Solutions to Selected Exercises from Chapter 11
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Andreas Alfons and Hanno Reuvers
Erasmus School of Economics, Erasmus Universiteit Rotterdam

Exercise 1
If $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \mathrm{N}(\mu, \sigma^2)$, then $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$. This implies that $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$ is a pivotal quantity. This pivotal quantity is used in parts (a)-(c).

(a) We have

$$\mathbb{P}\left(-z_{1-\frac{\alpha}{2}} < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < z_{1-\frac{\alpha}{2}}\right) = 1-\alpha,$$

and thus also

$$\mathbb{P}\left(\bar{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha.$$

With $z_{1-\frac{\alpha}{2}} = z_{0.95} = 1.645$ (see Table 3), a 90% confidence interval for μ is

$$\left(\bar{x} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = \left(19.3 - 1.645\frac{3}{\sqrt{16}}, 19.3 + 1.645\frac{3}{\sqrt{16}}\right) = (18.067, 20.534).$$

(b) By similar steps as in part (a), we have

$$\mathbb{P}\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < z_{1-\alpha}\right) = 1-\alpha, \quad \mathbb{P}\left(-z_{1-\alpha} < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = 1-\alpha,$$
$$\mathbb{P}\left(\bar{X} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}} < \mu\right) = 1-\alpha, \quad \mathbb{P}\left(\mu < \bar{X} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha.$$

With $z_{1-\alpha} = z_{0.90} = 1.282$ (see Table 3), one-sided 90% confidence limits for μ are

$$\ell(x_1, \dots, x_n) = \bar{x} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}} = 19.3 - 1.282\frac{3}{\sqrt{16}} = 18.339,$$
$$u(x_1, \dots, x_n) = \bar{x} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} = 19.3 + 1.282\frac{3}{\sqrt{16}} = 20.262.$$

(c) The length of the confidence interval is $2z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$. We need

$$2z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \le \lambda \qquad \Rightarrow \qquad \frac{1}{\sqrt{n}} \le \frac{\lambda}{2z_{1-\frac{\alpha}{2}}\sigma} \qquad \Rightarrow \qquad n \ge \left(\frac{2z_{1-\frac{\alpha}{2}}\sigma}{\lambda}\right)^2.$$

For the given numerical values, this evaluates to a required sample size of $n \ge (\frac{2\cdot 1.645 \cdot 3}{2})^2 = 24.354$. We round to $n = 25$.

(d) The pivotal quantity $\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t(n-1)$ yields

$$\mathbb{P}\left(-t_{1-\frac{\alpha}{2}} < \frac{\bar{X}-\mu}{s/\sqrt{n}} < t_{1-\frac{\alpha}{2}}\right) = 1-\alpha,$$

and

$$\mathbb{P}\left(\bar{X} - t_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \mu < \bar{X} + t_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{n}}\right) = 1-\alpha.$$

With $t_{1-\frac{\alpha}{2}}(n-1) = t_{0.95}(15) = 1.753$ (see Table 6), a 90% confidence interval for μ is

$$\left(\bar{x} - t_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{n}}\right) = \left(19.3 - 1.753\sqrt{\frac{10.24}{16}}, 19.3 + 1.753\sqrt{\frac{10.24}{16}}\right)$$
$$= (17.898, 20.702).$$

(e) The pivotal quantity $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ yields

$$\mathbb{P}\left(\chi^2_{\frac{\alpha}{2}} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{1-\frac{\alpha}{2}}\right) = 1-\alpha,$$

and

$$\mathbb{P}\left(\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}}\right) = 1-\alpha.$$

With $\chi^2_{\frac{\alpha}{2}}(n-1) = \chi^2_{0.005}(15) = 4.60$ and $\chi^2_{1-\frac{\alpha}{2}}(n-1) = \chi^2_{0.995}(15) = 32.80$ (see Table 4), a 99% confidence interval for σ^2 is obtained as

$$\left(\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}}, \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}}\right) = \left(\frac{15 \cdot 10.24}{32.80}, \frac{15 \cdot 10.24}{4.60}\right) = (4.683, 33.391)$$

