第6次作业

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4.1 要证 $P(X\geq\varepsilon)\geq\frac{1}{3}e^{-\frac{(\varepsilon+1)^2}{2}}$ 即证 $\frac{1}{\sqrt{2\pi}}\int_{\varepsilon}^{+\infty}e^{-\frac{t^2}{2}}dt\geq\frac{1}{3}e^{-\frac{(\varepsilon+1)^2}{2}}$

即证 $\frac{1}{\sqrt{2\pi}}\int_{\varepsilon}^{+\infty}e^{-\frac{t^2}{2}}dt-\frac{1}{3}e^{-\frac{(\varepsilon+1)^2}{2}}\geq 0$

设 $F(\varepsilon)=\frac{1}{\sqrt{2\pi}}\int_{\varepsilon}^{+\infty}e^{-\frac{t^2}{2}}dt-\frac{1}{3}e^{-\frac{(\varepsilon+1)^2}{2}}$

则有 $F'(\varepsilon)=\frac{1}{3}(\varepsilon+1)e^{-\frac{(\varepsilon+1)^2}{2}}-\frac{e^{-\frac{\varepsilon^2}{2}}}{\sqrt{2\pi}}=e^{-\frac{\varepsilon^2}{2}}\left(\frac{1}{3}(\varepsilon+1)e^{-\varepsilon-\frac{1}{2}}-\frac{1}{\sqrt{2\pi}}\right)$

设 $f(\varepsilon)=\frac{1}{3}(\varepsilon+1)e^{-\varepsilon-\frac{1}{2}}-\frac{1}{\sqrt{2\pi}}$

$f'(\varepsilon)=\frac{1}{3}(\varepsilon+1)e^{-\varepsilon-\frac{1}{2}}\times(-1)+\frac{1}{3}e^{-\varepsilon-\frac{1}{2}}$

$=-\frac{1}{3}\varepsilon e^{-\varepsilon-\frac{1}{2}}<0$

$\therefore f(\varepsilon)$ 在 $\varepsilon\in(0,+\infty)$ 上单调递减

$\therefore f(\varepsilon)<f(0)=\frac{1}{3}e^{-\frac{1}{2}}-\frac{1}{\sqrt{2\pi}}<0$

$\therefore F'(\varepsilon)<0$

$\Rightarrow F(\varepsilon)$ 在 $(0,+\infty)$ 单调递减

$\lim\limits_{\varepsilon\to\infty}F(\varepsilon)=\lim\limits_{\varepsilon\to\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{\varepsilon}^{+\infty}e^{-\frac{t^2}{2}}dt-\frac{1}{3}e^{-\frac{(\varepsilon+1)^2}{2}}\right)=0-0=0.$

$\Rightarrow F(\varepsilon)\geq 0$

故得证

$$f(x)=\begin{cases}\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}, & x>0\\ 0, & x\le 0\end{cases}$$

4.2. ① $E(X)=\int_0^{+\infty}x\cdot f(x)\,dx$

$$=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{+\infty}x^{\alpha}\cdot e^{-\frac{x}{\beta}}\,dx$$

设 $\frac{x}{\beta}=m$

$$E(X)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{+\infty}(\beta m)^{\alpha}\cdot e^{-m}\cdot\beta\,dm$$

$$=\frac{1}{\Gamma(\alpha)}\cancel{\int_0^{+\infty}}\Gamma(\alpha+1)\cdot\beta$$

$$=\alpha\beta.$$

② $(E(X))^2=(\alpha\beta)^2=\alpha^2\beta^2.$

$$E(X^2)=\int_0^{+\infty}x^2\cdot f(x)\,dx$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\frac{1}{\beta}x^2e^{-\frac{x}{\beta}}\left(\frac{x}{\beta}\right)^{\alpha-1}dx$$

$$=\frac{\beta^2}{\Gamma(\alpha)}\int_0^{+\infty}\frac{1}{\beta}e^{-\frac{x}{\beta}}\left(\frac{x}{\beta}\right)^{\alpha+1}dx$$

$$=\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\beta^2=\frac{\alpha(\alpha+1)\Gamma(\alpha)}{\Gamma(\alpha)}\beta^2=\alpha(\alpha+1)\beta^2$$

$$\Rightarrow Var(X)=E(X^2)-E(X)^2=\alpha(\alpha+1)\beta^2-\alpha^2\beta^2=\alpha\beta^2.$$

4.3

T26.

