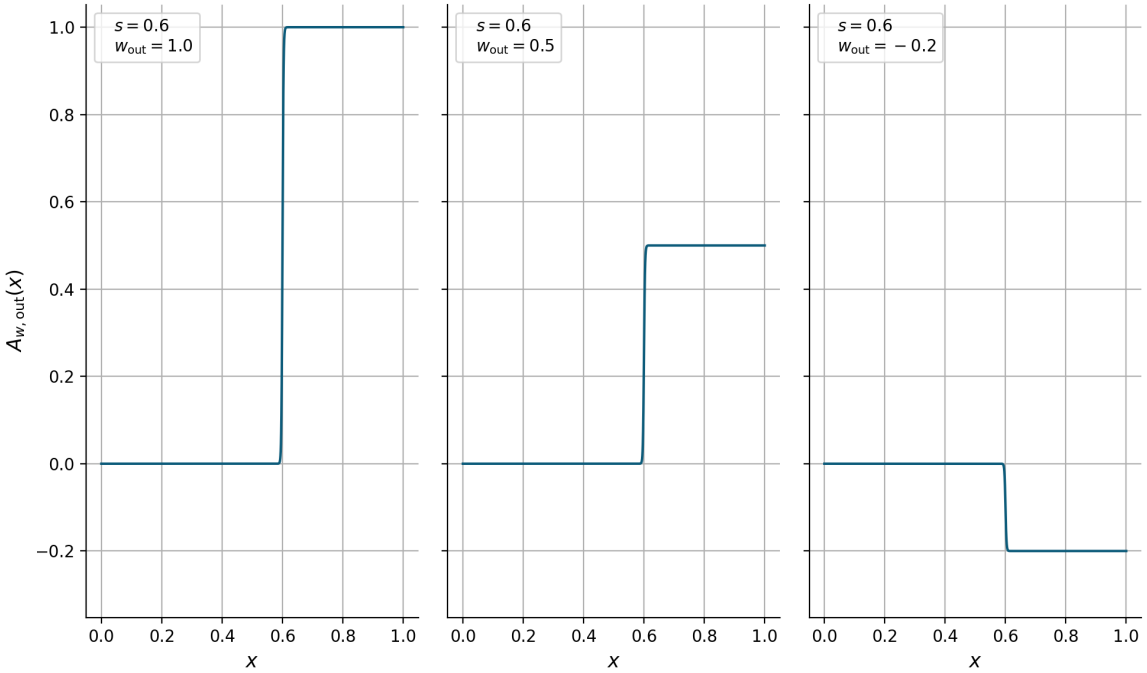
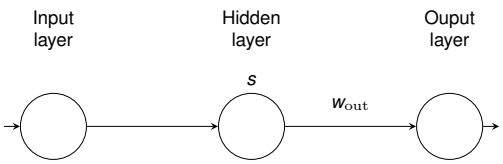
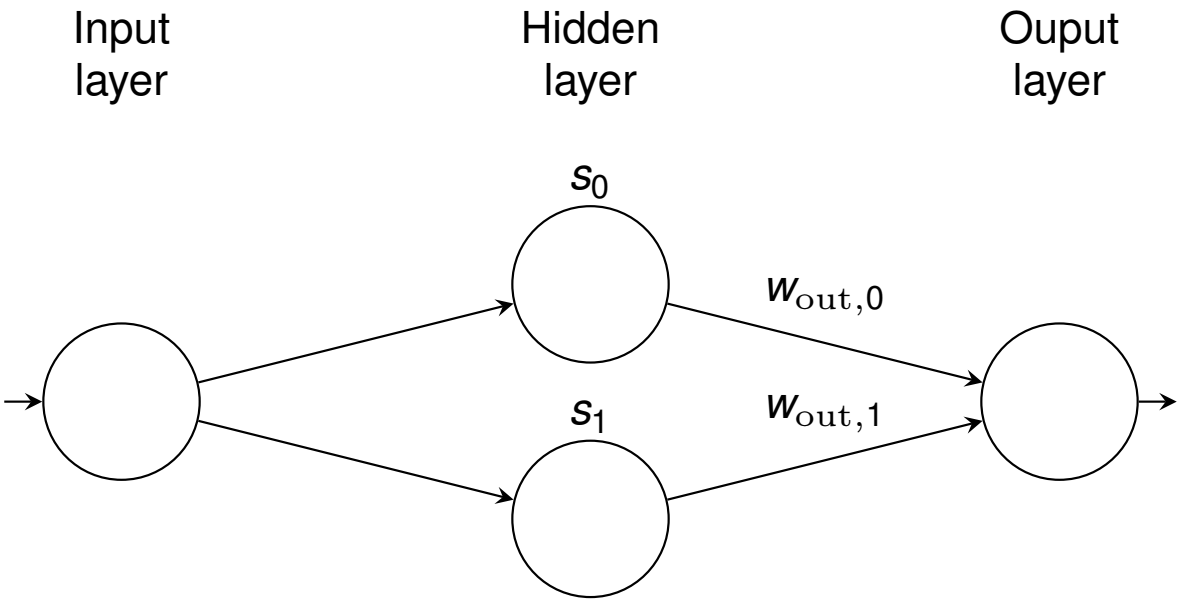
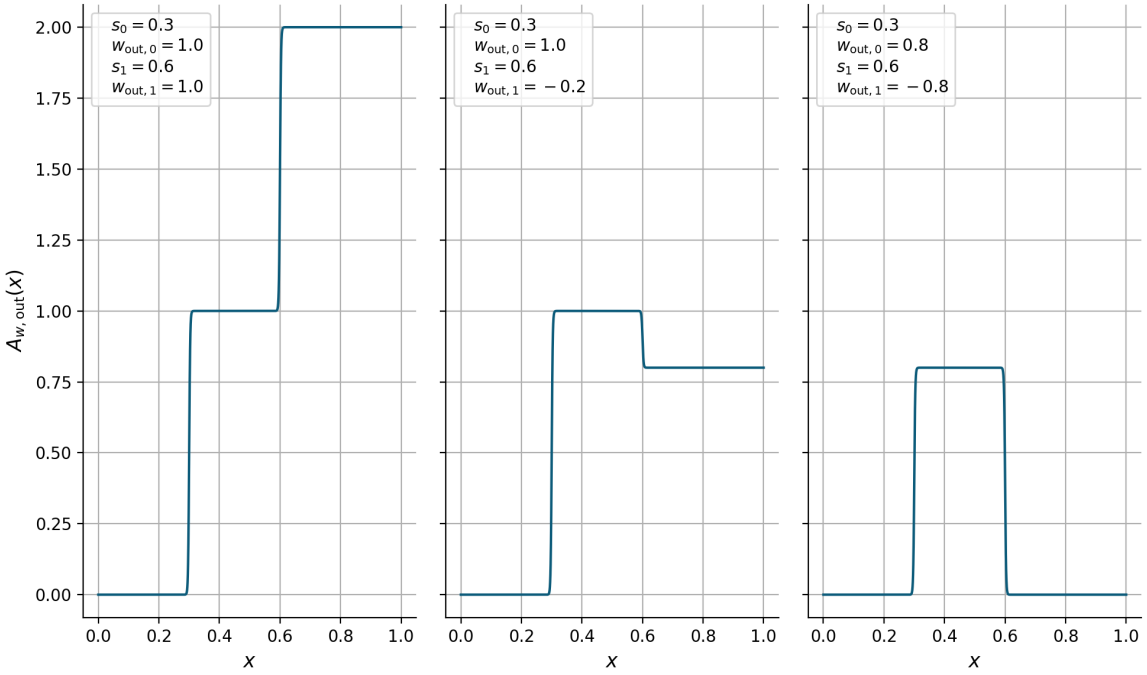


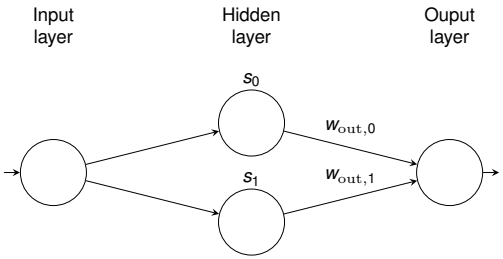


One hidden layer with two neurons and weighted output



Weighted output of one hidden layer with two neurons





Definition of h

In the case $w_{\mathrm{out},i} = -w_{\mathrm{out},j}$ and $s_i < s_j$ we can define the height of the resulting towers as

$$h_{ij} := w_{\mathrm{out},i} = -w_{\mathrm{out},j}. \tag{11}$$

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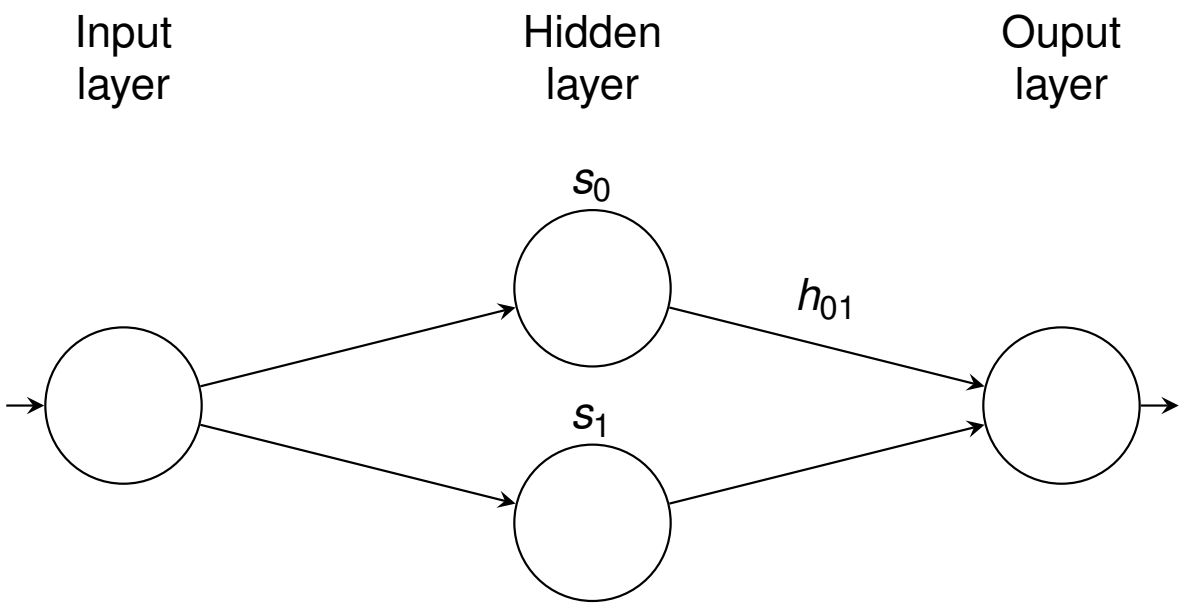
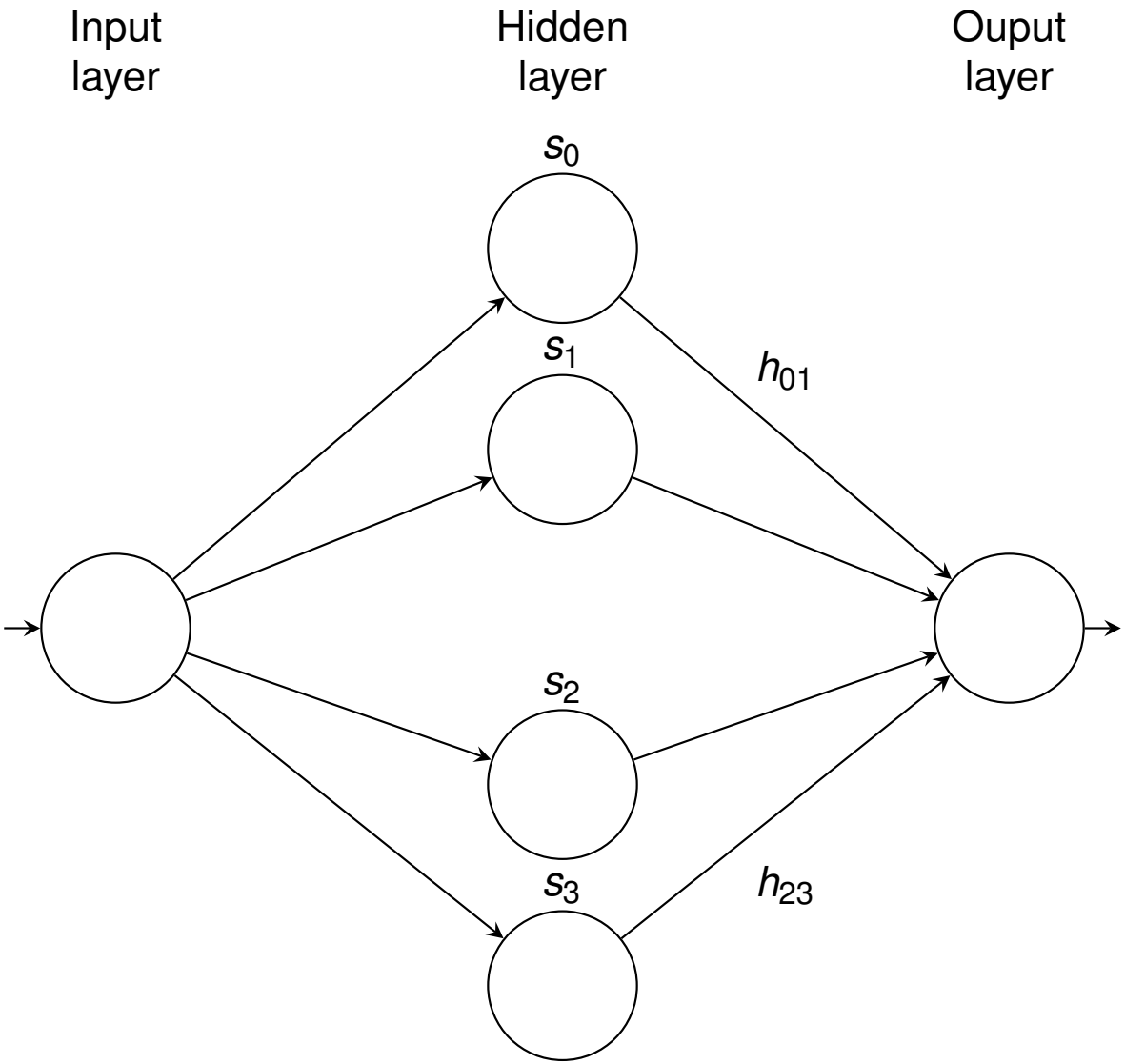
With that we get for our neural net :



