2.4 Hermite插值

Newton与Lagrange及分段线性插值: y=f(x),

其Newton，Lagrange及分段线性插值多项式$P_n(x)$，$N_n(x)$， $S_1(x)$

满足插值条件: $P_n(x_i)=N_n(x_i)=S_1(x_i)=f(x_i)$，i=0,1,2,…n

$$P_n(x)=\sum_{k=0}^{n} y_k l_k(x) \qquad l_k(x)=\frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}=\prod_{j=0,j\neq k}^{n}\frac{x-x_j}{x_k-x_j}$$

$$R_n(x)=\frac{f^{(n+1)}(x)}{(n+1)!}w_n(x)$$

$$N_n(x)=c_0+c_1(x-x_0)+c_2(x-x_0)(x-x_1)+\ldots+c_n(x-x_0)\ldots(x-x_{n-1}) \qquad c_i=f[x_0,\ldots,x_i]$$

$$R_n(x)=f[x,x_0,\ldots,x_n]w_n(x)$$

$$S_1(x)=y_i\frac{x-x_{i+1}}{x_i-x_{i+1}}+y_{i+1}\frac{x-x_i}{x_{i+1}-x_i}, \quad x\in[x_i,x_{i+1}]$$

$$|f(x)-S_1(x)|\le\frac{1}{8}Mh^2,\ x\in[a,b], M=\max_{x\in[a,b]}|f''(x)|$$

2.4 Hermite插值

Newton与Lagrange及分段线性插值的不足:

Lagrange，Newton及分段线性插值多项式$P_n(x)$，$N_n(x)$， $S_1(x)$

满足插值条件: $P_n(x_i)=N_n(x_i)=S_1(x_i)=f(x_i)$，i=0,1,2,…n

Lagrange，Newton与分段线性插值多项式与y= f(x)在插值节点具有相同的函数值----“过点”.

但在插值节点上y=f(x)与$y=P_n(x)$等一般不“相切”，$f'(x_i)\neq P_n'(x_i)$. ——光滑性较差

Hermite插值:

求与y=f(x)在插值节点 x_0，x_1，…，x_n 上有相同函数值及导数值(甚至高阶导数值)的插值多项式.

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Hermite插值

Chapter 2
插值方法

Problem2.5: 已知函数$y=f(x)$在插值节点$a\le x_0<x_1<\ldots<x_n\le b$上的函数值$f(x_i)$与导数值$f'(x_i)$,$i=0,1,2,\ldots n$. 求多项式$H(x)$，使:

$$H(x_i)=f(x_i),\quad H'(x_i)=f'(x_i)\quad i=0,1,2,\ldots n.$$

对于以上问题,可用两种方法求$H(x)$.

方法一:待定系数法.

由$2n+2$个插值条件，可唯一确定一个次数不超过$2n+1$次的多项式.

（1）$H(x)$是$2n+1$次多项式;

（2）令$H(x)=a_0+a_1x+\ldots+a_{2n+1}x^{2n+1}$;

（3）由$2n+2$个插值条件建立关于$a_0,a_1,\ldots a_{2n+1}$的线性方程组. 解得$H(x)$.

方法二:基函数法.

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Hermite插值

Chapter 2
插值方法

Problem: 已知$f(x_i)$，$f'(x_i)$，$i=0,1,\ldots n$. 求$H_{2n+1}(x)$:

$$H_{2n+1}(x_i)=f(x_i),\quad H'_{2n+1}(x_i)=f'(x_i),\ i=0,1,2,\ldots n.$$

基函数法:

（1）$2n+2$个已知量$f(x_i)$, $f'(x_i)$，$i=0,1,2,\ldots n$.

（2）构造$2n+2$个 基函数$\alpha_i(x)$, $\beta_i(x)$，$i=0,1,2,\ldots n$.

（3）使$H_{2n+1}(x)$为$2n+2$个基函数的线性组合:

$$H_{2n+1}=\alpha_0(x)f(x_0)+\alpha_1(x)f(x_1)+\ldots+\alpha_n(x)f(x_n)$$
$$+\beta_0(x)f'(x_0)+\beta_1(x)f'(x_1)+\ldots+\beta_n(x)f'(x_n).$$

这些基函数有什么限制？如何求呢?

