28. 要证X.Y不相互独立，即证$f(x,y)\neq f_X(x)f_Y(y)$

$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy$

当$-1\le x\le 1$时 $f_X(x)=\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\pi}dy=\frac{2}{\pi}\sqrt{1-x^2}$

$\Rightarrow f_X(x)=\begin{cases}\frac{2}{\pi}\sqrt{1-x^2}, & -1\le x\le 1\\ 0, & 其他\end{cases}$

同理可得 $f_Y(y)=\begin{cases}\frac{2}{\pi}\sqrt{1-y^2}, & -1\le y\le 1\\ 0, & 其他\end{cases}$

而由题意 $f(x,y)=\begin{cases}\frac{1}{\pi}, & x^2+y^2\le 1\\ 0, & 其他\end{cases}\neq f_X(x)f_Y(y)$

$\Rightarrow$ X.Y不相互独立.

由$f_X(x)$得 $E(X)=\int_{-\infty}^{+\infty}xf_X(x)dx=\int_{-1}^{1}\frac{2x}{\pi}\sqrt{1-x^2}dx$

$=\frac{2}{\pi}\int_{-1}^{1}x\sqrt{1-x^2}dx$

由于$y=x\sqrt{1-x^2}$为奇函数且$(-1,1)$对称

故$E(X)=0$.

同理可得$E(Y)=0$.

$E(XY)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf(x,y)dxdy$

$=\frac{1}{\pi}\int_{-1}^{1}xdx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}ydy=\frac{1}{\pi}\int_{-1}^{1}xdx\,\frac{1}{2}y^2\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$

$=\frac{1}{\pi}\int_{-1}^{1}x\times 0\,dx$

$\Rightarrow \rho_{XY}=\frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}=\frac{\overset{=0}{E(XY)}-E(X)E(Y)}{\sqrt{D(X)}\sqrt{D(Y)}}=0.$

$\Rightarrow$ X.Y不相关

$\Rightarrow$ X.Y是不相关的，但X和Y不是相互独立的.

30.

$E(X)=1\times P(A)+0\times P(\bar{A})=P(A)$

$E(Y)=1\times P(B)+0\times P(\bar{B})=P(B)$

$E(XY)=1\times P(AB)+0\times(P(\bar{A}B)+P(A\bar{B})+P(\bar{A}\bar{B}))=P(AB)$

由$\rho_{XY}=0$得 $E(XY)=E(X)E(Y)$

即$P(AB)=P(A)P(B)$

$\Rightarrow$ X和Y必定相互独立.

32. 由题意得

$$E(X)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f(x,y)\,dxdy$$
$$=\int_0^2\int_0^2 \frac{1}{8}x(x+y)\,dxdy$$
$$=\frac{1}{8}\int_0^2 \cancel{\int_0^2}\left[x^2y+\frac{1}{2}xy^2\Big|_0^2\right]dx$$
$$=\frac{1}{8}\int_0^2(2x^2+2x)\,dx$$
$$=\frac{1}{4}\times\left(\frac{1}{3}x^3+\frac{x^2}{2}\Big|_0^2\right)$$
$$=\frac{7}{6}$$

$$E(Y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y f(x,y)\,dxdy$$
$$=\int_0^2\int_0^2 \frac{1}{8}y(x+y)\,dxdy$$

同理 $E(Y)=\frac{7}{6}$

$$E(X^2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,y)\,dxdy$$
$$=\int_0^2\int_0^2 \frac{1}{8}x^2(x+y)\,dxdy$$
$$=\frac{1}{8}\int_0^2 \frac{1}{4}x^4+\frac{1}{3}yx^3\Big|_0^2\,dy$$
$$=\frac{1}{8}\int_0^2\left(4+\frac{8}{3}y\right)dy$$
$$=\frac{1}{8}\times\left(4y+\frac{4}{3}y^2\Big|_0^2\right)$$
$$=\frac{1}{8}\times\frac{40}{3}=\frac{5}{3}$$

$$\Rightarrow D(X)=E(X^2)-E(X)^2$$
$$=\frac{5}{3}-\frac{49}{36}$$
$$=\frac{11}{36}$$

