Math 174E
Lecture 13

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References

Hull
Chapter 13.2, 13.3, 13.4, 13.5, 13.6

One-Step Binomial Model: Key Formulas 1/2

Theorem 13.3

The one-step binomial tree model is **free of arbitrage** if and only if the parameters u, d, r satisfy

$$d < e^{rT} < u.$$

In this case, the **unique arbitrage-free price** of an option with payoff f_T is given by

$$f_0 = \mathbb{E}^*\left[e^{-rT} f_T\right] = e^{-rT}\left(p^* \cdot f^u + (1 - p^*) \cdot f^d\right) \qquad (\star)$$

with **risk-neutral probability**

$$p^* = \frac{e^{rT} - d}{u - d} \in (0, 1).$$

One-Step Binomial Model: Key Formulas 2/2

Theorem 13.3 (continued)

Moreover, the one-step binomial tree model is **complete**, i.e., *every contingent claim* f_T is perfectly **replicable** (**attainable**) by the replication strategy

$$\Delta_0 = \frac{f^u - f^d}{S_0 \cdot u - S_0 \cdot d}.$$

Comment: Note that the statement of Theorem 13.3 holds true for *any European option* with payoff of the form $f_T = h(S_T)$ which is a function h of the underlying stock S_T (e.g., call, put, ..., binary option, power option, ...).

Discussion: Risk-Neutral Modeling and Valuation 1/4

- pricing formula ($\star$) in Theorem 13.3 is referred to as the **risk-neutral pricing formula**
- it can be interpreted as an **expected value** of the discounted payoff $e^{-rT} f_T$ at maturity T where

$$p^* = \frac{e^{rT} - d}{u - d}$$

  denotes the probability of an up movement of the stock (and $1 - p^*$ the probability of a down movement)

- p^* = **risk-neutral probability** (or **pricing measure**)
- $\mathbb{E}^*$ = expected value computed with respect to p^*

- this principle of pricing derivatives is called **risk-neutral valuation**

Discussion: Risk-Neutral Modeling and Valuation 2/4

Expected return on the **stock** under p^*:

$$\mathbb{E}^*[S_T] = p^* \cdot S_0 \cdot u + (1-p^*) \cdot S_0 \cdot d = p^* \cdot S_0 \cdot (u-d) + S_0 \cdot d$$
$$= \frac{e^{rT} - d}{u - d} \cdot S_0 \cdot (u-d) + S_0 \cdot d = S_0 \cdot e^{rT}$$

Expected return on the **stock option** under p^*:

$$\mathbb{E}^*[f_T] = p^* \cdot f^u + (1-p^*) \cdot f^d = f_0 \cdot e^{rT}$$

- under p^* the **expected return on all risky assets** (which depend on the stock) in the one-step binomial tree model **equals the risk-free rate** $\boldsymbol{r}$
- this is why p^* is called **risk-neutral probability**
- there is no compensation for increased risk (stock and option vs. risk-free bank account)

Discussion: Risk-Neutral Modeling and Valuation 3/4

"Real world" vs. "risk-neutral world" modeling:

- p^* is interpreted as the probability of an up movement of the stock in a **"risk-neutral world"**
  - expected return on all risky assets (which depend on the stock) is equal to the risk-free rate
- in contrast, the probability p (see binomial tree on slide 16, Lecture 11) can be thought of as the model's "physical" or **"real world"** probability for an up movement of the stock
  - in the "real world" investors ask for a much *higher return* than the risk-free rate when investing in *risky* assets like stocks and options in order to be compensated for the additional risk (in contrast to the "risk-neutral" world where all investors are "risk-neutral")

Attention: The textbook by Hull uses the notation p for p^*, and p^* for p!

Discussion: Risk-Neutral Modeling and Valuation 4/4

Principle of risk-neutral valuation:

- in order to compute arbitrage-free prices for stock options written on the stock S_T, we set up the one-step binomial tree model for the evolution of stock price S_T *directly in a "risk-neutral" world* with probability p^*
- this allows us to readily compute arbitrage-free option prices via formula $(\star)$, i.e., as **expected futures payoffs** in a risk-neutral world ($\mathbb{E}^*$ computed with p^*) **discounted at the risk-free rate**:

$$\text{arbitrage-free price at time } 0 = \mathbb{E}^*[e^{-rT}h(S_T)]$$

- note that the "real world" probability p of an upward move of the stock does not play any role in computing the arbitrage-free stock option price in $(\star)$
- in other words, the stock's "real world" expected return is **irrelevant** for arbitrage-free option pricing

Example revisited

Example 13.4 (see Example 13.1)

Pricing a 3-month European call option with strike price \$21 on a stock whose current value is $S_0 = \$20$. The risk-free interest rate is 4% p.a. Suppose S_T at maturity $T = 3/12$ is either 18 or 22.