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Hermite插值基函数

如果： $\alpha_i(x_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ $\quad b_i(x_j) = 0$

$\alpha_i'(x_j) = 0$ $\quad \beta_i'(x_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$$\begin{aligned} H_{2n+1}(x_j) &= f(x_0)a_0(x_j) + \dots + f(x_j)a_j(x_j) + \dots + f(x_n)a_n(x_j) \\ &\quad + f'(x_0)b_0(x_j) + \dots + f'(x_j)b_j(x_j) + \dots + f'(x_n)b_n(x_j) \\ &= f(x_j) \end{aligned}$$

$$\begin{aligned} H'_{2n+1}(x_j) &= f(x_0)a'_0(x_j) + \dots + f(x_j)a'_j(x_j) + \dots + f(x_n)a'_n(x_j) \\ &\quad + f'(x_0)b'_0(x_j) + \dots + f'(x_j)b'_j(x_j) + \dots + f'(x_n)b'_n(x_j) \\ &= f'(x_j) \end{aligned}$$

Hermite插值基函数

$$\left.\begin{aligned} &\alpha_i(x_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \\ &\alpha_i'(x_j) = 0 \end{aligned}\right\} (\mathrm{I})$$

$l_i(x) = \prod_{j \neq i} \dfrac{(x - x_j)}{(x_i - x_j)}$

$a_i(x)$ ①degree=2n+1, ②有根 $x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ 且都是2重根

$\Rightarrow a_i(x) = (a_1 x + b_1) l_i^2(x)$ 因 $a_i(x_i) = 1, a_i'(x_i) = 0$

$$\Rightarrow \begin{cases} a_1 x_i + b_1 = 1 \\ a_1 l_i^2(x_i) + (a_1 x_i + b_1) \times 2 l_i(x_i) l_i'(x_i) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 x_i + b_1 = 1 \\ a_1 + 2 l_i'(x_i) = 0 \end{cases}$$

$$\alpha_i(x) = [1 - 2(x - x_i) \sum_{\substack{k=0 \\ k \neq i}}^{n} \frac{1}{x_i - x_k}] l_i^2(x)$$

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Hermite插值基函数

Chapter 2
插值方法

$$\left.\begin{aligned}\beta_i(x_j) &= 0\\ \beta_i'(x_j) &= \delta_{ij} = \begin{cases}0 & i \neq j\\ 1 & i = j\end{cases}\end{aligned}\right\}(\mathrm{II}) \quad b_i(x)$$

①degree=2n+1,

②有根 $x_0, \ldots, x_i, \ldots, x_n$

且除了x_i都是2重根

$$\Rightarrow b_i(x) = c(x - x_i)l_i^2(x) \quad \text{因} \ b_i'(x_i) = 1 \quad \Rightarrow c = 1$$

$$\Rightarrow b_i(x) = (x - x_i)l_i^2(x)$$

所求的Hermite插值多项式为

$$H_{2n+1}(x) = \sum_{i=0}^{n}\{f(x_i)[1 - 2(x - x_i)\sum_{\substack{k=0\\k\neq i}}^{n}\frac{1}{x_i - x_k}]l_i^2(x) + f'(x_i)(x - x_i)l_i^2(x)\}$$

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Hermite插值多项式的唯一性

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插值方法

$$H_{2n+1}(x) = \sum_{i=0}^{n}\{f(x_i)[1 - 2(x - x_i)\sum_{\substack{k=0\\k\neq i}}^{n}\frac{1}{x_i - x_k}]l_i^2(x) + f'(x_i)(x - x_i)l_i^2(x)\}$$

注：Hermite插值多项式是唯一的 （证：若$H_{2n+1}(x)$与 $G_{2n+1}(x)$都是所求的Hermite插值多项式，则$F(x)= H_{2n+1}(x)- G_{2n+1}(x)$有n+1个二重根$x_0$，$x_1$，…，$x_n$，，又$\deg(F(x)) \leqslant 2n+1$，故$F(x)= 0$.)

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Hermite插值余项$R_{2n+1}(x)=f(x)-H_{2n+1}(x)=?$

回顾: **lagrange插值余项**

$$R_n(x)=f(x)-P_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}w(x)$$

其中$W(x)=(x-x_0)(x-x_1)..(x-x_n)$

x_0，x_1，…，x_n为$R_n(x)$的根，$R_n(x)$有**n+1**阶零点.

显然，它们是Hermite插值余项$R_{2n+1}(x)$的二重根，

即$R_{2n+1}(x)$有2n+2阶零点.