One hidden layer with four neurons



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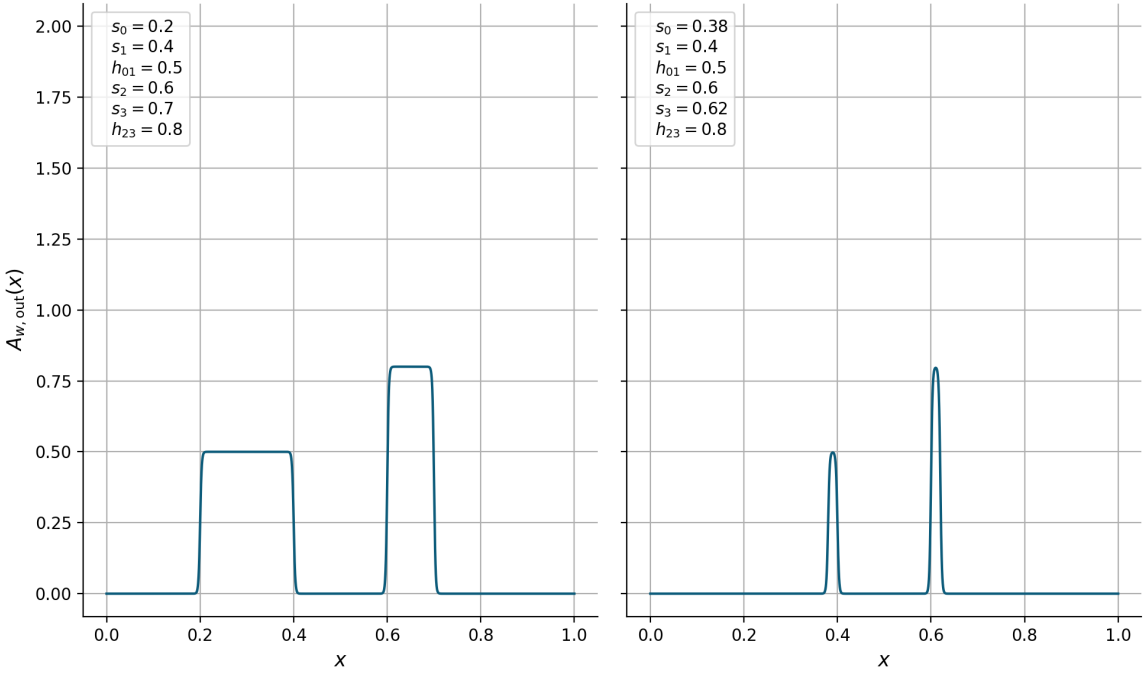
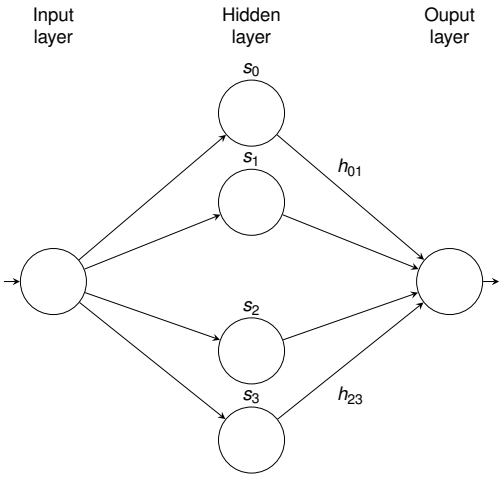
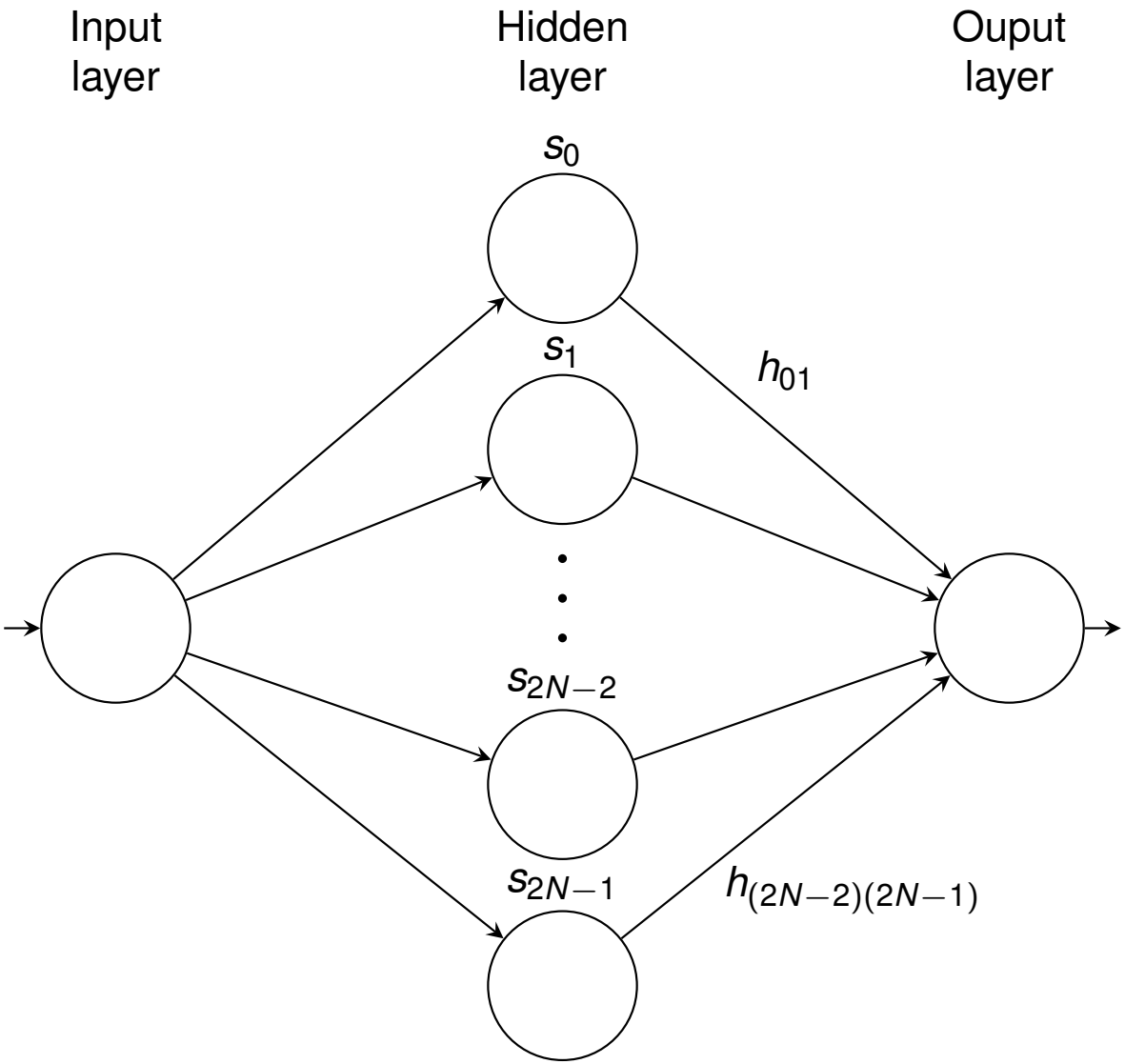
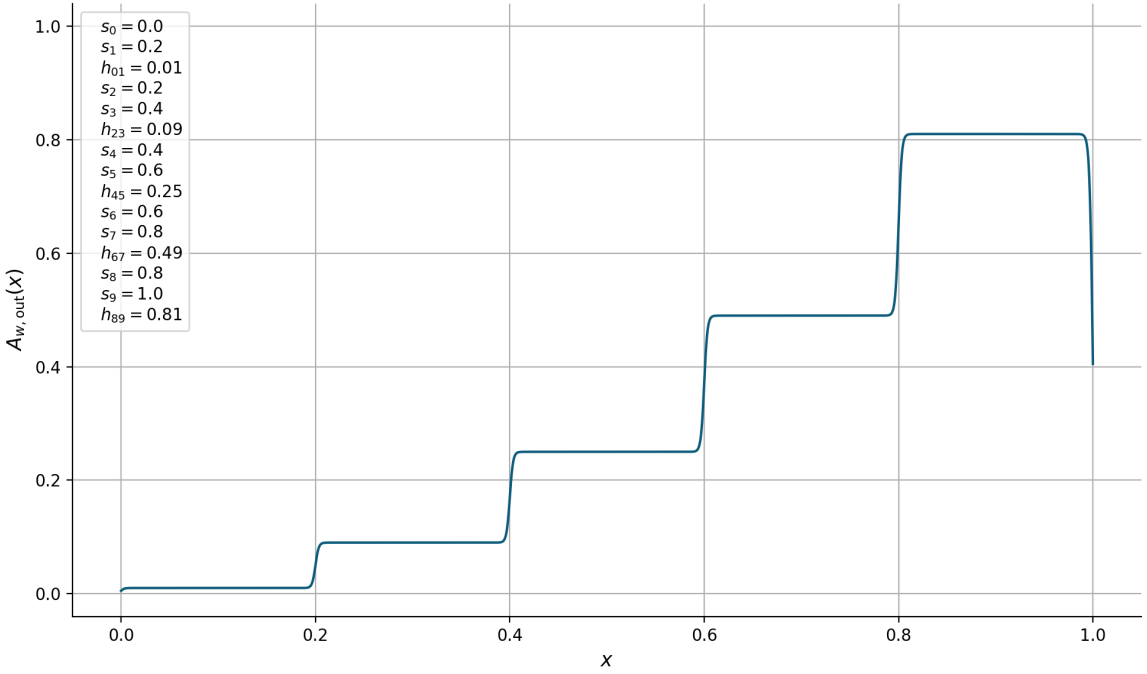


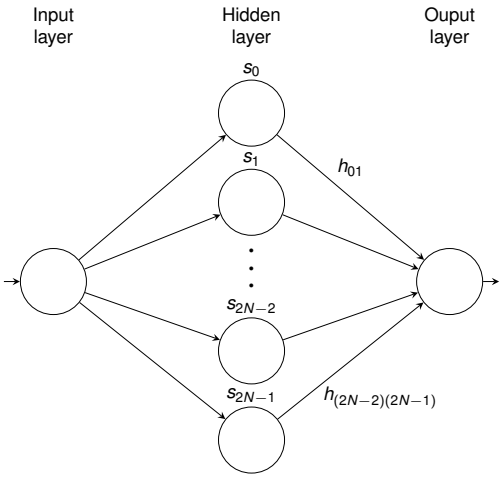


One hidden layer with $2N$ neurons

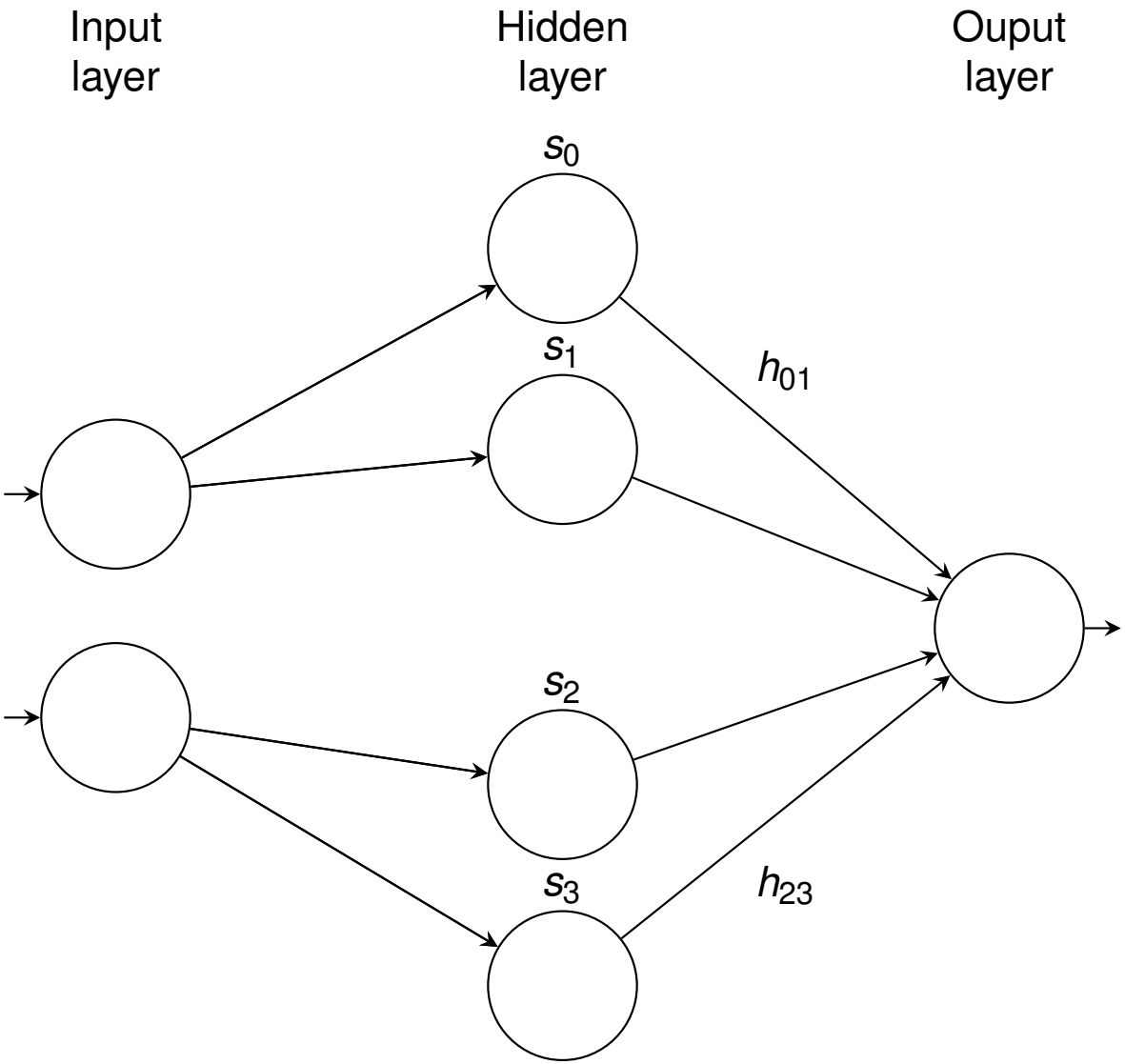


Weighted output of one hidden layer with ten neurons

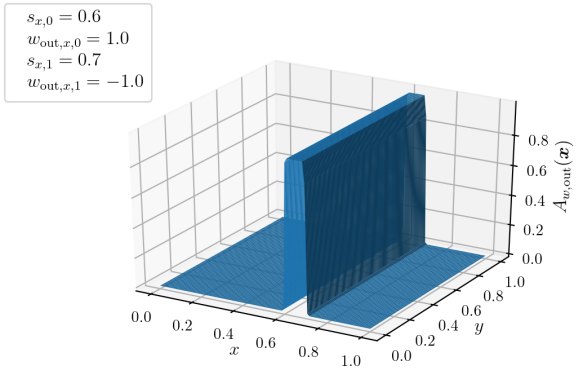


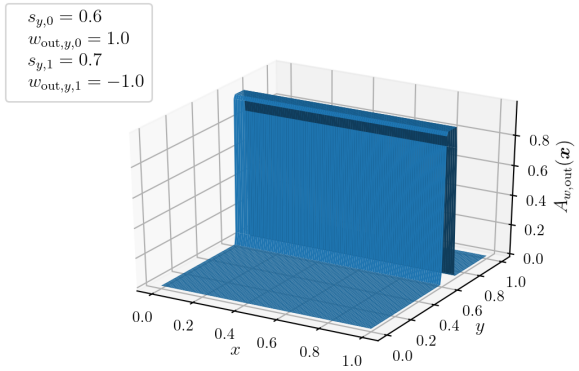


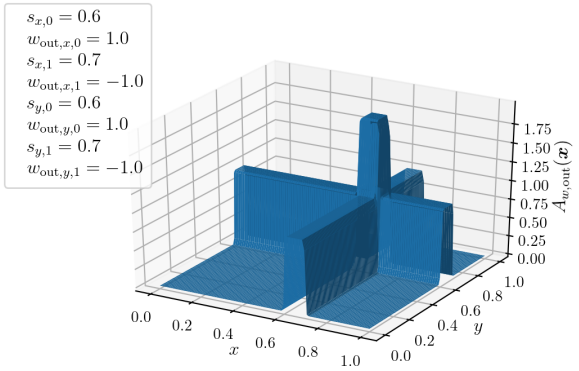
Two dimensional input

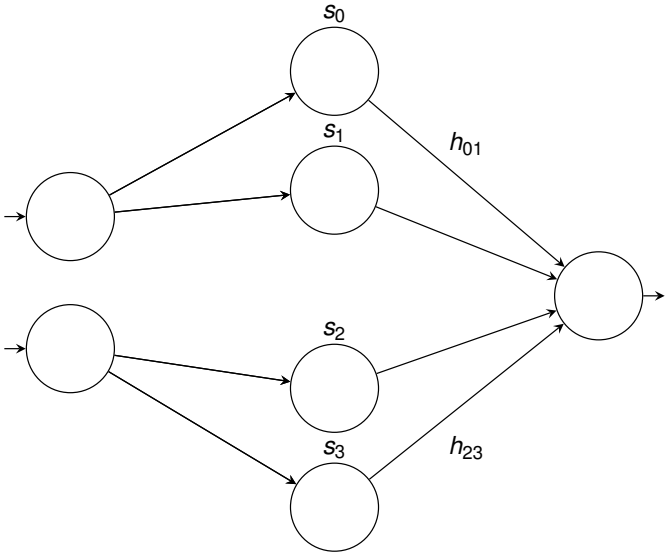


One dimensional towers in two dimensions



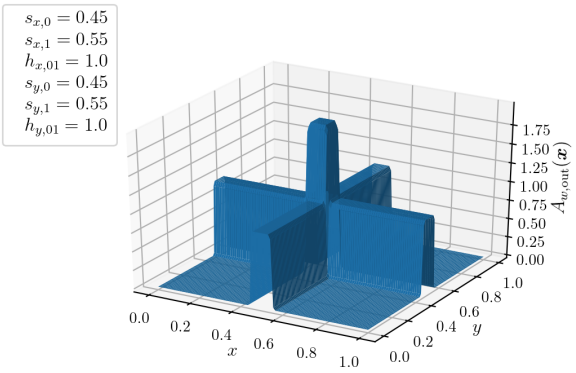


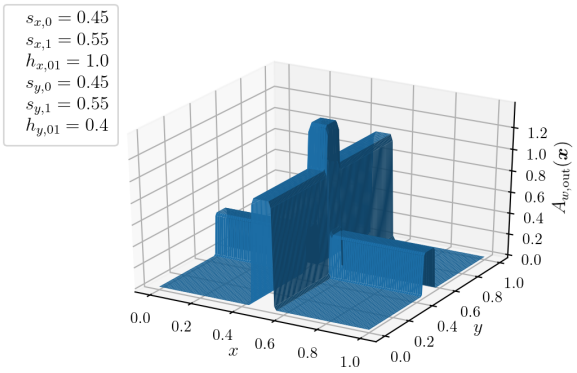


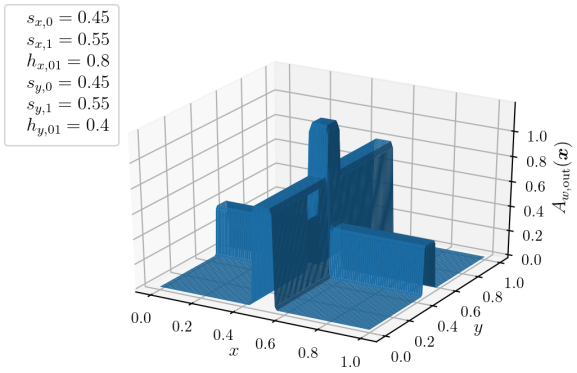


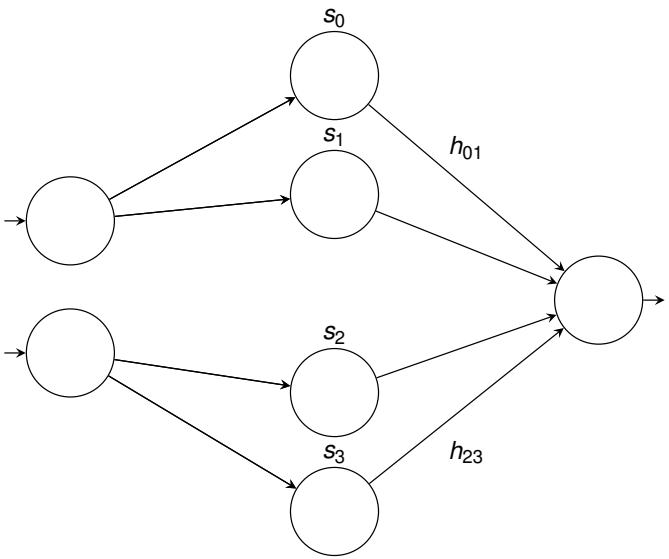
Vary heights two dimensions

Vary the *h* values :









Until now we have always looked at the linear function

$$A_{w,\mathrm{out}}(\boldsymbol{x}) = \sum_i w_{\mathrm{out},i} a_i(\boldsymbol{x}) \tag{12}$$

without taking a possible bias b_{out} into account.