$$D(Y)=\frac{11}{36}$$

$$E(XY)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f(x,y)\,dxdy$$
$$=\int_0^2\int_0^2 \frac{1}{8}xy(x+y)\,dxdy$$
$$=\frac{1}{8}\int_0^2\left(\frac{1}{2}x^2y^2+\frac{1}{3}xy^3\Big|_0^2\right)dx$$
$$=\frac{1}{8}\left(2x^2+\frac{8}{3}x\Big|_0^2\right)$$
$$=\frac{4}{3}$$

$$\Rightarrow Cov(X,Y)=E(XY)-E(X)E(Y)$$
$$=\frac{4}{3}-\frac{7}{6}\times\frac{7}{6}$$
$$=\frac{48}{36}-\frac{49}{36}$$
$$=-\frac{1}{36}$$

$$\rho_{XY}=\frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}=\frac{-\frac{1}{36}}{\frac{11}{36}}=-\frac{1}{11}$$

$$D(X+Y)=E\left(((X+Y)-E(X+Y))^2\right)$$
$$=E\left((X+Y)^2+(E(X)+E(Y))^2-2(X+Y)(E(X)+E(Y))\right)$$
$$=E(X^2)+E(Y^2)+2E(XY)+E(X)^2+E(Y)^2+2E(X)E(Y)-2E(X)^2-2E(Y)^2-4E(X)E(Y)$$
$$=E(X^2)-E(X)^2+E(Y^2)-E(Y)^2+2E(XY)-2E(X)E(Y)$$
$$=\frac{11}{36}+\frac{11}{36}-\frac{2}{36}$$
$$=\frac{20}{36}=\frac{5}{9}$$

34. (1) $E(W)=E((aX+3Y)^2)=D(aX+3Y)+(E(aX+3Y))^2$

$=a^2D(X)+9D(Y)+(E(aX)+E(3Y))^2+2Cov(aX,3Y)$

$=a^2D(X)+9D(Y)+a^2E(X)^2+9E(Y)^2+6aE(X)E(Y)$
$+6a\rho_{XY}\sqrt{D(X)D(Y)}$

$=4a^2+9\times16+6a\times(-0.5)\times\sqrt{4\times16}$

$=4a^2+24a+144$

$=(2a-6)^2+108$

当$a=3$时$E(W)$取最小值108.

(2) $Cov(W,V)=Cov(X-aY,X+aY)$

$=Cov(X,X)-a^2Cov(Y,Y)$

$=\sigma_X^2-a^2\sigma_Y^2$

当$a=\frac{\sigma_X}{\sigma_Y}$时$Cov(W,V)=0$

$\Rightarrow\rho_{XY}=0$

$\Rightarrow W$与V不相关.

35. (X,Y)服从二维正态分布，$X\sim N(0,3)$，$Y\sim N(0,4)$，$\rho_{XY}=-\frac{1}{4}$

$\Rightarrow(X,Y)\sim N(0,0,\sqrt{3},2,-\frac{1}{4})$

$\Rightarrow f(x,y)=\frac{1}{2\pi\times2\sqrt{3}\times\frac{\sqrt{15}}{4}}\exp\{-\frac{2}{\sqrt{15}}[\frac{x^2}{3}+\frac{1}{2}\times\frac{xy}{2\sqrt{3}}+\frac{y^2}{4}]\}$

$=\frac{\sqrt{5}}{15\pi}\exp[-\frac{8}{45}x^2-\frac{2xy}{15\sqrt{3}}-\frac{2}{15}y^2]$

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13. (1) 当$0<y\le1$时

$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}$

当$-\sqrt{y}<x<\sqrt{y}$时 $f_{X|Y}(x|y)=\frac{21}{4}x^2y\Big/\frac{7}{2}y^{\frac{5}{2}}=\frac{21}{4}\times\frac{2}{7}\times x^2y^{1-\frac{5}{2}}$

$=\frac{3}{2}x^2y^{-\frac{3}{2}}$

$\Rightarrow f_{X|Y}(x|y)=\begin{cases}\frac{3}{2}x^2y^{-\frac{3}{2}}, & -\sqrt{y}<x<\sqrt{y}\\ 0, & 其他.\end{cases}$

$Y=\frac{1}{2}$时

$f_{X|Y}(x|y=\frac{1}{2})=\begin{cases}3\sqrt{2}x^2, & -\frac{\sqrt{2}}{2}<x<\frac{\sqrt{2}}{2}\\ 0, & 其他\end{cases}$

