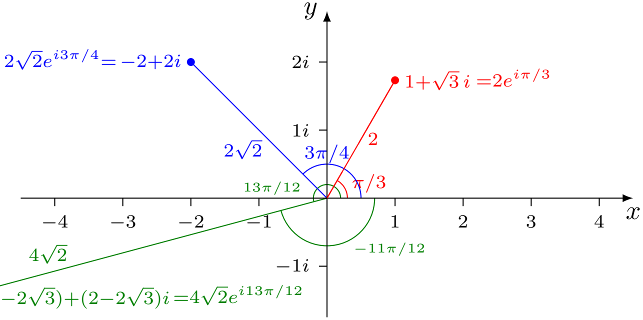
Math 4B: Differential Equations

Lecture 13: Complex Solutions

- More Geometry of Complex Numbers,
- A Return to the Mass & Spring,
- Complex Solutions,
- Examples & More!

Geometric Complex Numbers



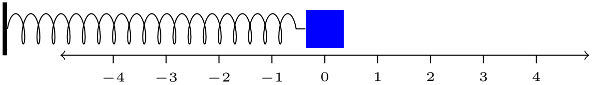
$$1+\sqrt{3}\,i = 2\left(\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right) = 2\left(\cos(\pi/3)+i\sin(\pi/3)\right) = 2e^{i\pi/3}$$

$$-2+2\,i = 2\sqrt{2}\left(\frac{-\sqrt{2}}{2}+i\,\frac{\sqrt{2}}{2}\right) = 2\left(\cos(3\pi/4)+i\sin(3\pi/4)\right) = 2\sqrt{2}e^{i3\pi/4}$$

$$\left(1+\sqrt{3}\,i\right)\left(-2+2\,i\right) = (-2-2\sqrt{3})+(2-2\sqrt{3})\,i \qquad \text{vs} \qquad \left(2e^{i\pi/3}\right)\left(2\sqrt{2}e^{i3\pi/4}\right) = 4\sqrt{2}e^{i13\pi/12}$$

Return to Mass & Spring

A mass on a spring:



$F = ma$ or rather $ma = \sum($ forces $)$.

Forces:

- Hooke's law says that the spring force is $F = -kx$ (where $k > 0$)
- Friction is proportional to velocity, then $F_{\text{friction}} = -\gamma v$ ($\gamma > 0$)
- External (non-constant?) forcing?

Thus

$$ma = F = F_{\text{spring}} + F_{\text{friction}} + F_{\text{external}} \qquad \Longrightarrow \qquad mx'' = -kx - \gamma x' + g(t)$$

or

$$mx'' + \gamma x' + kx = g(t).$$

Today's Plan

Today we'll focus on the homogeneous case (no external forcing):

$$mx'' + \gamma x' + kx = 0.$$

We saw to solve this we got $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where r_1, r_2 are roots of the ***characteristic equation***

$$mr^2 + \gamma r + k = 0 \qquad \Longrightarrow \qquad r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

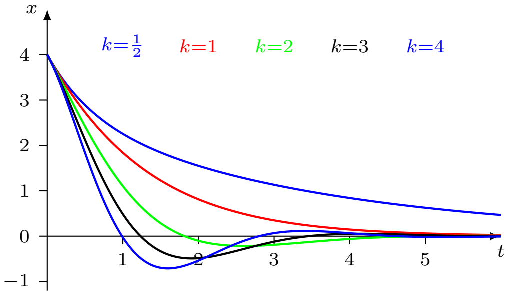
So there are three cases:

- $\gamma^2 - 4mk > 0 \implies$ 2 distinct real roots. Done this!
- $\gamma^2 - 4mk = 0 \implies$ 1 repeated real root. Next time!
- $\gamma^2 - 4mk < 0 \implies$ 2 complex (non-real) roots. Today!

Different Springs?

