Project 1: SQP Trajectory Optimization and NMPC for a 2D Quadrotor

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1 Model

1.1 Part 1: Trajectory Optimization Problem

To minimize cost function,

$$J(x(t), u(t), t_0, t_f) = S(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt$$

where we define Terminal Cost as,

$$S(x(t_f), t_f) = \frac{1}{2}(x(t_f) - x_{ref}(t_f))^T P(x(t_f) - x_{ref}(t_f))$$

and Running cost as,

$$L(x(t), u(t), t) = \frac{1}{2}(x(t) - x_{ref}(t))^T Q(x(t) - x_{ref}(t)) + \frac{1}{2}u(t)^T Ru(t)$$

Subject to constraints,

$$u_{min} < u < u_{max} \text{ for } u_{min} = 0, u_{max} = 10$$

for tuning matrices P, Q, and R, to incentivize the system to follow a reference trajctory x_{ref}.
The system dynamics is defined by,

$$\dot{x} = f(x(t), u(t), t)dt$$

$$\begin{aligned}
\dot{p_x} &= v_x \\
m\dot{v}_x &= -(u_1 + u_2)\sin\theta \\
\dot{p_y} &= v_y \\
m\dot{v}_y &= (u_1 + u_2)\cos\theta - mg \\
\dot{\theta} &= \omega \\
I\dot{\omega} &= r(u_1 - u_2)
\end{aligned}$$

for the state vector,

$$x = \begin{pmatrix} p_x \\ v_x \\ p_y \\ v_y \\ \theta \\ \omega \end{pmatrix}$$

1.1.1 1. Discrete Time Formulation

For Euler discretization $x_{t+1} = x_t + \Delta t f(x_t, u_t)$,

$$f(x_t, u_t) = \begin{pmatrix} v_{x,t} \\ -\frac{u_{1,t}+u_{2,t}}{m}\sin\theta_t \\ v_{y,t} \\ \frac{u_{1,t}+u_{2,t}}{m}\cos\theta_t - g \\ \omega_t \\ r\frac{u_{1,t}-u_{2,t}}{I} \end{pmatrix}$$

Discrete cost function,

$$\begin{aligned} J(x_t, x_T, u_t, t, T) &= S(x_T, T) + \sum_{t=0}^{T-1} L(x_t, u_t, t) \\ \text{for } S(x_T, T) &= \frac{1}{2}(x_T - x_{ref,T})^T P(x_T - x_{ref,T}) \\ \text{and } L(x_t, u_t, t) &= \frac{1}{2}(x_t - x_{ref,t})^T Q(x_t - x_{ref,t}) + \frac{1}{2}u_t^T R u_t \\ \text{subject to } u_{min} < u_t < u_{max} &\text{ for } u_{min} = 0, u_{max} = 10 \end{aligned}$$

1.1.2 2. Gradient and Hessian of the cost function

for a given state $\bar{x}$,

$$L(\bar{x}, \bar{u}) = \frac{1}{2}(\bar{x} - \bar{x}_{ref})^T Q(\bar{x} - \bar{x}_{ref}) + \frac{1}{2}\bar{u}^T R\bar{u}$$

Thus, the gradient,

$$\begin{aligned} \nabla_{\bar{x}} J &= Q(\bar{x} - \bar{x}_{ref}) \\ \nabla_{\bar{u}} J &= R\bar{u} \end{aligned}$$

and the hessian,

$$\nabla^2_{\bar{x}\bar{x}} J = Q$$

Or P for the terminal cost.

1.2 Quadratic subproblem

At iterate z^k, with step δz:

$$\min_{\delta z} \ \frac{1}{2}\delta z^T G^k \delta z + (g^k)^T \delta z \quad \text{s.t.} \quad A^k \Delta z = b^k, \ \ M\Delta z \le p.$$

1.3 Gradients and Hessians

Stage cost $L_t = \frac{1}{2}(x_t - x_t^{\text{ref}})^T Q(x_t - x_t^{\text{ref}}) + \frac{1}{2}u_t^T R u_t$:

$$\nabla_{x_t} L_t = Q(x_t - x_t^{\text{ref}}), \quad \nabla_{u_t} L_t = R u_t, \quad \nabla^2_{x_t x_t} L_t = Q, \quad \nabla^2_{u_t u_t} L_t = R, \quad \nabla^2_{x_t u_t} L_t = 0.$$

Terminal term $S_T = \frac{1}{2}(x_T - x_T^{\text{ref}})^T P(x_T - x_T^{\text{ref}})$:

$$\nabla_{x_T} S_T = P(x_T - x_T^{\text{ref}}), \quad \nabla^2_{x_T x_T} S_T = P.$$

Ignore second derivatives of constraints in H^k.

1.4 Filter line search

Merit pair (J, c). Constraint violation

$$c(z) = \sum_{t=0}^{T-1} \|x_{t+1} - x_t - \Delta t\, f(x_t, u_t)\|_1 + \sum_t \sum_{i=1}^{2} [u_{t,i} - 10]_+ + \sum_t \sum_{i=1}^{2} [0 - u_{t,i}]_+.$$

Initialize $J_{\text{best}} = \infty$, $c_{\text{best}} = \infty$. Backtracking with $\alpha \in (0, 1]$. Accept $z^{k+1} = z^k + \alpha \delta z$ if $J(z^{k+1}) < J_{\text{best}}$ or $c(z^{k+1}) < c_{\text{best}}$. Update the corresponding best value on acceptance.

1.5 Stopping

Stop on max iterations or small KKT residual:

$$\left\| \nabla J(z) + \nabla h(z)^T \lambda + \nabla g(z)^T \mu \right\|_\infty \leq \text{tol}.$$

A practical proxy uses stationarity, dynamics residual norm, and positive inequality parts.

2 Reference generation

Waypoints on (p_x, p_y, θ) with cubic splines. Evaluate on the grid to form x_t^{ref}. Encode flips by wrapping θ through $\pm 2\pi$. Maintain the same state ordering $[p_x, v_x, p_y, v_y, \theta, \omega]$.

3 Constraints

Apply control saturation $u_t = \text{clip}(u_t,\ 0,\ 10)$. Optional altitude $p_y \geq 0$ as a linear state inequality row. Keep consistency with the solver packing order.

4 NMPC

Receding horizon with horizon length T and step Δt. At time index t: slice $X_{\text{ref}}(t{:}t + T)$. Warm start from shifted (X^*, U^*). Solve SQP to get (X^*, U^*). Apply $u_t = \text{clip}(U_0^*, 0, 10)$. Shift warm start for the next call.

4.1 Warm-start shift

$$X^0_{0:T} \leftarrow X^*_{1:T}, \quad X^0_T \leftarrow X^*_T, \qquad U^0_{0:T-1} \leftarrow U^*_{1:T-1}, \quad U^0_{T-1} \leftarrow U^*_{T-1}.$$