Optimization Methods **Fall 2021**

Homework 3

Instructor: Lijun Zhang *Name:* Student name, *StudentId:* Student id

Notice

- The submission email is: **zhangzhenyao@lamda.nju.edu.cn**.
- Please use the provided LaTeX file as a template. If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: One inequality constraint

With $c \neq 0$, express the dual problem of

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & f(x) \leq 0 \end{array}$$

in terms of the conjugate $f^\star$.

Solution:

Write your solution here.

$\square$

Problem 2: KKT conditions

Consider the problem

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^2} & x_1^2 + x_2^2 \\ \text{s.t.} & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 2 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2 \end{array}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathbb{R}^2$.

(1) Write the Lagrangian for this problem.

(2) Does strong duality hold in this problem?

(3) Write the KKT conditions for this optimization problem.

Solution:

Write your solutions here.

$\square$

Problem 3: Equality Constrained Least-squares

Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|Ax - b\|_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbf{R}^{m \times n}$ with $\mathbf{rank}\ A = n$, and $G \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\ G = p$.

(1) Derive the Lagrange dual problem with Lagrange multiplier vector v.

(2) Derive expressions for the primal solution $x^\star$ and the dual solution $v^\star$.

Solution:
Write your solutions here.

□

Problem 4: Negative-entropy Regularization

Please show how to compute

$$\operatorname*{argmin}_{x \in \Delta^n} \quad b^\top x + c \cdot \sum_{i=1}^{n} x_i \ln x_i$$

where $\Delta^n = \{x | \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \cdots, n\}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Solution:
Write your solutions here.

□

Problem 5: Support Vector Machines
Consider the following optimization problem

$$\text{minimize} \quad \sum_{i=1}^n \max\left(0, 1 - y_i(w^T x_i + b)\right) + \frac{\lambda}{2}\|w\|_2^2$$

where $x_i \in \mathbf{R}^d, y_i \in \mathbf{R}, i = 1, \cdots, n$ are given, and $w \in \mathbf{R}^d, b \in \mathbf{R}$ are the variables.

(1) Derive an equivalent problem by introducing new variables $u_i, i = 1, \cdots, n$ and equality constraints

$$u_i = y_i(w^T x_i + b), i = 1, \cdots, n.$$

(2) Derive the Lagrange dual problem of the above equivalent problem.

(3) Give the Karush-Kuhn-Tucker conditions.

Hint: Let $\ell(x) = \max(0, 1 - x)$*. Its conjugate function* $\ell^*(y) = \sup_x (yx - \ell(x)) = \begin{cases} y, & -1 \leq y \leq 0 \\ \infty, & \textit{otherwise} \end{cases}$

Solution:
Write your solutions here.

□