Problem 1

$\lambda=0$ 时 $g(\lambda)=\inf c^Tx=-\infty$

$\lambda>0$ 时 $g(\lambda)=\inf(c^Tx+\lambda f(x))$

$=\lambda\inf((\frac{c}{\lambda})^Tx+f(x))$

$=-\lambda f_1^*(-c/\lambda)$

$\Rightarrow$ 对偶问题为

minimize $\quad -\lambda f_1^*(-\frac{c}{\lambda})$

subject to $\quad \lambda\geq 0.$

Problem 2.

(1) $L(x,\lambda)=x_1^2+x_2^2+\lambda_1[(x_1-1)^2+(x_2-1)^2-2]+\lambda_2[(x_1-1)^2+(x_2+1)^2-2]$

(2) $L'(x_1)=2x_1+2x_1\lambda_1-2\lambda_1+2x_1\lambda_2-2\lambda_2=0$ $\quad \lambda_1=-1$

$L'(x_2)=2x_2+2x_2\lambda_1-2\lambda_1+2x_2\lambda_2+2\lambda_2=0$ $\quad \lambda_2=\lambda_1$

$L'(\lambda_1)=x_1^2+x_2^2-2x_1-2x_2=0$ $\quad x_1=2$ 或 $x_1=0.$

$L'(\lambda_2)=x_1^2+x_2^2-2x_1+2x_2=0.$

解得 $\begin{cases}\lambda_1=\lambda_2=-1\\ x_1=2,\ x_2=0.\end{cases}$ 或 $\begin{cases}\lambda_1=\lambda_2=0\\ x_1=x_2=0.\end{cases}$

$L_{min}=0.$ 即 $d^*=0$

而 $p^*=0$

$\Rightarrow p^*=d^*$

$\Rightarrow$ 强对称性成立

(3) 令 $x^*=[x_1^*,x_2^*]^T$

$(x_1^*-1)^2+(x_2^*-1)^2\leq 2$

$(x_1^*-1)^2+(x_2^*+1)^2\leq 2$

$\lambda_i^*\geq 0,\ i=1,2$

$\lambda_1[(x_1-1)^2+(x_2-1)^2-2]=\lambda_2[(x_1^*-1)^2+(x_2+1)^2-2]=0.$

Problem 3

(1) $L(x,v)=\frac{1}{2}\|Ax-b\|_2^2+v^T(Gx-h)$

$$=\frac{1}{2}x^TA^TAx+(G^Tv-A^Tb)^Tx-v^Th+\frac{1}{2}b^Tb$$

当 $x=-(\frac{1}{2}A^TA)^{-1}(G^Tv-A^Tb)$

对偶函数 $g(v)=-\frac{1}{2}(G^Tv-2A^Tb)^T(A^TA)^{-1}(G^Tv-2A^Tb)-v^Th+\frac{1}{2}b^Tb$

$\Rightarrow$ 对偶问题为

$$\max \quad g(v)$$

(2) $\begin{cases} A^T(Ax^*-b)+G^Tv^*=0 & ① \\ Gx^*=h & ② \end{cases}$

$\Rightarrow x^*=(A^TA)^{-1}(A^Tb-G^Tv^*)$

$\Rightarrow G(A^TA)^{-1}A^Tb-G(A^TA)^{-1}G^Tv^*=h$

$\Rightarrow v^*=-(G(A^TA)^{-1}G^T)^{-1}(h-G(A^TA)^{-1}A^Tb)$

代入即可得到 x^*

Problem 4

由题意得，求原问题即求：

$$\operatorname{argmin} \quad b^T x + c\cdot\sum_{i=1}^{n} x_i \ln x_i$$
$$\text{s.t.} \quad -x_i \le 0$$
$$\sum_{i=1}^{n} x_i - 1 = 0$$

⇒ 得 Lagrangian：

$$L(x,\lambda,v) = b^T + c\cdot\sum_{i=1}^{n} x_i \ln x_i - \sum_{i=1}^{n}\lambda_i x_i + v\sum_{i=1}^{n} x_i - v$$

⇒ 对偶函数为

$$g(\lambda,v) = \inf\left(b^T x + c\cdot\sum_{i=1}^{n} x_i \ln x_i - \sum_{i=1}^{n}\lambda_i x_i + v\sum_{i=1}^{n} x_i - v\right)$$
$$= -c\sum_{i=1}^{n}\sup\left(\frac{1}{c}(\lambda_i - b_i - v)x_i - x_i\ln x_i\right) - v$$
$$= -ce^{-\frac{v}{c}-1}\sum_{i=1}^{n} e^{\frac{1}{c}(\lambda_i - b_i)} - v$$

