Name: Zongyi Han Perm #: 3203734

Section: Monday Li Chang Yuan

FINAL

PSTAT 160A – Spring 2022 (early exam)
Nils Detering

Instructions: Turn off and put away your cell phone. Do not turn the page until the exam starts.

- Open book.
- You will not be asked to algebraically simplify any complicated expressions. Answers that you feel are too complicated to be algebraically simplified can be left without simplification.
- **You must show all of your work and justify your answers to receive any credit**. You may justify your answers by either writing brief sentences explaining your reasoning or annotating your math work with brief explanations. Refer to the respective results, properties, theorems in class. When using a formula, always state the general form first before doing the actual calculation.
- You must write legibly and clearly mark the answers you want graded. If the work is illegible or it is unclear which answers you intend to be graded, credit may not be given. I strongly suggest circling, boxing, or writing a sentence indicating your answer.
- **Academic dishonesty:** Any student caught cheating will get **0 points** for the final. According to University rules, cheating must be reported to the College of Letters and Science. Further action may be taken.

Question:	1	2	3	4	5	6	7	8	Total
Points:	8	8	16	12	12	16	12	16	100
Score:									

1. (8 points) A fair coin is flipped 15 times. Let X denote the total number of heads, and Y the number of heads in the last 6 tosses. Derive $E[X|Y]$.

$X \sim Geo \Rightarrow \frac{1}{2}$ $\quad E(X) = \frac{1}{2} \cdot 15$

$Y \sim Geo \Rightarrow$

$Y = 1 \Rightarrow E(X|Y) = \frac{\frac{1}{2}(9) + 1}{15}$

$Y = 2 \Rightarrow E(X|Y) = \frac{\frac{1}{2}(9) + 2}{15}$

$Y = 3 \Rightarrow E(X|Y) = \frac{\frac{1}{2}(9) + 3}{15}$

$Y = 4 \Rightarrow E(X|Y) = \frac{\frac{1}{2}(9) + 4}{15}$

$Y = 5 \Rightarrow E(X|Y) = \frac{\frac{1}{2}(9) + 5}{15}$

$Y = 6 \Rightarrow E(X|Y) = \frac{\frac{1}{2}(9) + 6}{15}$

$E(X|Y)$

$\Rightarrow \frac{9}{2} + Y$

2. (8 points) Let X and Y be independent Poisson random variables with parameters λ and μ. Find the conditional distribution of Y given $X+Y=n$. You may use (without proof) that $X+Y$ is Poisson distributed with parameter $\lambda+\mu$.

$$e^{-\lambda}\frac{\lambda^k}{k!}$$

$$P(Y \mid X+Y)$$

$$= \frac{P(Y)\cdot P(X=n-Y)}{P(X+Y)}$$

$$= \frac{e^{-\mu}\frac{\mu^y}{y!}\cdot e^{\lambda}\frac{(\lambda)^{n-y}}{n-y!}}{e^{-\mu-\lambda}\cdot\frac{(\mu+\lambda)^n}{n!}}$$

$$= \frac{\mu^y\cdot(\lambda)^{n-y}\cdot n!}{Y!\cdot(\mu+\lambda)^n\cdot(n-y)!}$$

$$= \frac{n!}{Y!(n-y)!}\cdot\frac{\mu^y\cdot(\lambda)^{n-y}}{(\mu+\lambda)^n}$$

$$= \binom{n}{y}\frac{\mu^y\cdot(\lambda)^{n-y}}{(\mu+\lambda)^n}$$

3. (16 points) Let $(Y_n)_{n\geqslant 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}[Y_n = 1] = \mathbb{P}[Y_n = -1] = \frac{1}{4}$ and $\mathbb{P}[Y_n = 0] = \frac{1}{2}$. Define $X_n := \prod_{i=1}^{n} Y_i$ for all $n \geqslant 1$ and $X_0 = 1$.

   (a) Explain why $(X_n)_{n\geqslant 0}$ is a Markov chain and provide the corresponding transition matrix P and transition graph.