We have

$$S_0 u = 22 \qquad u = 1.1 \qquad f^u = 1$$

$$S_0 d = 18 \qquad d = 0.9 \qquad f^d = 0$$

and

$$p^* = \frac{e^{rT} - d}{u - d} = \frac{e^{(0.04)\frac{3}{12}} - 0.9}{1.1 - 0.9} = 0.5503.$$

Hence

$$f_0 = e^{-(0.04)\frac{3}{12}} (1 \cdot 0.5503 + 0 \cdot 0.4497) = 0.545.$$

and

$$\Delta_0 = \frac{f^u - f^d}{S_0 u - S_0 d} = \frac{1 - 0}{22 - 18} = \frac{1}{4}.$$

Two-Step Binomial Model: Notation

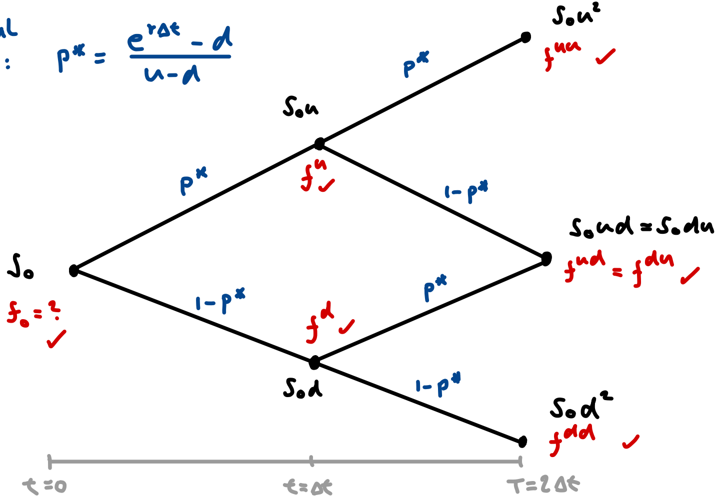
Compare with one-step binomial model (cf. slide 10).

- T = maturity (in years)
- r = risk-free interest rate p.a. (continuously compounded)
- **two equidistant time steps:** $t = 0, t = \Delta t, t = T = 2\Delta t$ (measured in years)

- S_0 = current stock price (today at time $t = 0$)
  - u = **one-step** factor for price upward move
  - d = **one-step** factor for price downward move
- risk-neutral probability for a **one-step** up movement of the stock price
$$p^* = \frac{e^{r\Delta t} - d}{u - d}$$
- f_0 = stock option price (today at time $t = 0$)
- f_T = stock option payoff at maturity T

Two-Step Binomial Model: Illustration 1/2

risk-neutral probability: $p^* = \dfrac{e^{r\Delta t} - d}{u - d}$



Risk-neutral pricing formula: arbitrage-free option price f_0

$$f_0 = \mathbb{E}^*\left[e^{-rT} f_T\right] = e^{-rT}\left((p^*)^2 f^{uu} + 2\,p^*(1-p^*)\,f^{ud} + (1-p^*)^2 f^{dd}\right)$$

Two-Step Binomial Model: Illustration 2/2

Backward induction:

$t=T$: f^{uu} f^{ud} f^{dd} option's payoff

$t=\Delta t$: $S^u = S_0 u$ $f^u = e^{-r\Delta t}\left(p^* f^{uu} + (1-p^*) f^{ud}\right)$

$S^d = S_0 d$ $f^d = e^{-r\Delta t}\left(p^* f^{du} + (1-p^*) f^{dd}\right)$

$t=0$: S_0 $f_0 = e^{-r\Delta t}\left(p^* f^u + (1-p^*) f^d\right)$

Two-Step Binomial Model: Key Formulas 1/3

Risk-neutral valuation (same idea as in the one-step model):

Theorem 13.5

The two-step binomial tree model is **free of arbitrage** if and only if the parameters u, d, r satisfy $d < e^{r\Delta t} < u$.

In this case, the **unique arbitrage-free price** of an option with payoff f_T is given by

$$
\begin{aligned}
f_0 &= \mathbb{E}^* \left[e^{-rT} f_T \right] \\
&= e^{-r \cdot 2\Delta t} \left((p^*)^2 \cdot f^{uu} + 2 \cdot p^* \cdot (1 - p^*) \cdot f^{ud} + (1 - p^*)^2 \cdot f^{dd} \right)
\end{aligned}
$$

with one-step **risk-neutral probability**

$$
p^* = \frac{e^{r\Delta t} - d}{u - d}.
$$

Two-Step Binomial Model: Replication Argument 1/2

Introduce:

V_0 = initial capital of portfolio

Δ_0 = number of shares to hold at time 0

Δ_1 = number of shares to hold at time 1

Δ_1^u = number of shares if stock price went up

Δ_1^d = number of shares if stock price went down

Two-Step Binomial Model: Replication Argument 2/2

Find V_0, Δ_0, Δ_1^u, Δ_1^d such that (compare w/ Lecture 12, slide 3)

(i) $\left(\left(V_0 - \Delta_0 S_0\right)e^{r\Delta t} + \Delta_0 S_0 u - \Delta_1^u S_0 u\right)e^{r\Delta t} + \Delta_1^u S_0 u^2 = f^{uu}$

(ii) $\left(\left(V_0 - \Delta_0 S_0\right)e^{r\Delta t} + \Delta_0 S_0 u - \Delta_1^u S_0 u\right)e^{r\Delta t} + \Delta_1^u S_0 u d = f^{ud}$

(iii) $\left(\left(V_0 - \Delta_0 S_0\right)e^{r\Delta t} + \Delta_0 S_0 d - \Delta_1^d S_0 d\right)e^{r\Delta t} + \Delta_1^d S_0 d u = f^{du}$

(iv) $\left(\left(V_0 - \Delta_0 S_0\right)e^{r\Delta t} + \Delta_0 S_0 d - \Delta_1^d S_0 d\right)e^{r\Delta t} + \Delta_1^d S_0 d^2 = f^{dd}$

Two-Step Binomial Model: Key Formulas 2/3

Replication argument (same idea as in the one-step model):

Theorem 13.5 (continued)

Moreover, the two-step binomial tree model is **complete**, i.e., *every contingent claim* f_T is perfectly **replicable** (**attainable**) by the **dynamic** replication strategy

$$\Delta_0 = \frac{f^u - f^d}{S_0 u - S_0 d}, \quad \Delta_1^u = \frac{f^{uu} - f^{ud}}{S_0 u^2 - S_0 ud}, \quad \Delta_1^d = \frac{f^{du} - f^{dd}}{S_0 du - S_0 d^2},$$

where Δ_1^u and Δ_1^d are the rebalanced share holdings after one time step when stock price went up or down, respectively

Two-Step Binomial Model: Key Formulas 3/3

Comments:

- note that the **delta (hedging) strategy** Δ_0, Δ_1 is not *static* but *dynamic* and changes over time (it depends on how the stock price changes)
- observe that the Δ ("delta") of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock

$$\Delta = \frac{\text{change in price of stock option}}{\text{change in price of underlying stock}}$$

- a portfolio which consists of a short position in a stock option and a long position in the stock option's Δ-strategy is called *delta-neutral* (= portfolio value does not change if stock price changes)
- the **replicating portfolio** which consists of the dynamic trading strategy Δ_0, Δ_1 in the underlying stock and a cash account is **self-financing**, i.e., there are no inflows/outflows.

Two-Step Binomial Model: A European Put Example

Example 13.6

Consider a 2-year European put option with a strike price of \$52 on a stock whose current price is \$50. The risk-free interest rate is 5% p.a.

Use a two-step binomial tree model to compute the arbitrage-free price of the European put option. Specifically, suppose that there are two time steps of 1 year, and in each time step the stock price moves up by 20% or moves down by 20%.

Two-Step Binomial Model: American Options

General procedure:

- work back through the tree from the end to the beginning as in the case of a European option (backward dynamic programming)
- at final nodes (at maturity): value of American option = payoff
- at each node prior to maturity: test if early exercise is optimal
  - compute...
    - ...**intrinsic value** (= payoff from early exercise) and
    - ...**continuation value** (= expected future value)

$$f = e^{-r\Delta t}\left(p^* \cdot f^u + (1 - p^*) \cdot f^d\right)$$

  - the value of the American option is then the **maximum** of the two values
  - early exercise optimal if

$$\text{intrinsic value} > \text{continuation value}$$

An American Put Example

Example 13.7

Consider the same situation as in Example 13.6 but suppose now the put option is American. Compute its arbitrage-free price today and determine at which node early exercise is optimal.

Matching Volatility

Three key parameters to construct a tree: u, d, p^*

- once u and d are specified, p^* must be chosen so that the expected return of the stock is the risk-free rate r

$$p^* = \frac{e^{r\Delta t} - d}{u - d}$$

- parameters u and d should be chosen to match the **volatility** of the stock
- **annualized volatility** $\boldsymbol{\sigma}$ = standard deviation of the asset's **log-returns** in a year (more details in Chapter 15)
- multiply annualized volatility σ with $\sqrt{\Delta t}$ to get the volatility for a period of length Δt (measured in years)
- choose

$$u = e^{\sigma\sqrt{\Delta t}} \qquad\qquad d = e^{-\sigma\sqrt{\Delta t}}$$

to match u and d with a given σ