⇒ dual problem is：

$$\operatorname{argmax} \quad g(\lambda,v)$$
$$\text{s.t.} \quad \lambda_i \ge 0,\ i=1,2,\cdots,n$$

⇒ 求导，得到 $v^* = c\ln\left(\sum_{i=1}^{n} e^{\frac{1}{c}(\lambda_i - b_i)}\right) - c$

⇒ 代入得 $\quad \operatorname{argmax} \ -c\ln\left(\sum_{i=1}^{n} e^{\frac{1}{c}(\lambda_i - b_i)}\right)$

$$\text{s.t.} \quad \lambda_i \ge 0,\ i=1,2,\cdots,n$$

⇒ 由于函数单递减

$\lambda_i^* = 0$

$$\Rightarrow v^* = c\ln\left(\sum_{i=1}^{n} e^{\frac{1}{c}(\lambda_i - b_i)}\right) - c = c\ln\left(\sum_{i=1}^{n} e^{-\frac{b_i}{c}}\right) - c$$

根据约束条件

$$\Rightarrow \frac{b_i}{c} + \ln x_i^* + \ln\left(\sum_{i=1}^{n} e^{-\frac{b_i}{c}}\right) = 0$$

$$\Rightarrow x_i = \frac{e^{-\frac{b_i}{c}}}{\sum_{i=1}^{n} e^{-\frac{b_i}{c}}},\quad i=1,2,\cdots,n$$

Problem 5.

(a) plug the equality constraints

$\Rightarrow$ minimize $\sum_{i=1}^{n} \max(0, 1-u_i) + \frac{\lambda}{2}\|w\|_2^2$

subject to $u_i = y_i(w^T x_i + b), i=1,\cdots,n$

(b) $L(w,b,u,v) = \sum_{i=1}^{n} \max(0,1-u_i) + \frac{\lambda}{2}\|w\|_2^2 + \sum_{i=1}^{n} v_i(u_i - y_i(w^T x_i + b))$

$\Rightarrow g(v) = \inf_{u} \sum_{i=1}^{n} [\max(0,1-u_i) + v_i u_i] + \frac{1}{2\lambda}\|\sum_{i=1}^{n} v_i x_i y_i\|^2 - \sum_{i=1}^{n} \frac{v_i y_i}{\lambda} \sum_{j=1}^{n} v_j x_j x_i y_j$

令 $l(x) = \max(0, 1-x)$

$\Rightarrow l^*(y) = \sup_{x}(yx - l(x)) = \begin{cases} y, & -1 \le y \le 0 \\ \infty, & \text{其他} \end{cases}$

$\Rightarrow g(v) = -\sup_{u} \sum_{i=1}^{n} [l(u_i) + v_i u_i] + \frac{1}{2\lambda}\|\sum_{i=1}^{n} v_i x_i y_i\|^2 - \sum_{i=1}^{n} \frac{v_i y_i}{\lambda} \sum_{j=1}^{n} v_j x_j^T x_i y_j$

$= -\sum_{i=1}^{n} \sup[-v_i u_i - l(u_i)] + \frac{1}{2\lambda}\|\sum_{i=1}^{n} v_i x_i y_i\|^2 - \sum_{i=1}^{n} \frac{v_i y_i}{\lambda} \sum_{j=1}^{n} v_j x_j^T x_i y_j$

$= \sum_{i=1}^{n} v_i + \frac{1}{2\lambda}\|\sum_{i=1}^{n} v_i x_i y_i\|^2 - \sum_{i=1}^{n} \frac{v_i y_i}{\lambda} \sum_{j=1}^{n} v_j x_j^T x_i y_j$

$\Rightarrow$ dual problem is maximize $g(v)$

subject to $\sum_{i=1}^{n} v_i y_i = 0$

$0 \le v_i \le 1$

c) $u_i^* = y_i(w^{*T} x_i + b^*), i=1 \ldots n$

$\nabla \sum_{i=1}^{n} \max(0, 1-u_i^*) + \frac{\lambda}{2}\|w^*\|_2^2 + \nabla \sum_{i=1}^{n} v_i(u_i^* - y_i(w^{*T} x_i + b^*)) = 0.$