Until now we have always looked at the linear function

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without taking a possible bias b_{out} into account.
By taking the bias b_{out} into account and still assuming that the neuron in the output layer is a linear neuron we get a activation of this neuron of

$$A_{wb,\mathrm{out}}(\boldsymbol{x}) := \sum_i w_{\mathrm{out},i} a_i(\boldsymbol{x}) + b_{\mathrm{out}}. \tag{13}$$

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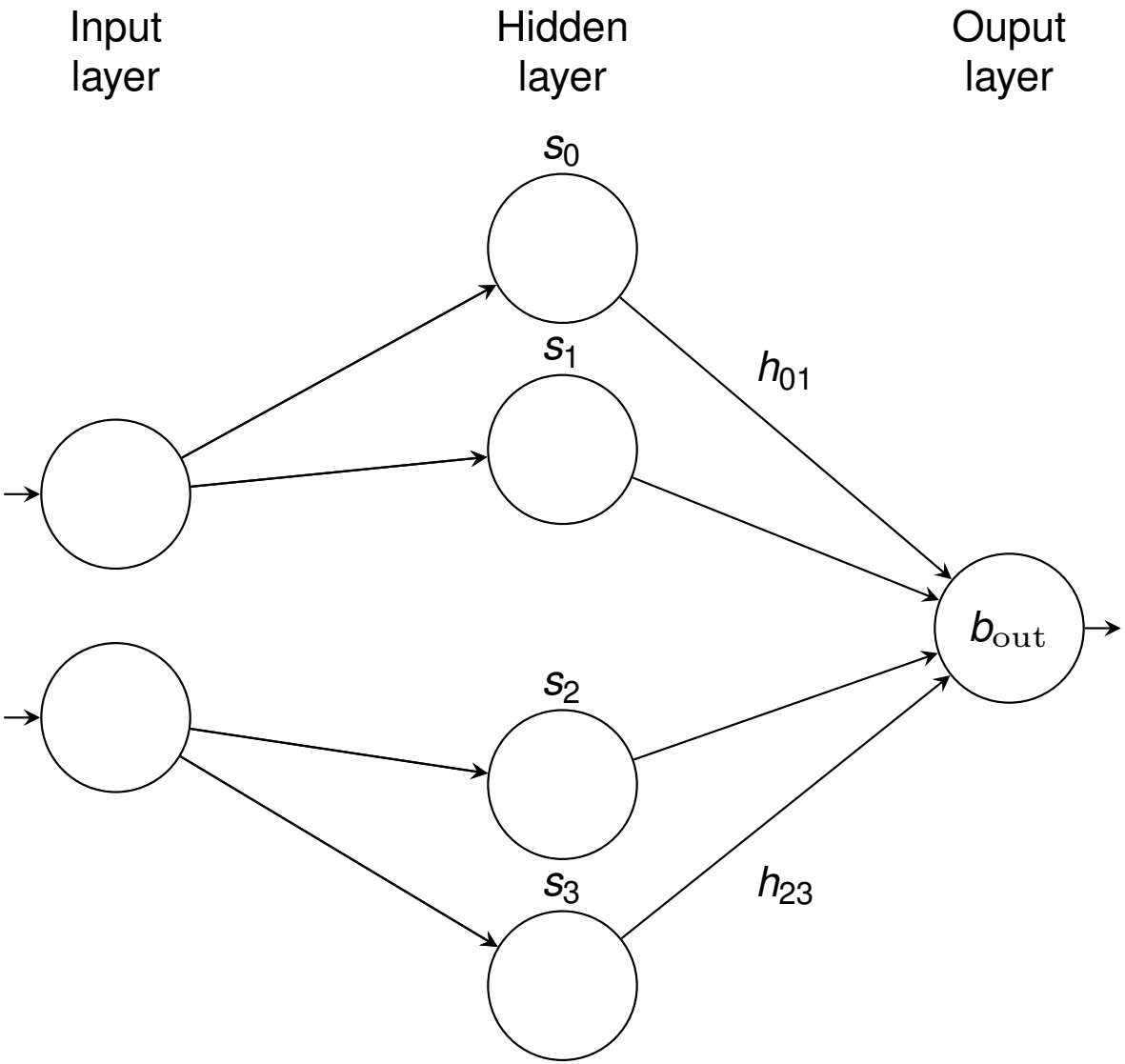
The output of the neuron in the output layer then becomes

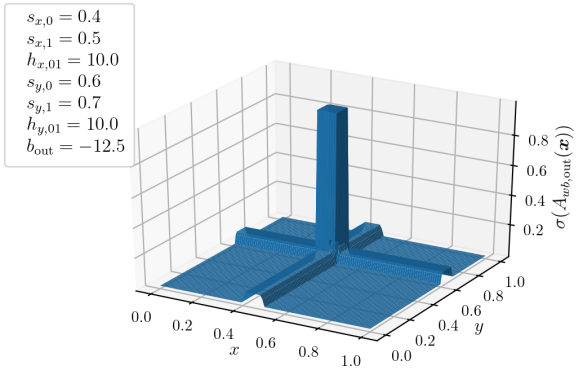
$$O = \sigma(A_{wb,\mathrm{out}}(\boldsymbol{x})). \tag{14}$$

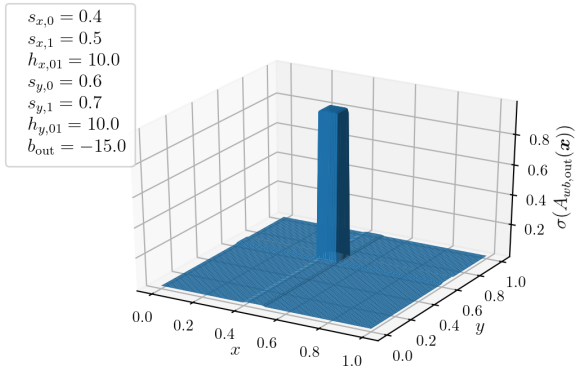
Two dimensional input with b_{out}

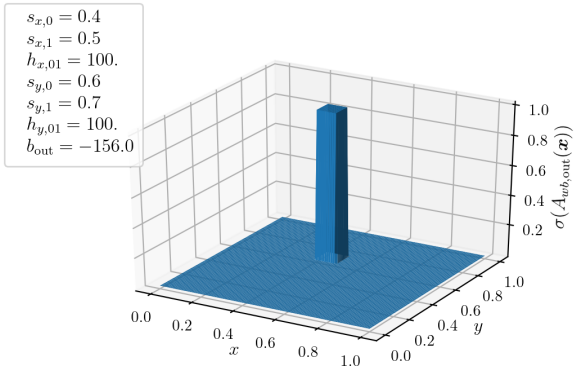


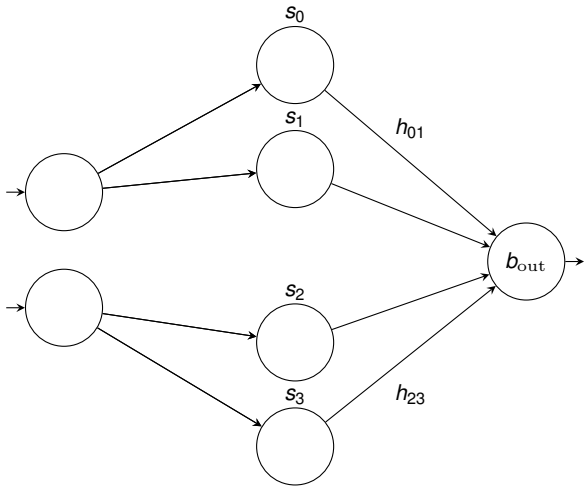
One tower in two dimensions

Vary the b_{out} values :









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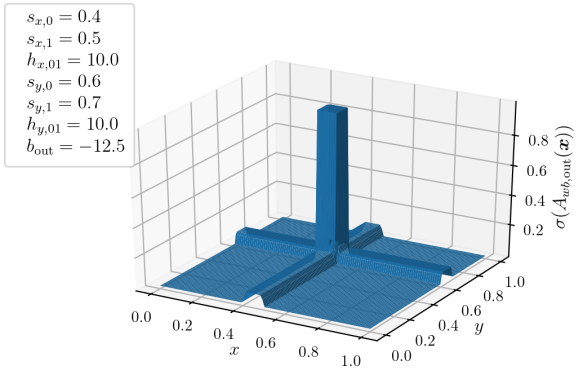

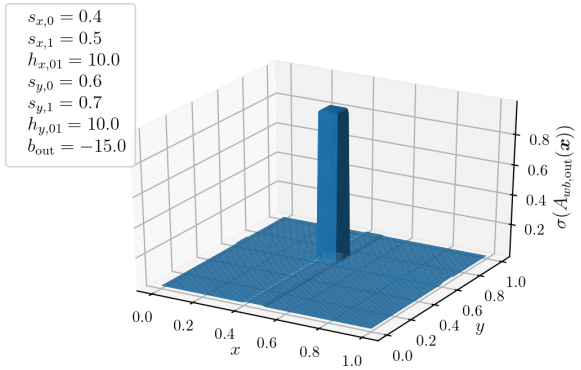

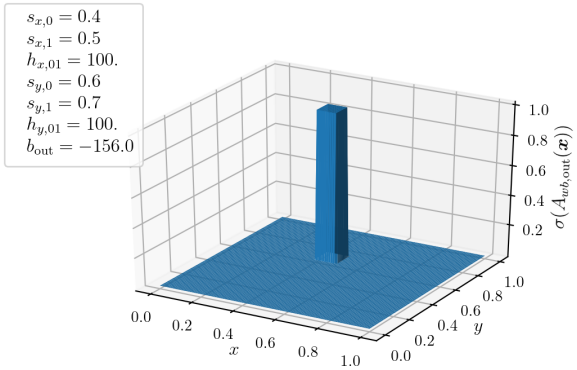

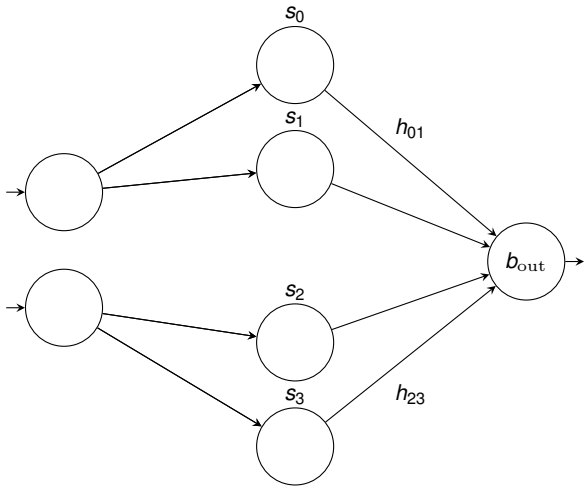

Best towers for

$$\frac{h_{x,01}}{b_{\text{out}}} \approx -\frac{2}{3} \approx \frac{h_{y,01}}{b_{\text{out}}}. \tag{15}$$

The parameter w'_{out}

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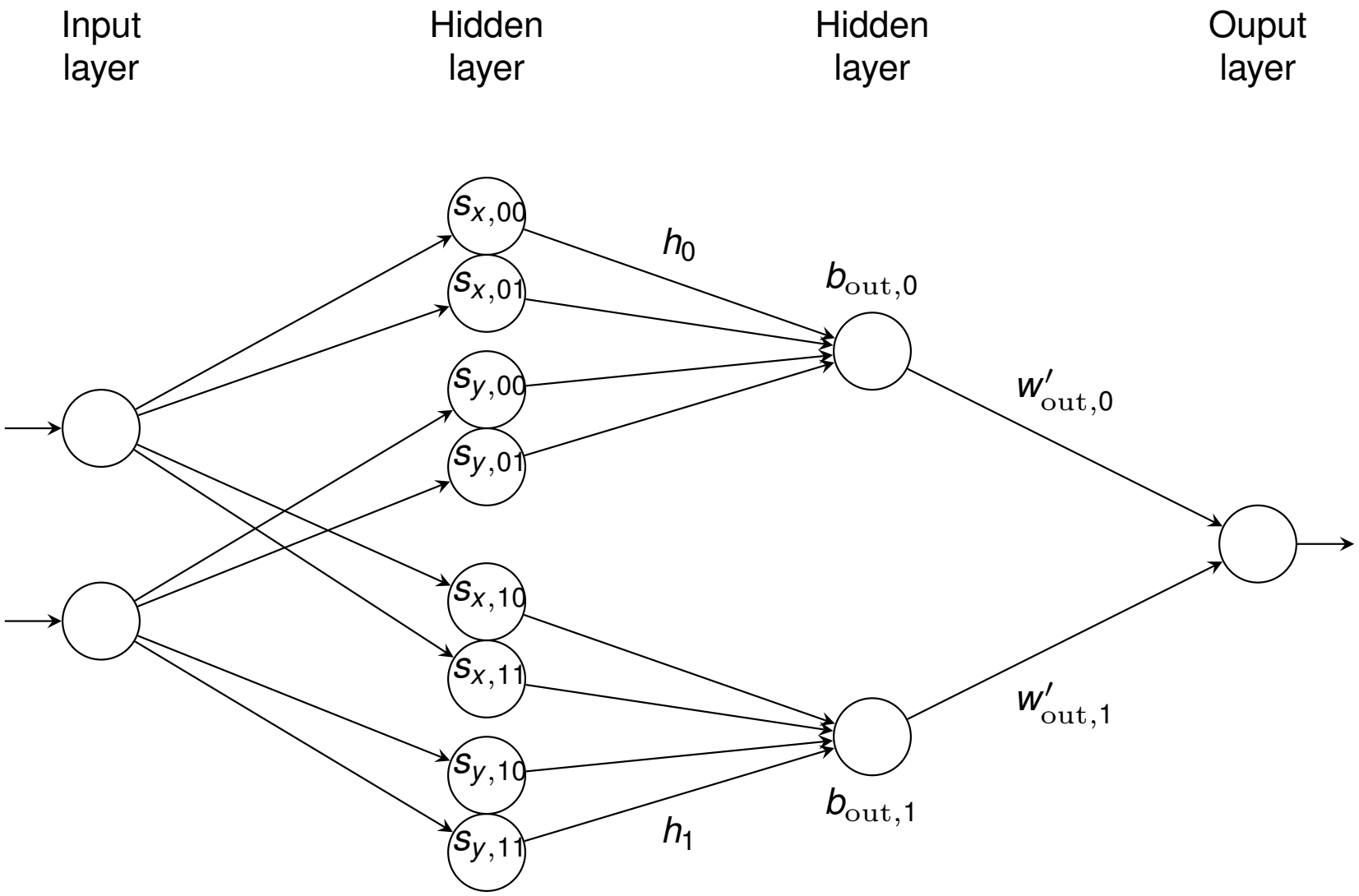
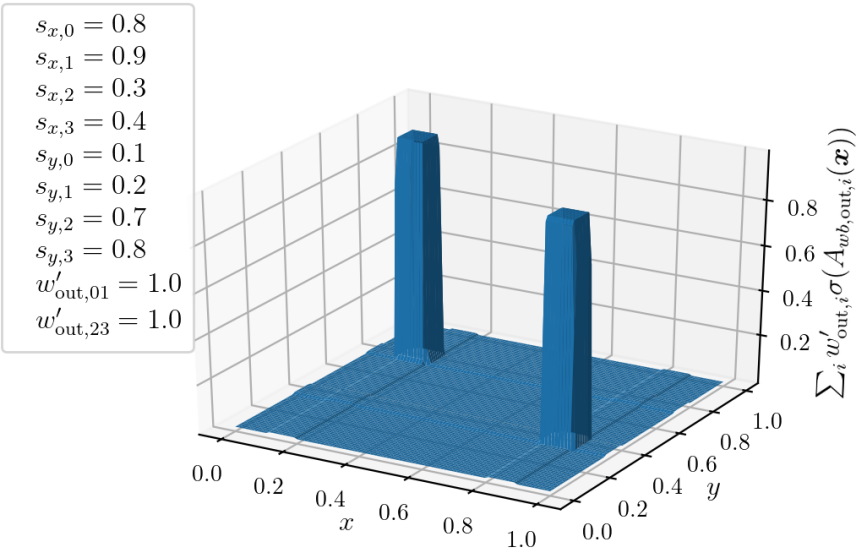
with the bias of the neuron in the new output layer $b'_{\text{out}} = 0$.

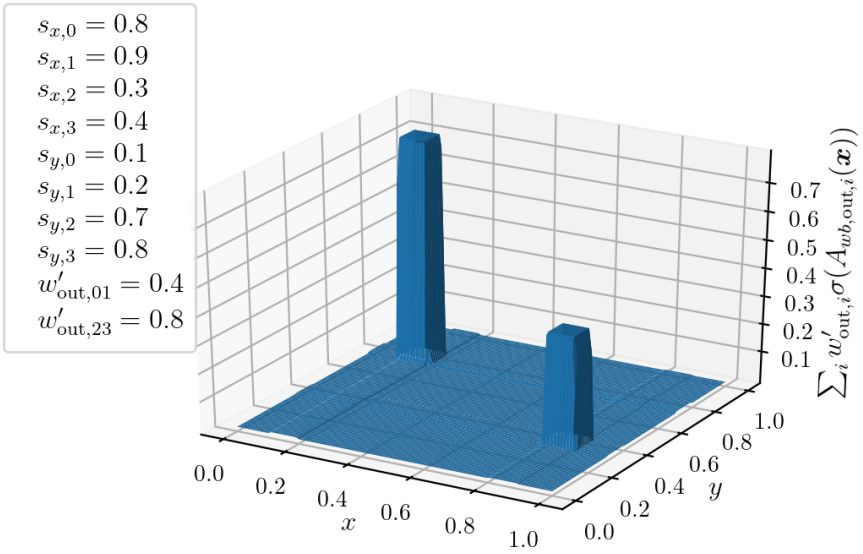
Two hidden layers with w'_{out}

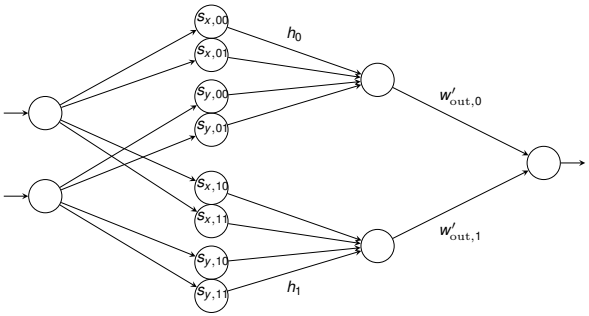


Two weighted towers in two dimensions

Vary the w'_{out} values with $\frac{h_0}{b_{\text{out},0}} = \frac{h_1}{b_{\text{out},1}} \approx -\frac{2}{3}$:







Web-application

Now in an interactive way in a web-application built with Streamlit .

