第十二次作业.

p173

2.(1).

求出总体矩:

$$E[X]=\int_{-\infty}^{+\infty}xf(x)dx=\int_{c}^{+\infty}x\theta c^{\theta}x^{-(\theta+1)}dx$$

$$=c^{\theta}\int_{c}^{+\infty}\theta x^{-\theta}dx$$

$$=c^{\theta}\int_{c}^{+\infty}\frac{\theta}{1-\theta}dx^{-\theta+1}$$

$$=\frac{\theta c^{\theta}}{1-\theta}\int_{c}^{+\infty}dx^{1-\theta}$$

$$=\frac{\theta c^{\theta}}{1-\theta}\ x^{1-\theta}\Big|_{c}^{+\infty}$$

$$=\frac{\theta c}{\theta-1}$$

求出样本矩: $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$

$\Rightarrow\ \bar{X}=E[X]$

$$\frac{\theta c}{\theta-1}=\frac{1}{n}\sum_{i=1}^{n}X_i=\bar{X}$$

$$\theta=\frac{\bar{X}}{\bar{X}-c}.$$

$\Rightarrow$ 矩估计量为 $\frac{\bar{X}}{\bar{X}-c}$, 矩估计值 $\frac{\bar{x}}{\bar{x}-c}$

(2) 求出总体矩

$$E[X]=\int_{-\infty}^{+\infty}xf(x)dx$$

$$=\int_{0}^{1}\sqrt{\theta}x^{\sqrt{\theta}}\ dx$$

$$=\int_{0}^{1}\frac{\sqrt{\theta}}{\sqrt{\theta}+1}dx^{\sqrt{\theta}+1}$$

$$=\frac{\sqrt{\theta}}{\sqrt{\theta}+1}\ x^{\sqrt{\theta}+1}\Big|_{0}^{1}$$

$$=\frac{\sqrt{\theta}}{\sqrt{\theta}+1}$$

求出样本矩: $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$

$\Rightarrow\ \bar{X}=E[X]$

$$\bar{X}=\frac{\sqrt{\theta}}{\sqrt{\theta}+1}$$

$$\theta=\left(\frac{\bar{X}}{1-\bar{X}}\right)^2$$

$\Rightarrow$ 矩估计量为 $\left(\frac{\bar{X}}{1-\bar{X}}\right)^2$, 矩估计值为 $\left(\frac{\bar{x}}{1-\bar{x}}\right)^2$

(3) 求出总体矩：

$$E[X]=\sum_{x=1}^{m}x\binom{m}{x}p^{x}(1-p)^{m-x}$$

$$=\sum_{x=1}^{m}\frac{m!}{x!(m-r)!}\times x\times p^{x}(1-p)^{m-x}$$

$$=mp\sum_{x=1}^{m}\frac{(m-1)!}{(x-1)!(m-r)!}p^{x-1}(1-p)^{m-x}$$

$$=mp.$$

求出样本矩：$\bar{X}$

$\Rightarrow mp=\bar{X}$

$p=\frac{\bar{X}}{m}$

$\Rightarrow$ 矩估计量为 $\frac{\bar{X}}{m}$，矩估计值为 $\frac{\bar{x}}{m}$

3.(1) 求出似然函数

$$L(\theta)=\prod_{i=1}^{n}f(x_i,\theta)$$

$$=\prod_{i=1}^{n}\theta\cdot c^{\theta}x_i^{-(\theta+1)}$$

$$=\theta^{n}c^{\theta n}\prod_{i=1}^{n}x_i^{-(\theta+1)}$$

两边取对数得

$$\ln L(\theta)=n\ln\theta+\theta n\ln c-(\theta+1)\sum_{i=1}^{n}\ln x_i$$

求θ的一阶偏导

$$\frac{d\ln L(\theta)}{d\theta}=\frac{n}{\theta}+n\ln c-\sum_{i=1}^{n}\ln x_i$$

令其为0

得到 $\hat{\theta}=\dfrac{1}{\frac{1}{n}\sum_{i=1}^{n}\ln x_i-\ln c}$

(2) 求出似然函数

$$L(\theta)=\prod_{i=1}^{n}f(x_i,\theta)$$

$$=\prod_{i=1}^{n}\sqrt{\theta}\,x_i^{\sqrt{\theta}-1}$$

$$=\theta^{\frac{n}{2}}\prod_{i=1}^{n}x_i^{\sqrt{\theta}-1}$$

两边取对数得

$$\ln L(\theta)=\frac{n}{2}\ln\theta+(\sqrt{\theta}-1)\sum_{i=1}^{n}\ln x_i$$

$$\frac{d\ln(L(\theta))}{d\theta}=\frac{n}{2\theta}+\frac{\sum_{i=1}^{n}\ln x_i}{2\sqrt{\theta}}$$

令其值为0.

