第九次作业

6.1

设随机变量 $Y=X-u$，那么有 $E(Y)=0, Var(Y)=\sigma^2$.

$$\Rightarrow P(X-u\le-\varepsilon)=P(Y\le-\varepsilon)$$
$$=P(Y-t\le-\varepsilon-t)$$
$$\le P((Y+t)^2\ge(\varepsilon+t)^2).$$ 由Markov不等式.
$$\le\frac{E((Y+t)^2)}{(\varepsilon+t)^2}=\frac{\sigma^2+t^2}{(\varepsilon+t)^2}.$$

$$\Rightarrow P(X-u\le-\varepsilon)\le\left(\frac{\sigma^2+t^2}{(\varepsilon+t)^2}\right)_{min}$$

$$\frac{\sigma^2+t^2}{(\varepsilon+t)^2}=\frac{(t+\varepsilon)^2-2\varepsilon(t+\varepsilon)+\sigma^2+\varepsilon^2}{(\varepsilon+t)^2}=1-2\varepsilon\times\frac{1}{\varepsilon+t}+\frac{\sigma^2+\varepsilon^2}{(\varepsilon+t)^2}$$

当 $\frac{1}{\varepsilon+t}=\frac{\varepsilon}{\sigma^2+\varepsilon^2}$ 时取最小值也即 $t=\frac{\sigma^2}{\varepsilon}$ 时取最小值 $\frac{\sigma^2}{\sigma^2+\varepsilon^2}$.

$$\Rightarrow P(X-u\le-\varepsilon)\le\frac{\sigma^2}{\varepsilon^2+\sigma^2}.$$

6.2.

解：$E(X)=2, E(Y)=-2 \Rightarrow E(X+Y)=0.$

$$\rho_{XY}=\frac{E[(X-E(X))(Y-E(Y))]}{\sqrt{Var(X)}\sqrt{Var(Y)}}=-\frac{1}{2}$$

$Var(X)=1, Var(Y)=4$

$$\Rightarrow -2E[(X-E(X))(Y-E(Y))]=\sqrt{1\times4}=2.$$

$$\Rightarrow Var(X+Y)=Var(X)+Var(Y)+2E[(X-E(X))(Y-E(Y))]$$
$$=4+1-2=3.$$

由Chebysheve不等式

$$Pr\{|X+Y|\ge6\}\le\frac{Var(X+Y)}{36}=\frac{1}{12}$$

6.3 由Chebyshev不等式

$$Pr(|\frac{1}{n}\sum_{i=1}^{n}X_i-u|\geq\varepsilon)\leq\frac{1}{\varepsilon^2}Var(\frac{1}{n}\sum_{i=1}^{n}X_i)$$

由$Var(\frac{1}{n}\sum_{i=1}^{n}X_i)=\frac{1}{n^2}Var(\sum_{i=1}^{n}X_i)=\frac{1}{n}Var(X_i)\leq\frac{\nu}{n}$

$$\Rightarrow Pr(|\frac{1}{n}\sum_{i=1}^{n}X_i-u|\geq\varepsilon)\leq\frac{1}{\varepsilon^2}Var(\frac{1}{n}\sum_{i=1}^{n}X_i)\leq\frac{\nu}{\varepsilon^2 n}.$$

6.4

阐述Chernoff方法：

利用矩生成函数和Markov不等式证明并得到，对任意随机变X和任意$t>0,\varepsilon>0$，

有$Pr[X\geq E[X]+\varepsilon]=Pr[e^{tX}\geq e^{tE[X]+t\varepsilon}]\leq e^{-t\varepsilon-tE[X]}E[e^{tX}]$

6.5

令$\bar{X}=\sum_{i=1}^{n}X_i$，$u=\sum_{i=1}^{n}E[X_i]=\sum_{i=1}^{n}p_i$

$$\begin{aligned}P[\frac{1}{n}\sum_{i=1}^{n}(X_i-E[X_i])\geq\varepsilon]&=P[\frac{1}{n}(\sum_{i=1}^{n}X_i-\sum_{i=1}^{n}E[X_i])\geq\varepsilon]\\&=P[\sum_{i=1}^{n}X_i\geq n\varepsilon+\sum_{i=1}^{n}E[X_i]]\\&=P[\bar{X}\geq n\varepsilon+u]\quad\text{由chernoff方法}\\&=P[e^{t\bar{X}}\geq e^{t(n\varepsilon+u)}]\\&\leq e^{-tn\varepsilon-tu}E[e^{t\bar{X}}]\end{aligned}$$