These all solve the IVP

$$\begin{aligned} x'' + 2x' + kx &= 0 \\ x(0) &= 4 \\ x'(0) &= -3. \end{aligned}$$



The Complex Case

If the roots of the characteristic polynomial are $r = \alpha \pm i\beta$ (with $\beta \neq 0$), then we'd ***expect*** a fundamental set of solutions to be

$$y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)$$
$$y_2 = e^{(\alpha-i\beta)t} = e^{\alpha t}\big(\cos(\beta t) - i\sin(\beta t)\big).$$

Complex Solutions Theorem

Suppose $y(t) = u(t) + iv(t)$ is a solution to the ODE

$$y'' + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are continuous real-valued function. Then the real part $u(t)$ and the imaginary part $v(t)$ are also solutions of this ODE.

Moral: If the roots of the characteristic polynomial are $r = \alpha \pm i\beta$ (with $\beta \neq 0$), then a fundamental set of solutions is

$$\left\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\right\}.$$

Wronskian!

Question: Can we check that

$$y_1 = e^{\alpha t}\cos(\beta t), \qquad y_2 = e^{\alpha t}\sin(\beta t)$$

is a fundamental set of solutions? Notice that

$$y_1' = e^{\alpha t}\big(\alpha\cos(\beta t) - \beta\sin(\beta t)\big) \text{ and } y_2' = e^{\alpha t}\big(\beta\cos(\beta t) + \alpha\sin(\beta t)\big).$$

Then the Wronskian is

$$\begin{aligned} W[y_1, y_2](t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\big(\alpha\cos(\beta t) - \beta\sin(\beta t)\big) & e^{\alpha t}\big(\beta\cos(\beta t) + \alpha\sin(\beta t)\big) \end{vmatrix} \\ &= \beta e^{2\alpha t}\big(\cos^2(\beta t) + \sin^2(\beta t)\big) = \beta e^{2\alpha t} \\ &\neq 0. \end{aligned}$$

Example 0

0. Find the general solution of the ODE

$$y'' + 4y = 0,$$

then find the particular solution satisfying the initial conditions $y(0) = 3$, $y'(0) = -2$.

Solution: The characteristic polynomial is $r^2 + 4$, which has roots $r = \pm 2i$. Thus the general solution is

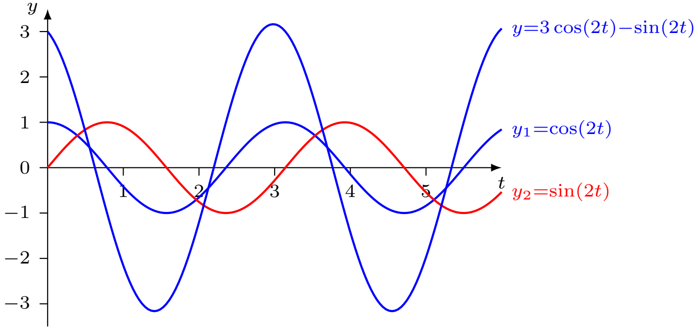
$$y = c_1 \cos(2t) + c_2 \sin(2t).$$

Since $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$, the initial conditions are

$$\begin{aligned} y(0) &= c_1 \cos(0) + c_2 \sin(0) = 3 \\ y'(0) &= -2c_1 \sin(0) + 2c_2 \cos(0) = -2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} c_1 &= 3 \\ 2c_2 &= -2 \end{aligned}$$

Thus the particular solution is $\boxed{y = 3\cos(2t) - \sin(2t)}$.

Pictures!



Example 1

1. Find the general solution of the ODE

$$x'' + 4x' + 13x = 0,$$

then find the particular solution satisfying the initial conditions $x(0) = 2$, $x'(0) = -2$.

