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1.) (a) $\nabla f(x,y,z) = (2x-2y,\ 2y-2x,\ -1)$

$\nabla f(1,-1,4) = (2(1)-2(-1),\ 2(-1)-2(1),\ -1)$

$= (4,\ -4,\ -1)$

(b) V: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$4(x-1)-4(y+1)-(z-4)=0$

$4x-4-4y-4-z+4=0$

$4x-4y-z=4$

(c) $L(t) = (1,-1,4)+t(4,-4,-1) \quad t\in\mathbb{R}$

2.) (a) $\lim_{\substack{(x,y)\to(0,0)\\ C_1}} \frac{x^2+x}{x} = \lim_{\substack{(x,y)\to(0,0)\\ C_1}} x+1 = 1$

(b) $\lim_{\substack{(x,y)\to(0,0)\\ C_2}} \frac{x^2+2x}{2x} = \lim_{\substack{(x,y)\to(0,0)\\ C_2}} \frac{x}{2}+1 = 1$

(c) $\lim_{\substack{(x,y)\to(0,0)\\ C_3}} \frac{x^2+x^2}{x^2} = 2$

(d) $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y}$ does not exists as $C_3 \neq C_2$ or C_1 there are 2 curves producing different limits.

3.) (a) $g(x,y) = y^3-x^2+3y^2+x$

$\cancel{\nabla g(x,y) = (-2x+1,\ 3y^2}$

$\frac{\partial}{\partial x} = -2x+1$, $\frac{\partial}{\partial y} = 3y^2+6y$

$\nabla g(x,y) = (-2x+1,\ 3y^2+6y)$

$\begin{cases} -2x+1=0 \\ 3y^2+6y=0 \end{cases} \Rightarrow \begin{cases} x=1/2 & y=0 \\ x=1/2 & y=-2 \end{cases}$

$\frac{\partial^2 f}{\partial y^2} = 6y+6$, $\frac{\partial^2 f}{\partial x^2} = -2$, $\frac{\partial^2 f}{\partial x \partial y} = 0$

$D(x,y) = (-2)(6y+6)-(0)^2 = -2(6y+6)$

for $(1/2, 0)$: $D(x,y)<0$ $\therefore$ $(1/2,0)$ is a saddle point

for $(1/2,-2)$: $D(x,y)>0$ and $\partial^2 f/\partial x^2 <0$ $\therefore$ $(1/2,-2)$ is a maximum

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so $g(x,y) = y^3 - x^2 + 3y^2 + x$ has 1 saddle point

(b) Let $\underline{r}$ be an R-R^n function and f be an R^n-R function if $\underline{r}$ is differentiable at t and f is differentiable at $\underline{r}(t)$ then $f \circ \underline{r}$ is differentiable at t and $(f \circ \underline{r})'(t) = \underline{\nabla} f(\underline{r}(t)) \cdot \underline{r}'(t)$

(c) $\underline{\nabla} f(x,y) = \left(\frac{1}{x}, \frac{1}{y}\right)$

$\underline{r}'(t) = \left(\frac{1}{\sqrt{t}}, 2t\right)$

$\underline{\nabla} f(\underline{r}(t)) = \left(\frac{1}{2\sqrt{t}}, \frac{1}{t^2}\right)$

$(f \circ \underline{r})'(t) = \left(\frac{1}{2\sqrt{t}}, \frac{1}{t^2}\right) \cdot \left(\frac{1}{\sqrt{t}}, 2t\right)$

$= \left(\frac{1}{2\sqrt{t}}\right)\left(\frac{1}{\sqrt{t}}\right) + \left(\frac{1}{t^2}\right)(2t)$

$(f \circ \underline{r})'(1) = \left(\frac{1}{2}\right)(1) + (1)(2)$

$= \frac{5}{2}$

4.) $f(x,y) = (x-2)^2 + (y-6)^2$

$g(x,y) = y - 2x + 1$

$\nabla f(x,y) = (2(x-2), 2(y-6))$

$\nabla g(x,y) = (-2, 1)$

$$\begin{cases} 2(x-2) = \lambda(-2) \\ 2(y-6) = \lambda(1) \\ y - 2x + 1 = 0 \end{cases}$$

$2y - 12 = \lambda \Rightarrow 2y = \lambda + 12 \Rightarrow y = \frac{\lambda + 12}{2}$

$2x - 4 = -2\lambda \Rightarrow x - 2 = -\lambda \Rightarrow x = -\lambda + 2$

$\frac{\lambda + 12}{2} - 2(-\lambda + 2) + 1 = 0$

$\lambda = -6/5$

$\therefore \quad y = 27/5, \quad x = \frac{16}{5}$

Point is $(16/5, 27/5)$

5.) $\int_C (xy+z)\, ds$

$\underline{r}(t) = (2,1,1) + t[(1,0,-1) - (2,1,1)] \quad t \in [0,1]$

$= (2,1,1) + t(-1,-1,-2)$

$= (2-t, 1-t, 1-2t)$

$\underline{r}'(t) = (-1, -1, -2)$

$\| \underline{r}'(t) \| = \sqrt{(-1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