$\mathbb{R}^n \to \mathbb{R}^m$

We only considered the case of approximating real-valued functions like

$$f : \mathbb{R}^n \to \mathbb{R}, \boldsymbol{x} \mapsto f(\boldsymbol{x}) \tag{19}$$

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with $n, m \in \mathbb{N}$ and $m > 1$?
We can simply approximate each ith component of the function $\tilde{f}$ with m different networks and combine them all together like

$$\tilde{f} = \begin{pmatrix} \tilde{f}_0 \\ \vdots \\ \tilde{f}_{m-1} \end{pmatrix} \tag{21}$$

Non-monotonic activation function

Does our activation function have to be monotonic ?

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With $z = wx + b$ we get

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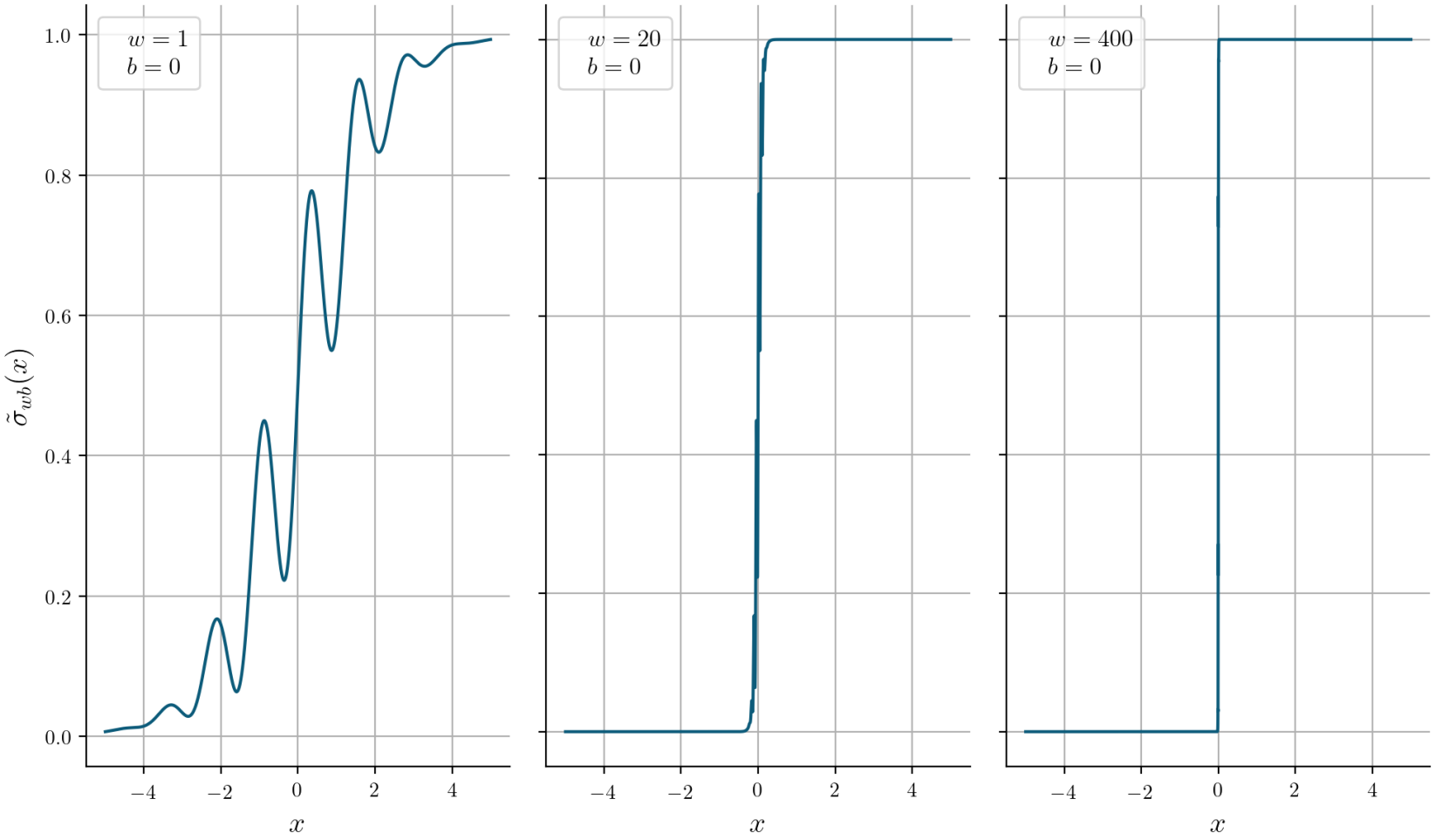
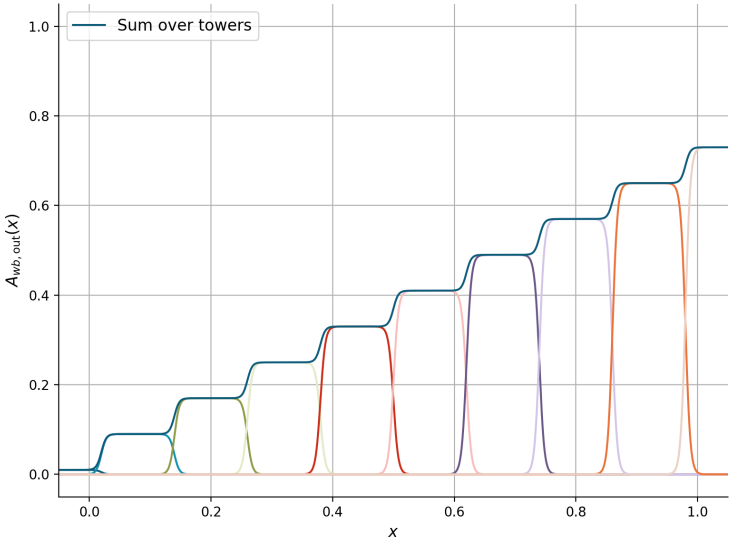
Non-monotonic activation function



Minimize error

One approximation of $\sigma^{-1} \circ f(x)$:



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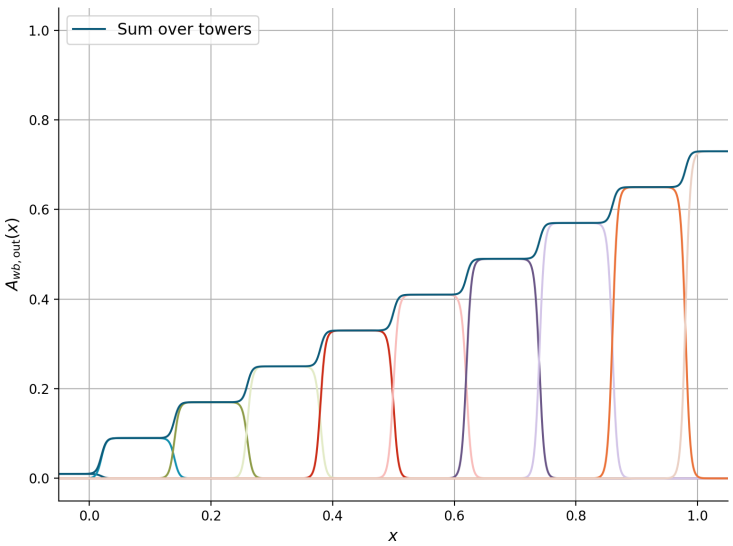
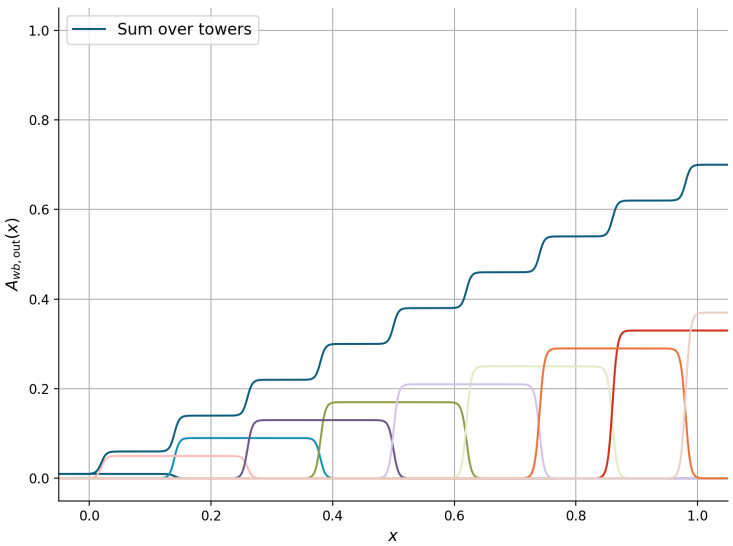
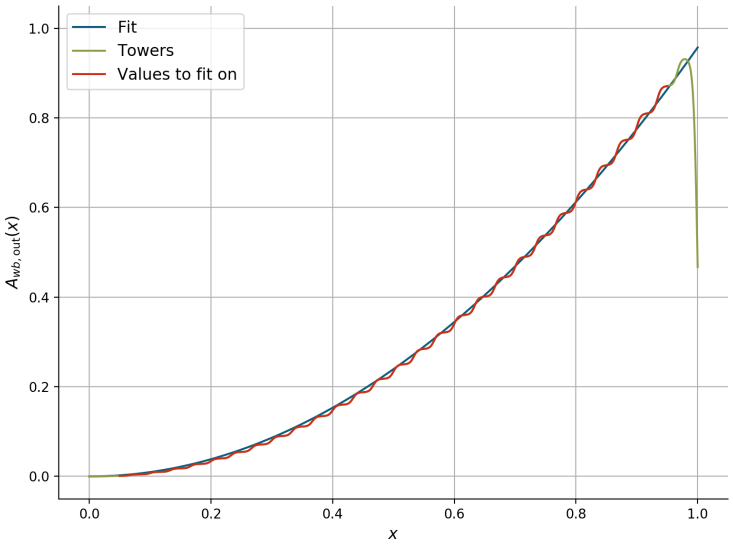


Two approximations of $\sigma^{-1} \circ f(x)/2$:



Minimize error example

One approximation with thirty towers.



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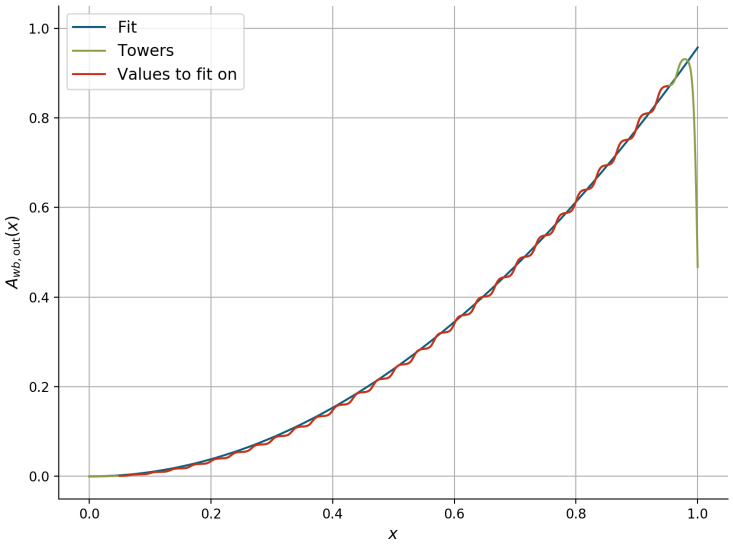

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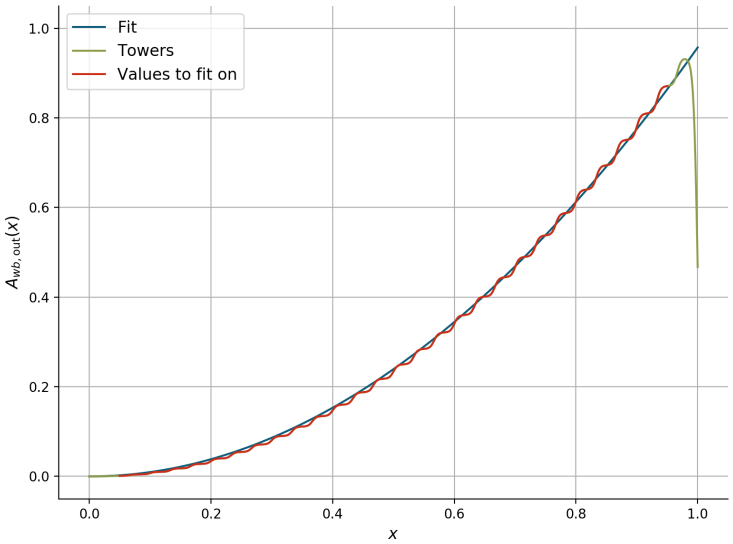
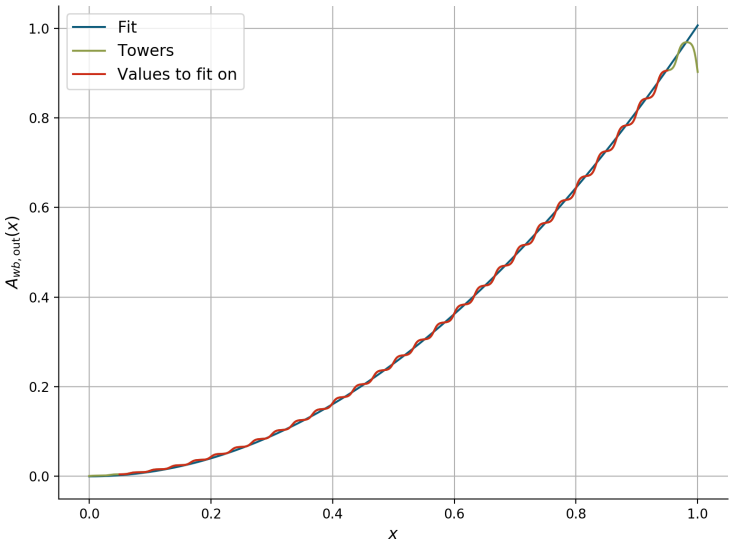


Six approximations with five towers each.



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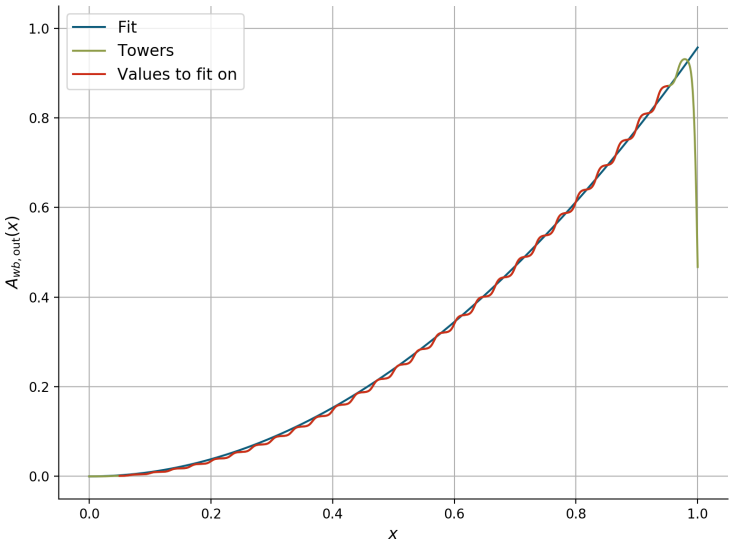

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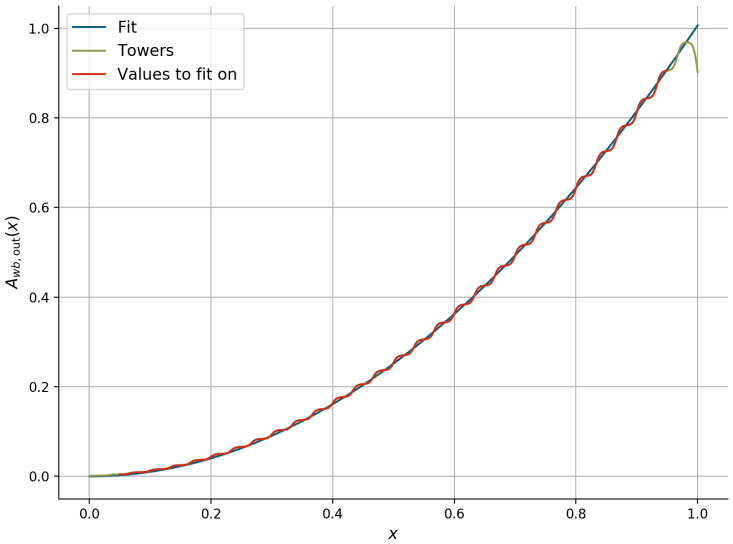

$$\sigma_{\mathrm{std}} \approx 1.31 \cdot 10^{-4} \tag{25}$$

Mathematical Approach

— A Brief Overview.