(1) 由 $X\sim N(3,2^2)$ 得 $Y=\frac{X-3}{2}\sim N(0,1)$

$P(2<X\le5)=P(X\le5)-P(X\le2)$

$=F_Y(\frac{5-3}{2})-F_Y(\frac{2-3}{2})$

$=F_Y(1)-F_Y(-\frac{1}{2})$

$=\int_{-\infty}^{1}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx-\int_{-\infty}^{-\frac{1}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

查标准正态分布表后可得 $P(2<X\le5)=0.841-(1-0.692)$

$=0.841-0.308=0.533$

$P(-4<X\le10)=F_Y(\frac{10-3}{2})-F_Y(\frac{-4-3}{2})$

$=F_Y(\frac{7}{2})-F_Y(-\frac{7}{2})=2F_Y(\frac{7}{2})-1$

$=2\times0.999761-1=0.999522$

$P(|X|>2)=P(X>2\cup X<-2)=1-P(-2\le X\le2)$

$=1-F_Y(\frac{2-3}{2})+F_Y(\frac{-2-3}{2})$

$=1-F_Y(-\frac{1}{2})+F_Y(-\frac{5}{2})$

$=1-(1-0.692)+(1-0.994)=0.698.$

$P(X>3)=1-P(X\le3)=1-F_Y(\frac{3-3}{2})=1-\frac{1}{2}=\frac{1}{2}.$

(2). $P(X>C)=1-P(X\le C)=P(X\le C)$

$\Rightarrow P(X\le C)=\frac{1}{2}\Rightarrow F_Y(\frac{C-3}{2})=\frac{1}{2}$

$\Rightarrow\frac{C-3}{2}=0\Rightarrow C=3.$

(3) $P(X>d) = 1 - P(X \leq d) \geq 0.9$

$$P(X \leq d) \leq 0.1$$

$$F_Y\left(\frac{d-3}{2}\right) \leq 0.1$$

$$\Rightarrow \frac{d-3}{2} < 0.$$

$$\Rightarrow F_Y\left(-\frac{d-3}{2}\right) \geq 0.9$$

$$\Rightarrow -\frac{d-3}{2} \cancel{\geq 1.29} \geq 1.281$$

$$\Rightarrow d \leq 0.436.$$

T32. $f(x).g(x)$ 是概率密度函数

$\Rightarrow f(x) \geq 0, g(x) \geq 0$

$\int_{-\infty}^{+\infty} f(x)dx = 1, \int_{-\infty}^{+\infty} g(x)dx = 1$

$0 \leq a \leq 1 \quad \Rightarrow 1-\alpha \geq 0.$

$\Rightarrow h(x) = af(x) + (1-\alpha)g(x) \geq 0.$

$$\int_{-\infty}^{+\infty} h(x)dx = \int_{-\infty}^{+\infty} \alpha f(x) + (1-\alpha)g(x)dx$$

$$= \alpha\int_{-\infty}^{+\infty} \cancel{\alpha} f(x)dx + (1-\alpha)\int_{-\infty}^{+\infty} g(x)dx$$

$$= \alpha + 1 - \alpha = 1$$

$\Rightarrow h(x)$ 是概率密度函数.

T34. (1) $F_Y(y) \overset{\triangle}{=} P(Y\le y)$

$X=\ln Y$

$\Rightarrow P(X\le \ln y)$

$\Rightarrow \int_0^{\ln y} f(x)dx$

$\Rightarrow \int_0^{\ln y} dx=\ln y$

$f_Y(y)=(\ln y)'=\frac{1}{y}\ (0<x<1\Rightarrow 1<y<e)$

$\Rightarrow f_Y(y)=\begin{cases}\frac{1}{y}, & 1<y<e\\ 0, & 其它.\end{cases}$

(2) $\cancel{Y=2X^2+1\Rightarrow X=}$

$Y=-2\ln X\Rightarrow X=e^{-\frac{y}{2}}$

$F_Y(y)=P(Y\le y)=P(X\ge e^{-\frac{y}{2}})$

$=\int_{e^{-\frac{y}{2}}}^{1} f(x)dx=\int_{e^{-\frac{y}{2}}}^{1} dx=1-e^{-\frac{y}{2}}$

$f_Y(y)=(1-e^{-\frac{y}{2}})'=\frac{1}{2}e^{-\frac{y}{2}}\ (y>0)$

$\Rightarrow f_Y(y)=\begin{cases}0, & y\le 0\\ \frac{1}{2}e^{-\frac{y}{2}}, & y>0\end{cases}$

T35. (1) $e^X>0\Rightarrow F_Y(y)=0\ (y\le 0)$.

当 $y>0$ 时

$F_Y(y)=P(Y\le y)$

由 $Y=e^X\Rightarrow e^X\le y$

$\Rightarrow X\le \ln y$

$\Rightarrow F_Y(y)=P(X\le \ln y)=\int_{-\infty}^{\ln y}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$

$\Rightarrow f_Y(y)=(F_Y(y))'=\frac{1}{\sqrt{2\pi}y}e^{-\frac{(\ln y)^2}{2}}$

故 $f_Y(y)=\begin{cases}0, & y\le 0\\ \frac{1}{\sqrt{2\pi}y}e^{-\frac{(\ln y)^2}{2}}, & y>0.\end{cases}$

(2) $F_Y(y)=P(Y\le y)$

由 $Y=2X^2+1$ 得 $2X^2+1\le y$

即 $-\sqrt{\frac{y-1}{2}}\le X\le \sqrt{\frac{y-1}{2}}$

$\Rightarrow F_Y(y)=P(-\sqrt{\frac{y-1}{2}}\le X\le \sqrt{\frac{y-1}{2}})$

$=P(X\le \sqrt{\frac{y-1}{2}})-P(X\le -\sqrt{\frac{y-1}{2}})$

$=2\int_{-\infty}^{\sqrt{\frac{y-1}{2}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt-1$

