Math 4B: Differential Equations

Lecture 12: More Second Order Theory

- Review from Last Time,
- Fundamental Sets of Solutions,
- Abel's Theorem,
- Complex Number Review & More!

Second Order Review

Last Time: We focused on the linear operator

$$L[y] = y'' + p(t)y' + q(t)y$$

where $p(t)$ and $q(t)$ are continuous on some interval I. When can we solve the IVP

$$\begin{cases} L[y] = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

where t_0 is an element of the interval I?

Existence-Uniqueness Theorem

The IVP above has a unique solution on all of I.

Second Order Review (cont'd)

We said we'd expect two solutions y_1 and y_2 to give us all solutions as $y = c_1y_1 + c_2y_2$. Does this work?

The Wronskian Theorem

Any solution to $L[y] = 0$ can be written as $y = c_1y_1 + c_2y_2$ at a point t_0 if and only if the Wronskian

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

is not zero at t_0.

New Terminology: We call $y = c_1y_1 + c_2y_2$ the ***general solution*** and we call $\{y_1, y_2\}$ a ***fundamental set*** of solutions.

Examples

Last time we saw that…

1. ODE: $y'' + 8y' + 12y = 0$
Fundamental set: $\{e^{-6t}, e^{-2t}\}$
Wronskian: $W[y_1, y_2] = 4e^{-8t} \neq 0$
General solution: $y = c_1 e^{-6t} + c_2 e^{-2t}$

2. ODE: $y'' + 4y = 0$
Fundamental set: $\{\sin(2t), \cos(2t)\}$
Wronskian: $W[y_1, y_2] = -2 \neq 0$
General solution: $y = c_1 \sin(2t) + c_2 \cos(2t)$

3. ODE: $y'' + 2y' + y = 0$
Fundamental set: $\{e^{-t}, te^{-t}\}$
Wronskian: $W[y_1, y_2] = e^{-2t} \neq 0$
General solution: $y = c_1 e^{-t} + c_2 te^{-t}$

More Examples

4. Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are a fundamental set of solutions for the ODE

$$2t^2 y'' + 3ty' - y = 0$$

for $t > 0$.

Solution: Need to show three things:

(1) $y_1 = \sqrt{t} = t^{1/2}$ is a solution of the ODE

(2) $y_2 = 1/t = t^{-1}$ is a solution of the ODE

(3) $W[t^{1/2}, t^{-1}] \neq 0$ for $t > 0$

4. (Continued)

$$\text{ODE:} \qquad 2t^2y'' + 3ty' - y = 0 \qquad (t > 0).$$

(1) $y_1 = t^{1/2}$, so $y_1' = \frac{1}{2}t^{-1/2}$ and $y_1'' = -\frac{1}{4}t^{-3/2}$. Then

$$\begin{aligned} 2t^2y_1'' + 3ty_1' - y_1 &= 2t^2\left(-\frac{1}{4}t^{-3/2}\right) + 3t\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2} \\ &= -\frac{1}{2}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} = 0. \end{aligned}$$

(2) $y_2 = t^{-1}$, so $y_2' = -t^{-2}$ and $y_2'' = +2t^{-3}$. Then

$$\begin{aligned} 2t^2y_2'' + 3ty_2' - y_2 &= 2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} \\ &= 4t^{-1} - 3t^{-1} - t^{-1} = 0. \end{aligned}$$

(3) $W[y_1, y_2] = \begin{vmatrix} t^{1/2} & t^{-1} \\ t^{-1/2}/2 & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}\,t^{-3/2}$

Another Theorem

Questions: Can we always find a fundamental set of solutions? Is there a unique fundamental set?

Existence of a Fundamental Set of Solutions

Suppose $p(t)$ and $q(t)$ are continuous on an interval I and $L[y] = y'' + p(t)y' + q(t)y$. Then there exists a fundamental set of solutions on I to $L[y] = 0$, namely $\{y_1, y_2\}$ where these are solutions

$$y_1 \text{ of } \begin{cases} L[y] = 0 \\ y(t_0) = 1 \\ y'(t_0) = 0 \end{cases} \qquad \text{and } y_2 \text{ of } \begin{cases} L[y] = 0 \\ y(t_0) = 0 \\ y'(t_0) = 1 \end{cases}$$

for some t_0 in I.

Not unique: $\{e^t, e^{-t}\}$ and $\{\cosh(t) = \frac{e^t + e^{-t}}{2}, \sinh(t) = \frac{e^t - e^{-t}}{2}\}$ are both fundamental sets for $y'' - y = 0$

Abel's Theorem

Abel's Theorem

Suppose $p(t)$ and $q(t)$ are continuous on an interval I and $L[y] = y'' + p(t)y' + q(t)y$. Then on the interval I,

$$W[y_1, y_2] = C \exp\left(-\int p(t)\, dt\right)$$

for some constant C (depending on y_1 and y_2, but not on t). In particular, $W[y_1, y_2]$ is either always zero or never zero.

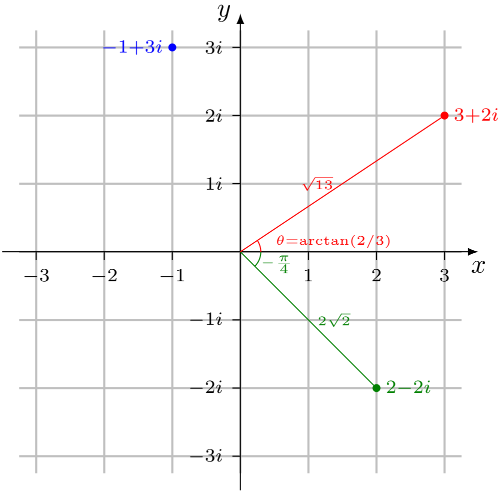
Example: $2t^2y'' + 3ty' - y = 0$ has $p(t) = \frac{3}{2t}$ and $q(t) = -\frac{1}{2t^2}$ and so

$$W[y_1, y_2] = C \exp\left(-\int \frac{3}{2t}\, dt\right) = C \exp\left(-\frac{3}{2}\ln(t)\right) = Ct^{-3/2}.$$

For example **4.** we had this ODE and $W[t^{1/2}, t^{-1}] = -\frac{3}{2}t^{-3/2}$.

A Quick Review of Complex

The ***complex numbers*** are the real numbers together with a new number $i = \sqrt{-1}$ (so $i^2 = -1$). We get a ***complex plane*** of numbers



Polar Form:

$$x+iy = r\cos(\theta)+ir\sin(\theta)$$

Use ***Euler's Formula***

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

to write this as

$$x + iy = re^{i\theta}.$$

Examples:

$$2 - 2i = 2\sqrt{2}\, e^{-i\pi/4}$$

$$3 + 2i = \sqrt{13}\, e^{i\theta}$$

Euler's Formula

So why is Euler's Formula true?

Usual explanation (which needs Taylor series):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

and so

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= \cos(x) + i\sin(x). \end{aligned}$$