Exercise 3

(a) The pivotal quantity $\frac{2n\bar{X}}{\theta} \sim \chi^2(2n)$ yields $\mathbb{P}\left(\frac{2n\bar{X}}{\theta} < \chi^2_\gamma\right) = \gamma$ and $\mathbb{P}\left(\frac{2n\bar{X}}{\chi^2_\gamma} < \theta\right) = \gamma$. With $\chi^2_\gamma(2n) = \chi^2_{0.95}(100) = 124.34$ (see Table 4), a one-sided lower 95% confidence limit for θ is obtained as

$$\ell(x_1, \ldots, x_n) = \frac{2n\bar{x}}{\chi^2_\gamma} = \frac{2 \cdot 50 \cdot 17.9}{124.34} = 14.396.$$

(b) Note that $e^{-t/\theta}$ is a monotone increasing transformation of θ. This implies that a lower confidence limit for θ can be transformed into a lower confidence limit for $e^{-t/\theta}$. The details are as follows:

$$0.95 = \mathbb{P}\left(\ell(X_1, \ldots, X_n) < e^{-\frac{t}{\theta}}\right) = \mathbb{P}\left(\ln \ell(X_1, \ldots, X_n) < -\frac{t}{\theta}\right)$$
$$= \mathbb{P}\left(-\frac{t}{\ln \ell(X_1, \ldots, X_n)} < \theta\right).$$

In part (a) we have found the lower bound of 14.396, hence $-\frac{t}{\ln \ell(x_1,\ldots,x_n)} = 14.396$. We find $\ell(x_1, \ldots, x_n) = e^{-\frac{t}{14.396}}$ as a one-sided lower 95% confidence limit for $\mathbb{P}(X > t) = e^{-\frac{t}{\theta}}$.

Note: The exercise states ‘where t is an arbitrary known value’. Note however that the choice $t = 0$ should be excluded.

Exercise 5

(a) The pdf of the EXP$(1,\eta)$ distribution is $f(x;\eta) = e^{-(x-\eta)}$ and we can integrate to find the cdf $F(x;\eta) = \int_\eta^x e^{-(t-\eta)}dt = 1 - e^{-(x-\eta)}$ for $x > \eta$ (and zero otherwise). The pdf for the minimum is thus

$$f_{X_{1:n}}(x;\eta) = n[1 - F(x;\eta)]^{n-1} f(x;\eta) = n\left[e^{-(x-\eta)}\right]^{n-1} e^{-(x-\eta)} = ne^{-n(x-\eta)}, \qquad x > \eta.$$

It is clear from the form of this pdf that η is a location parameter. The transformation $Q = X_{1:n} - \eta$ (with inverse transformation $X_{1:n} = Q + \eta$) yields

$$f_Q(x) = f_{X_{1:n}+\eta}(x;\eta) = ne^{-nx}, \qquad x > 0.$$

We see that $Q \sim \text{EXP}(1/n)$. This distribution does not depend on η and Q is thus a pivotal quantity.

(b) An $100\gamma\%$ equal tailed confidence interval is given by $\mathbb{P}(q_1 < Q < q_2) = \gamma$ where the quantiles q_1 and q_2 should satify $\mathbb{P}(Q \leq q_1) = F_Q(q_1) = \frac{1-\gamma}{2}$ and $\mathbb{P}(Q \geq q_2) = 1 - F_Q(q_2) = \frac{1-\gamma}{2}$. An explicit calculation of the cdf, $F_Q(x) = \int_0^x ne^{-nt}dt = 1 - e^{-nx}$, leads to

$$1 - e^{-nq_1} = \frac{1-\gamma}{2} \qquad\qquad e^{-nq_2} = \frac{1-\gamma}{2}$$
$$q_1 = -\frac{1}{n}\ln\left(\frac{1+\gamma}{2}\right) \qquad\qquad q_2 = -\frac{1}{n}\ln\left(\frac{1-\gamma}{2}\right)$$

