第十一次作业.

6.9

令 $\sum_{i=1}^{k} X_i^2 = \bar{X}$

$\Rightarrow Pr(\bar{X} \geq (1+\varepsilon)k)$ 由Chernoff方法

$= Pr(e^{t\bar{X}} \geq e^{t(1+\varepsilon)k})$

$\leq e^{-tk-t\varepsilon k} E(e^{t\bar{X}})$.

右式 $= e^{-tk(1+\varepsilon)} E(e^{t\sum_{i=1}^{k} X_i^2})$

$= e^{-tk(1+\varepsilon)} \prod_{i=1}^{k} E(e^{tX_i^2})$.

由 $X_i \sim N(0,1)$

$E(e^{tX_i^2}) = \int_{-\infty}^{+\infty} e^{tx^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$\Rightarrow e^{-tk(1+\varepsilon)} \prod_{i=1}^{k} E(e^{tX_i^2}) = e^{-tk(1+\varepsilon)} / (1-2t)^{\frac{k}{2}}$.

要证 $Pr(\sum_{i=1}^{k} X_i^2 \geq (1+\varepsilon)k) \leq exp(-k(\varepsilon^2-\varepsilon^3)/4)$

即证 $\frac{e^{-tk(1+\varepsilon)}}{(1-2t)^{\frac{k}{2}}} \leq e^{-k(\varepsilon^2-\varepsilon^3)/4}$

将t看作变量设 $y = \frac{e^{-t(1+\varepsilon)}}{(1-2t)^{\frac{k}{2}}}$ 令 $y'=0$ 得 $t = \frac{\varepsilon}{2+2\varepsilon}$.

此时 $y = e^{-\frac{\varepsilon k}{2}}(\varepsilon+1)^{\frac{k}{2}}$. $\Rightarrow e^{-\frac{\varepsilon k}{2}}(\varepsilon+1)^{\frac{k}{2}} \leq e^{\frac{\varepsilon^2(1-\varepsilon)}{4}}$

将ε看作变量设 $y = e^{k}(\varepsilon+1)^{\frac{k}{2}} - e^{\frac{\varepsilon(\varepsilon-1)}{2}} \frac{\varepsilon^2-\varepsilon}{2}$

$y' = e^{k} \times \frac{k}{2}(\varepsilon+1)^{\frac{k}{2}-1} - e^{\frac{\varepsilon(\varepsilon-1)}{2}} \times (\varepsilon - \frac{1}{2}) > 0$.

$\Rightarrow y > 0$.

故 $Pr(\sum_{i=1}^{k} X_i^2 \geq (1+\varepsilon)k) \leq exp(-k(\varepsilon^2-\varepsilon^3)/4)$ 可证.

3.10 证明Bennet不等式

由Chernoff方法:

$$\Pr\left[\frac{1}{n}\sum_{i=1}^{n}(X_n-u)\geq \varepsilon\right]\leq e^{-nt\varepsilon}E\left[\exp\left(\sum_{i=1}^{n}t(X_i-u)\right)\right]$$

$$=e^{-nt\varepsilon}\left(E\left[e^{t(X_1-u)}\right]\right)^n.$$

令$Y=X_1-u$

$$\ln E\left[e^{t(X_1-u)}\right]=\ln E\left[e^{tY}\right]$$

$$\leq E\left[e^{tY}\right]-1$$

$$=t^2E\left[\frac{e^{tY}-tY-1}{t^2Y^2}Y^2\right]$$

$$\leq t^2E\left[\frac{e^{t}-t-1}{t^2}Y^2\right]$$

$$=(e^t-t-1)\sigma^2.$$

令$f(x)=\frac{(e^x-x-1)}{x^2}$

$$f'(x)=-\frac{(e^x-x-1)}{x^4}\times 2x+\frac{x(e^x-1)}{x^3}$$

$$=\frac{xe^x-x-2e^x+2x+2}{x^3}$$

$$=\frac{x+2-xe^x-2e^x}{x^3}=(x+2)\frac{1-e^x}{x^3}$$

由$t>0$,故$y=\frac{e^t-t-1}{t^2}$单调增

$$\Rightarrow e^t-t-1\leq\frac{t^2}{2}\sum_{k=0}^{\infty}\left(\frac{t}{3}\right)^k=\frac{t^2}{2(1-\frac{t}{3})}.$$

