凸优化第二次作业

Problem 1: Convex functions

(a)

$f(x) = -\sum_{i=1}^{n} \log x_i$

由凸函数的定义一且R_{++}^n是凸集

对于任意$x, y \in dom f$和$0 \le \theta \le 1$.

有 $f(\theta x+(1-\theta)y) = -\sum_{i=1}^{n} \log(\theta x_i+(1-\theta)y_i)$

$= -[\log \prod_{i=1}^{n} (\theta x_i+(1-\theta)y_i)]$

根据不等式 $f(\theta x+(1-\theta)y) \le f(x)^\theta f(y)^{1-\theta} \Rightarrow \theta x_i+(1-\theta)y_i \le (x_i)^\theta (y_i)^{1-\theta}$.

$\prod_{i=1}^{n}(\theta x_i+(1-\theta)y_i) \le \prod_{i=1}^{n} (x_i)^\theta (y_i)^{1-\theta}$.

$\Rightarrow f(\theta x+(1-\theta)y) \le \theta f(x)+(1-\theta)f(y)$

$\Rightarrow f(x)$是严格凸的.

(b) ① $dom f$是凸集且f二阶可导

$\Rightarrow f(x) \ge f(y) + \nabla f(y)^T(x-y)$, $f(y) \ge f(x) + \nabla f(x)^T(y-x)$

$\Rightarrow f(x)-f(y)-\nabla f(y)^T(x-y) \ge 0$ ①, $f(y)-f(x)-\nabla f(x)^T(y-x) \ge 0$ ②.

①+② $\Rightarrow (\nabla f(x)-\nabla f(y))^T(x-y) \ge 0$.

② ~~取$g(t)=f(x+t(y-x))$~~

~~$g'(t)=\nabla f(x+t(y-x))^T(y-x)$~~

$f(y)=g(1) \ge g(0)+g'(0)$

$=f(x)+\nabla f(x)^T(y-x)$

$\Rightarrow f(x)$是凸函数

(c) f是凸函数

$\Rightarrow \forall x,y \in dom f, \theta \in [0,1], f(\theta x+(1-\theta)y) \le \theta f(x)+(1-\theta)f(y)$

且$dom f$为凸集

$\Rightarrow dom g = \{(x,t) \in R^{n+1}, x \in dom f, t>0\}$为凸集

② $\forall x,y \in dom, \theta \in [0,1]$

$g(\theta x+(1-\theta)y, t) = t f(\frac{\theta x+(1-\theta)y}{t})$

$\le t \times (\theta f(\frac{x}{t})+(1-\theta)f(\frac{y}{t}))$

$= \theta t f(\frac{x}{t})+(1-\theta)t f(\frac{y}{t})$

$= \theta g(x,t)+(1-\theta)g(y,t)$ $\Rightarrow$ g是凸函数.

Problem 2.

$$\frac{\partial f(x)}{\partial x_i} = \left(\sum_{i=1}^{n} x_i^{P}\right)^{(1-P)/P} x_i^{P-1} = \left(\frac{f(x)}{x_i}\right)^{1-P}$$

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{1-P}{x_i}\left(\frac{f(x)}{x_i}\right)^{-P}\left(\frac{f(x)}{x_j}\right)^{1-P}$$

$$= \frac{1-P}{f(x)}\left(\frac{f(x)^2}{x_i x_j}\right)^{1-P}$$

$$\frac{\partial^2 f(x)}{\partial x_i^2} = \frac{1-P}{f(x)}\left(\frac{f(x)^2}{x_i^2}\right)^{1-P} - \frac{1-P}{x_i}\left(\frac{f(x)}{x_i}\right)^{1-P}$$

$$y^T \nabla^2 f(x) y = \frac{1-P}{f(x)}\left(\left(\sum_{i=1}^{n} \frac{y_i f(x)^{1-P}}{x_i^{1-P}}\right)^2 - \sum_{i=1}^{n} \frac{y_i^2 f(x)^{2-P}}{x_i^{2-P}}\right)$$

由柯西-施瓦兹不等式，上式≤0.

$\Rightarrow f(x)$ 是凹函数.

Problem 3.

(a) ψ 是凸的 $\Rightarrow \psi(x) \geq \psi(y) + \nabla\psi(y)^T(x-y)$ 当且仅当 $x=y$ 时取"=".