Finally,

$$\mathbb{P}\left(-\frac{1}{n}\ln\left(\frac{1+\gamma}{2}\right) < X_{1:n} - \eta < -\frac{1}{n}\ln\left(\frac{1-\gamma}{2}\right)\right) = \gamma$$
$$\mathbb{P}\left(X_{1:n} + \frac{1}{n}\ln\left(\frac{1-\gamma}{2}\right) < \eta < X_{1:n} + \frac{1}{n}\ln\left(\frac{1+\gamma}{2}\right)\right) = \gamma$$

such that

$$\left(x_{1:n} + \frac{1}{n}\ln\left(\frac{1-\gamma}{2}\right), x_{1:n} + \frac{1}{n}\ln\left(\frac{1+\gamma}{2}\right)\right)$$

is a $100\gamma\%$ equal tailed confidence interval for η.

(c) It should be understood from the exercise (although this is not very clear) that the mileages are EXP(θ,η) distributed. If $X \sim \text{EXP}(\theta,\eta)$, then $Y = X/\theta$ $(\theta > 0)$ has the pdf

$$f_Y(y) = f_X(\theta y;\theta,\eta)\,|\theta| = \frac{1}{\theta}e^{-\frac{(\theta y-\eta)}{\theta}}\theta = e^{-(y-\frac{\eta}{\theta})}, \qquad y > \frac{\eta}{\theta}.$$

This is the pdf of an EXP$(1,\eta^*)$ distribution where $\eta^* = \frac{\eta}{\theta}$. We can use the result from part (b) to derive

$$\mathbb{P}\left(Y_{1:n} + \frac{1}{n}\ln\left(\frac{1-\gamma}{2}\right) < \frac{\eta}{\theta} < Y_{1:n} + \frac{1}{n}\ln\left(\frac{1+\gamma}{2}\right)\right) = \gamma,$$
$$\mathbb{P}\left(X_{1:n} + \frac{\theta}{n}\ln\left(\frac{1-\gamma}{2}\right) < \eta < X_{1:n} + \frac{\theta}{n}\ln\left(\frac{1+\gamma}{2}\right)\right) = \gamma.$$

A 90% confidence interval for η is obtained as

$$\left(x_{1:n} + \frac{\theta}{n}\ln\left(\frac{1-\gamma}{2}\right), x_{1:n} + \frac{\theta}{n}\ln\left(\frac{1+\gamma}{2}\right)\right) = \left(162 + \frac{850}{19}\ln(0.05), 162 + \frac{850}{19}\ln(0.95)\right)$$
$$= (27.980, 159.705).$$

Exercise 7

(a) We need to find the distribution of $Y = X^2$ when $X \sim \text{WEI}(\theta, 2)$. The transformation $Y = X^2$ has the inverse transformation $X = \sqrt{Y}$ such that the pdf of Y is given by

$$f_Y(y;\theta) = f_X(\sqrt{y}\,;\theta)\left|\frac{1}{2\sqrt{y}}\right| = \frac{2}{\theta^2}\sqrt{y}e^{-\left(\frac{\sqrt{y}}{\theta}\right)^2}\frac{1}{2\sqrt{y}} = \frac{1}{\theta^2}e^{-\frac{y}{\theta^2}}, \qquad y > 0.$$

We conclude that $Y \sim \text{EXP}(\theta^2)$. Using the distributional result from Example 11.2.1, we have $\frac{2\sum_{i=1}^n X_i^2}{\theta^2} = \frac{2n\bar{Y}}{\theta^2} \sim \chi^2(2n)$.

(b) From $\frac{2\sum_{i=1}^n X_i^2}{\theta^2} \sim \chi^2(2n)$, we obtain

$$\mathbb{P}\left(\chi^2_{\frac{1-\gamma}{2}} < \frac{2\sum_{i=1}^n X_i^2}{\theta^2} < \chi^2_{\frac{1+\gamma}{2}}\right) = \gamma \quad \Rightarrow \quad \mathbb{P}\left(\sqrt{\frac{2\sum_{i=1}^n X_i^2}{\chi^2_{\frac{1+\gamma}{2}}}} < \theta < \sqrt{\frac{2\sum_{i=1}^n X_i^2}{\chi^2_{\frac{1-\gamma}{2}}}}\right) = \gamma.$$

A $100\gamma\%$ confidence interval for θ is $\left(\sqrt{\frac{2\sum_{i=1}^n x_i^2}{\chi^2_{\frac{1+\gamma}{2}}}}, \sqrt{\frac{2\sum_{i=1}^n x_i^2}{\chi^2_{\frac{1-\gamma}{2}}}}\right)$.