1 Overview

2 Visual Approach
- Caveats
- Activation functions
- Construction
- The interactive way
- $\mathbb{R}^n \to \mathbb{R}^m$
- Non-monotonic activation functions
- Minimize error

3 Mathematical Approach
- Mathematical expression of a neural network
- What to prove ?
- Theorem I - Approximation of functions
- Lemma I - Sigmoidal functions are discriminatory

4 Back to Neural Networks

5 Appendix

Abstraction of a given problem

Neural Network $\longrightarrow$ Mathematical Expression

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Problem with n input parameters	$\longrightarrow$	Input-vector $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$

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$\downarrow$

Can g represent any $f \in \mathcal{C}(\boldsymbol{\mathcal{I}}_n)$?

Space of functions

Neural Network	$\longrightarrow$	Mathematical Expression
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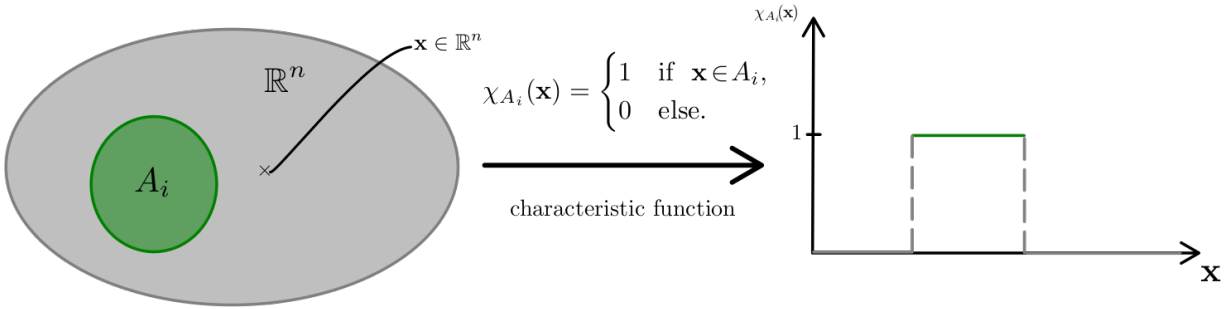

FIGURE – Characteristic function $\chi_{A_i}(\boldsymbol{x})$

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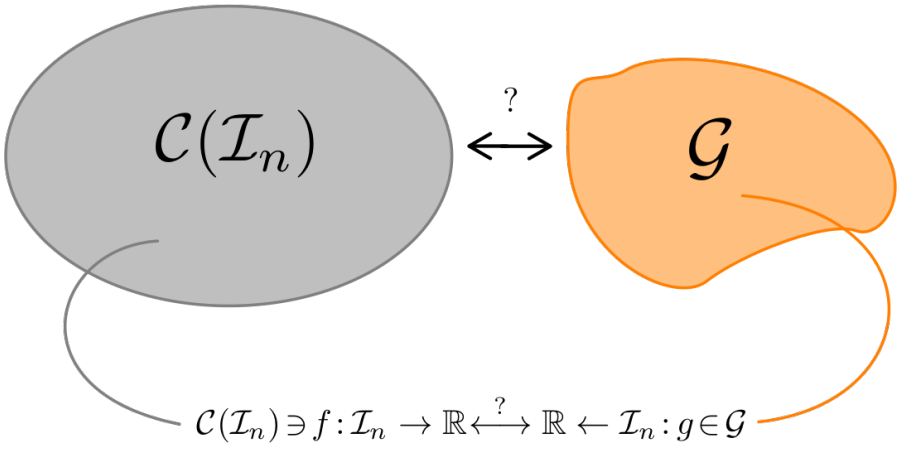

FIGURE – Function f we want to approximate with a function g.

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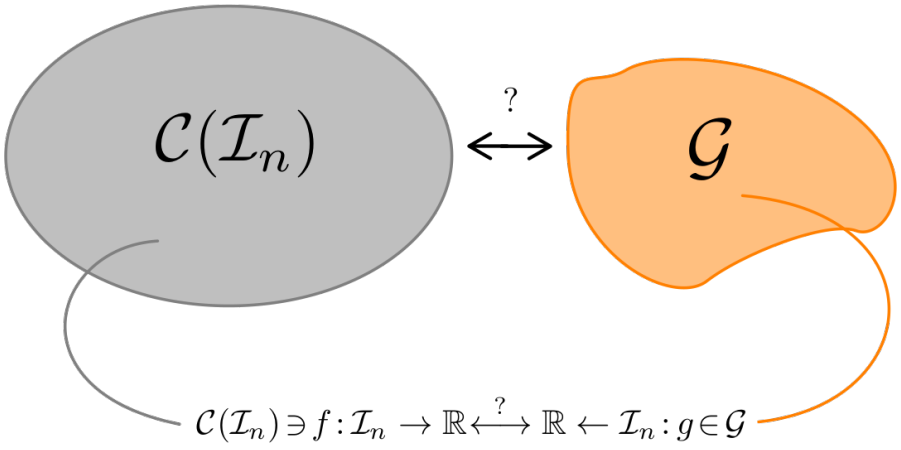

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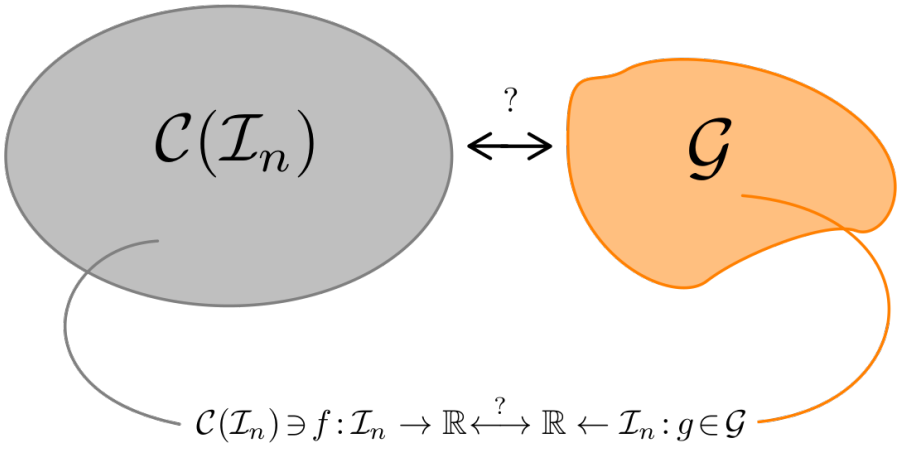

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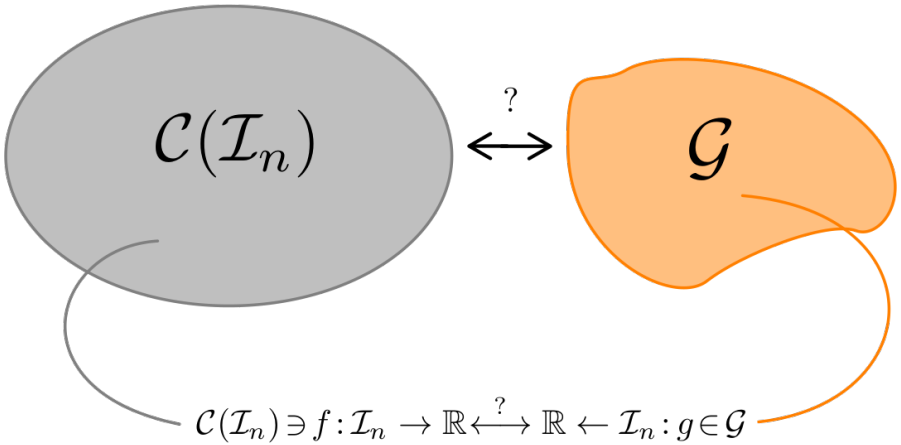

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or topologically speaking, is $\mathcal{G}$ dense in $\mathcal{C}(\boldsymbol{\mathcal{I}}_n)$?

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How do $\mathcal{G}$ and $\mathcal{C}(\mathcal{I}_n)$ topologically relate to each other ?

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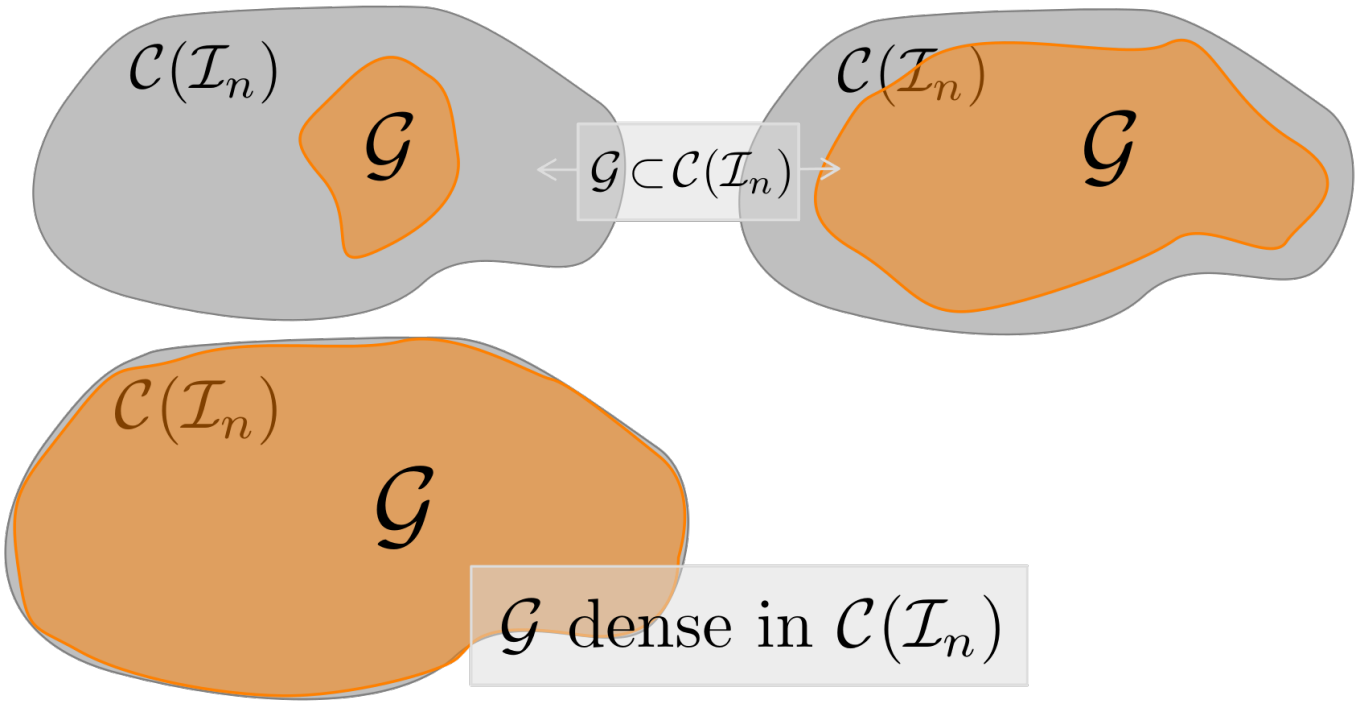

FIGURE – Topological relation of $\mathcal{G}$ and $\mathcal{C}(\boldsymbol{\mathcal{I}}_n)$.

Is $\mathcal{G}$ dense in $\mathcal{C}(\mathcal{I}_n)$?

How do $\mathcal{G}$ and $\mathcal{C}(\mathcal{I}_n)$ topologically relate to each other ?

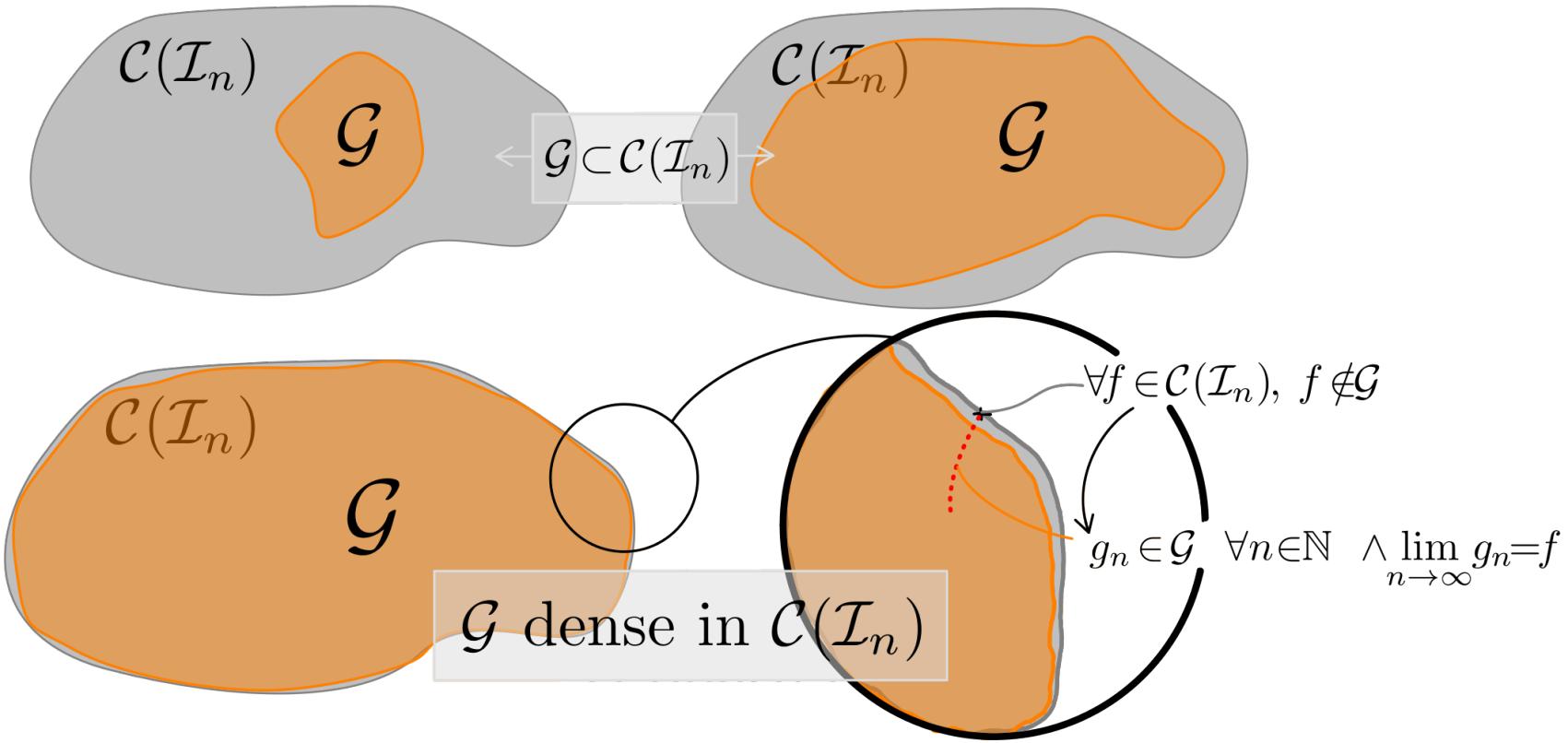

FIGURE – Topological relation of $\mathcal{G}$ and $\mathcal{C}(\mathcal{I}_n)$.