(2) 当 $-1<x<1$ 时

若 $x^2<y<1$

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{21}{4}x^2y\Big/\frac{21}{8}x^2(1-x^4)$$

$$=\frac{21}{4}x^2y\times\frac{8}{21}x^2(1-x^4)$$

$$=\frac{2y}{1-x^4}$$

$$\Rightarrow f_{Y|X}(y|x)=\begin{cases}\frac{2y}{1-x^4}, & x^2<y<1\\ 0, & 其他.\end{cases}$$

$$x=\frac{1}{3}时 f_{Y|X}(y|x=\tfrac{1}{3})=\begin{cases}\frac{81}{40}y, & \frac{1}{9}<y<1\\ 0, & 其他\end{cases}$$

$$x=\frac{1}{2}时 f_{Y|X}(y|x=\tfrac{1}{2})=\begin{cases}\frac{32}{15}y, & \frac{1}{4}<y<1\\ 0, & 其他.\end{cases}$$

(3) $P(Y\geq\frac{1}{4}|X=\frac{1}{2})=P(Y|X=\frac{1}{2})-P(Y<\frac{1}{4}|X=\frac{1}{2})$

$$=\int_{\frac{1}{4}}^{1}\frac{32}{15}y\,dy-\int_{\frac{1}{4}}^{\frac{1}{4}}\frac{32}{15}y\,dy$$

$$=\frac{32}{15}\times\frac{1}{2}\times\frac{15}{16}=1.$$

$P(Y\geq\frac{3}{4}|X=\frac{1}{2})=P(Y|X=\frac{1}{2})-P(Y<\frac{3}{4}|X=\frac{1}{2})$

$$=1-\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{32}{15}y\,dy$$

$$=1-\frac{32}{15}\times\frac{1}{2}\times\frac{8}{16}=\frac{7}{15}$$

14. 当 $0<x<1$ 时

$$f_X(x)=\int_{-x}^{x}1\,dy=2x$$

$$\Rightarrow f_X(x)=\begin{cases}2x, & 0<x<1\\ 0, & 其他.\end{cases}$$

当 $-x<y<x$ 时

$$f_Y(y)=\int_{|y|}^{1}1\,dx=1-|y|$$

$$\Rightarrow f_Y(y)=\begin{cases}1-|y|, & -1<y<1\\ 0, & 其他.\end{cases}$$

当 $0<x<1$ 时

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x,y)}=\frac{1}{2x}$$

$$\Rightarrow f_{Y|X}(y|x)=\begin{cases}\frac{1}{2x}, & |y|<x\\ 0, & 其他.\end{cases}$$

当 $-1<y<1$ 时

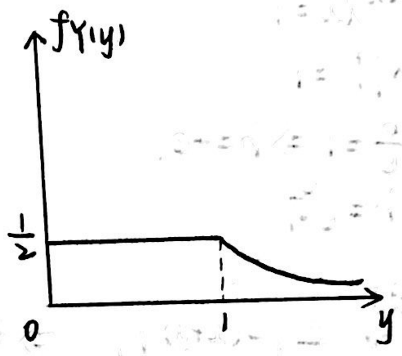
$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_Y(y)}=\frac{1}{1-|y|}$$

$$\Rightarrow f_{Y|X}(y|x)=\begin{cases}\frac{1}{1-|y|}, & |y|<x<1\\ 0, & \text{其他}.\end{cases}$$

15. (1) $f_{Y|X}(y|x)=\begin{cases}x, & 0<y<\frac{1}{x}\\ 0, & \text{其他}.\end{cases}$

$$f(x,y)=f_X(x)f_{Y|X}(y|x)=\begin{cases}x, & 0<y<\frac{1}{x},\ 0<x<1\\ 0, & \text{其他}\end{cases}$$

(2) $f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx$



当 $0<y<1$ 时 $f_Y(y)=\int_0^1 x\,dx=\frac{1}{2}$

当 $y\geq 1$ 时 $f_Y(y)=\int_0^{\frac{1}{y}} x\,dx=\frac{1}{2y^2}$

$$\Rightarrow f_Y(y)=\begin{cases}\frac{1}{2}, & 0<y<1\\ \frac{1}{2y^2}, & y\geq 1\\ 0, & \text{其他}.\end{cases}$$