(c) Note that $\exp\left[-(t/\theta)^2\right]$ is an increasing function in θ^2. A lower confidence limit for θ^2 can thus be manipulated into a lower confidence limit for $\mathbb{P}(X > t) = \exp\left[-(t/\theta)^2\right]$. We find this lower confidence limit from

$$\gamma = \mathbb{P}\left(\frac{2\sum_{i=1}^n X_i^2}{\theta^2} < \chi^2_\gamma\right) = \mathbb{P}\left(\frac{\theta^2}{2\sum_{i=1}^n X_i^2} > \frac{1}{\chi^2_\gamma}\right) = \mathbb{P}\left(\theta^2 > \frac{2\sum_{i=1}^n X_i^2}{\chi^2_\gamma}\right).$$

The remaining steps of the calculation are as follow

$$\gamma = \mathbb{P}\left(\frac{1}{\theta^2} < \frac{\chi^2_\gamma}{2\sum_{i=1}^n X_i^2}\right) = \mathbb{P}\left(\frac{-t^2}{\theta^2} > \frac{-t^2\chi^2_\gamma}{2\sum_{i=1}^n X_i^2}\right) = \mathbb{P}\left(\exp\left[-(t/\theta)^2\right] > \exp\left(\frac{-t^2\chi^2_\gamma}{2\sum_{i=1}^n X_i^2}\right)\right),$$

where we used $t > 0$ (the case $t = 0$ should be excluded because $\mathbb{P}(X > 0) = 1$). A lower $100\gamma\%$ confidence limit for $\exp\left[-(t/\theta)^2\right]$ is $\ell(x_1, \ldots, x_n) = \exp\left(\frac{-t^2\chi^2_\gamma}{2\sum_{i=1}^n x_i^2}\right)$.

(d) We first need to compute the p^{th} percentile for the given Weibull distribution. If we denote this percentile by x_p, then x_p satisfies the equation $\mathbb{P}\left(X \le x_p\right) = \frac{p}{100}$. With

$$F(x;\theta) = \int_0^x \frac{2}{\theta^2}te^{-\left(\frac{t}{\theta}\right)^2}dt = -\int_0^x \left(-\frac{2t}{\theta^2}\right)e^{-\frac{t^2}{\theta^2}}dt = -e^{-\frac{t^2}{\theta^2}}\Big|_0^x = 1 - e^{-\frac{x^2}{\theta^2}}, \qquad x > 0,$$

the pth percentile is obtained as

$$1 - e^{-\frac{x_p^2}{\theta^2}} = \frac{p}{100} \qquad \Rightarrow \qquad x_p = \sqrt{-\theta^2\ln\left(1 - \frac{p}{100}\right)}.$$

The expression $\sqrt{-\theta^2 \ln\left(1-\frac{p}{100}\right)}$ is again an increasing function in θ^2. An upper confidence limit for θ^2 will thus imply an upper confidence limit for the p^{th} percentile of the distribution. We have

$$\gamma = \mathbb{P}\left(\chi^2_{1-\gamma} < \frac{2\sum_{i=1}^n X_i^2}{\theta^2}\right) = \mathbb{P}\left(\theta^2 < \frac{2\sum_{i=1}^n X_i^2}{\chi^2_{1-\gamma}}\right)$$

and by noting that $-\ln\left(1-\frac{p}{100}\right)$ is a *positive* quantity

$$\gamma = \mathbb{P}\left(-\theta^2 \ln\left(1-\frac{p}{100}\right) < \frac{-2\ln\left(1-\frac{p}{100}\right)\sum_{i=1}^n X_i^2}{\chi^2_{1-\gamma}}\right) = \mathbb{P}\left(x_p < \sqrt{\frac{-2\ln\left(1-\frac{p}{100}\right)\sum_{i=1}^n X_i^2}{\chi^2_{1-\gamma}}}\right).$$

An upper $100\gamma\%$ confidence limit for the p^{th} percentile is thus $\sqrt{\frac{-2\ln\left(1-\frac{p}{100}\right)\sum_{i=1}^n x_i^2}{\chi^2_{1-\gamma}}}$.

