4.1

因 $X \sim N(0,1)$ 故其概率密度函数 $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, x\in R.$

则 $P(X\geqslant \varepsilon)=\int_{\varepsilon}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

$$=\left(\int_{\varepsilon}^{\varepsilon+1}+\int_{\varepsilon+1}^{+\infty}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$\geqslant\int_{\varepsilon}^{\varepsilon+1}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$\geqslant(\varepsilon+1-\varepsilon)\frac{1}{\sqrt{2\pi}}e^{-\frac{(\varepsilon+1)^2}{2}}$$

$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{(\varepsilon+1)^2}{2}}$$

$$\geqslant\frac{1}{3}e^{-\frac{(\varepsilon+1)^2}{2}}$$

4.2

$E(X)=\int_0^{+\infty}\frac{x^{\alpha}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}dx=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{+\infty}x^{\alpha}e^{-\frac{x}{\beta}}dx$. 令 $t=\frac{x}{\beta}$.

则上式 $=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{+\infty}(\beta t)^{\alpha}e^{-t}d(\beta t)$

$$=\frac{\beta}{\Gamma(\alpha)}\int_0^{+\infty}t^{\alpha}e^{-t}dt$$

$$=\beta\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}=\beta\alpha$$

$E(X^2)=\int_0^{+\infty}\frac{x^{\alpha+1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}dx=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{+\infty}x^{\alpha+1}e^{-\frac{x}{\beta}}dx$. 令 $t=\frac{x}{\beta}$.

则上式 $=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{+\infty}(\beta t)^{\alpha+1}e^{-t}d(\beta t)$

$$=\frac{\beta^2}{\Gamma(\alpha)}\int_0^{+\infty}t^{\alpha+1}e^{-t}dt$$

$$=\beta^2\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}=\alpha(\alpha+1)\beta^2.$$

则 $Var(X)=E(X^2)-[E(X)]^2=\alpha(\alpha+1)\beta^2-\alpha^2\beta^2=\alpha\beta^2$

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26.

因 $X \sim N(3,2^2)$，则 $\frac{X-3}{2} \sim N(0,1)$.

(1) $P(2<X\leq 5) = P(-\frac{1}{2}<\frac{X-3}{2}\leq 1) = \Phi(1)-\Phi(-\frac{1}{2}) = 0.532$

$P(-4<X\leq 10) = P(-\frac{7}{2}<\frac{X-3}{2}\leq\frac{7}{2}) = \Phi(\frac{7}{2})-\Phi(-\frac{7}{2}) = 0.999$

$P(|X|>2) = P(\frac{X-3}{2}>-\frac{1}{2}$ 或 $\frac{X-3}{2}<-\frac{5}{2}) = 1-\Phi(-\frac{1}{2})+\Phi(-\frac{5}{2}) = 0.698$

$P(X>3) = P(\frac{X-3}{2}>0) = 1-\Phi(0) = 0.5.$

(2) $P(X>c) = P(\frac{X-3}{2}>\frac{c-3}{2}) = 1-\Phi(\frac{c-3}{2})$

$P(X\leq c) = P(\frac{X-3}{2}\leq\frac{c-3}{2}) = \Phi(\frac{c-3}{2})$

则 $1-\Phi(\frac{c-3}{2}) = \Phi(\frac{c-3}{2})$，则 $\Phi(\frac{c-3}{2}) = \frac{1}{2}$，$\frac{c-3}{2}=0$，$c=3$.

(3) $P(X>d) = P(\frac{X-3}{2}>\frac{d-3}{2}) = 1-\Phi(\frac{d-3}{2})\geqslant 0.9 \Longrightarrow \Phi(\frac{d-3}{2})\leq 0.1$

得 $d\leq 0.436$.

32.

因 $f(x)$, $g(x)$ 都是概率密度函数，故 $\int_{-\infty}^{+\infty}f(x)dx = \int_{-\infty}^{+\infty}g(x)dx = 1$ 且 $f(x), g(x)\geqslant 0$.