$$\Rightarrow \Pr\left[\frac{1}{n}\sum_{i=1}^{n}(X_n-u)\geq\varepsilon\right]\leq\exp\left(-nt\varepsilon+\frac{nt^2\sigma^2}{2(1-\frac{t}{3})}\right).$$

$$\Rightarrow t=\frac{\varepsilon}{\sigma^2+\frac{\varepsilon}{3}}$$

$$\Rightarrow \Pr\left[\frac{1}{n}\sum_{i=1}^{n}(X_n-u)\geq\varepsilon\right]\leq\exp\left(-\frac{n\varepsilon^2}{2\sigma^2+\frac{2\varepsilon}{3}}\right).$$

6.11 证明 Berstein不等式.

同6.10可以得到:

$$Pr[\frac{1}{n}\sum_{i=1}^{n}(X_n-u)\geq\varepsilon]\leq exp(-nt\varepsilon+\frac{nt^2\sigma^2}{2(1-\frac{t}{\delta})})$$

取 $t=\frac{\varepsilon}{\sigma^2+b\varepsilon}$

$$\Rightarrow Pr[\frac{1}{n}\sum_{i=1}^{n}(X_n-u)\geq\varepsilon]\leq exp(-\frac{n\varepsilon^2}{2\sigma^2+2b\varepsilon})$$

6.12

令 $Pr[\frac{1}{n}\sum_{i=1}^{n}(X_n-u)\geq\varepsilon]\leq exp(-\frac{n\varepsilon^2}{2\sigma^2+2b\varepsilon})=\delta$

由 $e^{-\frac{n\varepsilon^2}{2\sigma^2+2b\varepsilon}}=\delta$ 方程

$n\varepsilon^2=\ln\frac{1}{\delta}(2\sigma^2+2b\varepsilon)$

$n\varepsilon^2-2b\ln\frac{1}{\delta}\varepsilon-2\sigma^2\ln\frac{1}{\delta}=0$

$\Delta>0.$

$$\Rightarrow 解得\ \varepsilon=\frac{2b\ln\frac{1}{\delta}+\sqrt{4b^2\ln^2\frac{1}{\delta}+8\sigma^2\ln\frac{1}{\delta}}}{2n}$$

$$=\frac{b\ln\frac{1}{\delta}+\sqrt{b\ln\frac{1}{\delta}+2\sigma^2\ln\frac{1}{\delta}n}}{n}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}X_i\leq u+\frac{b\ln\frac{1}{\delta}+\sqrt{b\ln\frac{1}{\delta}+2\sigma^2\ln\frac{1}{\delta}n}}{n}$$

6.13. 若 $E[X_i]=0$.

对 $\forall t>0$, 由Jensen不等式

$$exp(tE[\max_{i\in[n]}X_i])\leq E[exp(t\max_{i\in[n]}X_i)]$$

$$=E[\max_{i\in[n]}X_i]\leq\frac{\ln n}{t}+\frac{bt}{2}$$

令右式 $y=\frac{\ln n}{t}+\frac{bt}{2}$

$y'=-\frac{\ln n}{t^2}+\frac{b}{2}$

$=\frac{bt^2-\ln n}{2t^2}$ 令 $y'=0$ 得 $t=\sqrt{\frac{2\ln n}{b}}$

$$\Rightarrow E[\max_{i\in[n]}X_i]\leq\sqrt{2b\ln n}$$

由本题题意 $X_i\sim N(u,\sigma^2)$ 得 $E[X_i]=u$.

且高斯变量是参数为 σ^2 的 sub-Guassian随机变量.

$$\Rightarrow 本题\ E[\max_{i\in[n]}X_i]\leq\sqrt{2\sigma^2\ln n}+u.$$