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To prove

$\mathcal{G}$ is dense in $\mathcal{C}(\mathcal{I}_n)$ if $\forall f \in \mathcal{C}(\mathcal{I}_n), f \notin \mathcal{G}$ and $\varepsilon > 0$, $\exists$ a function $g \in \mathcal{G}$ with,

$$\begin{aligned} &|g(\boldsymbol{x}) - f(\boldsymbol{x})| < \varepsilon \quad , \forall \boldsymbol{x} \in \mathcal{I}_n \\ &\Leftrightarrow \overline{\mathcal{G}} = \mathcal{C}(\mathcal{I}_n) \quad , \overline{\mathcal{G}} \text{ closure of } \mathcal{G}. \end{aligned} \tag{27}$$

Further Definitions

Definition (Sigmoidal Functions)

A continuous function $\sigma \in \mathcal{C}(\mathcal{I}_n)$ is called **sigmoidal** if,

$$\sigma(z) \to \begin{cases} 1 & \text{if } z \to \infty \\ 0 & \text{if } z \to -\infty \end{cases} \tag{28}$$

Definition (Discriminatory)

A function σ is called **discriminatory** if for a measure $\mu \in \mathcal{M}(\mathcal{I}_n)$,

$$\int_{\mathcal{I}_n} \sigma(\boldsymbol{w}^T \boldsymbol{x} + b) \mathrm{d}\mu(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{w} \in \mathbb{R}^n, b \in \mathbb{R}$$
$$\Rightarrow \mu = 0 \tag{29}$$

$\mathcal{C}(\mathcal{I}_n)$	Continuous functions
$\mathcal{G} \subset \mathcal{C}(\mathcal{I}_n)$	Functions of form $g(\boldsymbol{x})$
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Let σ be any **discriminatory** continuous function. Then $\mathcal{G}$, i.e. the space functions of the form

$$g(\boldsymbol{x}) = \sum_{i=1}^{N} \alpha_i \sigma(\boldsymbol{w}_i^T \boldsymbol{x} + b_i) \quad \alpha_i, b_i \in \mathbb{R} \;\; \boldsymbol{w}_i \in \mathbb{R}^n, \tag{30}$$

is dense in $\mathcal{C}(\mathcal{I}_n)$, i.e. given any function $f \in \mathcal{C}(\mathcal{I}_n)$ and $\mathbb{R} \ni \varepsilon > 0 \Rightarrow \exists \;\; g \in \mathcal{G}$ for which,

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Proof - in a visual way

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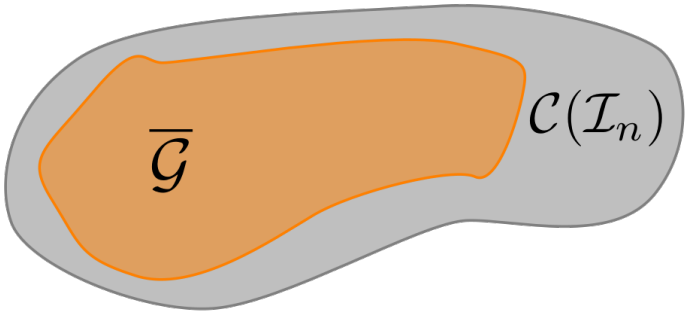

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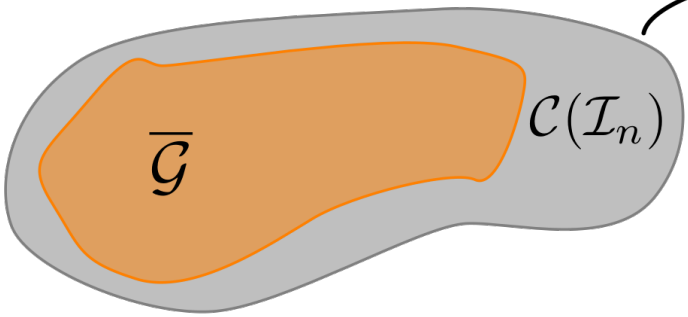

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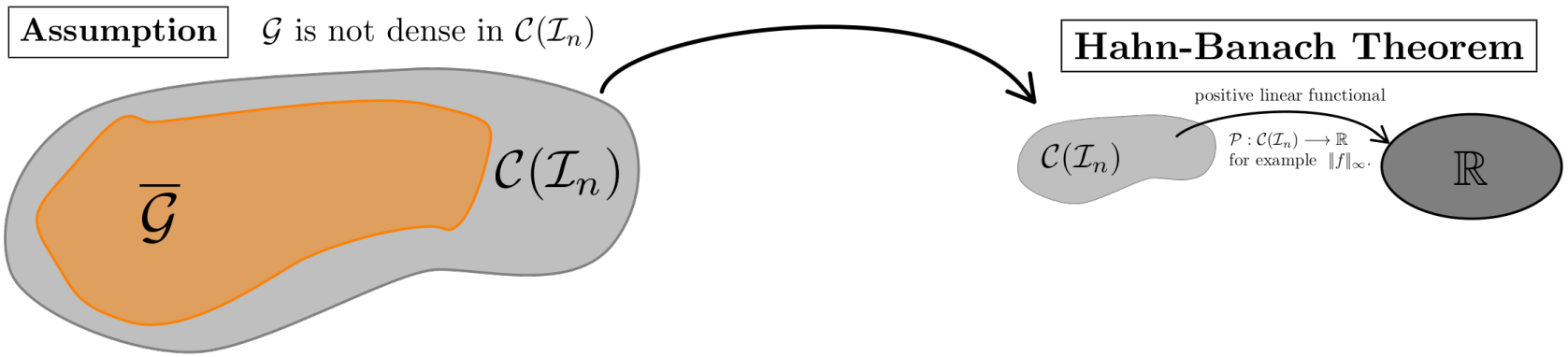

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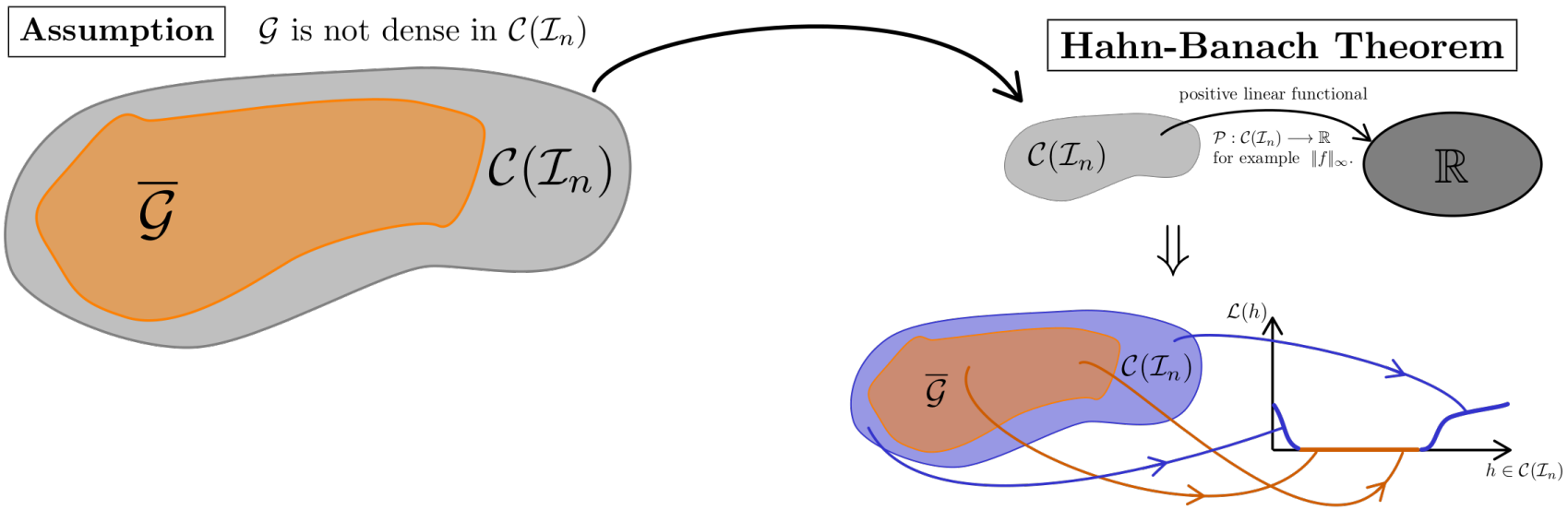

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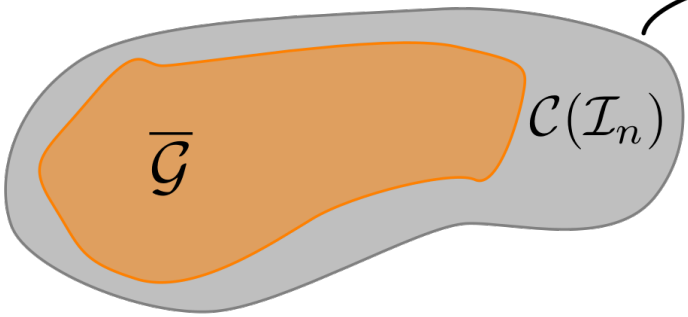

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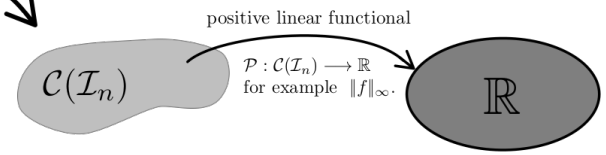

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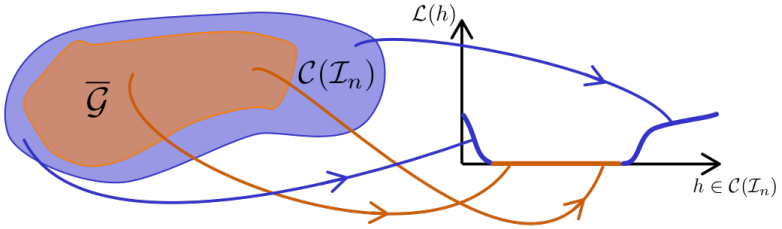

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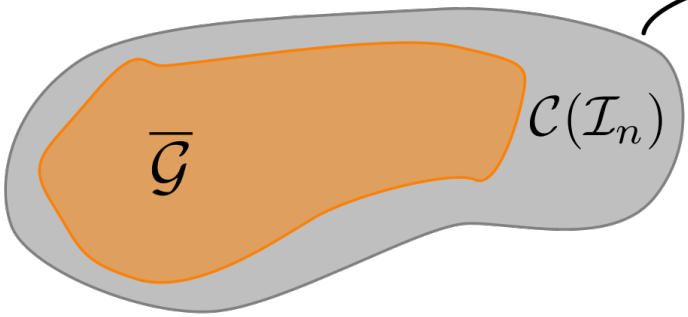

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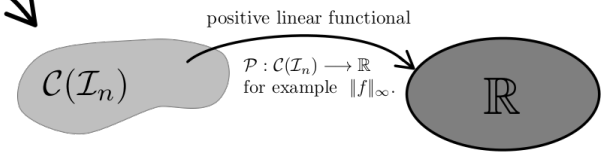

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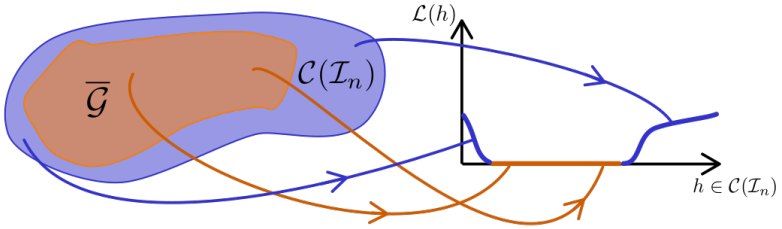

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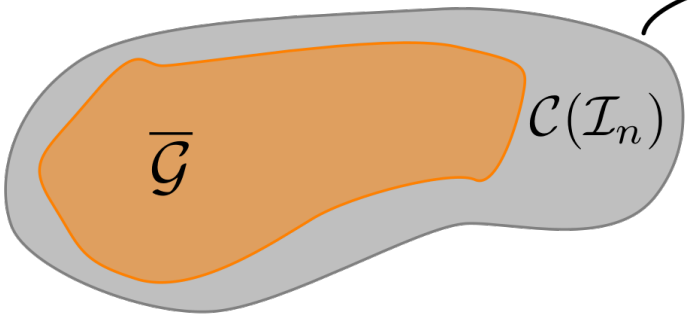

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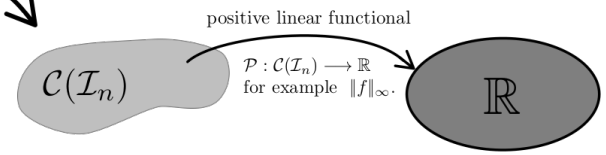

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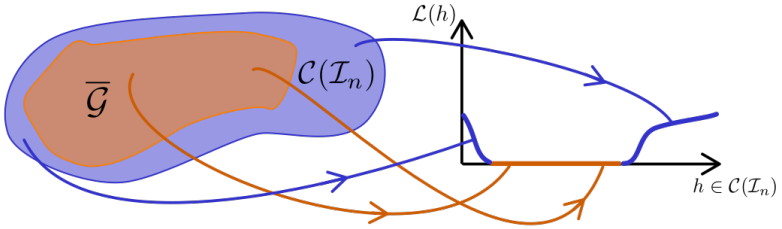

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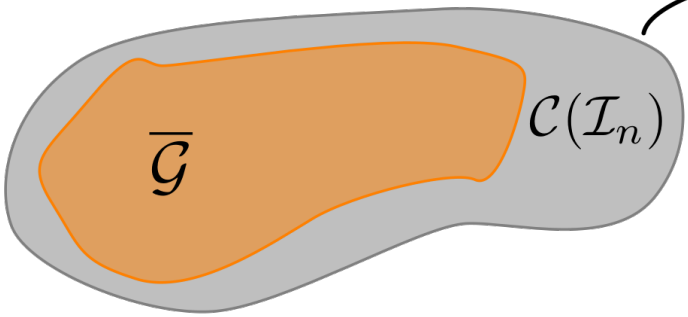

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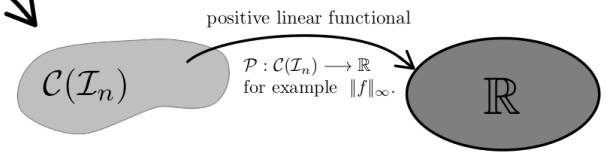

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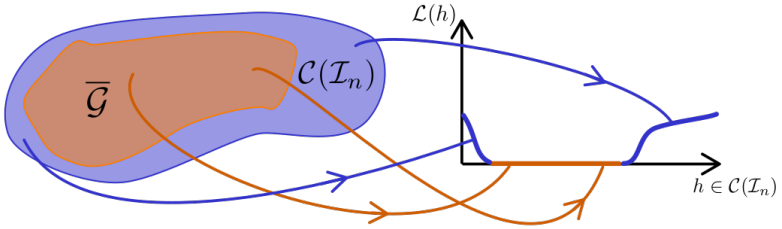

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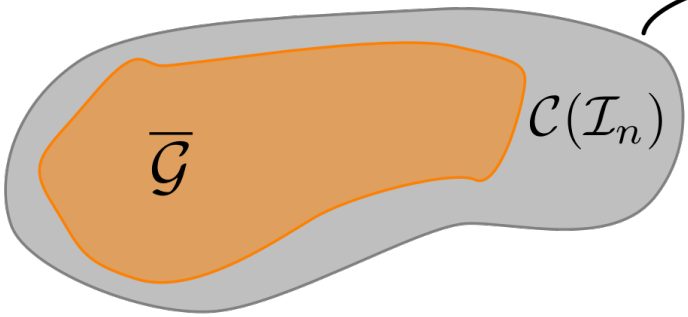

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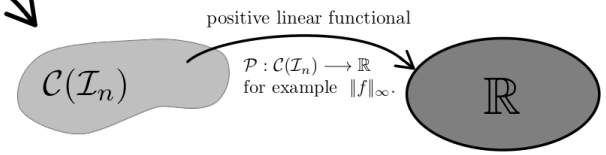

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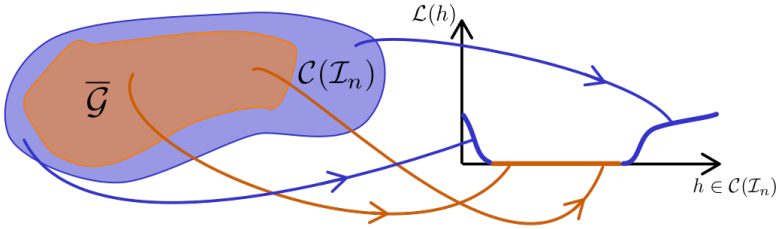

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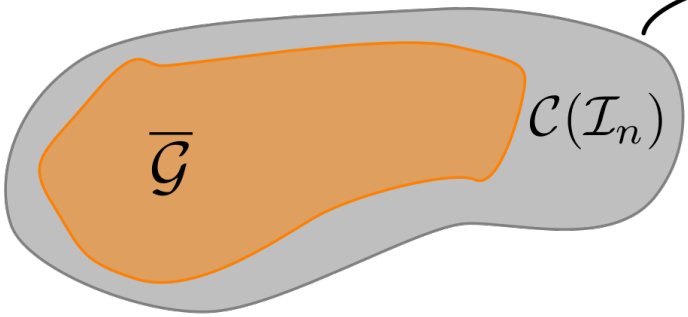

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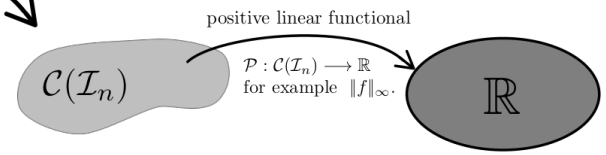

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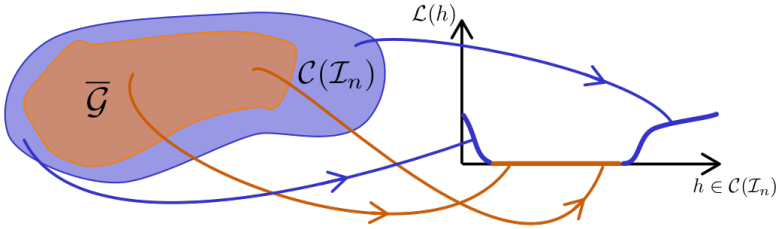

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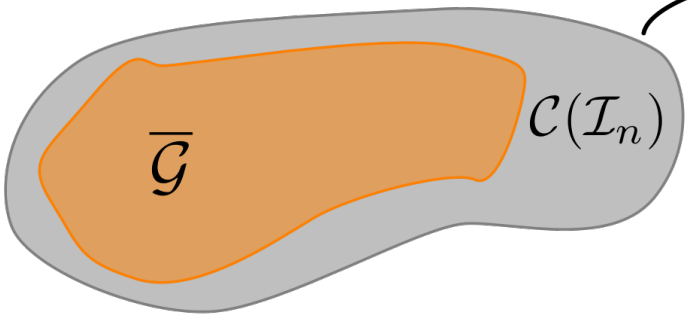

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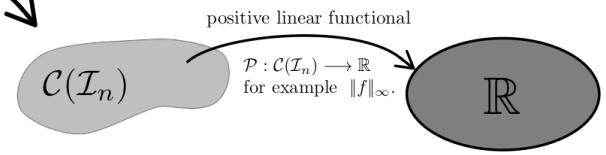

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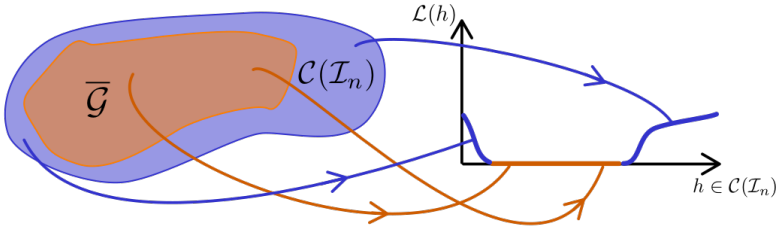

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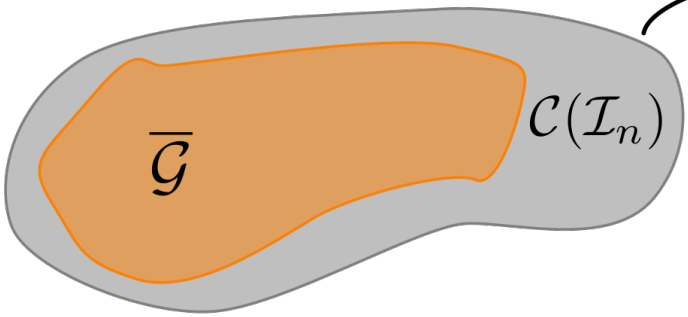

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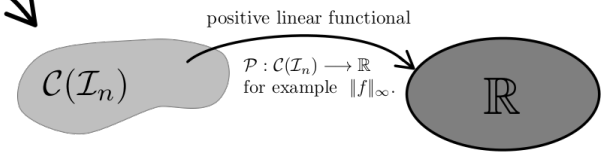

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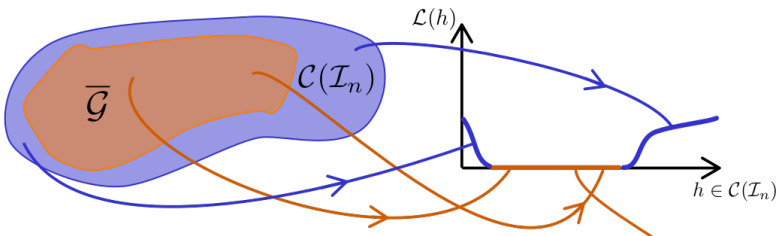

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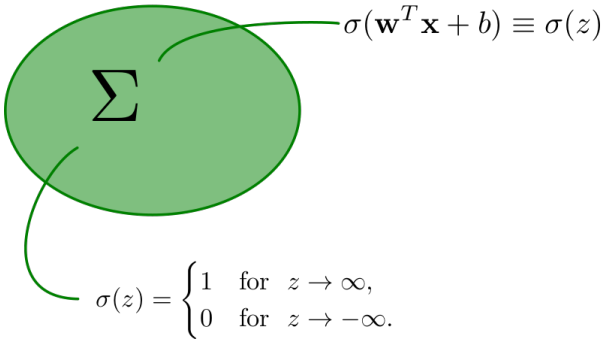
Are all sigmoidal functions discriminatory ?