得 $\hat{\theta}=\dfrac{n^2}{(\sum\limits_{i=1}^{n}\ln x_i)^2}$.

(3) 求出似然函数

$$L(p)=\prod_{i=1}^{n}\binom{m}{x_i}p^{x_i}(1-p)^{m-x_i}$$
$$=p^{\sum\limits_{i=1}^{n}x_i}(1-p)^{nm-\sum\limits_{i=1}^{n}x_i}\prod_{i=1}^{n}\binom{m}{x_i}$$

$$\ln L(p)=\sum_{i=1}^{n}x_i\ln p+(nm-\sum_{i=1}^{n}x_i)\ln(1-p)+\sum_{i=1}^{n}\ln\binom{m}{x_i}$$

$$\frac{d(\ln L(p))}{dp}=\frac{\sum_{i=1}^{n}x_i}{p}-\frac{nm-\sum\limits_{i=1}^{n}x_i}{1-p}=0$$

得 $\hat{p}=\dfrac{\frac{1}{n}\sum\limits_{i=1}^{n}x_i}{m}=\dfrac{\bar{x}}{m}$

4. (1) $E[X]=1\times\theta^2+2\times2\theta(1-\theta)+3(1-\theta)^2$
$=3-2\theta.$
$=\bar{X}$

$\Rightarrow\hat{\theta}=\dfrac{3-\bar{X}}{2}=\dfrac{3-\frac{1}{3}(1+2+1)}{2}=\dfrac{5}{6}$

$$L(\theta)=\theta^2\times2\theta(1-\theta)\times\theta^2$$
$$=2\theta^5(1-\theta)$$

$$\ln L(\theta)=\ln2+5\ln\theta+\ln(1-\theta)$$

$$\frac{d(\ln L(\theta))}{d\theta}=\frac{5}{\theta}-\frac{1}{1-\theta}=0$$

得 $\hat{\theta}=\frac{5}{6}$.

(2) $L(\lambda)=\prod\limits_{i=1}^{n}f(x_i,\lambda)$
$$=\prod_{i=1}^{n}\frac{\lambda^{x_i}e^{-\lambda}}{x_i}=\frac{\lambda^{\sum\limits_{i=1}^{n}x_i}e^{-\lambda n}}{\prod\limits_{i=1}^{n}x_i!}$$

$$\ln L(\lambda)=\sum_{i=1}^{n}x_i\ln\lambda-\lambda n-\sum_{i=1}^{n}\ln x_i!$$

$$\frac{d\ln L(\lambda)}{d\lambda}=\frac{\sum\limits_{i=1}^{n}x_i}{\lambda}-n=0$$

$$\lambda=\frac{\sum\limits_{i=1}^{n}x_i}{n}$$

$\Rightarrow\hat{\lambda}=\bar{X}$

$$E[X]=\sum_{x=0}^{+\infty}\frac{\lambda^x e^{-\lambda}}{x!}x=\sum_{x=1}^{+\infty}\frac{\lambda^x e^{-\lambda}}{x!}x$$

$$=\sum_{x=1}^{+\infty}\frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

$$=\sum_{x=0}^{+\infty}\frac{\lambda\lambda^{x-1}e^{-\lambda}}{(x-1)!}$$

$$=\lambda$$

$$\Rightarrow \hat{\lambda}=\bar{X}.$$

(3) $L(p)=\prod_{i=1}^{n}\binom{x_i-1}{r-1}p^r(1-p)^{x_i-r}$

$$=\left[\prod_{i=1}^{n}\binom{x_i-1}{r-1}\right]p^{nr}(1-p)^{\sum_{i=1}^{n}x_i-nr}$$

$$\ln L(p)=\ln\left[\prod_{i=1}^{n}\binom{x_i-1}{r-1}\right]+nr\ln p+\left(\sum_{i=1}^{n}x_i-nr\right)\ln(1-p)$$

$$\frac{d\ln L(p)}{dp}=\frac{nr}{p}-\frac{\sum_{i=1}^{n}x_i-nr}{1-p}=0$$

$$p=\frac{r}{\frac{1}{n}\sum_{i=1}^{n}x_i}$$

$$\Rightarrow \hat{p}=\frac{r}{\bar{x}}$$

8(1).