$\Rightarrow \psi(x) - \psi(y) - \langle \nabla\psi(y), (x-y) \rangle \geq 0.$

$\Rightarrow \Delta\psi(x,y) \geq 0$，当且仅当 $x=y$ 时取"=".

(b) 令 $F(x) = L(x) + \Delta\psi(x, x_0)$

由 $x^* = \arg\min_{x \in C} L(x) + \Delta\psi(x, x_0)$

$\Rightarrow$ 存在次梯度 $d \in \partial F(x^*)$. $\langle d, x - x^* \rangle \geq 0$

由，$\partial F(x^*) = \{ g + \nabla_{x=x^*}\Delta\psi(x, x_0) \mid g \in \partial L(x^*) \}$.

$\Rightarrow$ 存在次梯度 $g \in L(x^*)$

$\langle g + \nabla\psi(x^*) - \nabla\psi(x_0), x - x^* \rangle \geq 0.$

$\Rightarrow L(y) \geq L(x^*) + \langle g, y - x^* \rangle$

$\geq L(x^*) + \langle \nabla\psi(x_0) - \nabla\psi(x^*), y - x^* \rangle.$

$= L(x^*) - \langle \nabla\psi(x_0), x^* - x_0 \rangle + \psi(x^*) - \psi(x_0) + \langle \nabla\psi(x_0), y - x_0 \rangle - \psi(y) + \psi(x_0)$

$- \langle \nabla\psi(x^*), y - x^* \rangle + \psi(y) - \psi(x^*)$.

$= L(x^*) + \Delta\psi(x^*, x_0) - \Delta\psi(y, x_0) + \Delta\psi(y, x^*)$.

$\Rightarrow L(y) + \Delta\psi(y, x_0)$

$\geq L(x^*) + \Delta\psi(x^*, x_0) + \Delta\psi(y, x^*)$.

Problem 4.

令 $f(x)=\frac{1}{2}\|\Pi_X(x)-\Pi_X(y)\|_2^2$ $\nabla f(x)=\Pi_X(x)-\Pi_X(y)$

$\nabla^2 f(x)=1>0$

$\Rightarrow f(x)$ 为凸函数

$\Rightarrow f(y)\geq f(x)+\langle\nabla f(x),(y-x)\rangle$

$\Rightarrow \|\Pi_X(x)-\Pi_X(y)\|_2^2\leq(\Pi_X(x)-\Pi_X(y))(x-y)$.

(b) 由(a)结论

$\langle\Pi_X(x)-\Pi_X(y),x-y\rangle=\|\Pi_X(x)-\Pi_X(y)\|_2\|x-y\|_2\cdot\cos\theta$.

$\Rightarrow\|\Pi_X(x)-\Pi_X(y)\|_2\leq\|x-y\|_2\cos\theta\leq\|x-y\|_2$.

Problem 5.

(a) $f^*(y)=\sup_{x\in\mathrm{dom}f}(y^Tx-f(x))$.

当 $0>1-x$ 即 $x>1$ 时 $f(x)=0$.

$f^*(y)=\sup_{x>1} y^Tx$.

当 $x>1$ 时无上确界

当 $0\leq 1-x$ 即 $x\leq 1$ 时 $f(x)=1-x$

$f^*(y)=\sup_{x\leq 1}(xy-1+x)$.

$=y$.

(b) ~~手~~令 $g(x)=\ln(1+e^{-x})$

$\cancel{g'(x)=\frac{1}{1+e^{-x}}\times e^{-x}(-1)}$

$\cancel{=-\frac{e^{-x}}{1+e^{-x}}<0}$.

$f^*(y)=\sup(xy-\ln(1+e^{-x}))$.

$g(x)=xy-\ln(1+e^{-x})$

$g'(x)=y+\frac{e^{-x}}{1+e^{-x}}$

$g'(x)\geq 0$ $y+e^{-x}y+e^{-x}\geq 0$

$e^{-x}\geq-\frac{y}{y+1}$

$x\leq-\ln(-\frac{y}{y+1})$.

$\Rightarrow f^*(y)$

$=-y\ln(-\frac{y}{y+1})-\ln(1+\ln(-\frac{y}{y+1}))$