第六次作业 **201300015** 李丹悦

4.1

$P(X \geq \epsilon) = \int_{\epsilon}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \geq \int_{\epsilon}^{\epsilon+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, 由积分中值定理可得

$\int_{\epsilon}^{\epsilon+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} > \frac{1}{3} e^{-\frac{\xi^2}{2}} \geq \frac{1}{3} e^{-\frac{(\epsilon+1)^2}{2}}$,

所以$P(X \geq \epsilon) \geq \frac{1}{3} e^{-\frac{(\epsilon+1)^2}{2}}$.

4.2

Gamma Function: $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$, 其递推公式为$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_0^{+\infty} \frac{x^{\alpha}}{\beta^{\alpha}\Gamma(\alpha)} e^{-x/\beta} dx$, 令$t = x/\beta$, 所以$x = t\beta$,

我们有: $E(X) = \frac{\beta}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha} e^{-t} dt = \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1) = \frac{\beta}{\Gamma(\alpha)} \alpha\Gamma(\alpha) = \alpha\beta$.

$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx = \int_0^{+\infty} \frac{x^{\alpha+1}}{\beta^{\alpha}\Gamma(\alpha)} e^{-x/\beta} dx$, 令$t = x/\beta$, 所以$x = t\beta$,

$E(X) = \frac{\beta^2}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha+1} e^{-t} dt = \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2) = \frac{\beta^2}{\Gamma(\alpha)} (\alpha+1)\Gamma(\alpha+1) = \beta^2\alpha(\alpha+1)$

$Var(X) = E(X^2) - E^2(X) = \beta^2\alpha(\alpha+1) - \alpha^2\beta^2 = \alpha\beta^2$.

4.3

$T26$

(1)

因为$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = 1$, 所以

$\Phi(x) + \Phi(-x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt + \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$

$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt + \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = 1$

$P(2 < X \leq 5) = P(X \leq 5) - P(X \leq 2) = \Phi(\frac{5-3}{2}) - \Phi(\frac{2-3}{2}) = \Phi(1) - \Phi(-0.5)$

$= \Phi(1) - 1 + \Phi(0.5) = 0.5328$

同理,

$P(-4 < X \leq 10) = \Phi(3.5) - \Phi(-3.5) = 2\Phi(3.5) - 1 = 0.9996$

$P(|X| > 2) = 1 - P(-2 \leq X \leq 2) = \Phi(0.5) + 1 - \Phi(2.5) = 0.6977$

$P(X > 3) = 1 - P(X \leq 3) = 1 - \Phi(0) = 0.5$

(2)

$P(X > c) = 1 - P(X \leq c) = 1 - \Phi(\frac{c-3}{2}) = P(X \leq c) = \Phi(\frac{c-3}{2})$

所以$\Phi(\frac{c-3}{2}) = 0.5, c = 3.$

(3)

$P(X > d) = 1 - P(X \leq d) = 1 - \Phi(\frac{d-3}{2}) \geq 0.9$

$\Phi(\frac{d-3}{2}) \leq 0.1$, 经过查表计算得$d$至多为0.436.

$T32$
非负性:

因为$f(x),\ g(x)$都是概率密度函数, 所以$f(x) \geq 0,\ g(x) \geq 0$, 又因为$0 \leq \alpha \leq 1$,

所以显然有$h(x) \geq 0$.

规范性:

因为$f(x),\ g(x)$都是概率密度函数, 所以$\int_{-\infty}^{+\infty} f(t)dt = 1, \int_{-\infty}^{+\infty} g(t)dt = 1.$

那么我们有$\int_{-\infty}^{+\infty} h(t)dt = \alpha \int_{-\infty}^{+\infty} f(t)dt + (1-\alpha) \int_{-\infty}^{+\infty} g(t)dt = 1$, 得证.