类似得 $$R_{2n+1}(x)=K(x)w^2(x)=\frac{f^{(2n+2)}(\xi)}{(2n+2)!}w^2(x)$$

Hermite插值余项$R_{2n+1}(x)=f(x)-H_{2n+1}(x)$

定理**2.4** 设区间[a,b]含有互异节点$x_0, x_1, \ldots x_n$,而f(x)在[a,b]内存在直到2n+2阶导数,则满足插值条件:

$$H_{2n+1}(x_i)=f(x_i),\ H'_{2n+1}(x_i)=f'(x_i),\quad i=0,1,\ldots n$$

的Hermite插值多项式$H_{2n+1}(x)$的余项

$$R_{2n+1}(x)=f(x)-H_{2n+1}(x)=\frac{f^{(2n+2)}(\xi)}{(2n+2)!}w^2(x)$$

其中，ξ ∈ [a,b]且与x的位置有关，$W(x)=(x-x_0)(x-x_1)..(x-x_n)$

证明:

由插值条件：$H_{2n+1}(x_i)=f(x_i)$, $H'_{2n+1}(x_i)=f'(x_i)$，i=0,1,…n，则

$R_{2n+1}(x_i)=H_{2n+1}(x_i)-f(x_i)=0$；$R'_{2n+1}(x_i)=H'_{2n+1}(x_i)-f'(x_i)=0$，

则可令$R_{2n+1}(x)=K(x)W^2(x)$，构造辅助函数并应用Rolle定理证明。

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插值方法

定理2.4的证明:

（1）在插值节点x_0~x_n处,$R_{2n+1}(x_i)=0$,余项公式显然成立.

（2）对于[a,b]中异于插值节点x_0~x_n的x,考虑辅助函数

$$F(t)=f(t)-H_{2n+1}(t)-K(x)w^2(t)=R_{2n+1}(t)-K(x)w^2(t)$$

∵ $F(x_0)=F(x_1)=F(x_2)=\ldots=F(x_n)=F(x)=0$

由Rolle定理,存在$\xi_0\in(x_0,x_1)$,使$F'(\xi_0)=0$

类似,共有n+1个互异点ξ_0,ξ_1,…,ξ_n使$F'(t)=0$

∵ $\dfrac{dw^2(t)}{dt}=2w(t)w'(t)$ ∴ $F'(x_0)=F'(x_1)=F'(x_2)=\ldots=F'(x_n)=0$

$F'(t)$有2n+2个互异根$\xi_0,\xi_1,\ldots,\xi_n$, $x_0, x_1,\ldots,x_n$, 由Rolle定理,

则存在$\xi\in(a,b)$. 使: $F^{(2n+2)}(\xi)=f^{(2n+2)}(\xi)-K(x)(2n+2)!=0.$

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插值方法

注:当n=1时，满足插值条件

$$H_3(x_i)=f(x_i),\quad H'_3(x_i)=f'(x_i),\quad i=0,1$$

的插值公式:

$$\alpha_0(x)=\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2,\quad \alpha_1(x)=\left(1+2\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2,$$

$$\beta_0(x)=(x-x_0)\left(\frac{x-x_1}{x_0-x_1}\right)^2,\qquad \beta_1(x)=(x-x_1)\left(\frac{x-x_0}{x_1-x_0}\right)^2,$$

$$H_3(x)=f(x_0)\alpha_0(x)+f(x_1)\alpha_1(x)+f'(x_0)\beta_0(x)+f'(x_1)\beta_1(x).$$

$$R_3(x)=f(x)-H_3(x)=\frac{f^{(4)}(\xi)}{4!}(x-x_0)^2(x-x_1)^2,\ x_0<\xi<x_1.$$

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例题2.7 依据下列数据表构造插值多项式

X	Y	Y′
0	0	3
1	1	9

解:

$$H_3(x) = 0\alpha_0(x) + 1\alpha_1(x) + 3b_0(x) + 9b_1(x)$$

$$= (1 + 2\frac{x-1}{0-1})(\frac{x-0}{1-0})^2 + 3(x-0)(\frac{x-1}{0-1})^2 + 9(x-1)(\frac{x-0}{1-0})^2$$

$$= -2x^3 + 3x^2 + 3x(x^2 - 2x + 1) + 9x^2(x-1)$$

$$= 10x^3 - 12x^2 + 3x$$

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-0)^2(x-1)^2 \quad ,$$

$0 < \xi < 1$ and depending on x.

