第十一次作业

1. 设 $X_1, X_2, \dots X_m$ 和 $X_1', X_2', \dots, X_m'$ 分别为来自总体 $X \sim N(u, a^2)$ 的两个样本

根据正态分布的性质有

$\overline{X_1} = \frac{1}{m}\sum_{i=1}^{m} X_i \sim N(u, \frac{a^2}{m})$ ， $\overline{X_2} = \frac{1}{n}\sum_{i=1}^{n} X_i' \sim N(u, \frac{a^2}{n})$

进一步根据正态分布的性质有：

$\overline{X_1} - \overline{X_2} \sim N(0, \frac{a^2}{m} + \frac{a^2}{n})$

于是可得 $Pr(|\overline{X_1} - \overline{X_2}| > \epsilon) = 2 - 2\Phi(\epsilon / \sqrt{\frac{a^2}{m} + \frac{a^2}{n}})$

2. 设随机变量 $Z = X + Y$

有 $f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\,dx$

$= \int_0^z \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\lambda x} \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} (z-x)^{\alpha_2 - 1} e^{-\lambda(z-x)}\,dx$

$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} \int_0^z x^{\alpha_1-1}(z-x)^{\alpha_2-1}\,dx$

令变量替换 $x = zt$ 有

$\int_0^z x^{\alpha_1-1}(z-x)^{\alpha_2-1}\,dx = z^{\alpha_1+\alpha_2-1}\int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\,dt = z^{\alpha_1+\alpha_2-1} B(\alpha_1, \alpha_2)$

利用Beta函数的性质有：

$B(\alpha_1, \alpha_2) = \int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\,dt = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$

代入完成证明

3. 设随机变量 $Y = X^2$

当 $y \le 0$ 时 ~~则~~ 分布函数 $F_Y(y) = 0$

当 $y > 0$ 时

$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{y}}^{\sqrt{y}} e^{-\frac{x^2}{2}}\,dx$

$= \frac{1}{\sqrt{2\pi}} \times 2\int_0^{\sqrt{y}} e^{-\frac{x^2}{2}}\,dx$

$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sqrt{y}} e^{-\frac{y}{2}}$

$\Rightarrow X^2 \sim \Gamma(\frac{1}{2}, \frac{1}{2})$

当 k 为偶数时

$E(X^k) = \int_{-\infty}^{+\infty} \frac{x^k}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = -\int_{-\infty}^{+\infty} \frac{x^{k-1}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = (k-1)\int_{-\infty}^{+\infty} \frac{x^{k-2}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$

$= -(k-1)\int_{-\infty}^{+\infty} \frac{x^{k-3}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = (k-1)(k-3) \frac{x^{k-4}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$

$= \dots = (k-1)(k-3)\dots 1 = \int_{-\infty}^{+\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = (k-1)!!$

当K为奇数时

$E(X^k)=\int_{-\infty}^{+\infty}\frac{x^k}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=\cdots=(k-1)!!\int_{-\infty}^{+\infty}\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

由$(-\infty,+\infty)$关于y轴对称，且$\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$为奇函数

$\Rightarrow E(X^k)=0.$

$\Rightarrow E(X^k)=\begin{cases}(k-1)!! & K\text{为偶数}\\ 0 & K\text{为奇数}.\end{cases}$

$(0,a^2)$

4. 根据正态分布的性质有：

$X_1+2X_2+\cdots+nX_n\sim N(0,a^2+4a^2+\cdots+n^2a^2)$

即$X_1+2X_2+\cdots+nX_n\sim N(0,\frac{n(n+1)(2n+1)}{6}a^2)$ 令$1\sim n$的平方和为t

即$\frac{X_1+2X_2+\cdots+nX_n}{\sqrt{t}a}\sim N(0,1)$.

同理可得$\frac{Y_1+2Y_2+\cdots+mY_m}{\sqrt{S}a}\sim N(0,1)$，$1\sim m$的平方和为$S$.

$\Rightarrow$当$a=\frac{1}{n(n+1)(2n+1)a^2}$，$b=\frac{1}{m(m+1)(2m+1)a^2}$时服从$\chi^2$分布

且自由度为2，即$Y\sim\chi^2(2)$成立.

$\Rightarrow Y$的概率密度函数为

$f_Y(y)=\begin{cases}\frac{\frac{1}{2}}{\Gamma(1)}y^0e^{-\frac{y}{2}}, & y>0\\ 0, & y\leq 0.\end{cases}$ 即$f_Y(y)=\begin{cases}\frac{1}{2}e^{-\frac{y}{2}}, & y>0\\ 0, & y\leq 0.\end{cases}$

5. $Y_1,\cdots,Y_n$是$X\sim N(0,n)$的样本，由正态分布的性质得

$\Rightarrow\frac{Y_1}{\sqrt{n}},\cdots,\frac{Y_n}{\sqrt{n}}\sim N(0,1)$.

$\Rightarrow\sum_{i=1}^{n}(\frac{Y_i}{\sqrt{n}})^2=\frac{1}{n}\sum_{i=1}^{n}Y_i^2\sim\chi^2(n)$

$X_1,\cdots,X_n$是$X\sim N(0,n)$的样本，由正态分布的性质得

$\Rightarrow\frac{X_1}{\sqrt{n}},\cdots,\frac{X_n}{\sqrt{n}}$是$X'\sim N(0,1)$的样本

$\Rightarrow\sum_{i=1}^{n}\frac{X_i}{\sqrt{n}}\sim N(0,1)$.即$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_i\sim N(0,1)$.

