Math 4B: Differential Equations

Lecture 11: Second Order ODE Theory

- Linear Differetial Operators,
- An Existence / Uniqueness Theorem,
- The Wronskian & More!

Differential Operators

Today: The theory behind (solutions of) the linear second order ODEs and IVPs based on

$$y'' + p(t)y' + q(t)y = g(t).$$

Differential Operators

Given continuous functions $p(t)$ and $q(t)$ on a (possibly infinite) interval $\alpha < t < \beta$, we define a ***differential operator*** $L[y]$ for any twice-differentiable function on (α, β) by

$$L[y] = y''(t) + p(t)y'(t) + q(t)y(t).$$

Notice that both y and $L[y]$ are functions of t.

Example: $L[y] = y'' + \dfrac{1}{t^2 - 1}\, y' + \dfrac{1}{t^2 + 1}\, y$ is a differential operator on $(-1, +1)$ or $(-\infty, -1)$ or $(1, \infty)$

Examples

1. If $L[y] = y'' + 3y' + 2y$, then we know how to solve the IVP

$$\begin{cases} L[y] = 0 \\ y(0) = a \\ y'(0) = b \end{cases}$$

from last time.

$$\boxed{y = c_1 e^{-2t} + c_2 e^{-t}}$$

General Solution: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where r_1, r_2 solve the characteristic equation $r^2 + 3r + 2 = 0$

2. If $L[y] = y'' + 3y' + 2y$, then we don't know how to solve the IVP

$$\begin{cases} L[y] = 1 \\ y(0) = a \\ y'(0) = b \end{cases}$$

even though that's a really "simple" nonhomogeneous second order ODE.

Existence / Uniqueness Theorem

The Existence and Uniqueness Theorem

Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on the interval I given by $\alpha < t < \beta$ containing t_0. Let $L[y] = y'' + p(t)y' + q(t)y$. Then there is a unique solution $y = \phi(t)$ to the IVP

$$\begin{cases} L[y] = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases} \qquad \text{or} \qquad \begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_0'. \end{cases}$$

that is defined for all t in I.

Warning! Solutions exist and defined, but are not necessarily easy to (or *possible* to!) find.

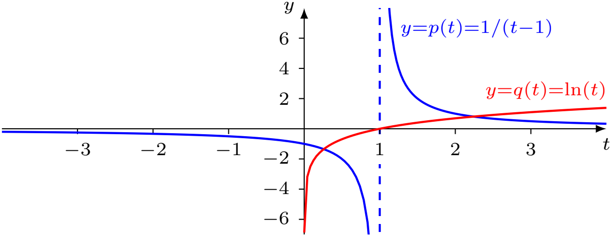
Examples

3. $ay'' + by' + cy = 0$

Here $p(t) = b/a$ and $q(t) = c/a$, so both are continuous for all t (provided $a \neq 0$). So solutions exist and are unique for all t.

4. $y'' + \dfrac{1}{t-1}\, y' + \ln(t)\, y = 0$

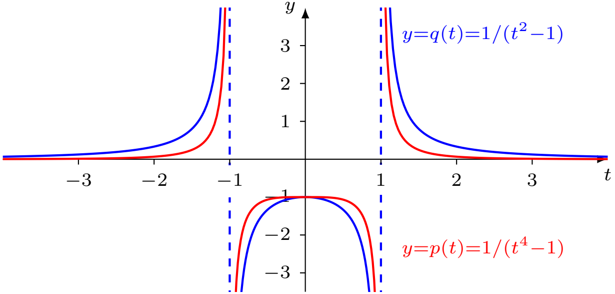
Here $p(t) = 1/(t-1)$ is continuous for $t \neq 1$ and $q(t) = \ln(t)$ is continuous for $t > 0$. So solutions exist and are unique for all t in the intervals $(0, 1)$ and $t > 1$.



Examples

5. $(t^2-1)y'' + \dfrac{1}{t^2+1}\,y' + y = 0$

In standard form this is $y'' + \dfrac{1}{(t^2+1)(t^2-1)}\,y' + \dfrac{1}{t^2-1}\,y = 0$, so $p(t) = \frac{1}{(t^2-1)(t^2+1)}$ is continuous for $t \neq \pm 1$ and $q(t) = \frac{1}{t^2-1}$ is continuous for $t \neq \pm 1$. So solutions exist and are unique for all t in the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.



Properties of $L[\,\cdot\,]$

Claim 1: $L[c \cdot y] = c \cdot L[y]$

$$\text{Idea:}\quad \begin{aligned} L[c\cdot y] &= (cy)'' + p(t)(cy)' + q(t)(cy) \\ &= cy'' + p(t)cy' + q(t)cy \\ &= c\big(y'' + p(t)y' + q(t)y\big) = c \cdot L[y]. \end{aligned}$$

Claim 2: $L[y_1 + y_2] = L[y_1] + L[y_2]$

$$\text{Idea:}\quad \begin{aligned} L[y_1+y_2] &= (y_1+y_2)'' + p(t)(y_1+y_2)' + q(t)(y_1+y_2) \\ &= y_1'' + y_2'' + p(t)y_1' + p(t)y_2' + q(t)y_1 + q(t)y_2 \\ &= \big(y_1'' + p(t)y_1' + q(t)y_1\big) + \big(y_2'' + p(t)y_2' + q(t)y_2\big) \\ &= L[y_1] + L[y_2]. \end{aligned}$$

$L[\,\cdot\,]$ is a Linear Operator

Theorem: Differential Operators are Linear

Suppose L is a differential operator given by $L[y] = y'' + p(t)y' + q(t)y$. Then, for constants c and functions y,

- $L[cy] = c\,L[y]$
- $L[y_1 + y_2] = L[y_1] + L[y_2]$
- $L[c_1y_1 + c_2y_2] = c_1\,L[y_1] + c_2\,L[y_2]$

That is, L takes a linear combination of y_1 and y_2 to the same linear combination of $L[y_1]$ and $L[y_2]$.

