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1) (1.1) $\frac{dy}{dx} + \ln x = y \ln x$

$\frac{dy}{y-1} = \ln x \, dx$

$\int \frac{1}{y-1} \, dy = \int \ln x \, dx$, $y=1$ trivial solution consider when $y \neq 1$

$\ln(y-1) = x \ln x - x + C$

$y = C e^{x \ln x - x} + 1$

$y=1$ is part of the solution at $C=0$

(1.2) $(2xy + x^2 - 1)\,dy + (y+x)^2\,dx = 0$

$M(x,y) = 2xy + x^2 - 1$, $N(x,y) = (y+x)^2$

$M(x,y)'_x = N(x,y)'_y = 2y + 2x$ $\therefore$ Exact

$dF(x,y) = M(x,y)\,dy + N(x,y)\,dx$

$F(x,y) = \int (y+x)^2 \, dx$

$= \frac{(x+y)^3}{3} + C_y$

$\left(\frac{(x+y)^3}{3}\right)' = (x+y)^2$

$C_y = \int M(x,y) - \left(\frac{(x+y)^3}{3}\right)'_y \, dy$

$= \int 2xy - (x+y)^2 + x^2 - 1 \, dy$

$= -\frac{(y+x)^3}{3} + xy^2 + x^2y - y$

$F(x,y) = -\frac{(x+y)^3}{3} + xy^2 + x^2y - y + \frac{(x+y)^3}{3} = C$

$C = -xy^2 - x^2y + y$

(1.3)(a) $x\frac{dy}{dx} - (1+x)y = xy^2$

$x\,dy = (xy^2 + xy + y)\,dx$

$xy' = xy^2 + (x+1)y$

$y' - \frac{(x+1)y}{x} = y^2$

$\frac{y'}{y^2} - \frac{x+1}{xy} = 1$, $y=0$ trivial solution consider when $y \neq 0$

Sub $u = \frac{1}{y}$ $\quad u' = -\frac{y'}{y^2}$

$y = \frac{1}{u}$ $\quad y' = -u'y^2$

$-u' - \frac{u(x+1)}{x} = 1$

$u' + u\left(\frac{1}{x} + 1\right) = -1$

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(b) $u' + u\left(\frac{1}{x}+1\right) = -1$

$= 0$

$u' = -u\left(\frac{1}{x}+1\right)$

$du = -u\left(\frac{1}{x}+1\right)dx$

$\frac{du}{u} = \left(-\frac{1}{x}-1\right)dx$, trivial solution $u=0$ consider when $u \neq 0$

$\int \frac{1}{u}\,du = \int -\frac{1}{x} - 1\,dx$

$\ln u = -\ln x - x + c$

$u = \frac{C}{xe^x}$

Sub $C = v(x$

$u = \frac{v}{xe^x}$

$\frac{v'}{xe^x} = -1 \Rightarrow v' = -xe^x$

$dv = -xe^x\,dx$

$\int 1\,dv = \int -xe^x\,dx$

$v = (1-x)e^x + C$

$u = \frac{(1-x)e^x + C}{xe^x}$

$\frac{1}{y} = \frac{C}{xe^x} + \frac{1-x}{x}$

$y(x) = -\frac{xe^x}{(x-1)e^x + C}$

2.) $(x+1)y'' - (2-x)y' + y = 0$, $y(0)=2$, $y'(0) = -1$

$y = \sum_{k=0}^{\infty} a_k x^k$, $y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$, $y'' = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$

$(x+1)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - (2-x)\sum_{k=1}^{\infty} k a_k x^{k-1} + \sum_{k=0}^{\infty} a_k x^k = 0$

$= (x+1)\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - (2-x)\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^k$

$= \sum_{k=0}^{\infty}\left[(k+2)(k+1)a_{k+2} - (k+1)a_{k+1} + a_k\right]x^k = 0$

$(k+2)(k+1)a_{k+2} - (k+1)a_{k+1} + a_k = 0$

$(k+2)(k+1)a_{k+2} = (k+1)a_{k+1} - a_k$

$a_{k+2} = \frac{(k+1)a_{k+1} - a_k}{(k+2)(k+1)}$

$a_0 = 2$

$a_1 = -1$

$a_2 = \frac{(0+1)(-1) - 2}{(0+2)(0+1)} = -\frac{3}{2}$

$a_3 = \frac{(1+1)(-3/2) - (-1)}{(1+2)(1+1)} = -\frac{1}{3}$

$a_4 = \frac{(2+1)(-1/3) - (-3/2)}{(2+2)(2+1)} = \frac{1}{24}$

$a_5 = \frac{(3+1)(1/24) - (-1/3)}{(3+2)(3+1)} = \frac{1}{40}$

$\vdots$

$y(x) = \sum_{k=0}^{\infty} a_k x^k = 2 - x - \frac{3}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{24}x^4 + \frac{1}{40}x^5 \ldots$

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3.) (3.1) $y'' - y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 3$