令随机变量$T=\frac{\sum_{i=1}^{n}X_i}{\sqrt{\sum_{i=1}^{n}Y_i^2}}$ 也即$T=\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_i}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}Y_i^2}}$

$\Rightarrow T\sim t(n)$

6. 由 $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

而 $\sigma^2 = \frac{1}{4}$

$\Rightarrow 4(n-1)S^2 \sim \chi^2(n-1)$

由 $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$

$\Rightarrow 4\sum_{i=1}^{n}(X_i-\bar{X})^2$

$\Rightarrow Pr[\sum_{i=1}^{n}(X_i-\bar{X})^2 \geq \varepsilon]$

$= Pr[4\sum_{i=1}^{n}(X_i-\bar{X})^2 \geq 4\varepsilon]$

$= 1 - Pr[4\sum_{i=1}^{n}(X_i-\bar{X})^2 \leq 4\varepsilon]$

令 $4\sum_{i=1}^{n}(X_i-\bar{X})^2 = Y$，分布函数为 $F(y) = \int_0^{+\infty} \frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} e^{-\frac{y}{2}}$

$\Rightarrow Pr[\sum_{i=1}^{n}(X_i-\bar{X})^2 \geq \varepsilon] = 1 - F(4\varepsilon)$

7. $X_1, \cdots, X_n$ 是 $X \sim N(12, \sigma^2)$ 的样本. 由正态分布的性质

$\Rightarrow \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-12)}{\sigma} \sim N(0,1)$

而又有 $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

$\Rightarrow$ 有随机变量 $T = \frac{\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-12)}{\sigma}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-12)}{S} = \frac{\frac{1}{n}\sum_{i=1}^{n}X_i-12}{S} \sim t(n)$

$\Rightarrow Pr[\frac{1}{n}\sum_{i=1}^{n}X_i \geq \varepsilon] = Pr[T \geq \frac{\varepsilon-12}{S}]$

$= 1 - Pr[T \leq \frac{\varepsilon-12}{S}]$

令 T 的分布函数为 $F(t)$

$1 - Pr[T \leq \frac{\varepsilon-12}{S}] = 1 - F(\frac{\varepsilon-12}{S})$

8. 书147-148页 2(2), 4(2), 7, 9

2(2).

$P\{max(X_1,X_2,X_3,X_4,X_5)>15\}$

$=1-P\{max(X_1,X_2,X_3,X_4,X_5)\le 15\}$

$=1-P(X_1\le 15, X_2\le 15, X_3\le 15, X_5\le 15)$

$=1-\prod_{i=1}^{5}P(X_i\le 15)$

$=1-\prod_{i=1}^{5}P\{\frac{X_i-12}{2}\geq\frac{10-12}{2}\}$

$=1-\prod_{i=1}^{5}P\{\frac{X_i-12}{2}\geq -1\}.$

$=1-[1-\Phi(-1)]^5$

$=1-[\Phi(1)]^5=0.5785.$

4(2)

$X_1, X_2, \cdots, X_5$ 是总体 $X\sim N(0,1)$ 的样本

$\Rightarrow X_1+X_2\sim N(0,2)$, $X_3^2+X_4^2+X_5^2\sim \chi^2(3)$

$\Rightarrow \frac{X_1+X_2}{\sqrt{2}}\sim N(0,1)$

又由 X_1+X_2, $X_3^2+X_4^2+X_5^2$ 相互独立

$\Rightarrow \frac{\frac{X_1+X_2}{\sqrt{2}}}{\sqrt{X_3^2+X_4^2+X_5^2/3}}\sim t(3)$

$\Rightarrow \sqrt{\frac{3}{2}}\frac{X_1+X_2}{\sqrt{X_3^2+X_4^2+X_5^2}}\sim t(3)$

$\Rightarrow C=\sqrt{\frac{3}{2}}$

7. $X\sim \chi^2(n)$

$\Rightarrow E(X)=n$, $D(X)=2n$

而 $\bar{X}=\frac{1}{10}\sum_{i=1}^{10}X_i$

$\Rightarrow E(\bar{X})=E(\frac{1}{10}\sum_{i=1}^{10}X_i)=\frac{1}{10}\sum_{i=1}^{10}E(X_i)=E(X)=n$

$D(\bar{X})=D(\frac{1}{10}\sum_{i=1}^{10}X_i)=\frac{1}{100}\sum_{i=1}^{10}D(X_i)=\frac{D(X)}{10}=\frac{n}{5}$

$E(S^2)=E[\frac{1}{9}\sum_{i=1}^{10}(X_i-\bar{X})^2]=D(X)=2n$

9 (1)

由题意

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

当 $n=16$ 时

$$\frac{15S^2}{\sigma^2} \sim \chi^2(15)$$

$$\Rightarrow P\left\{\frac{S^2}{\sigma^2} \le 2.041\right\} = P\left\{\frac{15S^2}{\sigma^2} \le 2.041 \times 15\right\}$$

$$= P\left\{\frac{15S^2}{\sigma^2} \le 30.615\right\}$$

$$= 1 - P\left\{\frac{15S^2}{\sigma^2} > 30.615\right\}$$

$$= 0.99$$

(2) 由卡方分布的性质

$$D\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{(n-1)^2}{\sigma^4} D(S^2) = 2(n-1)$$

$$\Rightarrow D(S^2) = 2(n-1) \times \frac{\sigma^4}{(n-1)^2}$$

$$= \frac{2\sigma^4}{n-1}$$

当 $n=16$ 时 $D(S^2) = \frac{2}{15}\sigma^4$