Exercise 11

The setting corresponds to a random sample from the $\mathrm{BIN}(1,p)$ distribution. If $X \sim \mathrm{BIN}(1,p)$, then $\mathbb{E}(X) = p$ and $\mathbb{V}\mathrm{ar}(X) = p(1-p)$. The CLT implies that

$$\frac{\sqrt{n}(\bar{X}-p)}{\sqrt{p(1-p)}} \xrightarrow{d} Z \sim \mathrm{N}(0,1).$$

Now note that $\hat{p} = \bar{X}$ is a consistent estimator for p such that also $\frac{\sqrt{n}(\bar{X}-p)}{\sqrt{\hat{p}(1-\hat{p})}} \xrightarrow{d} Z \sim \mathrm{N}(0,1)$. Hence, for large n, we find

$$\mathbb{P}\left(-z_{1-\frac{\alpha}{2}} < \frac{\sqrt{n}(\hat{p}-p)}{\sqrt{\hat{p}(1-\hat{p})}} < z_{1-\frac{\alpha}{2}}\right) \approx 1-\alpha,$$

$$\mathbb{P}\left(\hat{p} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) \approx 1-\alpha.$$

With $\hat{p} = \frac{5}{40} = \frac{1}{8}$ and $z_{1-\frac{\alpha}{2}} = z_{0.95} = 1.645$ (see Table 3), an approximate 90% confidence interval for p is obtained as

$$\left(\hat{p} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = \left(\frac{1}{8} - 1.645\sqrt{\frac{\frac{1}{8}\cdot\frac{7}{8}}{40}}, \frac{1}{8} + 1.645\sqrt{\frac{\frac{1}{8}\cdot\frac{7}{8}}{40}}\right)$$
$$= (0.039, 0.211).$$

Exercise 12

(a) Equation (11.3.20) implies that $\mathbb{P}\left(\frac{\bar{X}-\mu}{\sqrt{\frac{\mu}{n}}} < z_\gamma\right) \approx \gamma$ for large n. We need to manipulate the inequality inside the probability. Having this in mind, we define $\theta = \sqrt{\mu}$, such that

$$\gamma \approx \mathbb{P}\left(\frac{\bar{X}-\mu}{\sqrt{\frac{\mu}{n}}} < z_\gamma\right) = \mathbb{P}\left(\frac{\bar{X}-\theta^2}{\frac{\theta}{\sqrt{n}}} < z_\gamma\right) = \mathbb{P}\left(\theta^2 + \frac{z_\gamma}{\sqrt{n}}\theta - \bar{X} > 0\right).$$

The solutions of the quadratic equation $\theta^2 + \frac{z_\gamma}{\sqrt{n}}\theta - \bar{X} = 0$ are $\theta_1 = -\frac{z_\gamma}{2\sqrt{n}} - \sqrt{\frac{z_\gamma^2}{4n} + \bar{X}}$ and $\theta_2 = -\frac{z_\gamma}{2\sqrt{n}} + \sqrt{\frac{z_\gamma^2}{4n} + \bar{X}}$. Since $\theta > 0$ and $\theta_1 < 0$, it always holds that $\theta - \theta_1 > 0$, and the above can be written as

$$\begin{aligned}\gamma &\approx \mathbb{P}\Big((\theta-\theta_1)(\theta-\theta_2) > 0\Big) = \mathbb{P}(\theta - \theta_2 > 0) = \mathbb{P}(\theta_2 < \theta) = \mathbb{P}(\theta_2^2 < \mu) \\ &= \mathbb{P}\left(\left(-\frac{z_\gamma}{2\sqrt{n}} + \sqrt{\frac{z_\gamma^2}{4n} + \bar{X}}\right)^2 < \mu\right).\end{aligned}$$