   (b) Determine the communication classes and their periodicity.

   (c) Argue that $P_{i,j}^{(n)}$ converges for $n \to \infty$ for all i and j in $\mathcal{S}$, and determine the corresponding limits $\lim_{n\to\infty} P_{i,j}^{(n)}$.

   (d) Find a stationary distribution for this Markov Chain.

a). $X_n = \prod_{i=1}^{n} Y_i = Y_1 \cdot Y_2 \cdots Y_n$

$$\begin{array}{c|ccc} & 1 & -1 & 0 \\ \hline 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array}$$

$P(X_{n+1} = j \mid X_n = j, X_{n-1} = i_{n-1} \cdots X_0 = i_0)$

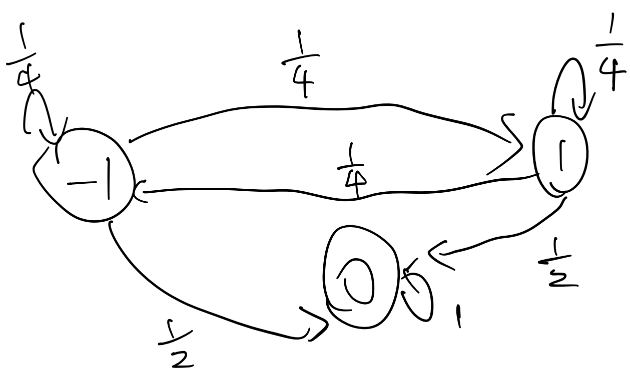
$= P(Y_1 \cdots Y_n \cdot Y_{n+1} = j \mid Y_1 \cdots Y_n = i, \cdots)$

$= P(i \cdot Y_{n+1} = j \mid Y_1, \cdots Y_n)$

$= P(i \cdot Y_1 = j) = P(X_1 = j \mid X_0 = i)$

Markov property and time homogenous hold

$\Rightarrow$ markov chain.



b) $\{1, -1, 0\}$, $d(1) = d(-1) = d(0) = \gcd(1, 2, 3 \ldots)$
$= 1$

c). $\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 = \pi_1$

d) $\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 = \pi_2$

$1\pi_3 = \pi_3$

$\pi_1 = \pi_2 = 0$

$\pi_3 = 1$

$\alpha^T = [0, 0, 1]$

if there is limiting distribution, it's stationary dist.

4. (12 points) Let X be a random variable with moment generating function (m.g.f.)

$$m_X(t) = \frac{1}{\pi}e^{-5t} + \frac{1}{\pi}e^{5t} + (1 - \frac{2}{\pi}) = +1 - \frac{2}{\pi}$$

(a) Can you obtain the probability mass function of X? Is it unique?

(b) What is $\mathbb{E}[X]$?

a)

X	$P(X)$
-5	$\frac{1}{\pi}$
5	$\frac{1}{\pi}$
0	$1-\frac{2}{\pi}$

$$P(X=x) \begin{cases} \frac{1}{\pi} & X=-5 \\ \frac{1}{\pi} & X=5 \\ 1-\frac{2}{\pi} & X=0 \\ 0 & \text{o.w.} \end{cases}$$

b)

$$E(x) = 0 \cdot \left(\frac{-2}{\pi}\right) + (1) \cdot 0 + (-5) \cdot \frac{1}{\pi} + 5 \cdot \frac{1}{\pi}$$

$$= 0$$

5. (12 points) Probability bounds:

(a) You are given that a positive valued random variable X has mean $\mathbb{E}[X] = 4$. Use Markov's Inequality to bound the probability $\mathbb{P}(X \geqslant 5)$.