例: 用 Hermite插值求满足下列条件的四次多项式$H_4(x)$与余项。

$$H_4(0) = 0, H_4(1) = 1, H_4(2) = 1, H_4'(0) = 0, H_4'(1) = 1.$$

分析: 考虑$x_0=0, x_1=1, x_2=2$ 的插值问题。

解: 基函数法

设 $H_4(x)=f(x_0)a_0(x)+f(x_1)a_1(x)+f(x_2)a_2(x)+f'(x_0)b_0(x)+f'(x_1)b_1(x)$

$H_4(x)=a_1(x)+a_2(x)+b_1(x)$ 其中

$$\begin{cases} a_1(0) = a_1(2) = 0, a_1(1) = 1 \\ a_1'(0) = a_1'(1) = 0 \end{cases} \qquad \begin{cases} a_2(0) = a_2(1) = 0, a_2(2) = 1 \\ a_2'(0) = a_2'(1) = 0 \end{cases}$$

$$\begin{cases} b_1(0) = b_1(1) = b_1(2) = 0 \\ b_1'(0) = 0, b_1'(1) = 1 \end{cases}$$

$a_1(x)=x^2(x-2)^2$

$a_1(x)$ $a_1(x) = (ax+b)(x-0)^2(x-2)$

又: $a_1(1)=1, a_1'(1)=0 \Rightarrow$ a=1, b=-2

$$H_4(0)=0, H_4(1)=1, H_4(2)=1, H_4^{'}(0)=0, H_4^{'}(1)=1.$$

$$\begin{cases} a_2(0)=a_2(1)=0, a_2(2)=1 \\ a_2^{'}(0)=a_2^{'}(1)=0 \end{cases} \qquad \begin{cases} b_1(0)=b_1(1)=b_1(2)=0 \\ b_1^{'}(0)=0, b_1^{'}(1)=1 \end{cases}$$

$a_2(x)$ $\quad a_2(x)=c(x-0)^2(x-1)^2,\ a_2(2)=1 \Rightarrow c=\frac{1}{4} \Rightarrow a_2(x)=\frac{1}{4}x^2(x-1)^2$

$b_1(x)$ $\quad b_1(x)=d(x-0)^2(x-1)(x-2)$ $\quad \Rightarrow b_1(x)=-x^2(x-1)(x-2)$

$b_1^{'}(1)=1 \Rightarrow d=-1$

$$\therefore H_4(x)=\frac{1}{4}x^4-\frac{3}{2}x^3+\frac{9}{4}x^2$$

$$R_4(x)=f(x)-H_4(x)=K(x)(x-x_0)^2(x-x_1)^2(x-x_2),$$

$$K(x)=\frac{f^{(5)}(x_x)}{5!}, 0<x_x<2$$

$$H_4(0)=0, H_4(1)=1, H_4(2)=1, H_4^{'}(0)=0, H_4^{'}(1)=1.$$

方法二（基于承袭性）：

考虑$x_0=0$, $x_1=1$ 的标准Hermite插值问题

$$H_3(0)=0, H_3(1)=1, H_3^{'}(0)=0, H_3^{'}(1)=1 \Rightarrow H_3(x)=-x^3+2x^2$$

if ： $H_4(x)=H_3(x)+A(x-0)^2(x-1)^2$ and $H_4(2)=1$

$$\Rightarrow A=\frac{1}{4}$$

@ 求Hermite多项式的基本步骤：

• 写出相应于条件的 $a_i(x)$，$b_i(x)$ 的组合式；

‚ 对每一个 $a_i(x)$，$b_i(x)$ 找出尽可能多的条件给出的根；

ƒ 根据多项式的总次数和根的个数写出表达式；

„ 根据尚未利用的条件解出表达式中的待定系数；

… 最后完整写出 $H(x)$。

HW:
p.53 #16，23

分段三次（Hermite）插值

分段线性插值：具有一致收敛性，折线不光滑。

$$f(x) \approx S_1(x) = y_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + y_{i+1} \frac{x - x_i}{x_{i+1} - x_i}, \quad x \in [x_i, x_{i+1}]; \quad i = 0, 1, \mathbf{L}, n-1.$$

$|f(x)-S_1(x)| \le Mh^2/8; \quad x \in [a,b]$

三次Hermite插值：两条曲线在插值节点相切，光滑但不收敛

$$H_3(x) = f(x_0)a_0(x) + f(x_1)a_1(x) + f'(x_0)\beta_0(x) + f'(x_1)\beta_1(x).$$

$$a_0(x) = (1 + 2\frac{x - x_0}{x_1 - x_0})(\frac{x - x_1}{x_0 - x_1})^2, \quad a_1(x) = (1 + 2\frac{x - x_1}{x_0 - x_1})(\frac{x - x_0}{x_1 - x_0})^2,$$

$$\beta_0(x) = (x - x_0)(\frac{x - x_1}{x_0 - x_1})^2, \qquad \beta_1(x) = (x - x_1)(\frac{x - x_0}{x_1 - x_0})^2,$$

$$R_3(x) = f(x) - H_3(x) = \frac{f^{(4)}(\xi)}{4!}(x - x_0)^2(x - x_1)^2 . x_0 < \xi < x_1.$$