With $z_\gamma = z_{0.90} = 1.282$ (see Table 3), an approximate one-sided lower 90% confidence limit for μ is obtained as

$$\ell(x_1,\ldots,x_n) = \left(-\frac{z_\gamma}{2\sqrt{n}} + \sqrt{\frac{z_\gamma^2}{4n} + \bar{x}}\right)^2 = \left(-\frac{1.282}{2\sqrt{45}} + \sqrt{\frac{1.282^2}{4\cdot 45} + 1.7}\right)^2 = 1.468.$$

(b) For large n, Equation (11.3.21) yields

$$\gamma \approx \mathbb{P}\left(\frac{\bar{X}-\mu}{\sqrt{\frac{\bar{X}}{n}}} < z_\gamma\right) = \mathbb{P}\left(\bar{X} - z_\gamma\sqrt{\frac{\bar{X}}{n}} < \mu\right).$$

With $z_\gamma = z_{0.90} = 1.282$ (see Table 3), an approximate one-sided lower 90% confidence limit for μ is obtained as

$$\ell(x_1,\ldots,x_n) = \bar{x} - z_\gamma\sqrt{\frac{\bar{x}}{n}} = 1.7 - 1.282\sqrt{\frac{1.7}{45}} = 1.451$$

Exercise 19

For the pdf of the $\mathrm{N}(\mu_1,\sigma_1^2)$ distribution with μ_1 known, we have $f(x;\sigma_1^2) = (2\pi\sigma_1^2)^{-1/2}\exp\left(-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}\right)$. This pdf is a member of the REC with $t(x) = (x-\mu_1)^2$. $S_1 = \sum_{i=1}^{n_1}(X_i-\mu_1)^2$ is thus a sufficient statistic for σ_1^2. Similarly, $S_2 = \sum_{j=1}^{n_2}(Y_j-\mu_2)^2$ is a sufficient statistic for σ_2^2.

The mean and standard deviation of the normal distribution are location-scale parameters. It is therefore easily shown that $\frac{X_1-\mu_1}{\sigma_1},\ldots,\frac{X_{n_1}-\mu_1}{\sigma_1} \sim \mathrm{N}(0,1)$ and $\frac{Y_1-\mu_2}{\sigma_2},\ldots,\frac{Y_{n_2}-\mu_2}{\sigma_2} \sim \mathrm{N}(0,1)$ and this implies both $\frac{S_1}{\sigma_1^2} = \sum_{i=1}^{n_1}(\frac{X_i-\mu_1}{\sigma_1})^2 \sim \chi^2(n_1)$ and $\frac{S_2}{\sigma_2^2} = \sum_{j=1}^{n_2}(\frac{Y_j-\mu_2}{\sigma_2})^2 \sim \chi^2(n_2)$. By taking ratios and rescaling we can find the pivotal quantity:

$$\frac{n_2\sigma_2^2}{n_1\sigma_1^2}\frac{S_1}{S_2} = \frac{\left(\frac{S_1}{\sigma_1^2}\right)/n_1}{\left(\frac{S_2}{\sigma_2^2}\right)/n_2} \overset{d}{=} \frac{\chi^2(n_1)/n_1}{\chi^2(n_2)/n_2} \sim F(n_1,n_2),$$

where $\overset{d}{=}$ is used to denote equivalence in distribution. Denoting the α quantile of the $F(n_1,n_2)$ distribution by f_α, we obtain

$$\mathbb{P}\left(f_{\frac{\alpha}{2}} < \frac{n_2 S_1\sigma_2^2}{n_1 S_2\sigma_1^2} < f_{1-\frac{\alpha}{2}}\right) = 1-\alpha \quad\Rightarrow\quad \mathbb{P}\left(f_{\frac{\alpha}{2}}\frac{n_1 S_2}{n_2 S_1} < \frac{\sigma_2^2}{\sigma_1^2} < f_{1-\frac{\alpha}{2}}\frac{n_1 S_2}{n_2 S_1}\right) = 1-\alpha.$$

A $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$ is $\left(f_{\frac{\alpha}{2}}\frac{n_1 s_2}{n_2 s_1}, f_{1-\frac{\alpha}{2}}\frac{n_1 s_2}{n_2 s_1}\right)$.