Math 4B: Differential Equations

Lecture 27: Nonhomogeneous Systems

- Uncoupled Systems,
- Exploiting Diagonalization,
- Generalizing Methods from Nonhomogeneous ODEs, & More!

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Nonhomogeneous Linear Systems

Question: Can we find the general solution to a nonhomogeneous system of ODEs like

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + \mathbf{g}(t)?$$

Solution: If A is particularly simple, then yes:

$$\mathbf{x}'(t) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}.$$

This ***decouples*** and we are left to just solve two first-order ODEs:

$$\begin{aligned} x_1' &= \lambda_1 x_1 + g_1(t) \\ x_2' &= \lambda_2 x_2 + g_2(t). \end{aligned}$$

We've solved these kinds of equations in Chapter 3.

An Example

1. Solve the linear system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} e^t - 5t \\ e^t + 5t \end{pmatrix}.$$

Solution: This means we're solving the linear systems

$$x_1' = x_1 + e^t - 5t \qquad \text{and} \qquad x_2' = 3x_2 + e^t + 5t.$$

Putting these in standard form and using the usual integrating factor, we get

$$e^{-t}x_1' - e^{-t}x_1 = e^{-t}\left(e^t - 5t\right) \qquad \text{and} \qquad e^{-3t}x_2' - 3e^{-3t}x_2 = e^{-3t}\left(e^t + 5t\right)$$

or

$$\left(e^{-t}x_1\right)' = 1 - 5te^{-t} \qquad \text{and} \qquad \left(e^{-3t}x_2\right)' = e^{-2t} + 5te^{-3t}.$$

Solving, we get $\mathbf{x}(t) = \begin{pmatrix} 5t + te^t + c_1e^t + 5 \\ -\frac{5}{3}\,t - \frac{1}{2}\,e^t + c_2e^{3t} - \frac{5}{9} \end{pmatrix}$.

A More General Case

We'll assume that A has a basis of eigenvectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$; let S be the matrix whose kth column is $\boldsymbol{\xi}_k$:

$$S = \begin{pmatrix} | & | & & | \\ \boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 & \cdots & \boldsymbol{\xi}_n \\ | & | & & | \end{pmatrix}.$$

Remember that this means $A = SDS^{-1}$ (and $S^{-1}AS = D$), where

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Pick $\mathbf{y}$ such that $\mathbf{x} = S\mathbf{y}$; that is, set $\mathbf{y} = S^{-1}\mathbf{x}$. Then

$$\begin{aligned} \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) \quad &\Longrightarrow \quad S\mathbf{y}' = AS\mathbf{y} + \mathbf{g}(t) \\ &\Longrightarrow \quad \mathbf{y}' = S^{-1}AS\mathbf{y} + S^{-1}\mathbf{g}(t) \\ &\Longrightarrow \quad \mathbf{y}' = D\mathbf{y} + \mathbf{h}(t) \qquad \text{where } \mathbf{h}(t) = S^{-1}\mathbf{g}(t). \end{aligned}$$

An Example

2. Find the general solution to

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10t \\ 2e^t \end{pmatrix}.$$

Solution: We need to find the eigenvalues (for D) and eigenvectors (for S) of A.

Eigenvalues: These are the roots of

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3.$$

Thus $\lambda_1 = 1$ and $\lambda_2 = 3$.

Eigenvectors: We get

$$\operatorname{Null}(A - 1I) = \operatorname{Null}\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix} = \operatorname{Null}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{Span}\left\{ \boldsymbol{\xi}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

$$\operatorname{Null}(A - 3I) = \operatorname{Null}\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} = \operatorname{Null}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{Span}\left\{ \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Example (continued)

We have $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$, and $S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ with $A = SDS^{-1}$ and $D = S^{-1}AS$. Then

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) \qquad \Longrightarrow \qquad \mathbf{y}' = D\mathbf{y} + \mathbf{h}(t)$$

where $\mathbf{x} = S\mathbf{y}$ and $\mathbf{h}(t) = S^{-1}\mathbf{g}(t)$. Here this means

$$\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{y} + \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 10t \\ 2e^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} e^t - 5t \\ e^t + 5t \end{pmatrix}.$$

This was Example 1, so we get $\mathbf{y} = \begin{pmatrix} 5t + te^t + c_1 e^t + 5 \\ -\frac{5}{3}t - \frac{1}{2}e^t + c_2 e^{3t} - \frac{5}{9} \end{pmatrix}$ and so

$$\begin{aligned} \mathbf{x} = S\mathbf{y} &= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5t + te^t + c_1 e^t + 5 \\ -\frac{5}{3}t - \frac{1}{2}e^t + c_2 e^{3t} - \frac{5}{9} \end{pmatrix} \\ &= c_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} -t - \frac{1}{2} \\ t - \frac{1}{2} \end{pmatrix} + \frac{10}{9} \begin{pmatrix} -6t - 5 \\ 3t + 4 \end{pmatrix}. \end{aligned}$$

General Solution of Nonhomogeneous Linear Systems

The general solution of

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t) + \mathbf{g}(t) \tag{$*$}$$

can be found via the following steps.