利用随机变量的独立性以及$1+x\leq e^x$有

$$\begin{aligned}E[e^{t\bar{X}}]=E[e^{t\sum_{i=1}^{n}X_i}]&=\prod_{i=1}^{n}E[e^{tX_i}]\\&=\prod_{i=1}^{n}[(1-p_i)+p_ie^t]=\prod_{i=1}^{n}[1+p_i(e^t-1)]\\&\leq\exp(\sum_{i=1}^{n}p_i(e^t-1))=\exp(u(e^t-1))\end{aligned}$$

$$\Rightarrow P[\frac{1}{n}\sum_{i=1}^{n}(X_i-E[X_i]\geq\varepsilon]\leq e^{-tn\varepsilon-tu}\exp(u(e^t-1))=e^{-tn\varepsilon-tu+u(e^t-1)}$$

求右式的最小值.

令$F(t)=ue^t-(n\varepsilon+u)t-u$

$F'(t)=ue^t-n\varepsilon-u$

令$F'(t)=0$得$e^t=\frac{n\varepsilon+u}{u}\Rightarrow t=\ln\frac{\varepsilon n+u}{u}$

当$t=\ln\frac{\varepsilon n+u}{u}$时$F(t)=\varepsilon n+u-(n\varepsilon+u)\ln\frac{n\varepsilon+u}{u}-u$

$$=\varepsilon n-(n\varepsilon+u)\ln\frac{n\varepsilon+u}{u}$$

$$\Rightarrow P[\frac{1}{n}\sum_{i=1}^{n}(X_i-E[X_i]\geq\varepsilon]\leq\exp[\varepsilon n-(\varepsilon n+u)\ln\frac{n\varepsilon+u}{u}]$$

同理可得：$P[\frac{1}{n}\sum_{i=1}^{n}(X_i-E[X_i])\le-\varepsilon]\le\exp[-\varepsilon n+(\varepsilon n-u)\ln\frac{\varepsilon n+u}{u}]$

6.6.

令 $Y_i=X_i-\frac{a+b}{2}$

$P(X_i=a)=P(X_i=b)=\frac{1}{2}\Rightarrow P(Y_i=\frac{a-b}{2})=P(Y_i=\frac{b-a}{2})=\frac{1}{2}$.

$$E[e^{tY}]=E[e^{t(X_i-\frac{a+b}{2})}]=\frac{1}{2}e^{\frac{a-b}{2}t}+\frac{1}{2}e^{\frac{b-a}{2}t}$$
$$=e^{\frac{a-b}{2}}(\frac{1}{2}e^{t}+\frac{1}{2}e^{-t})$$
$$\le e^{\frac{(b-a)t^2}{4}}$$

$$P[\frac{1}{n}\sum_{i=1}^{n}(X_i-\frac{a+b}{2})\ge\varepsilon]=P[\frac{1}{n}\sum_{i=1}^{n}Y_i\ge\varepsilon]$$
$$=P[\sum_{i=1}^{n}Y_i\ge n\varepsilon]$$

由Chernoff方法得 $P[\sum_{i=1}^{n}Y_i\ge n\varepsilon]\le\exp(-nt\varepsilon)E[\exp(\sum_{i=1}^{n}tY_i)]$
$$=\exp(-nt\varepsilon)\prod_{i=1}^{n}E[\exp(\sum_{i=1}^{n}tY_i)]$$
$$\le\exp(-nt\varepsilon+\frac{(b-a)n}{4}t^2)$$

当 $t=\frac{2\varepsilon}{b-a}$ 时右式取最小值 $\exp(-\frac{2\varepsilon^2n}{b-a}+\frac{n\varepsilon^2}{b-a})=\exp(\frac{-n\varepsilon^2}{b-a})$.