$\lambda^2 - \lambda - 12 = 0$

$(\lambda - 4)(\lambda + 3) = 0$

$\lambda_1 = 4 \qquad \lambda_2 = -3$

multiplicity of roots: $\lambda_1 \Rightarrow 1, \quad \lambda_2 \Rightarrow 1$

Summand of roots: $\lambda_1 \Rightarrow C_1 e^{4x}, \quad \lambda_2 \Rightarrow \frac{C_2}{e^{3x}}$

$y(x) = C_1 e^{4x} + \frac{C_2}{e^{3x}}$

$$\begin{cases} y = C_1 e^{4x} + \frac{C_2}{e^{3x}} \\ y' = 4C_1 e^{4x} - \frac{3C_2}{e^{3x}} \end{cases}$$

at $x = 0, \ y = 1, \ y' = 3$

$$\begin{cases} 1 = C_2 + C_1 \\ 3 = 4C_1 - 3C_2 \end{cases} \Rightarrow \begin{cases} C_1 = 6/7 \\ C_2 = 1/7 \end{cases}$$

$y(x) = \frac{6e^{4x}}{7} + \frac{1}{7e^{3x}}$

(3.2) $y'' - 4y' + 3y = e^x - 2x$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda_1 = 3, \quad \lambda_2 = 1$

Multiplicity: $\lambda_1 = 1, \quad \lambda_2 = 1$

Summand: $\lambda_1 = C_1 e^{3x}, \quad \lambda_2 = C_2 e^x$

$y_h = C_1 e^{3x} + C_2 e^x$

$y = e^{3x} C_1(x) + e^x C_2(x)$

$$\begin{cases} C_1'(x) y_1 + C_2'(x) y_2 = 0 \\ C_1'(x) y_1' + C_2'(x) y_2' = \frac{f(x)}{a_0} \end{cases}$$

$y_1 = e^{3x}, \quad y_2 = e^x$

$y_1' = 3e^{3x}, \quad y_2' = e^x$

$a_0 y'' = 1, \quad f(x) = e^x - 2x$

$$\begin{cases} e^{3x} C_1'(x) + e^x C_2'(x) = 0 \\ 3e^{3x} C_1'(x) + e^x C_2'(x) = e^x - 2x \end{cases}$$

2B

$$W = \begin{vmatrix} e^{3x} & e^{x} \\ 3e^{3x} & e^{x} \end{vmatrix} = -2e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & e^{x} \\ e^{x}-2x & e^{x} \end{vmatrix} = 2xe^{x} - e^{2x}$$

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & e^{x}-2x \end{vmatrix} = e^{4x} - 2xe^{3x}$$

$$C_1'(x) = \frac{1}{2e^{2x}} - \frac{x}{e^{3x}}$$

$$C_2'(x) = \frac{x}{e^{x}} - \frac{1}{2}$$

$$C_1(x) = \int C_1'(x)\,dx = C_3 - \frac{9e^{x} - 12x - 4}{36e^{3x}}$$

$$C_2(x) = \int C_2'(x)\,dx = -\frac{x+1}{e^{x}} - \frac{x}{2} + C_4$$

$$y(x) = C_3 e^{3x} - \frac{xe^{x}}{2} + C_4 e^{x} - \frac{e^{x}}{4} - \frac{2x}{3} - \frac{8}{9}$$

4.) $\cancel{x^3 \frac{\partial u}{\partial x} + y^3 \frac{\partial u}{\partial y} = 0}$, $\cancel{u(x,x)=1}$ ~~and~~ $\cancel{u(1,1/2) = e^3}$

~~Assume~~ $\cancel{u(x,y) = X(x)Y(y)}$

$\cancel{x^3 X'(x)Y(y) + y^3 X(x)Y'(y) = 0}$

$\cancel{\frac{X'(x)}{X(x)} = -\frac{y^3 Y'(y)}{x^3 Y(y)}}$

$\cancel{\frac{X'(x)}{X(x)} = k, \quad -\frac{y^3 Y'(y)}{x^3 Y(y)} = k}$

$\cancel{\begin{cases} X'(x) = kX(x) \\ Y'(y) = -ky^3 Y(y) \end{cases} \Rightarrow \begin{cases} X(x) = Ae^{kx} \\ Y(y) = Be^{-k\frac{y^4}{4}} \end{cases}}$

$\cancel{u(x,y) = Ae^{kx} \cdot Be^{-k\frac{y^4}{4}}}$

$\cancel{u(x,y) = ABe^{kx - k\frac{y^4}{4}}}$

$\cancel{u(x,x) = ABe^{kx - k\frac{x^4}{4}} = 1}$

$\cancel{u(1, 1/2) = ABe^{k - k\frac{1}{64}} = e^3}$

$\cancel{\begin{cases} \frac{1}{X}X' = k \\ \frac{1}{Y}Y' = -ky^3 \end{cases} \Rightarrow \begin{cases} \ln|X| = kx + c \end{cases}}$

