Math 4B: Differential Equations

Lecture 20: Solving IVPs via Laplace

- Overview from Last Time,
- Inverse Laplace Transforms,
- Solving IVPs & More!

Solving IVPs via Laplace Transforms

Our application of Laplace transforms to IVPs is a three-step process:

1. Take the Laplace transform of an equation in t to get an algebraic equation in s.
2. Solve the algebraic equation to find the Laplace transform of the solution to the IVP.
3. Undo the Laplace transform (take the ***inverse Laplace transform***) to find the solution to the original IVP.

Last Time: Found some Laplace transforms
Discovered some properties of Laplace transforms

Today: Solve a few IVPs, and move on.

An Example

Suppose we are trying to solve

$$\begin{cases} ay'' + by' + cy = f(t) \\ y(0) = y_0 \\ y'(0) = y'_0 \end{cases}$$

By linearity of the Laplace Transform, we can turn this equation into

$$a\mathscr{L}[\,y''\,] + b\mathscr{L}[\,y'\,] + c\mathscr{L}[\,y\,] = \mathscr{L}[\,f(t)\,].$$

Let's write $Y(s) = \mathscr{L}[\,y\,]$ and $F(s) = \mathscr{L}[\,f(t)\,]$. Then our questions are

- Can we compute $F(s)$?
- Can we write $\mathscr{L}[\,y''\,]$ and $\mathscr{L}[\,y'\,]$ in terms of $Y(s) = \mathscr{L}[\,y\,]$?
- Once we solve for $Y(s)$, can we find $y = \mathscr{L}^{-1}[\,Y(s)\,]$?

Can We Compute $F(s)$?

In fact, we've found ***lots*** of Laplace Transforms, but here's a table. See Table 6.2.1 in your book.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}[f(t)]$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n > 0$ an integer	$\frac{n!}{s^{n+1}}$, $s > 0$
$t^n e^{at}$, $n > 0$ an integer	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$	$\frac{e^{-cs}}{s}$, $s > 0$

Can We Find $\mathscr{L}[\,y''\,]$ and $\mathscr{L}[\,y'\,]$?

In fact, we found the Laplace transforms of all derivatives. They were...

$$\mathscr{L}[\,y'\,] = s\mathscr{L}[\,y\,] - y_0$$

$$\mathscr{L}[\,y''\,] = s^2\mathscr{L}[\,y\,] - sy_0 - y_0'$$

$$\mathscr{L}[\,y'''\,] = s^3\mathscr{L}[\,y\,] - s^2y_0 - sy_0' - y''(0)$$

and so on to

$$\mathscr{L}[\,y^{(n)}\,] = s^n\mathscr{L}[\,y\,] - s^{n-1}y_0 - s^{n-2}y_0' - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0).$$

Back to IVPs

So let's return to the IVP

$$\begin{cases} ay'' + by' + cy = f(t) \\ y(0) = y_0 \\ y'(0) = y'_0 \end{cases}$$

Taking the Laplace transform, we get

$$a\mathscr{L}[\,y''\,] + b\mathscr{L}[\,y'\,] + c\mathscr{L}[\,y\,] = \mathscr{L}[\,f(t)\,].$$

Again writing $Y(s) = \mathscr{L}[\,y\,]$ and $F(s) = \mathscr{L}[\,f(t)\,]$, we get

$$a\big(s^2Y(s) - sy_0 - y'_0\big) + b\big(sY(s) - y_0\big) + cY(s) = F(s)$$

or

$$Y(s) = \frac{F(s) + (as+b)y_0 + ay'_0}{as^2 + bs + c}$$

Can We Find $y = \mathscr{L}^{-1}\left[\,Y(s)\,\right]$?

Well, sort of. There ***is*** a way to compute the inverse Laplace transform using complex analysis. We'll just use a table:

$f(t) = \mathscr{L}^{-1}\left[\,F(s)\,\right]$	$F(s) = \mathscr{L}\left[\,f(t)\,\right]$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n > 0$ an integer	$\frac{n!}{s^{n+1}}$, $s > 0$
$t^n e^{at}$, $n > 0$ an integer	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$	$\frac{e^{-cs}}{s}$, $s > 0$

Fact: If $f(t)$ and $g(t)$ are continuous with $\mathscr{L}[\,f\,] = \mathscr{L}[\,g\,]$, then $f = g$. This is a *uniqueness* result.

Examples

1. Find $\mathscr{L}^{-1}\left[\dfrac{1}{s^2+9}\right]$.

Solution: Remember that $\mathscr{L}^{-1}\left[\dfrac{b}{s^2+b^2}\right]=\sin(bt)$. With $b=3$, we get

$$\mathscr{L}^{-1}\left[\frac{1}{s^2+9}\right]=\mathscr{L}^{-1}\left[\frac{1}{3}\cdot\frac{3}{s^2+9}\right]=\frac{1}{3}\mathscr{L}^{-1}\left[\frac{3}{s^2+9}\right]=\frac{1}{3}\sin(3t).$$

We've used the fact that $\mathscr{L}^{-1}[\,\cdot\,]$ is ***linear:***

$$\mathscr{L}^{-1}[\,c_1F_1(s)+c_2F_2(s)\,]=c_1\mathscr{L}^{-1}[\,F_1(s)\,]+c_2\mathscr{L}^{-1}[\,F_2(s)\,].$$

More Examples

2. Find $\mathscr{L}^{-1}\left[\dfrac{1}{s^2+2s+10}\right]$.

Solution: Remember that $\mathscr{L}^{-1}\left[\dfrac{b}{(s-a)^2+b^2}\right]=e^{at}\sin(bt)$.