This leads to...

The Principle of Superposition

If y_1 and y_2 are solutions to $L[y] = 0$, then so are $c_1y_1 + c_2y_2$ for any constants c_1 and c_2. $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$.

Questions!

Because

- we have two derivatives in a second order ODE, and
- we have the principle of superposition

we expect that it is enough to find two solutions y_1 and y_2 of $L[y] = 0$ to find *all* solutions.

3. How can we tell that y_1 and y_2 are "linearly independent" (as in Math 4A)?

4. Is it really true the case that two solutions are enough to find all of them?

Answer to Question 4.

Question 4. asks if two solutions are enough to find the (unique!) solutions to every IVP. Given Existence/Uniqueness, we can find solutions

$$y_1 \text{ of } \begin{cases} L[y] = 0 \\ y(t_0) = 1 \\ y'(t_0) = 0 \end{cases} \qquad \text{and } y_2 \text{ of } \begin{cases} L[y] = 0 \\ y(t_0) = 0 \\ y'(t_0) = 1. \end{cases}$$

This means the (unique!) solution to

$$\begin{cases} L[y] = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

is $y = y_0 \cdot y_1 + y_0' \cdot y_2$.

Check this!

Linear Algebra Review

Math 4A Question

Here's a Math 4A question: When can we always solve the system

$$ax + by = s \qquad\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$
$$cx + dy = t \quad \text{or}$$

for x and y?

Answers:

(1) When $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two pivots when row reduced (so RREF is I).

(2) When $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is not zero.

Linear Algebra Review

The Basic Question

Given two solutions y_1 and y_2 to a second order linear ODE $L[y] = 0$, can we tell if they are "linearly independent" and thus find all solutions as $c_1 y_1 + c_2 y_2$?

Given initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$, we'd need to solve a system of linear equations for c_1 and c_2:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

Question: When can I *always* solve this system? (Think 4A!)

Answer: When the ***Wronskian***

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$$

The Wronskian Theorem

Theorem

Suppose y_1 and y_2 are solutions to the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Suppose we are trying to solve this equation $L[y] = 0$ with the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Then it is always possible to choose constants c_1, c_2 so that

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and initial conditions if and only if the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

is not zero at the point t_0.

Examples

5. Remember that $y'' + 8y' + 12y = 0$ has solutions $y_1 = e^{-6t}$ and $y_2 = e^{-2t}$ (since $r^2 + 8r + 12$ has roots $r_1 = -6$ and $r_2 = -2$). Can every solution be written as $c_1 e^{-6t} + c_2 e^{-2t}$?

Solution: The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-6t} & e^{-2t} \\ -6e^{-6t} & -2e^{-2t} \end{vmatrix}$$

$$= (-2 + 6)e^{-8t} = 4e^{-8t} \neq 0.$$

Thus ***any*** solution can be written as a linear combination of e^{-6t} and e^{-2t}.

Note: This is true for any second order linear homogeneous ODE with constant coefficients that has two distinct (different) real roots.

Examples

6. Remember we saw that $y'' + 4y = 0$ has solutions $y_1 = \sin(2t)$ and $y_2 = \cos(2t)$. Can every solution be written as $c_1 \sin(2t) + c_2 \cos(2t)$?

Solution: The Wronskian is

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{vmatrix} \\ &= -2\sin^2(2t) - 2\cos^2(2t) \\ &= -2\big(\sin^2(2t) + \cos^2(2t)\big) \\ &= -2 \neq 0. \end{aligned}$$

Thus ***any*** solution can be written as a linear combination of $\sin(2t)$ and $\cos(2t)$.

Note: This is true for any ODE like this: $y'' + k^2 y = 0$ with $k \neq 0$.

Examples

7. It turns out that $y'' + 2y' + y = 0$ has solutions $y_1 = e^{-t}$ and $y_2 = te^{-t}$. Can every solution be written as $c_1 e^{-t} + c_2 te^{-t}$?

Solution: The derivatives of y_2 are

$$y_2' = (1-t)e^{-t} \qquad \text{and} \qquad y_2'' = (t-2)e^{-t},$$

so you can check that

$$y_2'' + 2y_2' + y_2 = (t-2)e^{-t} + 2(1-t)e^{-t} + te^{-t} = 0.$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix} = (1-t+t)e^{-2t} = e^{-2t} \neq 0.$$

Thus ***any*** solution can be written as a linear combination of e^{-t} and te^{-t}.

Note: Again, similar ODEs will behave similarly.