Math 4B: Differential Equations

Lecture 15: Nonhomogeneous ODEs

- Solving Nonhomogeneous ODEs,
- The Method of Undetermined Coefficients,
- Some examples & More!

Solutions of Nonhomogeneous ODEs

Today we're going to talk about solutions to the nonhomogeneous second order linear ODE

$$y'' + p(t)y' + q(t)y = g(t). \tag{$*$}$$

Two Solutions of Equation $(*)$

Suppose Y_1 and Y_2 are solutions to the nonhomogeneous second order linear ODE $(*)$. Then $Y_1 - Y_2$ is a solution to the corresponding homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0. \tag{$**$}$$

Thus, if $\{y_1, y_2\}$ is a fundamental set of solutions to $(**)$, then $Y_1 = Y_2 + c_1y_1 + c_2y_2$ for some constants c_1 and c_2.

Idea:

$$\begin{aligned}&(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2)\\ &= \big(Y_1'' + p(t)Y_1' + q(t)Y_1\big) - \big(Y_2'' + p(t)Y_2' + q(t)Y_2\big)\\ &= g(t) - g(t) = 0.\end{aligned}$$

General Solution of Nonhomogeneous Second Order Linear ODEs

The general solution of

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

can be found via the following steps.

1. Find the general solution of the corresponding homogeneous equation

   $$y'' + p(t)y' + q(t)y = 0. \qquad (**)$$

   as $c_1 y_1 + c_2 y_2$. This is often called the ***complementary solution*** y_c.

2. Find a particular solution y_p of $(*)$.

3. The general solution of $(*)$ is then

   $$y = y_p + y_c \qquad \text{or} \qquad y = y_p + c_1 y_1 + c_2 y_2.$$

Finding a Particular Solution

Question: How do we find a particular solution of

$$y'' + p(t)y' + q(t)y = g(t)?$$

When

- $p(t)$, $q(t)$ are constant, and
- $g(t)$ is a polynomial or exponential or sine or cosine

then we use the ***method of undetermined coefficients*** (also known as ***guess and check***):

- Guess the form of the solution, and
- Check which coefficients work.

Example 1

1. Find the general solution of the ODE

$$y'' + 5y' + 6y = 3t^2.$$

Solution:

1. The complementary solution is $\boxed{c_1 e^{-3t} + c_2 e^{-2t}}$.

2. For the particular solution, we guess

$$y_p = At^2 + Bt + C.$$

Then $y_p' = 2At + B$ and $y_p'' = 2A$, so

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= 2A + 5\big(2At + B\big) + 6\big(At^2 + Bt + C\big) \\ &= 6At^2 + \big(10A + 6B\big)t + \big(2A + 5B + 6C\big) \\ &= 3t^2 + 0t + 0. \end{aligned}$$

So $A = 1/2$, $B = -5/6$, $C = 19/36$. This means $\boxed{y_p = \frac{1}{2}t^2 - \frac{5}{6}t + \frac{19}{36}}$.

3. Thus $\boxed{y = \dfrac{1}{2}t^2 - \dfrac{5}{6}t + \dfrac{19}{36} + c_1 e^{-3t} + c_2 e^{-2t}}$.

Example 2

2. Find the general solution of the ODE

$$y'' + 5y' + 6y = 4e^{-t}.$$

Solution:

1. The complementary solution is again $\boxed{c_1 e^{-3t} + c_2 e^{-2t}}$.

2. For the particular solution, we guess

$$y_p = Ae^{-t}$$

Then $y_p' = -Ae^{-t}$ and $y_p'' = Ae^{-t}$, so

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= Ae^{-t} - 5Ae^{-t} + 6Ae^{-t} \\ &= 2Ae^{-t} = 4e^{-t}. \end{aligned}$$

So $A = 2$ and $\boxed{y_p = 2e^{-t}}$.