(3). $P(X>Y)=\int_y^1\int_0^1 x\,dx\,dy=\int_0^1\left(\frac{1}{2}x^2\Big|_y^1\right)dy$

$$=\int_0^1\left(\frac{1}{2}-\frac{1}{2}y^2\right)dy$$

$$=\frac{1}{2}y-\frac{1}{6}y^3\Big|_0^1$$

$$=\frac{1}{3}.$$

20. (1) X和Y相互独立 $\Rightarrow f(x,y)=f_X(x)f_Y(y)=\begin{cases}\lambda u e^{-\lambda x-uy}, & x>0,y>0.\\ 0, & \text{其他}.\end{cases}$

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=f_X(x)=\begin{cases}\lambda e^{-\lambda x}, & x>0\\ 0, & \text{其他}.\end{cases}$$

(2) $P(Z=1)=P(X\leq Y)=\int_0^{+\infty}\int_x^{+\infty}f(x,y)\,dx\,dy$

$$=\int_0^{+\infty}\int_x^{+\infty}\lambda\, e^{-\lambda x-uy}\,dx\,dy$$

$$=\int_0^{+\infty}\lambda\cdot e^{-(\lambda+u)x}dx$$

$$=-\frac{\lambda}{\lambda+u}\times(0-1)$$

$$=\frac{\lambda}{\lambda+u}.$$

$P(Z=0)=1-\frac{\lambda}{\lambda+u}$

$$F(z)=\begin{cases}0, & z<0\\ \frac{u}{\lambda+u}, & 0\le z<1\\ 1, & z\ge 1\end{cases}$$

29. (1) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy=1$

$\Rightarrow \int_0^1\int_0^{+\infty} be^{-(x+y)}\,dx\,dy=1$

$-b\int_0^1 \left(e^{-(x+y)}\Big|_0^{+\infty}\right)dx=1$

$-b\int_0^1 -e^{-x}\,dx=1$

$b\int_0^1 e^{-x}\,dx=1$

$-b\ e^{-x}\Big|_0^1=1$

$-b(e^{-1}-1)=1$

$b=\frac{e}{e-1}$

(2). 当 $0<x<1$ 时

$f_X(x)=\int_0^{+\infty}\frac{e}{e-1}e^{-(x+y)}dy$

$=\frac{e}{e-1}\times(-1)\,e^{-(x+y)}\Big|_0^{+\infty}$

$=\frac{e}{1-e}(0-e^{-x})$

$=\frac{e^{1-x}}{e-1}$

$\Rightarrow f_X(x)=\begin{cases}\frac{e^{1-x}}{e-1}, & 0<x<1\\ 0, & 其他.\end{cases}$

当 $y>0$ 时

$f_Y(y)=\int_0^1\frac{e}{e-1}e^{-(x+y)}dx$

$=\frac{e}{e-1}\times(-1)\times e^{-(x+y)}\Big|_0^1$

$=e^{-y}$

$\Rightarrow f_Y(y)=\begin{cases}e^{-y}, & y>0\\ 0, & 其他.\end{cases}$

(3) $F_X(x)=\int_0^x\frac{e^{1-x}}{e-1}dx=\frac{e}{e-1}\int_0^x e^{-x}dx$

$=\frac{e}{e-1}\times -e^{-x}\Big|_0^x$

$=\frac{e}{e-1}(1-e^{-x})$

$$\Rightarrow F_X(x)=\begin{cases}0, & x<0\\ \frac{e}{e-1}(1-e^{-x}), & 0\le x<1\\ 1, & x\ge 1.\end{cases}$$

$F_Y(y)=\int_0^y e^{-y}dy=-e^{-y}\Big|_0^y=1-e^{-y}$

$$\Rightarrow F_Y(y)=\begin{cases}1-e^{-y}, & y>0\\ 0, & y\le 0.\end{cases}$$

$$F_U(u) = P(U \le u) = P(\max(X,Y) \le u)$$
$$= P(X \le u, Y \le u)$$
$$= P(X \le u)P(Y \le u)$$
$$= F_X(u)F_Y(u)$$

$u \le 0$时 $F_U(u) = 0 \times 0 = 0$

$0 < u < 1$时 $F_U(u) = \frac{e}{e-1}(1-e^{-u})(1-e^{-u})$

$u \ge 1$时 $F_U(u) = 1 \times (1-e^{-u}) = 1-e^{-u}$