$ds = \sqrt{6}\, dt$

$f(\underline{r}(t)) = f(2-t, 1-t, 1-2t)$

$= (2-t)(1-t) + (1-2t)$

$= t^2 - 5t + 3$

$\int_C (xy+z)\, ds = \sqrt{6} \int_0^1 t^2 - 5t + 3\, dt$

$= \sqrt{6} \left[\int_0^1 t^2 - \int_0^1 5t + \int_0^1 3 \right]$

$= \sqrt{6} \left(\frac{1}{3} - \frac{5}{2} + 3 \right)$

$= \sqrt{6}\, \frac{5}{6} = \frac{5}{\sqrt{6}}$

6.) $\underline{F}(x,y) = \begin{bmatrix} x^2 - 6 \\ xy + x \end{bmatrix}$

$J = \begin{bmatrix} 2x & 0 \\ y+1 & x \end{bmatrix}$

$\det(J) = (2x)(x) = 2x^2$

$2x^2$ is not equal to 0 when $x \neq 0$

$\therefore$ $\underline{F}$ is invertible everywhere except on the y axis where $x=0$

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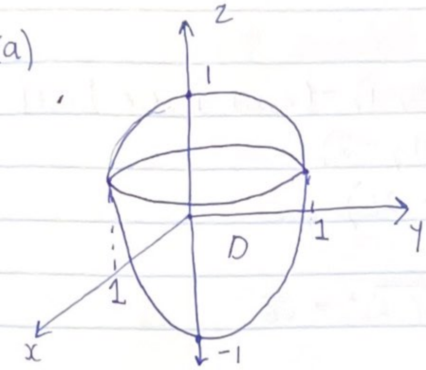
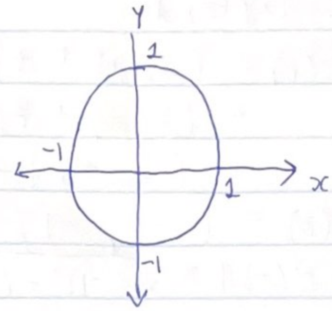
7.) (a)





(b)

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq \rho \leq 1$

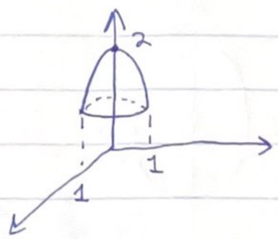
$\iiint_D z \, dv = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \left(\sqrt{1-\rho\sin\phi} - \rho\sin\phi\right) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$=$

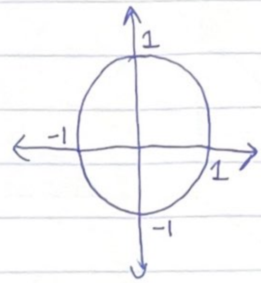
8.) (a) Let D be a closed, bounded region in R^3 with a piecewise smooth outer surface S orientated outward. If $\underline{F} = (M, N, P)$ is a smooth 3D vector field defined on S then

$$\iiint_D \operatorname{div} \underline{F} \, dV = \iint_S \underline{F} \cdot \underline{n} \, dS$$

(b) $\operatorname{div} \underline{F} = z + (-z) + 2 = 2$



$1 = 2 - x^2$



$0 \leq r \leq \sqrt{2-z}$

$0 \leq \theta \leq 2\pi$

$1 \leq z \leq 2$

$$\cancel{\int_0^{2\pi} \int_0^{\sqrt{2-z}} \int_1^{2} 2r \, dr \, d\theta \, dz}$$

$$\cancel{\int_0^{2\pi} \int_0^{\sqrt{2-z}} \int_1^{2} 2r \, dz \, dr \, d\theta}$$

$$\cancel{\int_1^{2} \int_0^{2\pi} \int_0^{\sqrt{2-z}} 2r \, dr \, d\theta \, dz}$$

$$\cancel{= \int_1^{2} \int_0^{2\pi} -z+2 \, d\theta}$$

$$\int_0^{2\pi} \int_1^{2} \int_0^{\sqrt{2-z}} 2r \, dr \, dz \, d\theta$$

$$= \int_0^{2\pi} \int_1^{2} -z+2 \, dz \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta$$

$$= \pi$$

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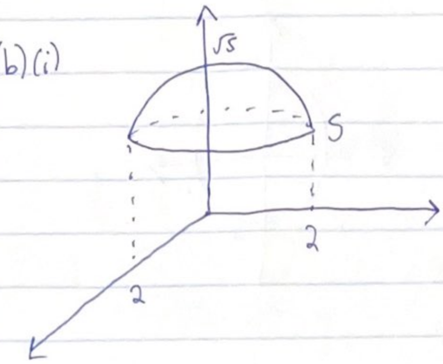
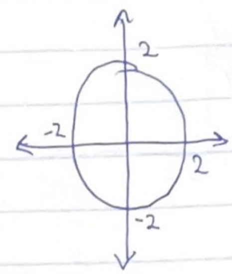
9.) (a) Let S be a piecewise smooth oriented surface whose oriented boundry C is a piecewise smooth simple closed curve if $\underline{F} = (M, N, P)$ is a smooth 3D vector field whose domain is a region in $\mathbb{R}^3$ that contains S then

$$\iint_S (\underline{\text{curl}}\ \underline{F}) \cdot \underline{n}\ dS = \oint_C \underline{F} \cdot d\underline{r}$$

(b)(i)





(ii) $\underline{r}(t) = (2\cos t, 2\sin t, \sqrt{5})$ $t \in [0, 2\pi]$

(iii) $\underline{F}(\underline{r}(t)) = (-2\sin t, 2\cos t, -\sqrt{5})$

$\underline{r}'(t) = (-2\sin t, 2\cos t, 0)$

$$\oint_C \underline{F} \cdot d\underline{r} = \int_C \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$$

$$= \int_0^{2\pi} (-2\sin t, 2\cos t, -\sqrt{5}) \cdot (-2\sin t, 2\cos t, 0)\, dt$$

$$= \int_0^{2\pi} 4\sin^2 t + 4\cos^2 t\ dt$$

$$= \int_0^{2\pi} 4\ dt$$

$$= 8\pi$$