$$L(\theta)=\prod_{i=1}^{n}\theta x_i^{\theta-1}$$

$$=\theta^n\prod_{i=1}^{n}x_i^{\theta-1}$$

$$\ln L(\theta)=n\ln\theta+(\theta-1)\sum_{i=1}^{n}\ln x_i$$

令 $\frac{d\ln L(\theta)}{d\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\ln x_i=0$

得 $\theta=-\cancel{\frac{\sum_{i=1}^{n}\ln x_i}{n}}\ \frac{n}{\sum_{i=1}^{n}\ln x_i}$

而 $U=e^{\frac{1}{\theta}}$

$$\Rightarrow \hat{U}=e^{-\frac{\sum_{i=1}^{n}\ln x_i}{n}}$$

(2) $X_1,X_2,\cdots X_n$ 是 $X\sim N(u,1)$ 的样本

$\Rightarrow X_i-u\sim N(0,1)$

$\Rightarrow P\{X>2\}=1-P(X\leq 2)$

$$=1-\Phi(2-u)$$

$$=1-\Phi(2-\bar{X}).$$

(3) $X \sim b(m, \theta)$

$\Rightarrow \hat{\theta} = \dfrac{\bar{x}}{m}$

由 $\theta = \frac{1}{3}(1+\beta)$ 得 $\beta = 3\theta - 1$

$\Rightarrow \hat{\beta} = \dfrac{3\bar{x}}{m} - 1$

9(1) $S_1^2 = \dfrac{1}{n_1-1}\sum_{i=1}^{n_1}(X_i - \bar{X})^2$

$S_2^2 = \dfrac{1}{n_2-1}\sum_{i=1}^{n_2}(Y_i - \bar{Y})^2$

$\Rightarrow E[S_1^2] = E[S_2^2] = \sigma^2$

$\Rightarrow E[S_w^2] = \dfrac{1}{n_1+n_2-2} E[(n_1-1)S_1^2 + (n_2-1)S_2^2]$

$= \dfrac{1}{n_1+n_2-2}[(n_1-1)E[S_1^2] + (n_2-1)E[S_2^2]]$

$= \dfrac{1}{n_1+n_2-2} \times (n_1+n_2-2)\sigma^2 = \sigma^2$

$\Rightarrow S_w^2$ 为 σ^2 的无偏估计.

(2) $E(X_i) = E(X) = u$.

$E\left(\dfrac{\sum_{i=1}^n a_iX_i}{\sum_{i=1}^n a_i}\right) = \dfrac{1}{\sum_{i=1}^n a_i} E\left(\sum_{i=1}^n a_iX_i\right)$

$= \dfrac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i u$

$= u$.

$\Rightarrow \dfrac{\sum_{i=1}^n a_iX_i}{\sum_{i=1}^n a_i}$ 是u的无偏估计量.

10(1). $E[c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2] = cE[\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2]$

$E[\sum_{i=1}^{n}(X_{i+1}-X_i)^2] = D[\sum_{i=1}^{n}(X_{i+1}-X_i)] + E[\sum_{i=1}^{n}(X_{i+1}-X_i)]^2$

$= D(X_{i+1}) + D(X_i)$

$= 2\sigma^2$.

$\Rightarrow E[c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2] = c(n-1)2\sigma^2 = \sigma^2$

$\Rightarrow c = \dfrac{1}{2(n-1)}$

(2). $E[(\bar{X})^2-cS^2]$

$=E[(\bar{X})^2]-cE[S^2]$

$=E[(\bar{X})^2]-cE[\frac{1}{n-1}(\sum_{i=1}^{n}X_i^2-n\bar{X}^2)]$

$=\frac{\sigma^2}{n}+\mu^2-\frac{c}{n-1}E[\sum_{i=1}^{n}X_i^2]-\frac{cn}{n-1}E[\bar{X}^2]$

$=\frac{\sigma^2}{n}+\mu^2-c\sigma^2.$

$=\mu^2$

$\Rightarrow c=\frac{1}{n}.$

11.(1). $L(\theta)=\prod_{i=1}^{n}\frac{1}{\theta}x_i^{\frac{1-\theta}{\theta}}$

$=(\frac{1}{\theta})^n\prod_{i=1}^{n}x_i^{\frac{1-\theta}{\theta}}$

$\ln L(\theta)=-n\ln\theta+\frac{1-\theta}{\theta}\sum_{i=1}^{n}\ln x_i$

$=-n\ln\theta+\frac{1}{\theta}\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\ln x_i$

令 $\frac{d\ln L(\theta)}{d\theta}=-\frac{n}{\theta}-\frac{1}{\theta^2}\sum_{i=1}^{n}\ln x_i=0$

得 $\hat{\theta}=-\frac{1}{n}\sum_{i=1}^{n}\ln X_i$

(2) 要证 $\hat{\theta}$ 是 θ 的无偏估计量.