3. Thus $\boxed{y = 2e^{-t} + c_1 e^{-3t} + c_2 e^{-2t}}$.

Example 3

3. Find the general solution of the ODE

$$y'' + 5y' + 6y = 78\sin(3t)$$

Solution:

1. The complementary solution is again $\boxed{c_1 e^{-3t} + c_2 e^{-2t}}$.

2. For the particular solution, we guess

$$y_p = A\sin(3t)$$

Then $y_p' = 3A\cos(3t)$ and $y_p'' = -9A\sin(3t)$, so

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= -9A\sin(3t) + 15A\cos(3t) + 6A\sin(3t) \\ &= -3A\sin(3t) + 15A\cos(3t) = 78\sin(3t). \end{aligned}$$

But this doesn't work!!!!

Example 3 Again

3. Find the general solution of the ODE

$$y'' + 5y' + 6y = 78\sin(3t)$$

Solution:

1. The complementary solution is again $\boxed{c_1e^{-3t} + c_2e^{-2t}}$.

2. For the particular solution, we NOW guess

$$y_p = A\sin(3t) + B\cos(3t)$$

Then $y_p' = 3A\cos(3t) - 3B\sin(3t)$ and $y_p'' = -9A\sin(3t) - 9B\cos(3t)$, so

$$y_p'' + 5y_p' + 6y_p = (-3A - 15B)\sin(3t) + (15A - 3B)\cos(3t) = 78\sin(3t).$$

So we get the linear system

$$\begin{aligned} -3A - 15B &= 78 \\ 15A - 3B &= 0, \end{aligned} \qquad \Longrightarrow \qquad A = -1,\ B = -5.$$

Thus $\boxed{y = c_1e^{-3t} + c_2e^{-2t} - \sin(3t) - 5\cos(3t)}$.

General Approach

Right-hand side ($g(t)$)	Guess
Polynomial	Polynomial of the same degree
e^{kt}	Ae^{kt} [See Notes!]
$\sin(\beta t)$ or $\cos(\beta t)$	$A\sin(\beta t) + B\cos(\beta t)$
$e^{\alpha t}\sin(\beta t)$ or $e^{\alpha t}\cos(\beta t)$	$Ae^{\alpha t}\sin(\beta t) + Be^{\alpha t}\cos(\beta t)$

Notes:

1. If e^{kt} is a solution to the homogeneous equation (that is, if k is a root of the characteristic polynomial), the guess should be te^{kt}.

2. If e^{kt} and te^{kt} are solutions to the homogeneous equation (that is, if k is a double root of the characteristic polynomial), the guess should be t^2e^{kt}.

Second-to-Last Comment

Question: How can we find the solution to

$$y'' + 5y' + 6y = 4e^{-t} + 78\sin(3t)?$$

Answer: Let $L[y] = y'' + 5y' + 6y$. Then we know that

$$\begin{aligned} L[e^{-3t}] &= 0 \\ L[e^{-2t}] &= 0 \\ L[2e^{-t}] &= 4e^{-t} \\ L[-\sin(3t) - 5\cos(3t)] &= 78\sin(3t). \end{aligned}$$

So

$$L\left[c_1e^{-3t} + c_2e^{-2t} + 2e^{-t} - \sin(3t) - 5\cos(3t)\right] = 4e^{-t} + 78\sin(3t).$$

Moral: To solve $L[y] = g_1(t) + g_2(t)$, solve $L[y] = g_1(t)$ and $L[y] = g_2(t)$ first.

This is the same as Linear Algebra!

Linear Algebra	**Diff'l Eq'ns**
How do we solve $A\mathbf{v} = \mathbf{b}$?	How do we solve $L[y] = g(t)$?

1. Find ***one*** solution to the equation.

Find $\mathbf{v}_p$ with $A\mathbf{v}_p = \mathbf{b}$.	Find y_p with $L[y_p] = g(t)$.

2. Find ***all*** solutions to the corresponding homogeneous equation.

Find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of the Null Space; that is, all solutions of $A\mathbf{v} = \mathbf{0}$.	Find a fundamental set of solutions $\{y_1, y_2\}$ of the homogeneous equation $L[y] = 0$.

3. Write down ***all*** solutions to the nonhomogeneous equation.

$\mathbf{v} = \mathbf{v}_p + c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$	$y = y_p + c_1 y_1 + c_2 y_2$