$\Rightarrow f_Y(y)=(F_Y(y))'=2\times\frac{1}{\sqrt{2\pi}}e^{-\frac{y-1}{4}}\times(\sqrt{\frac{y-1}{2}})'$

$=\frac{1}{2\sqrt{\pi(y-1)}}e^{-\frac{y-1}{4}}$

故 $f_Y(y)=\begin{cases}\frac{1}{2\sqrt{\pi(y-1)}}e^{-\frac{y-1}{4}}, & y\ge 1\\ 0, & y\le 1\end{cases}$

(3). $F_Y(y)=P(Y\le y)$

由 $Y=|X|$ 得 $|X|\le y$ 即 $-y\le X\le y$

$\Rightarrow F_Y(y)=P(-y\le X\le y)=P(X\le y)-P(X\le -y)$

$=2\int_{-\infty}^{y}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt-1$

$\cancel{=2\times}$

$\Rightarrow f_Y(y)=(F_Y(y))'=2\times\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}=\sqrt{\frac{2}{\pi}}e^{-\frac{y^2}{2}}$.

$\Rightarrow f_Y(y)=\begin{cases}0, & y\le 0\\ \sqrt{\frac{2}{\pi}}e^{-\frac{y^2}{2}}, & y>0.\end{cases}$

T36. (1) $F_Y(y)=P(Y\le y)$

由 $Y=X^3$ 得 $X^3\le y$ 即 $X\le y^{\frac{1}{3}}$

$\Rightarrow F_Y(y)=P(Y\le y)=P(X\le y^{\frac{1}{3}})$

$=\int_{-\infty}^{y^{\frac{1}{3}}}f(x)\,dx$

$\cancel{=\int_{-\infty}^{y^{\frac{1}{3}}}f(y^{\frac{1}{3}})\,dy^{\frac{1}{3}}}$

$\cancel{=\frac{1}{3}\int_{-\infty}^{y^{\frac{1}{3}}}y^{-\frac{2}{3}}f}$

$\Rightarrow f_Y(y)=(F_Y(y))'=\frac{1}{3}y^{-\frac{2}{3}}f(y^{\frac{1}{3}})$

(2). $F_Y(y)=P(Y\le y)$

由 $Y=X^2$ 得 $X^2\le y$ 即 $-\sqrt{y}\le X\le\sqrt{y}$

$\Rightarrow F_Y(y)=P(-\sqrt{y}\le X\le\sqrt{y})=P(X\le y)-P(X\le -y)$

$=\int_{-\infty}^{\sqrt{y}}f(x)\,dx$

$\Rightarrow f_Y(y)=(F_Y(y))'$

$=e^{-\sqrt{y}}\times(\sqrt{y})'\ (y\ge 0)$

$\Rightarrow f_Y(y)=\begin{cases}0, & y<0\\ \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, & y\ge 0.\end{cases}$

$\cancel{\Rightarrow f_Y(y)=(F_Y(y))}$

T37. $F_Y(y)=P(Y\le y)$

由 $Y=\sin X$ 得 $\sin X\le y$ 即 $X\le\arcsin y$ or $X\ge\pi-\arcsin y$

由 $x\in(0,\pi)$ 得 $y\in(0,1)$

$\Rightarrow F_Y(y)=P(Y\le y)=P(X\le\arcsin y)+P(X\ge\pi-\arcsin y)$

$=\int_{0}^{\arcsin y} f(x)dx+\int_{\pi-\arcsin y}^{1} f(x)dx$

$=\int_{0}^{\arcsin y}\frac{2x}{\pi^2}dx+\int_{\pi-\arcsin y}^{\pi} f(x)dx$

$=\frac{x^2}{\pi^2}\Big|_{0}^{\arcsin y}+\frac{x^2}{\pi^2}\Big|_{\pi-\arcsin y}^{\pi}$

$=\frac{(\arcsin y)^2}{\pi^2}+1-\frac{(\pi-\arcsin y)^2}{\pi^2}$

$=\frac{2}{\pi}\arcsin y$

$\Rightarrow f_Y(y)=(F_Y(y))'=\frac{2}{\pi}\times\frac{1}{\sqrt{1-y^2}}=\frac{2}{\pi\sqrt{1-y^2}}\ (y\in(0,1))$.

故 $f_Y(y)=\begin{cases}\frac{2}{\pi\sqrt{1-y^2}}, & 0<y<1\\ 0, & \text{其它}.\end{cases}$

4.4 $F(x,y)=P(X\le x,Y\le y)$

$\cancel{=P(X\le x)P(Y\le y)}$

求 $P(X>x,Y>y)=P(x<X<+\infty,y<Y<+\infty)=1-P(Y\le y)-P(X\le x)+P(X\le x,Y\le y)$

$=1-F(+\infty,y)-F(x,+\infty)+F(x,y)$

$=\cancel{1-P(Y\le y)-P(X\le x)+F(x,y)}$