Completing the square, we get $s^2+2s+10=(s+1)^2+3^2$. So with $a=-1$ and $b=3$, we get

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{s^2+2s+10}\right]&=\mathscr{L}^{-1}\left[\frac{1}{(s+1)^2+3^2}\right]\\&=\frac{1}{3}\cdot\mathscr{L}^{-1}\left[\frac{3}{(s+1)^2+3^2}\right]\\&=\frac{1}{3}e^{-t}\sin(3t).\end{aligned}$$

More Examples

3. Find $\mathscr{L}^{-1}\left[\dfrac{s+4}{s^2+2s+10}\right]$.

Solution: Now we write this as

$$\mathscr{L}^{-1}\left[\frac{s+4}{s^2+2s+10}\right]=\mathscr{L}^{-1}\left[\frac{s+4}{(s^2+2s+1)+9}\right]=\mathscr{L}^{-1}\left[\frac{s+4}{(s+1)^2+9}\right]$$
$$=\mathscr{L}^{-1}\left[\frac{s+1}{(s+1)^2+3^2}\right]+\mathscr{L}^{-1}\left[\frac{3}{(s+1)^2+3^2}\right].$$

Remember that

$$\mathscr{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right]=e^{at}\sin(bt) \text{ and } \mathscr{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos(bt).$$

Thus (with $a=-1$ and $b=3$ again), we get

$$\mathscr{L}^{-1}\left[\frac{s+4}{s^2+2s+10}\right]=e^{-t}\cos(3t)+e^{-t}\sin(3t).$$

More Examples

4. Find $\mathscr{L}^{-1}\left[\dfrac{s+5}{s^2+4s+5}\right]$.

Solution: This is just like the previous problem:

$$\mathscr{L}^{-1}\left[\frac{s+5}{s^2+4s+5}\right]=\mathscr{L}^{-1}\left[\frac{s+2}{(s+2)^2+1^2}\right]+3\cdot\mathscr{L}^{-1}\left[\frac{1}{(s+2)^2+1^2}\right].$$

Remember that

$$\mathscr{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right]=e^{at}\sin(bt)$$

and

$$\mathscr{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos(bt).$$

Thus (with $a=-2$ and $b=1$), we get

$$\mathscr{L}^{-1}\left[\frac{s+5}{s^2+4s+5}\right]=e^{-2t}\cos(t)+3\,e^{-2t}\sin(t).$$

IVP #1

5. Solve the IVP

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Solution: We take the Laplace transform of the ODE and find

$$\mathscr{L}[\,y''\,] + 9\mathscr{L}[\,y\,] = \mathscr{L}[\,0\,] \qquad \text{or} \qquad \left(s^2 Y(s) - 0s - 1\right) + 9Y(s) = 0.$$

Thus $Y(s) = \dfrac{1}{s^2+9}$. So

$$y(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2+9}\right] = \frac{1}{3}\sin(3t)$$

by our previous work.

IVP #2

6. Solve the IVP

$$\begin{cases} y'' + 2y' + 10y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Solution: We take the Laplace transform of the ODE and find

$$\mathscr{L}[\,y''\,] + 2\mathscr{L}[\,y'\,] + 10\mathscr{L}[\,y\,] = \mathscr{L}[\,0\,]$$

or

$$\left(s^2Y(s) - 0s - 1\right) + 2\left(sY(s) - 0\right) + 10Y(s) = 0.$$

Thus $Y(s) = \dfrac{1}{s^2 + 2s + 10}$. So

$$y(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2 + 2s + 10}\right] = \frac{1}{3}\,e^{-t}\sin(3t)$$

by our previous work.

IVP #3: Forcing

7. Solve the IVP

$$\begin{cases} y'' + 9y = 3\cos(4t) \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Solution: We take the Laplace transform of the ODE and find

$$\mathscr{L}[\,y''\,] + 9\mathscr{L}[\,y\,] = 3\mathscr{L}[\,\cos(4t)\,]$$

or

$$\left(s^2 Y(s) - 0s - 1\right) + 9Y(s) = \frac{3s}{s^2 + 4^2}.$$

Thus

$$Y(s) = \frac{1}{s^2 + 9} + \frac{3s}{(s^2 + 16)(s^2 + 9)}.$$

Given $Y(s) = \mathscr{L}[\,y\,]$, can we find y?

Inverses using Partial Fractions

We've found that

$$Y(s) = \frac{1}{s^2+9} + \frac{3s}{(s^2+16)(s^2+9)}.$$

We'll use partial fractions to write

$$\frac{3s}{(s^2+16)(s^2+9)} = \frac{As+B}{s^2+16} + \frac{Cs+D}{s^2+9}.$$

Clearing the denominator, we get

$$3s = (As+B)(s^2+9) + (Cs+D)(s^2+16).$$

By plugging in different values of s, we find that

$$Y(s) = \frac{1}{s^2+9} + \frac{3}{7}\,\frac{s}{s^2+9} - \frac{3}{7}\,\frac{s}{s^2+16}.$$

So
$$\begin{aligned} y(t) &= \frac{1}{3}\,\mathscr{L}^{-1}\left[\frac{3}{s^2+9}\right] + \frac{3}{7}\,\mathscr{L}^{-1}\left[\frac{s}{s^2+9}\right] - \frac{3}{7}\,\mathscr{L}^{-1}\left[\frac{s}{s^2+16}\right] \\ &= \frac{1}{3}\sin(3t) + \frac{3}{7}\cos(3t) - \frac{3}{7}\cos(4t). \end{aligned}$$