(b) You are given that a real valued random variable X has mean $\mathbb{E}[X] = 20$ and variance $\mathrm{Var}(X) = 25$. Use Chebyshev inequality to bound the probability $\mathbb{P}(X \leqslant 8)$.

a) $P(|X| > c) \leqslant \frac{E(x)}{c}$

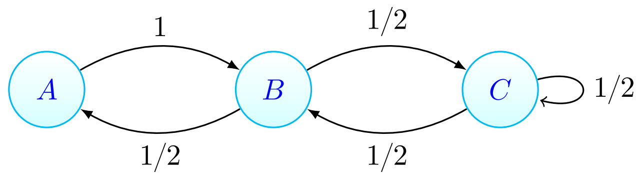
$P(x > 5) \leqslant \frac{4}{5}$

b) $P(|X - \mu| \geqslant c) \leqslant \frac{\sigma^2}{c^2}$

$P(|X - 20| \geqslant 8) \leqslant \frac{25}{c^2}$

$P(X - 20 \leqslant -12) \leqslant \frac{25}{144}$

6. (16 points) A Markov chain $X_0, X_1, X_2, \ldots$ has the following transition graph:



(a) Provide the transition matrix for the Markov chain.

(b) Determine the set of stationary distributions.

(c) If $\pi^T = (\pi_1, \pi_2, \pi_3)$ is a stationary distribution and the distribution of X_0 (initial distribution of the chain). What do you know about $\mathbb{P}(X_1 = i)$ for $i \in \{1, 2, 3\}$? What do you know about $\mathbb{P}(X_k = i)$ for $i \in \{1, 2, 3\}$ and $k > 1$?

(d) Is there a limiting distribution? If so, determine it. If not, explain why.

a)

$$\begin{array}{c|c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

b)

$$\left.\begin{array}{l} \pi_2 = \pi_1 \\ \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 = \pi_2 \\ \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 = \pi_3 \end{array}\right\} \alpha^T = \left[\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right]$$

c) $i=1 \Rightarrow \alpha^T P^1 = \left[\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right]$

$i=2 \Rightarrow \alpha^T P^2 = \left[\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right]$

$i=3 \Rightarrow \alpha^T P^3 = \left[\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right]$

it's stationary dist but also limiting dist.

d) Yes there is.

Irreducible, 1 class, finite class space. Aperoidic

limitig = stationary distribution, $\left[\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right]$

7. (12 points) For each of the below listed descriptions, provide the transition graph of a Markov Chain that satisfies them. Explain in a few sentences why your example has the properties. If such an example does not exist, explain why:

   (a) A 2 state Markov Chain without limiting distribution but with at least one stationary distribution.

   (b) A 2 state Markov Chain with limiting distribution. Determine it.

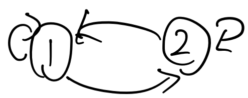
   (c) An irreducible 3 state Markov Chain which is not aperiodic.

   (d) An irreducible 5 state Markov Chain which is aperiodic.

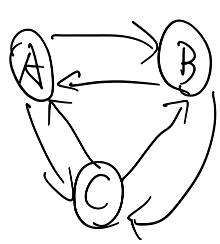
   (e) An irreducible 4 state Markov Chain which is aperiodic and such that the expected return time to state 1 is infinite (i.e. $\mathbb{E}[T_1|X_0 = 1] = \infty$.).

   (f) A Markov Chain with more than one communication class and a limiting distribution.

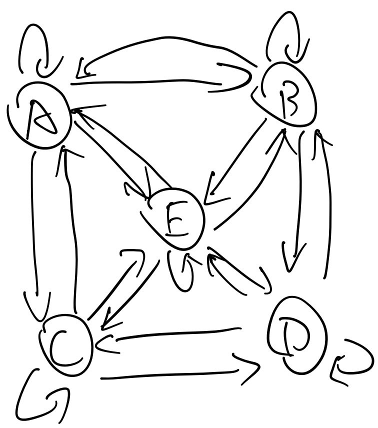
a) $d^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \pi_1 = \pi_2 = \frac{1}{2}$, but no limiting, as Periodic=2. So it exists

b) 

$\Rightarrow$ theris is limiting distribution.
As it's Aperiodic, irreducible, 1 class, finite class space, limiting = stationary distribution.
So it exists

c) 

$d(C) = d(B) = d(A) = 2$, So it exists

d). 