$\Rightarrow P[\frac{1}{n}\sum_{i=1}^{n}(X_i-\frac{a+b}{2})\ge\varepsilon]\le\exp(\frac{-n\varepsilon^2}{b-a})$

同理可得 $P[\frac{1}{n}\sum_{i=1}^{n}(X_i-\frac{a+b}{2})\le-\varepsilon]\le\exp(\frac{-n\varepsilon^2}{b-a})$

6.7

令 $\bar{X}=\sum_{i=1}^{n}X_i$，$u=\sum_{i=1}^{n}E[X_i]=\sum_{i=1}^{n}p_i$

$P[\sum_{i=1}^{n}X_i\ge(1+\varepsilon)\sum_{i=1}^{n}p_i]$

$=P[\bar{X}\ge(1+\varepsilon)u]$ 由Chenoff方法

$=P[e^{t\bar{X}}\ge e^{t(u+\varepsilon u)}]$

$\le e^{-tu-t\varepsilon u}E[e^{t\bar{X}}]$

利用随机变量的独立性以及 $1+x\le e^x$ 有

$E[e^{t\bar{X}}]=\exp(u(e^t-1))$

$\Rightarrow P[\bar{X}\ge(1+\varepsilon)u]\le e^{-tu-t\varepsilon u}\times e^{ue^t-u}=e^{-u(t+t\varepsilon-e^t+1)}$

令 $f(t)=t+t\varepsilon-e^t+1$，$f'(t)=1+\varepsilon-e^t$

令 $f'(t)=0$ 得 $t=\ln(1+\varepsilon)$.

$\Rightarrow P[\bar{X}\ge(1+\varepsilon)u]\le\exp(-u\ln(1+\varepsilon)(1+\varepsilon)-1-\varepsilon+1$
$$=\exp(-u(1+\varepsilon)\ln(1+\varepsilon)-\varepsilon)=(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}})^{u}\le e^{-\frac{u\varepsilon^2}{3}}$$

此时 $\varepsilon\in(0,1)$ 令 $g(\varepsilon)=\ln(\frac{e}{(1+\varepsilon)^{1+\varepsilon}})+\frac{\varepsilon}{3}$
$$=-(1+\varepsilon)\ln(1+\varepsilon)+\frac{\varepsilon}{3}$$

令 $g'(\varepsilon)=0$ 得 $\varepsilon=e^{-\frac{2}{3}}-1$

$g(\varepsilon)$ 在 $(0,1)$ $\downarrow$

$g(\varepsilon)<g(0)<0$

即证右式 $\leq \frac{\varepsilon^2 u}{3}$

$\Rightarrow$ 结论得证.

6.8.

令 $Y=\frac{X-a}{b-a}$ 即 $X=(b-a)Y+a$. $E(Y)=u$.

$E[e^{tX}]=E[e^{t((b-a)Y+a)}]=e^{at}E[e^{(b-a)tY}]$

由凸函数性质得

$e^{(b-a)tY}=e^{t(b-a)Y+(b-a)(1-Y)0}\leq Ye^{(b-a)t}+(1-Y)e^{0}$...

两边同时取期望有 $E(e^{tY})\leq 1-u+ue^{(b-a)t}$

$=\exp(\ln(1-u+ue^{(b-a)t})$.

令 $f(t)=\ln(1-u+ue^{(b-a)t})$.

$f'(t)=\frac{u(b-a)e^{(b-a)t}}{1-u+ue^{(b-a)t}}\Rightarrow f'(0)=u(b-a)$

$f''(t)=\frac{u(b-a)^2e^{(b-a)t}}{1-u+ue^{(b-a)t}}-\frac{u^2(b-a)^2e^{2(b-a)t}}{(1-u+ue^{(b-a)t})^2}\leq\frac{1}{4}(b-a)^2$

由Taylor中值定理得

$f(t)=f(0)+tf'(0)+f''(\xi)t^2/2\leq(b-a)tu+(b-a)^2t^2/8$.

$\Rightarrow E(e^{tX})\leq\exp(ut+t^2(b-a)^2/8)$.