Lemma I

If σ is a continuous bounded *sigmoidal* function, then σ is *discriminatory*.

Proof - in a visual way

Proof of Lemma I



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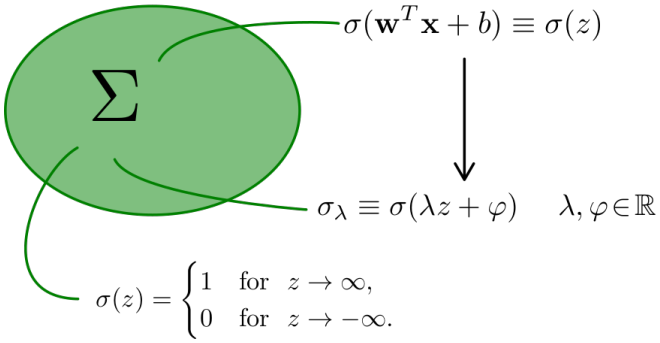

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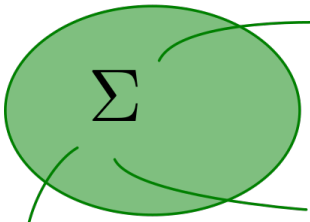

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$$\sigma(z) = \begin{cases} 1 & \text{for } z \to \infty, \\ 0 & \text{for } z \to -\infty. \end{cases}$$

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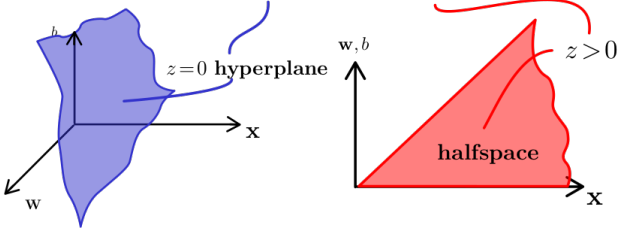

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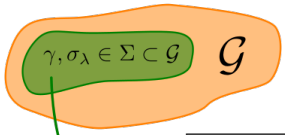

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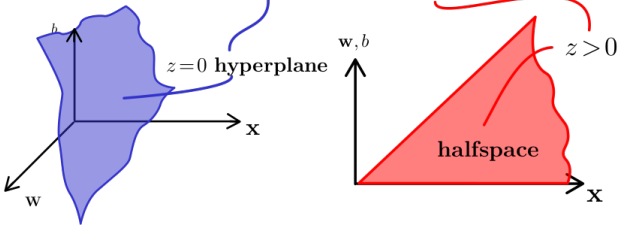

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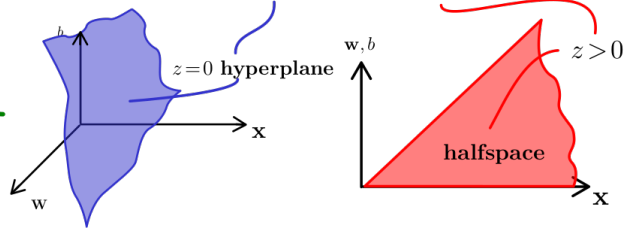

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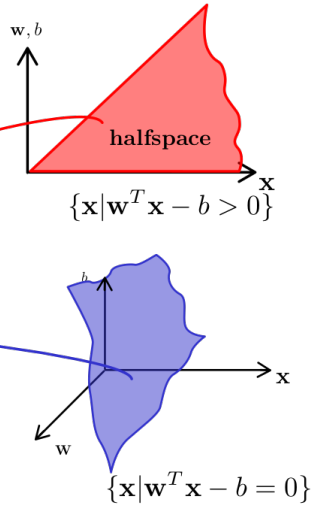

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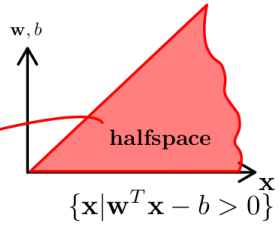

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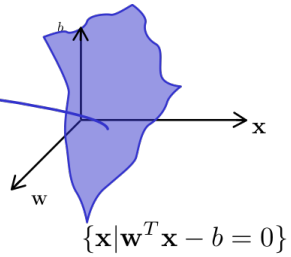

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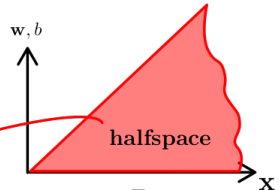

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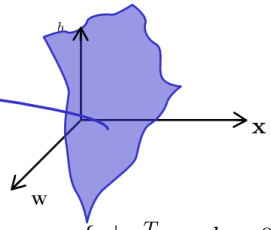

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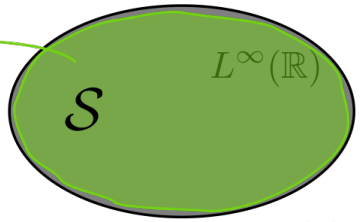

$\mathcal{S}$ is dense in $L^\infty(\mathbb{R})$

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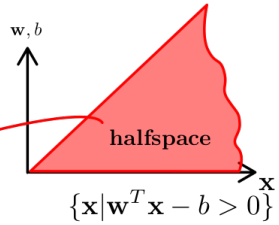

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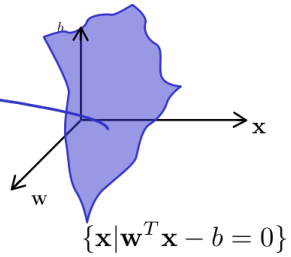

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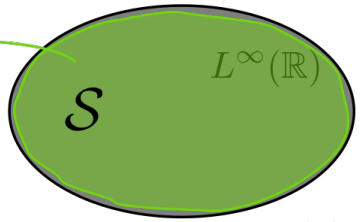

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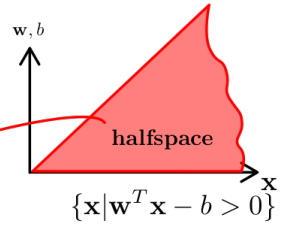

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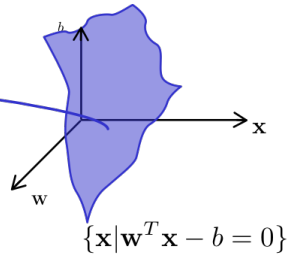

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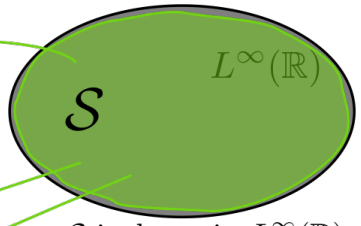

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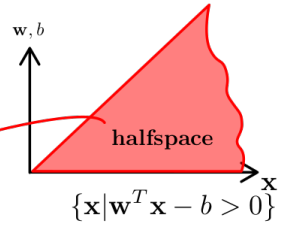

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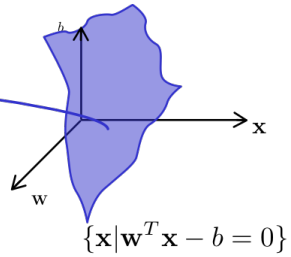

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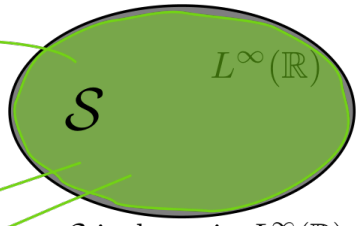

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□

Conclusion

Any *sigmoidal* function σ is *discriminatory*.

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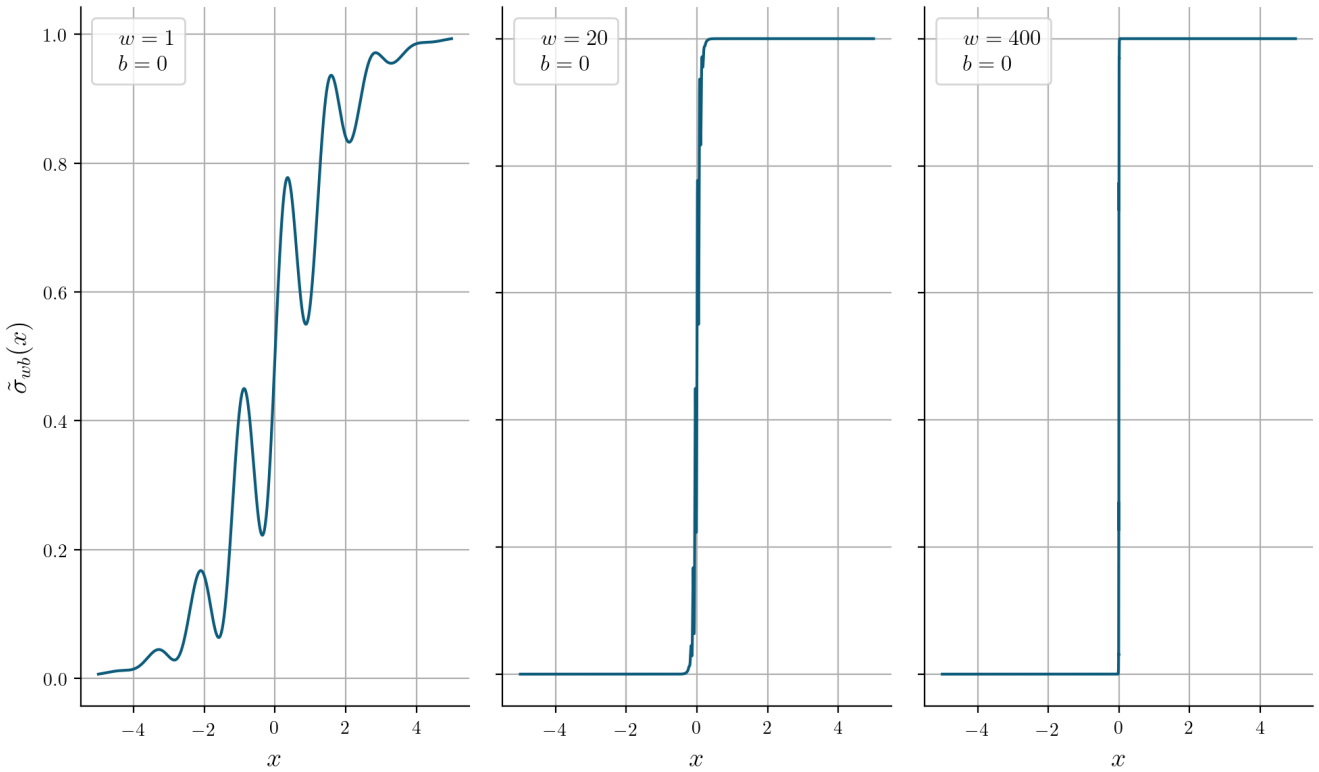

FIGURE – Proven for continuous bounded sigmoidal function σ.

Solving any function

What we have proven :

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1. $\mathcal{G}$ is dense in $\mathcal{C}(\boldsymbol{\mathcal{I}}_n)$, hence for any $f \in \mathcal{C}(\boldsymbol{\mathcal{I}}_n)$ and $\varepsilon > 0 \ \exists \ g\left(\sigma(\boldsymbol{x})\right)$ with

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Problem of Machine Learning solved — ?

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We dont know the function, we want to approximate.

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The Idea

Find the algorithm (learning system, e.g. *Back-Propagation*) that will approximate the *correct* or rather *correctest* function to a given Problem, with the knowledge that any function can be approximated !

Thank you for your attention !

Are there any questions ?

Activation functions

Step function :

$$\sigma(z) = \begin{cases} 0 & : z < 0 \\ 1 & : z \geq 0 \end{cases} \tag{31}$$

Sigmoid function :

$$\sigma(z) = \frac{1}{1 + e^{-z}} \tag{32}$$

ReLU function :

$$\sigma(z) = \begin{cases} 0 & : z < 0 \\ z & : z \geq 0 \end{cases} \tag{33}$$

Parametrised ReLU function :

$$\sigma(z) = \begin{cases} \alpha_1 z & : z < 0 \\ \alpha_2 z & : z \geq 0 \end{cases} \tag{34}$$

Swish function :

$$\sigma(z) = \frac{z}{1 + e^{-z}} \tag{35}$$

Linear function :

$$\sigma(z) = \alpha z \tag{36}$$

Tanh function :

$$\sigma(z) = \frac{2}{1 + e^{-2z}} - 1 \tag{37}$$

Leaky ReLU function :

$$\sigma(z) = \begin{cases} 0.1z & : z < 0 \\ z & : z \geq 0 \end{cases} \tag{38}$$

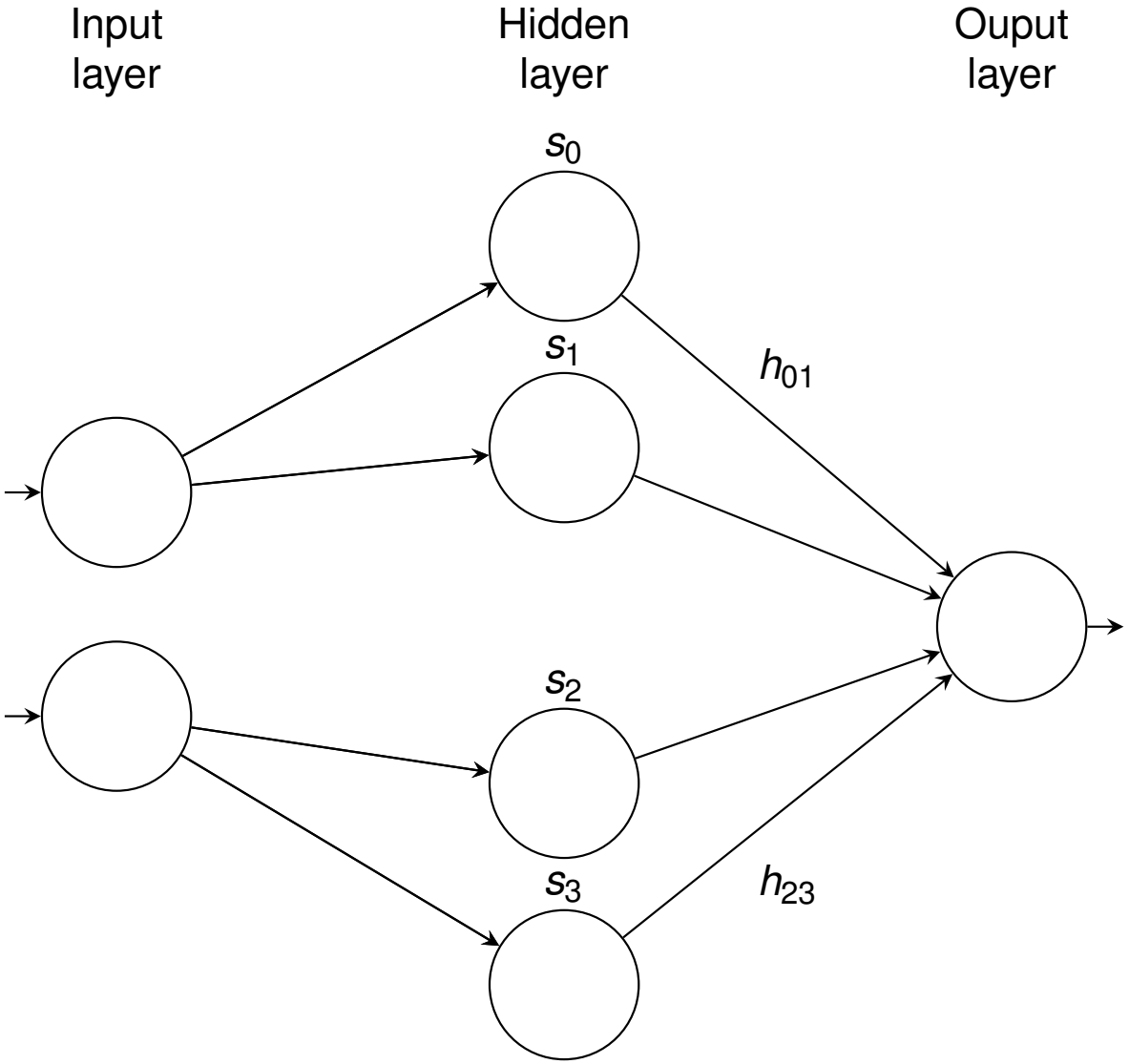
ELU function :

$$\sigma(z) = \begin{cases} \alpha(e^z - 1) & : z < 0 \\ z & : z \geq 0 \end{cases} \tag{39}$$