$d(a) = d(b) = d(C) = d(E) = 1$
Aperiodic
So it exists

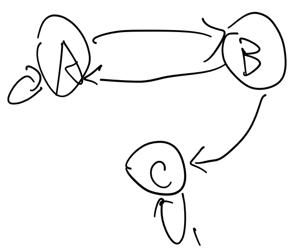
E). No doesn't exist.

$\mu_i = E[T_j \mid X_0 = j] = \frac{1}{\pi_j}$

irreducible + Aperiodic + finite class space $\Rightarrow$ there is π_j

$\frac{1}{\text{Finite}}$ = finite number.

f).



if there is absorbing state then it exists.

8. (16 points) Let $(S_n)_{n\geqslant 0}$ be a simple random walk starting in 0 (i.e. $S_0 = 0$) with $p = 0.4$ and $q = 1 - p = 0.6$. Compute the following probabilities:

(a) $\mathbb{P}(S_1 = 1 | S_2 = 0)$

(b) $\mathbb{P}(S_2 = 2, S_5 = 1)$,

(c) $\mathbb{P}(S_2 = 2, S_4 = 3, S_5 = 1)$,

(d) $\mathbb{P}(M_7 \leqslant -5, S_7 = -5)$, where $M_n = \min_{0\leqslant i\leqslant n} S_i$.

$$\binom{n}{\frac{1}{2}(n+k)} p^{\frac{1}{2}(n+k)} q^{\frac{1}{2}(n-k)}$$

a) $$\frac{P(S_1=1, S_2=0)}{P(S_2=0)} = \frac{P(S_2-S_1=0-1)\,P(S_1-S_0=1-0)}{P(S_2-S_0=0)}$$

$$= \frac{qp}{\binom{2}{1}pq} = \frac{1}{2}$$

b) $P(S_5-S_2=1-2)\,P(S_2-S_0=2-0)$

$$= \left[\binom{3}{1}p^1q^2\right]\left[\binom{2}{2}p^2\right]$$

$$= 3p^3q^2 = 3\cdot(0.4)^3(0.6)^2 = 0.06912$$

c) $P(S_4=3)=0 \Rightarrow P(S_2=2, S_4=3, S_5=1)=0$

d) $P(S_7=-5) \Rightarrow P(M_7\leqslant -5)=1$

$$P(S_7=-5) = P(S_7-S_0=-5-0) = \binom{7}{1}p^1q^6 = 0.1306$$

TABLES OF RANDOM VARIABLES

Discrete R.V.

Name	abbrev.	pmf	$\mathbb{E}(X)$	$\mathrm{Var}(X)$	MGF
Binomial	$\mathrm{Bin}(n,p)$	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$	$[(1-p)+pe^t]^n$
Poisson	$\mathrm{Pois}(\lambda)$	$e^{-\lambda}\dfrac{\lambda^k}{k!}$	λ	λ	$\exp(\lambda(e^t-1))$
Geometric	$\mathrm{Geom}(p)$	$(1-p)^{k-1}p$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$ for $t < -\ln(1-p)$

Continuous R.V.

Name	abbrev.	pdf	$\mathbb{E}(X)$	$\mathrm{Var}(X)$	MGF
Uniform	$\mathrm{Unif}(a,b)$	$\begin{cases} \dfrac{1}{b-a} & a \leqslant t \leqslant b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Exponential	$\mathrm{Exp}(\lambda)$	$\begin{cases} \lambda e^{-\lambda t} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$ for $t < \lambda$
Normal	$\mathrm{N}(\mu,\sigma^2)$	$\left(\dfrac{1}{\sqrt{2\pi\sigma^2}}\right)\exp\left(\dfrac{-(t-\mu)^2}{2\sigma^2}\right)$	μ	σ^2	$e^{\mu t}e^{\sigma^2 t^2/2}$