Solution: The characteristic polynomial is $r^2 + 4r + 13$, which has roots $r = -2 \pm 3i$. Thus the general solution is

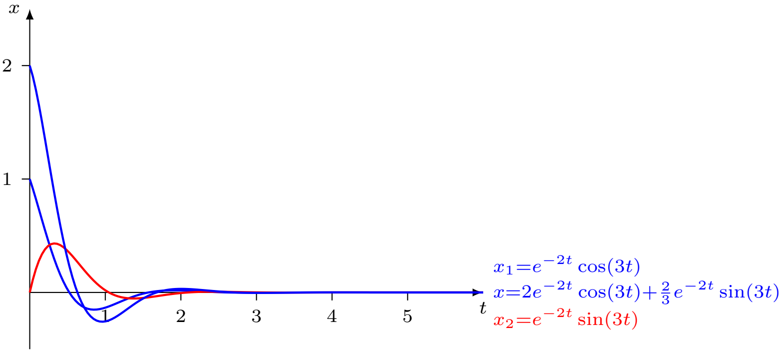
$$x = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t).$$

Since $x' = \big(-3c_1 - 2c_2 \big) e^{-2t} \sin(2t) + (-2c_1 + 3c_2) e^{-2t} \cos(3t)$, the initial conditions are

$$\begin{aligned} x(0) &= c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = 2 \\ x'(0) &= (-3c_1 - 2c_2) e^0 \sin(0) + (-2c_1 + 3c_2) e^0 \cos(0) = -2 \end{aligned}$$

or $c_1 = 2$ and $-2c_1 + 3c_2 = -2$. Thus the particular solution is $\boxed{x = 2e^{-2t} \cos(3t) + \frac{2}{3} e^{-2t} \sin(2t)}$.

Pictures!



This is a little disappointing.

Example 2

2. Find the general solution of the ODE

$$16x'' + 8x' + 257x = 0,$$

then find the particular solution satisfying the initial conditions $x(0) = 4$, $x'(0) = -5$.

Solution: The characteristic polynomial is $16r^2 + 8r + 257$, which has roots $r = -\frac{1}{4} \pm 4i$. Thus the general solution is

$$x = c_1 e^{-t/4} \cos(4t) + c_2 e^{-t/4} \sin(4t).$$

Since $x' = \big(-4c_1 - \frac{1}{4}c_2 \big) e^{-t/4} \sin(4t) + (-\frac{1}{4}c_1 + 4c_2) e^{-2t} \cos(4t)$, the initial conditions are

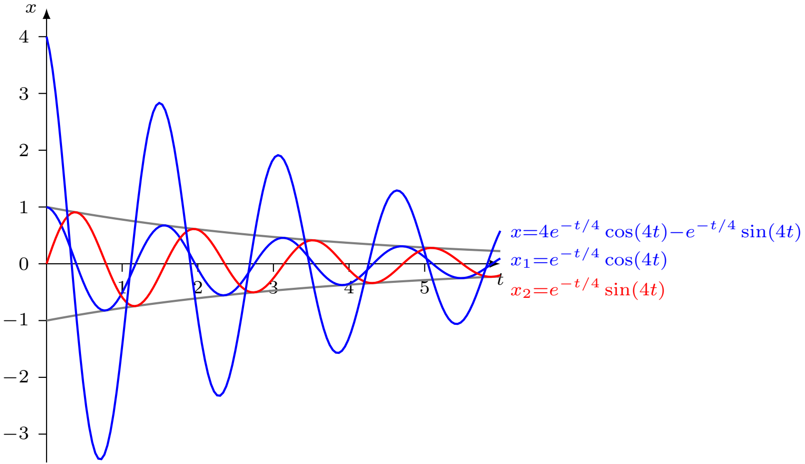
$$x(0) = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = 4$$

$$x'(0) = \left(-4c_1 - \frac{1}{4}c_2 \right) e^0 \sin(0) + \left(-\frac{1}{4}c_1 + 4c_2 \right) e^0 \cos(0) = -5$$

or $c_1 = 4$ and $-\frac{1}{4}c_1 + 4c_2 = -5.$ Thus the particular solution is

$$\boxed{x = 4e^{-t/4} \cos(4t) - e^{-t/4} \sin(4t)}.$$

Pictures!



Much better!