故 $h(x) = \alpha f(x)+(1-\alpha)g(x)\geqslant 0$,

$\int_{-\infty}^{+\infty}h(x)dx = \alpha\int_{-\infty}^{+\infty}f(x)dx+(1-\alpha)\int_{-\infty}^{+\infty}g(x)dx = 1$

故 $h(x)$ 是概率密度函数.

34.

$$F_X(x)=\begin{cases}0, & x\le 0\\ x, & x\in(0,1)\\ 1, & x\ge 1.\end{cases}$$

(1) $F_Y(y)=P(e^X\le y)=\begin{cases}0, & y\le 1\\ \ln y, & y\in(1,e)\\ 1, & y\ge e\end{cases}$

则 $f(y)=F_Y'(y)=\begin{cases}\frac{1}{y}, & y\in(1,e)\\ 0, & \text{其它}.\end{cases}$

(2) $F_Y(y)=P(-2\ln X\le y)=\begin{cases}0, & y\le 0.\\ 1-e^{-\frac{y}{2}}, & y>0\end{cases}$

则 $f(y)=F_Y'(y)=\begin{cases}0, & y\le 0.\\ \frac{1}{2}e^{-\frac{y}{2}}, & y>0.\end{cases}$

35.

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

(1) 当 $y\le 0$ 时显然 $F_Y(y)=0$.

当 $y>0$ 时 $F_Y(y)=P(e^X\le y)=P(X\le \ln y)=\int_{-\infty}^{\ln y}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

则 $f(y)=F_Y'(y)=\begin{cases}0, & y\le 0\\ \frac{1}{\sqrt{2\pi}\,y}e^{-\frac{\ln^2 y}{2}}, & y>0.\end{cases}$

(2) 当 $y\le 1$ 时显然 $F_Y(y)=0$.

当 $y>1$ 时 $F_Y(y)=P(2X^2+1\le y)=P\left(-\sqrt{\frac{y-1}{2}}\le X\le\sqrt{\frac{y-1}{2}}\right)=\int_{-\sqrt{\frac{y-1}{2}}}^{\sqrt{\frac{y-1}{2}}}f(x)dx$

则 $f(y)=F_Y'(y)=\begin{cases}0, & y\le 1\\ \frac{1}{2\sqrt{(y-1)\pi}}e^{-\frac{y-1}{4}}, & y>1\end{cases}$

(3) 当 $y\le 0$ 时显然 $F_Y(y)=0$.

当 $y>0$ 时 $F_Y(y)=P(|X|\le y)=P(-y\le X\le y)=\int_{-y}^{y}f(x)dx$

则 $f(y)=F_Y'(y)=\begin{cases}0, & y\le 0\\ \frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}, & y>0.\end{cases}$

36.

(1) $F_Y(y)=P(X^3\le y)=P(X\le y^{\frac{1}{3}})=\int_{-\infty}^{y^{\frac{1}{3}}}f(x)dx.$

则 $f(y)=F_Y'(y)=\frac{1}{3}y^{-\frac{2}{3}}f(y^{\frac{1}{3}}),\ y\neq 0.$

(2) 当 $y\le 0$ 时显然 $F_Y(y)=0$.

当 $y>0$ 时 $F_Y(y)=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})=\int_0^{\sqrt{y}}e^{-x}dx$

则 $f(y)=F_Y'(y)=\begin{cases}0, & y<0.\\ \frac{1}{2\sqrt{y}}e^{-\sqrt{y}}, & y\geq 0.\end{cases}$

37.

显然当 $y\le 0$ 时 $F_Y(y)=0$，当 $y\geq 1$ 时 $F_Y(y)=1$.

当 $y\in(0,1)$ 时 $F_Y(y)=P(\sin X\le y)=P(0<X\le \arcsin y$ 或 $\pi-\arcsin y\le X<\pi)$

$=\int_0^{\arcsin y}f(x)dx+\int_{\pi-\arcsin y}^{\pi}f(x)dx$

则 $f(y)=F_Y'(y)=\begin{cases}\frac{2}{\pi\sqrt{1-y^2}}, & y\in(0,1)\\ 0, & \text{其它}.\end{cases}$

4.4

$P(X>x,Y>y)=1-F(x,+\infty)-F(+\infty,y)+F(x,y).$