* Please see last Page for Question 4 *

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5.) (5.1) $f(t)=0$ does not contribute to the transform so we only consider

$$f(t)=4 \quad , \quad t \geq 1$$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$
$$= \int_0^\infty e^{-st} 4 \, dt$$
$$= \left[4 \cdot \int e^{-st} \, dt\right]\Big|_0^\infty$$
$$= \left[4\left(-\frac{1}{s} e^{-st} + c\right)\right]\Big|_0^\infty$$
$$= \left(-e^{-st} \frac{4}{s} + c\right)\Big|_0^\infty$$

$$\lim_{t \to 0_+} \left(-e^{-st} \frac{4}{s}\right) = -\frac{4}{s}$$
$$\lim_{t \to \infty} \left(-e^{-st} \frac{4}{s}\right) = 0$$

$$0 - \left(-\frac{4}{s}\right)$$
$$= \frac{4}{s}$$

(5.2) $\mathcal{L}\{e^{-2t} \cos 4t\}$

$$= \mathcal{L}\{\cos(4t)\}(s+2)$$
$$= \left(\frac{s}{s^2+16}\right)(s+2)$$
$$= \frac{s+2}{(s+2)^2+16}$$

(5.3) $y''(t) - y'(t) + 1 = 0 \quad , \quad y(0)=1 \quad , \quad y'(0)=1$

$$y'' - y' = -1$$
$$\mathcal{L}\{y'' - y'\} = \mathcal{L}\{-1\}$$
$$s^2 Y(s) - s y(0) - y'(0) - (sY(s) - y(0)) = -\frac{1}{s}$$
$$s^2 Y(s) - s Y(s) - s = -\frac{1}{s}$$
$$Y(s) = \frac{s+1}{s^2}$$
$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2}\right.$$
$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2} + \frac{1}{s^2}\right\}$$
$$y(t) = 1 + t$$

6.) $\frac{dN}{dt} \propto b\,N(t)$

$\Rightarrow \frac{dN}{dt} \propto \cos^2 \pi t \cdot N(t)$

$\frac{dN}{dt} = k(\cos^2(\pi t))\,N$

for some proportionality constant k

$\frac{1}{N}\frac{dN}{dt} = k\cos^2 \pi t$

$\int \frac{1}{N}\,dN = \int k\cos^2 \pi t\,dt$

$\ln|N(t)| = k\left(\frac{t}{2} + \frac{\sin 2\pi t}{4\pi} + c\right)$

$N(t) = e^{kc}\,e^{k\left(t/2 + \frac{\sin 2\pi t}{4\pi}\right)}$

$= N_0\,e^{k\left(t/2 + \frac{\sin 2\pi t}{4\pi}\right)} \quad N_0 > 0$

Consider when $k < 0$ then and then $e^{k\left(t/2 + \frac{\sin 2\pi t}{4\pi}\right)} \to 0$ as $t \to \infty$. the population will decrease to 0 and since $\frac{dN}{dt} \leq 0$ for all t the max population is the initial N_0.

Consider when $k = 0$ the population will stay at the maximum N_0.

Consider when $k > 0$ then $N_0 > 0$ and then $e^{k\left(t/2 + \frac{\sin 2\pi t}{4\pi}\right)} \to \infty$ as $t \to \infty$, this means the population growth is unbounded therefore there is no maximum for $N(t)$.

* 4.) $x^3 \frac{\partial u}{\partial x} + y^3 \frac{\partial u}{\partial y} = 0$

Assume $u(x,y) = X(x)\,Y(y)$

$$x^3 \frac{dX}{dx} Y + y^3 X \frac{dY}{dy} = 0$$

$$\begin{cases} \frac{1}{x^3} \frac{dX}{dx} \frac{1}{X} = k \\ \frac{1}{y^3} \frac{dY}{dy} \frac{1}{Y} = -k \end{cases} \Rightarrow \begin{cases} X(x) = c_1 e^{\frac{k}{4} x^4} \\ Y(y) = c_2 e^{-\frac{k}{4} y^4} \end{cases}$$

The absolute values can be ignored if assumed $c \in \mathbb{R}$

$$u(x,y) = c_1 c_2 e^{\frac{k}{4} x^4 - \frac{k}{4} y^4}$$

$$c_1 c_2 e^{k/4\, x^4 - \frac{k}{4} x^4} = 1$$

$$c_1 c_2 e^{k/4 - k/4 (1/2)^4} = e^3$$

$$c_1 c_2 = 1$$

$$e^{k/4 - k/64} = e^3$$

$$k = 64/5$$

$$c_1 c_2 e^{\frac{64/5}{4} - \frac{64/5}{4}(1/2)^4} = e^3$$

$c_1 = \frac{1}{Y}$, $c_2 \in \mathbb{R}$; $c_2 \neq 0$

$\therefore\ u(x,y) = \frac{1}{Y} c\, e^{64/5\, x^4 - \frac{64/5}{4} y^4}$, where $c \in \mathbb{R}$ and $c \neq 0$