$T34$
(1)

$$f_X(x) = \begin{cases} 1,\ 0 < x < 1 \\ 0,\ otherwise \end{cases}$$

因为$y = e^x$处处可导且严格单调增, $x = h(y) = \ln y$, 所以有

$$f_Y(y) = \begin{cases} \frac{1}{y},\ 1 < y < e \\ 0,\ otherwise \end{cases}$$

(2)

同上, 因为$y = -2\ln x$在区间上处处可导且严格单调减, $x = h(y) = e^{\frac{-y}{2}}$,

$|h'(y)| = |\frac{-1}{2}e^{\frac{-y}{2}}| = \frac{1}{2}e^{\frac{-y}{2}}, g(0) = +\infty, g(1) = 0,$ 所以有

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{\frac{-y}{2}},\ 0 < y < +\infty \\ 0,\ otherwise \end{cases}$$

$T35$

(1)

同上一题做法, 因为$y = e^x$处处可导且单调递增, 所以$x = h(y) = \ln y, h'(y) = \frac{1}{y}$

$y > 0$时, $f_Y(y) = f_X(\ln y)h'(y) = \frac{1}{\sqrt{2\pi}y}e^{-(\ln y)^2/2}$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}y}e^{-(\ln y)^2/2},\ y > 0 \\ 0,\ otherwise \end{cases}$$

(2)

易知, 当$y \le 1$时, $f_Y(y) = 0$

当$y > 1$时, $F_Y(y) = P(Y \le y) = P(2X^2 + 1 \le y) = P(-\sqrt{\frac{y-1}{2}} \le X \le \sqrt{\frac{y-1}{2}})$

$F_Y(y) = \Phi(\sqrt{\frac{y-1}{2}}) - \Phi(-\sqrt{\frac{y-1}{2}}) = 2\Phi(\sqrt{\frac{y-1}{2}}) - 1$

$f_Y(y) = F_Y'(y) = 2 \cdot \frac{1}{2}\frac{1}{2\sqrt{\frac{y-1}{2}}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-(y-1)/4} = \frac{1}{2\sqrt{\pi(y-1)}} \cdot e^{-(y-1)/4}$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{\pi(y-1)}} \cdot e^{-(y-1)/4},\ y > 1 \\ 0,\ otherwise \end{cases}$$

(3)

易知, 当$y \leq 0$时, $f_Y(y) = 0$

当$y > 0$时, $F_Y(y) = P(Y \leq y) = P(|X| \leq y) = P(-y \leq X \leq y) = 2\Phi(y) - 1$

$f_Y(y) = F_Y'(y) = 2 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-y^2/2}$

$$f_Y(y) = \begin{cases} \sqrt{\frac{2}{\pi}} \cdot e^{-y^2/2}, \ y > 0 \\ 0, \ otherwise \end{cases}$$

$T36$

(1)

$y = x^3$处处可导且单调增, $x = h(y) = y^{1/3}$

$f_Y(y) = f_X(h(y))h'(y) = \frac{1}{3}y^{-2/3}f(y^{1/3})$

$$f_Y(y) = \begin{cases} \frac{1}{3}y^{-2/3}f(y^{1/3}), \ y \neq 0 \\ 0, \ y = 0 \end{cases}$$

(2)

易知, $y \leq 0$时, $f_Y(y) = 0$

当$y > 0$时, $y = x^2(x > 0)$处处可导且严格单调增, $x = h(y) = y^{1/2}$

$f_Y(y) = f_X(h(y))h'(y) = \frac{1}{2}y^{-1/2}e^{-y^{1/2}}$

$$f_Y(y) = \begin{cases} \frac{1}{2}y^{-1/2}e^{-y^{1/2}}, \ y > 0 \\ 0, \ y = 0 \end{cases}$$

$T37$
易知, 当$y \leq 0$或者$y \geq 1$时, $f_Y(y) = 0$

当$0 < y < 1$时,
$F_Y(y) = P(Y \leq y) = P(sinX \leq y) = P(0 < X \leq \arcsin y) + P(\pi - \arcsin y \leq X \leq \pi)$

$F_Y(y) = \int_0^{\arcsin y} \frac{2t}{\pi^2}dt + \int_{\pi - \arcsin y}^{\pi} \frac{2t}{\pi^2}dt = \frac{2\arcsin y}{\pi}$

$$f_Y(y) = F'_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}$$

$$f_Y(y) = \begin{cases} \frac{2}{\pi\sqrt{1-y^2}},\ 0 < y < 1 \\ 0,\ otherwise \end{cases}$$

4.4

$$P(X > x, Y > y) = 1 - F(x, +\infty) - F(+\infty, y) + F(x, y)$$