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三次（Hermite）插值

$$\alpha_0(x)=\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2,\quad \alpha_1(x)=\left(1+2\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2,$$

$$\beta_0(x)=(x-x_0)\left(\frac{x-x_1}{x_0-x_1}\right)^2,\qquad \beta_1(x)=(x-x_1)\left(\frac{x-x_0}{x_1-x_0}\right)^2,$$

$$H_3(x)=f(x_0)\alpha_0(x)+f(x_1)\alpha_1(x)+f'(x_0)\beta_0(x)+f'(x_1)\beta_1(x).$$

$$y(x)=y(x_0+th)\Rightarrow y'(t)=y'(x)\cdot x'(t)=hy'$$

$x=x_0+th,\ h=x_1-x_0;\ t\in[0,1]$			
$x=x_0$	x_1	$t=0$	1
$y=y_0$	y_1	y_0	y_1
$y'=y_0'$	y_1'	hy_0'	hy_1'

$$\begin{cases}\alpha_0(t)=(t-1)^2(2t+1)\\ \alpha_1(t)=t^2(-2t+3)\end{cases}\quad \begin{cases}\beta_0(t)=t(t-1)^2\\ \beta_1(t)=t^2(t-1)\end{cases}$$

$$H_3(x)=y_0\alpha_0\left(\frac{x-x_0}{h}\right)+y_1\alpha_1\left(\frac{x-x_0}{h}\right)+hy_0'\beta_0\left(\frac{x-x_0}{h}\right)+hy_1'\beta_1\left(\frac{x-x_0}{h}\right)$$

两条曲线f(x)与$H_3(x)$在插值节点相切，光滑但不收敛。

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分段三次（Hermite）插值

- 已知划分 D 的每个节点 x_i 处对应的 y_i 和y_i'，求作具有划分 D 的分段三次多项式 $S_3(x)$，满足:

$$S_3(x_i)=y_i,\quad S_3'(x_i)=y_i'\qquad i=0,1,\ldots,n$$

$S_3(x)$ 在每个小区间 $[x_i, x_{i+1}]$ 上是一个三次 Hermite 插值多项式，且:

$$\begin{cases}S_3^{[i]}(x_i)=y_i\\ S_3'^{[i]}(x_i)=y_i'\end{cases}\qquad \begin{cases}S_3^{[i]}(x_{i+1})=y_{i+1}\\ S_3'^{[i]}(x_{i+1})=y_{i+1}'\end{cases}$$

分段三次（Hermite）插值（续）

$$H_3(x) = y_0\alpha_0\left(\frac{x-x_0}{h}\right) + y_1\alpha_1\left(\frac{x-x_0}{h}\right) + hy_0'\beta_0\left(\frac{x-x_0}{h}\right) + hy_1'\beta_1\left(\frac{x-x_0}{h}\right)$$

$$\begin{cases} \alpha_0(t) = (t-1)^2(2t+1) \\ \alpha_1(t) = t^2(-2t+3) \end{cases} \qquad \begin{cases} \beta_0(t) = t(t-1)^2 \\ \beta_1(t) = t^2(t-1) \end{cases}$$

$$S_3^{[i]}(x) = y_i\alpha_0\left(\frac{x-x_i}{h_i}\right) + y_{i+1}\alpha_1\left(\frac{x-x_i}{h_i}\right) + h_i y_i'\beta_0\left(\frac{x-x_i}{h_i}\right) + h_i y_{i+1}'\beta_1\left(\frac{x-x_i}{h_i}\right)$$

$x \in [x_i, x_{i+1}]$, $h_i = x_{i+1} - x_i$, $i = 0, 1, \cdots, n-1$

分段三次（Hermite）插值（续）

分段三次 Hermite 插值的插值余项:

$$\left|f(x) - S_3(x)\right| \le \frac{1}{384} h^4 \max_{a \le x \le b}\left|f^{(4)}(x)\right| \qquad h = \max h_i$$

- h 足够小（例如小于 1）时，分段三次 Hermite 插值的插值余项远小于分段线性插值的插值余项，因此前者的插值精度更高。
- 分段三次 Hermite 插值的插值曲线比分段线性插值的曲线更光滑，但光滑度仍不够: $S_3(x) \in C^1[a,b]$.
- 三次样条插值：在插值节点处连续，一阶与二阶导数也连续，属于 $C^2[a,b]$ 函数类。