即证 $E[\hat{\theta}]=\theta.$

$E[-\frac{1}{n}\sum_{i=1}^{n}\ln X_i]=-\frac{1}{n}E[\sum_{i=1}^{n}\ln X_i]$

$E[\ln X]=\int_0^1\ln x\times\frac{1}{\theta}x^{\frac{1-\theta}{\theta}}dx$

$=\int_0^1\ln x\,dx^{\frac{1}{\theta}}$

$=x^{\frac{1}{\theta}}\ln x\Big|_0^1-\int_0^1 x^{\frac{1}{\theta}}d\ln x$

$=0-\int_0^1 x^{\frac{1}{\theta}-1}dx$

$=-\int_0^1\theta\,dx^{\frac{1}{\theta}}=-\theta.$

$\Rightarrow E[-\frac{1}{n}\sum_{i=1}^{n}\ln X_i]=-\frac{1}{n}\times(-n\theta)=\theta.$

$\Rightarrow\hat{\theta}$ 是 θ 的无偏估计量

12. (1) $E[T_1] = \frac{1}{6}\times 2\theta + \frac{1}{3}\times 2\theta = \theta$.

$E[T_2] = \frac{10\theta}{5} = 2\theta$.

$E[T_3] = \frac{4\theta}{4} = \theta$.

$\Rightarrow$ T_1和T_3是θ的无偏估计量.

(2). $D[T_1] = \frac{1}{36}(\theta^2+\theta^2) + \frac{1}{9}(2\theta^2)$

$= \frac{1}{18}\theta^2 + \frac{2}{9}\theta^2 = \frac{5}{18}\theta^2$.

$D[T_3] = \frac{1}{16}\times 4\theta^2 = \frac{1}{4}\theta^2$

由此可见 $D[T_3] < D[T_1]$

$\Rightarrow$ T_3更有效.

13. (1) $E[\hat\theta^2]$

$= D[\hat\theta] + E[\hat\theta]^2$ 由$\hat\theta$是θ的无偏估计.

$= D[\hat\theta] + \theta^2$.

$> \theta^2$

$\Rightarrow$ $\hat\theta^2$不是θ^2的无偏估计

(2). $L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \frac{1}{\theta^n}$

$\ln L(\theta) = n\ln\frac{1}{\theta} = -n\ln\theta$.

由 $\frac{d\ln(L(\theta))}{d\theta} = -\frac{n}{\theta} < 0$. $\downarrow$.

使$L(\theta)$最小. $\hat\theta = \max X_i$.

令 $Y = \max X_i$

$f(y) = n\frac{y^{n-1}}{\theta^n}$

$E(Y) = \int_0^\theta ny\frac{y^{n-1}}{\theta^n}dy = \frac{n}{\theta^n}\int_0^\theta y^n dy$

$= \frac{n}{\theta^n(n+1)}\int_0^\theta dy^{n+1}$

$= \frac{n\theta}{n+1}$

$\Rightarrow$最大似然估计量不是无偏的

14. $E[Y]=E[a\bar{X}_1+b\bar{X}_2]$

$=aE[\bar{X}_1]+bE[\bar{X}_2]$

$=au+bu$

$=(a+b)u$

由$a+b=1$

$\Rightarrow E[Y]=u$

$\Rightarrow Y$是u的无偏估计量.

$D[Y]=E[Y^2]-E[Y]^2$

$=E[(a\bar{x}_1+b\bar{x}_2)^2]-E[a\bar{x}_1+b\bar{x}_2]^2$

$=a^2E[\bar{X}_1^2]+b^2E[\bar{X}_2^2]+2abE[\bar{X}_1\bar{X}_2]-u^2$

$=a^2(\frac{\sigma^2}{n_1}+u^2)+b^2(\frac{\sigma^2}{n_2}+u^2)+2abu^2-u^2$

$=\frac{(n_1\sigma^2+n_2\sigma^2)b^2-2n_2\sigma^2b+n_2\sigma^2}{n_1n_2}$

当$b=\frac{n_2}{n_1+n_2}$时取最小.

此时$a=\frac{n_1}{n_1+n_2}$.