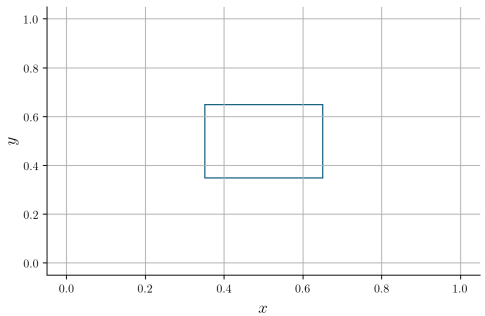
Sine ReLU function :

$$\sigma(z) = \begin{cases} \alpha(\sin(z) - \cos(z)) & : z < 0 \\ z & : z \geq 0 \end{cases} \tag{40}$$

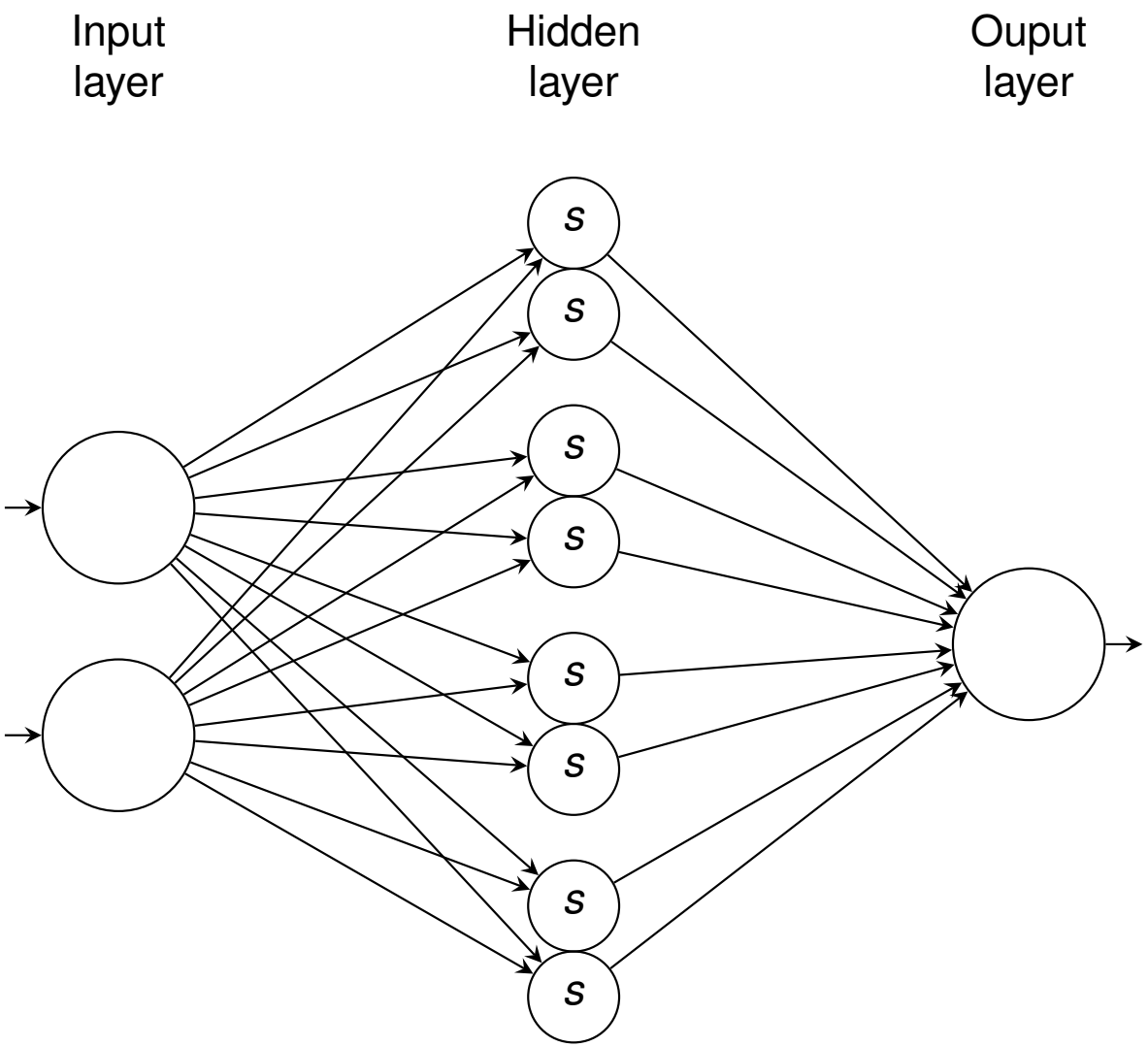
Two dimensional input - rectangle



Corresponding area :



Two dimensional input - two rectangles



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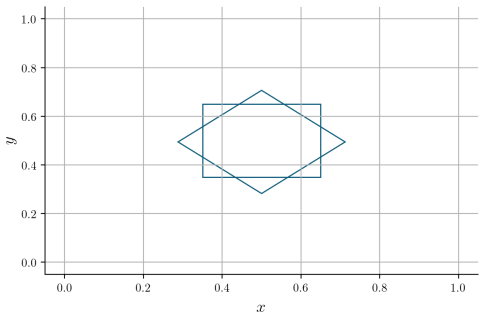
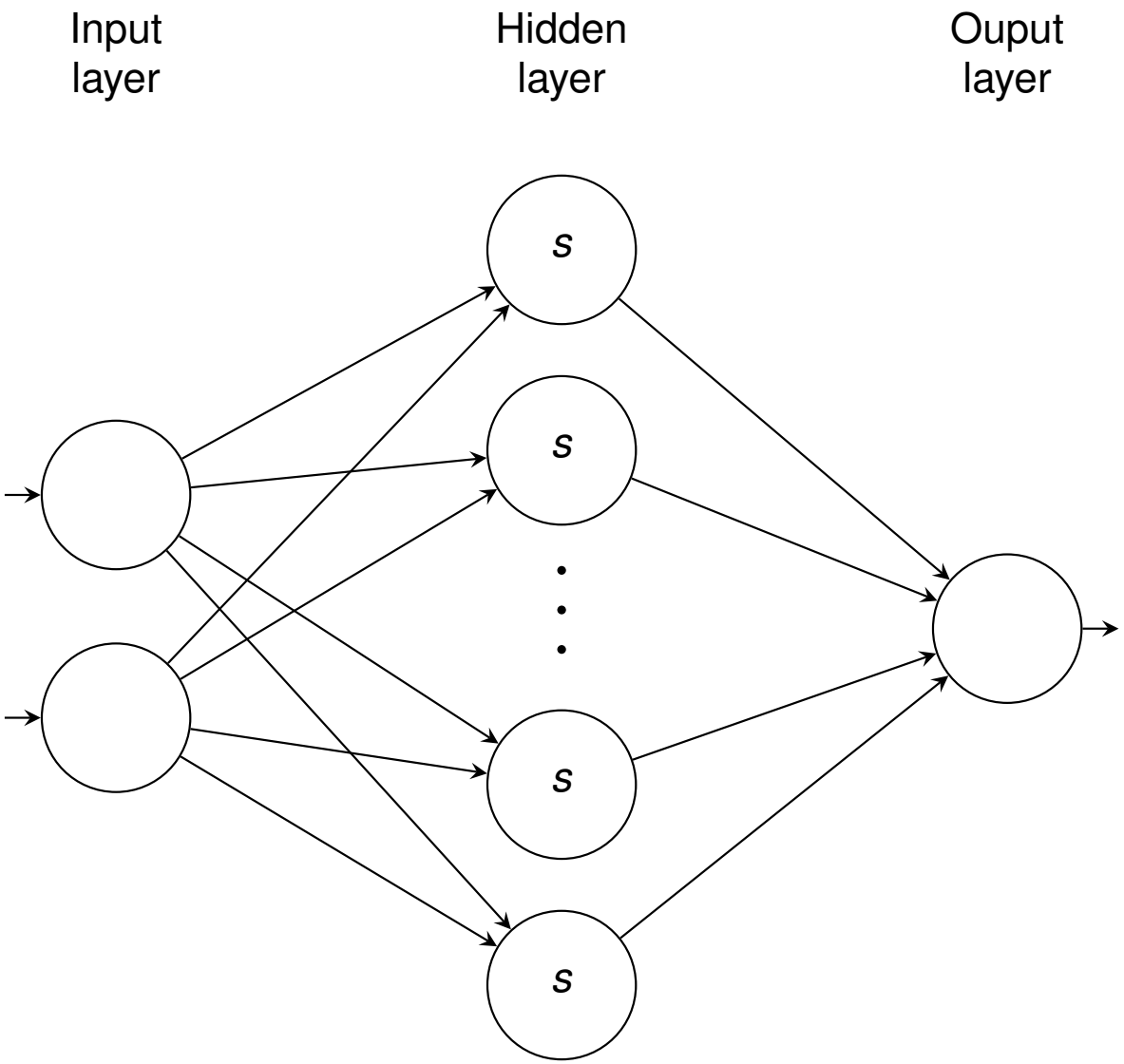


Two dimensional input - circle



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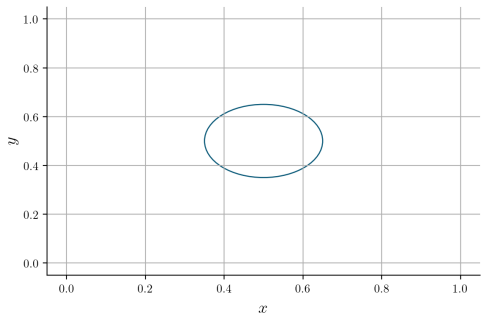

Towers built out of ReLU functions

ReLU function :

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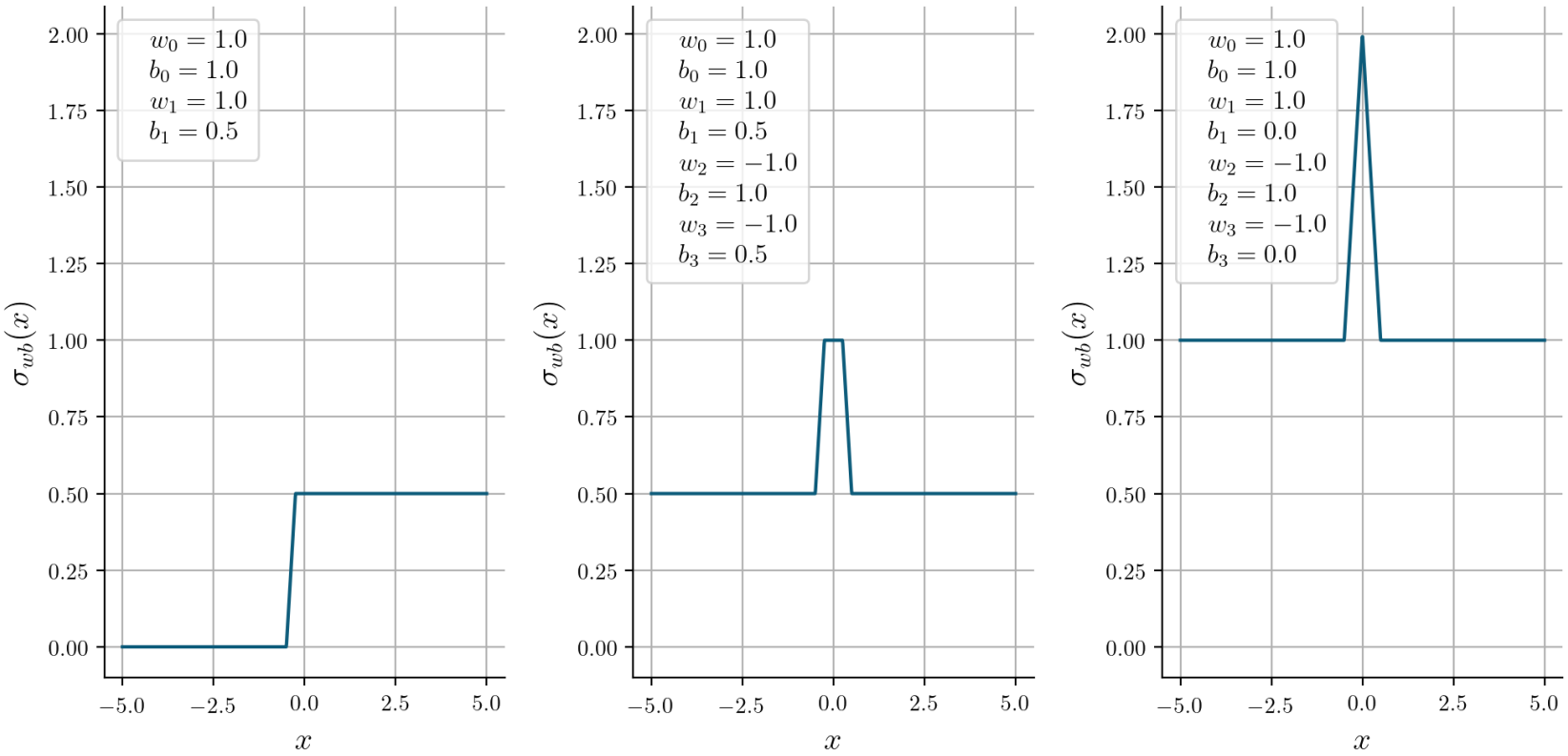

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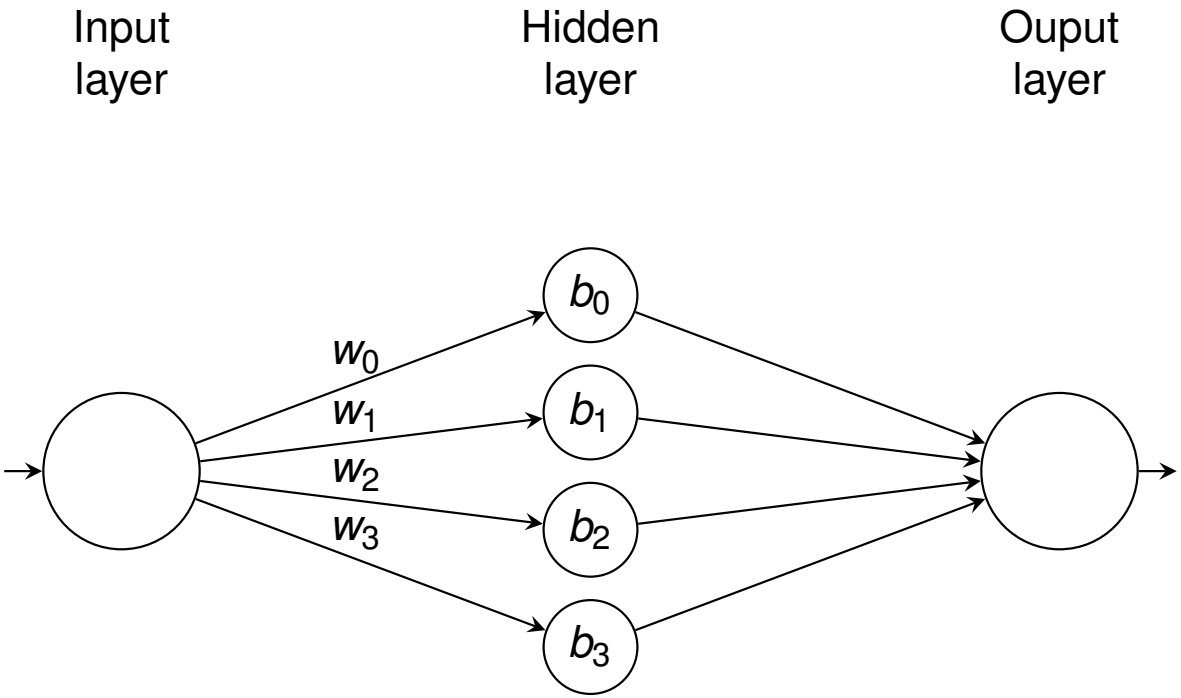

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proof by contradiction

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$\perp$ **Contradiction to the assumption** $\overline{\mathcal{G}} \subset \mathcal{C}(\mathcal{I}_n)$. $\square$

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$$\sigma\left(\lambda z + \varphi\right) \to \begin{cases} 1 & \text{if } z > 0 \ \wedge \ \lambda \to \infty, \\ 0 & \text{if } z < 0 \ \wedge \ \lambda \to \infty, \\ \sigma(\varphi) & \text{if } z = 0 \quad \forall \lambda. \end{cases} \tag{45}$$

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Lebesgue Bounded Convergence $\Pi_{\boldsymbol{w},b} := \{\boldsymbol{x} | \boldsymbol{w}^T\boldsymbol{x} + b = 0\}$ and the half-space $\mathcal{H}_{\boldsymbol{w},b} := \{\boldsymbol{x} | \boldsymbol{w}^T\boldsymbol{x} + b > 0\}$.

$$\begin{aligned} 0 &= \int_{I_n} \sigma(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}) \\ &= \int_{I_n} \gamma(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}) \\ &= \sigma(\varphi)\mu\left(\Pi_{\boldsymbol{w},b}\right) + \mu\left(\mathcal{H}_{\boldsymbol{w},b}\right) \end{aligned}$$

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