1. Find the general solution of the corresponding homogeneous system
$$\mathbf{x}'(t) = P(t)\mathbf{x}(t). \tag{$**$}$$
as $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$. We will write this as $\mathbf{x}_c$.
2. Find a particular solution $\mathbf{x}_p$ of $(*)$.
3. The general solution of $(*)$ is then

$$\mathbf{x}(t) = \mathbf{x}_p + \mathbf{x}_c \qquad \text{or} \qquad \mathbf{x}(t) = \mathbf{x}_p + c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

An Example: Undetermined Coefficients

3. Find the general solution to

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} + t \begin{pmatrix} 10 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Solution: This is the same equation as before! We'll solve the homogeneous version as

$$\mathbf{x}_c(t) = c_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(remembering the eigensystem from the previous problem).
We'll ***guess*** the form of a particular solution. Based on the form of the $\mathbf{g}(t)$ term, we'll guess

$$\mathbf{x}_p(t) = \mathbf{a}t + \mathbf{b} + \mathbf{c}e^t$$

for some constant vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. This won't work, as we have e^t in our solution to the homogeneous version.

More Undetermined Coefficients

Our second guess is

$$\mathbf{x}_p(t) = \mathbf{a}t + \mathbf{b} + \mathbf{c}te^t$$

for some constant vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. This won't work either. Instead, we need a guess of the form

$$\mathbf{x}_p(t) = \mathbf{a}t + \mathbf{b} + \mathbf{c}te^t + \mathbf{d}e^t$$

for some constant vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$.

Let's Try It:

$$\mathbf{x}_p'(t) = \mathbf{a} + \mathbf{c}te^t + \mathbf{c}e^t + \mathbf{d}e^t$$

$$A\mathbf{x}_p + t\begin{pmatrix}10\\0\end{pmatrix} + e^t\begin{pmatrix}0\\2\end{pmatrix} = \left[A\mathbf{a} + \begin{pmatrix}10\\0\end{pmatrix}\right]t + A\mathbf{b} + A\mathbf{c}te^t + \left[A\mathbf{d} + \begin{pmatrix}0\\2\end{pmatrix}\right]e^t$$

So

$$A\mathbf{a} + \begin{pmatrix}10\\0\end{pmatrix} = \mathbf{0}, \qquad A\mathbf{b} = \mathbf{a} \qquad A\mathbf{c} = \mathbf{c} \qquad A\mathbf{d} + \begin{pmatrix}0\\2\end{pmatrix} = \mathbf{c} + \mathbf{d}$$

Our Particular Solution

So our particular solution $\mathbf{x}_p(t) = \mathbf{a}t + \mathbf{b} + \mathbf{c}te^t + \mathbf{d}e^t$ of the linear system

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} + t \begin{pmatrix} 10 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

satisfies...

$$\begin{aligned}
A\mathbf{a} + \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \mathbf{0} \qquad & \Longrightarrow \ \mathbf{a} = \frac{10}{3}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \\
A\mathbf{b} = \mathbf{a} \qquad & \Longrightarrow \ \mathbf{b} = \frac{10}{9}\begin{pmatrix} -5 \\ 4 \end{pmatrix} \\
A\mathbf{c} = \mathbf{c} \qquad & \Longrightarrow \ \mathbf{c} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{(or any multiple)} \\
A\mathbf{d} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \mathbf{c} + \mathbf{d} \qquad & \Longrightarrow \ \mathbf{d} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{(or many others)}
\end{aligned}$$

That is, one solution is

$$\mathbf{x}_p(t) = \frac{10}{3}t\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \frac{10}{9}\begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^t.$$

Undetermined Coefficients–Conclusion

Our old solution (when we called this Example 2) was

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} -t - \frac{1}{2} \\ t - \frac{1}{2} \end{pmatrix} + \frac{10}{9} \begin{pmatrix} -6t - 5 \\ 3t + 4 \end{pmatrix}.$$

Compare this to

$$\mathbf{x}_p(t) = \frac{10}{3}\, t \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \frac{10}{9} \begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^t.$$

These are the same when $c_1 = \frac{1}{2}$ and $c_2 = 0$ and our general solution (from Example 3) is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \mathbf{x}_p(t) \\ &= c_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{10}{9} \begin{pmatrix} -6t - 5 \\ 3t + 4 \end{pmatrix} + e^t \begin{pmatrix} -t - 1 \\ t \end{pmatrix}. \end{aligned}$$