Math 4B: Differential Equations

Lecture 18: Higher Order ODEs

- Higher Order Linear ODEs,
- Some Theory,
- A Few Examples & More!

Linear Differential Equations

Weeks 1–3: First order ODEs, including linear ODEs

Weeks 4–5: Second order linear ODEs

Today: Nth order linear ODEs

- Solving homogeneous ODEs with constant coefficients
- Linear independence of solutions
- Wronskians
- Solutions of inhomogeneous ODEs

nth Order Linear ODEs

Today we study ***n*th order linear differential equations**; that is, equations of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t).$$

We assume these coefficients $\{P_0(t), P_1(t), \ldots, P_n(t), G(t)\}$ are continuous (and $P_0(t) \neq 0$) on some interval $I = (\alpha, \beta)$.

We will often divide through by $P_0(t)$ to get the equation

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t).$$

For initial conditions, we'll usually have

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

for some t_0 in the interval $I = (\alpha, \beta)$.

Existence / Uniqueness Theorem

The Existence and Uniqueness Theorem

Suppose $p_1(t), p_2(t), \ldots, p_{n-1}(t)$, and $g(t)$ are continuous on the interval I given by $\alpha < t < \beta$ containing t_0. Then there is a unique solution $y = \phi(t)$ to the IVP

$$\begin{cases} y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \\ \vdots \\ y^{(n-1)}(t_0) = y_0^{(n-1)} \end{cases}$$

that is defined for all t in I.

Warning! As we saw with second (and first!) order ODEs, solutions exist and are defined, but are not necessarily even *possible* to find.

Fundamental Sets?

If we have some solutions to our homogoeneous nth order linear differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = 0,$$

how can we tell if we have ***enough***? That is, how many ***linearly independent*** solutions form a ***fundamental set***?

Answer: We need n solutions $\{y_1, \ldots, y_n\}$ so that we can always solve

$$\begin{aligned}
c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\
c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\
&\vdots \\
c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)}
\end{aligned}$$

for the constants $c_1, \ldots, c_n$. We need a Wronskian.

Wronskians

The Wronskian Theorem

Any solution to

$$L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

can be written as $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ at a point t_0 if and only if the Wronskian

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is not zero at t_0. As in the $n = 2$ case, it turns out that (for solutions) we have either $W[y_1, \ldots, y_n]$ is either identically zero on I or ***never*** zero on I. Such a set of solutions $\{y_1, y_2, \ldots, y_n\}$ is a ***fundamental set*** if and only if when $W[y_1, \ldots, y_n] \neq 0$.

Summary

A set of solutions $\{y_1, \ldots, y_n\}$ of our linear homogeneous ODE

$$L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

is. . .

- a ***fundamental set*** if any solution y can be written as

  $$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

  for some constants c_n.

- This happens exactly when $W[y_1, \ldots, y_n] \neq 0$ on the interval I (on which $p_1, \ldots, p_n$ and q are continuous). On this interval W is either ***always zero*** or ***never zero***.

- In the case $W \neq 0$, we say that the set $\{y_1, \ldots, y_n\}$ is ***linearly independent***. This is equivalent to the linear algebra definition: if

  $$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

  for some constants, then $c_1 = c_2 = \cdots = c_n = 0$.

How Do We Find Solutions?

To solve linear homogeneous ODEs **with constant coefficients**, we proceed as in the $n = 2$ case. If

$$L(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0,$$

then

$$L\left(e^{rt}\right) = e^{rt}\left(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n\right) = 0.$$

Thus we get n possible solutions e^{rt}, one for each root r of

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$$

but again we have to deal with

- distinct real roots,
- repeated real roots, and
- complex (non-real) roots.

We do this in ***the same way*** as in the $n = 2$ case.

Examples

1. Find a fundamental set of solutions for

$$y'''' - 10y'' + 9y = 0.$$

Hint: $r^4 - 10r^2 + 9 = (r+1)(r-1)(r+3)(r-3)$

Solution: A fundamental set of solutions is $\{e^{-t}, e^{t}, e^{-3t}, e^{3t}\}$.

2. Find a fundamental set of solutions for

$$y''' - 6y'' + 12y' - 8y = 0.$$

Hint: $r^3 - 6r^2 + 12r - 8 = (r-2)^3$

Solution: A fundamental set of solutions is $\{e^{2t}, te^{2t}, t^2e^{2t}\}$.

Examples

3. Find a fundamental set of solutions for

$$y^{(4)} + 4y''' + 6y'' + 4y' + 5y = 0.$$

Hint: $r^4 + 4r^3 + 6r^2 + 4r + 5 = (r^2 + 1)(r^2 + 4r + 5)$

Solution: We get roots $r = \pm i$ and $r = -2 \pm i$, so a fundamental set of solutions is $\{\cos(t), \sin(t), e^{-2t}\cos(t), e^{-2t}\sin(t)\}$.

Examples

4. Find a fundamental set of solutions for

$$y'''' + y = 0.$$

Hint: $r^4 + 1 = 0$, so $r^4 = -1 = e^{i\pi} = e^{i(\pi+2n\pi)}$ for any integer n

Solution: The four roots are

$$r_1 = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i, \qquad r_2 = e^{i3\pi/4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i,$$

$$r_3 = e^{i5\pi/4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i, \quad \text{and} \qquad r_4 = e^{i7\pi/4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i.$$

So a fundamental set of solutions is

$$\left\{ e^{t/\sqrt{2}} \cos\left(\tfrac{t}{\sqrt{2}}\right), e^{t/\sqrt{2}} \sin\left(\tfrac{t}{\sqrt{2}}\right), e^{-t/\sqrt{2}} \cos\left(\tfrac{t}{\sqrt{2}}\right), e^{-t/\sqrt{2}} \sin\left(\tfrac{t}{\sqrt{2}}\right) \right\}.$$

Nonhomogeneous Linear ODEs

What about solutions to the nonhomogeneous nth order linear ODE

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)? \qquad (*)$$

Two Solutions of Equation $(*)$

Suppose Y_1 and Y_2 are solutions to the nonhomogeneous nth order linear ODE $(*)$. Then $Y_1 - Y_2$ is a solution to the corresponding homogeneous ODE

$$L[y] = 0. \qquad (**)$$

Thus, if $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions to $(**)$, then $Y_1 = Y_2 + c_1y_1 + c_2y_2 + \cdots + c_ny_n$ for some constants $c_1, c_2, \ldots, c_n$.

Idea:

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0.$$

Method of Undetermined Coefficients

Again, we guess and check to find particular solutions.

5. Find the general solution of

$$y''' - 6y'' + 12y' - 8y = 24e^{2t}$$

Hint: The general solution to the corresponding homogeneous equation is

$$y_h = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t}.$$

Solution: To get $24e^{2t}$ our guess should be $y_p = At^3e^{2t}$. We get

$$y_p''' - 6y_p'' + 12y_p' - 8y_p = 6Ae^{2t},$$

so we want $A = 4$. Thus the general solution we want is

$$\boxed{y = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + 4t^3e^{2t}}.$$

Method of Undetermined Coefficients

One more example:

6. Find the general solution of

$$y^{(4)} + 4y''' + 6y'' + 4y' + 5y = 32\sin(t).$$

Hint: The general solution to the corresponding homogeneous equation is

$$y_c = c_1\cos(t) + c_2\sin(t) + c_3e^{-2t}\cos(t) + c_4e^{-2t}\sin(t).$$

Solution: To get $32\sin(t)$ our guess should be $y_p = At\sin(t) + Bt\cos(t)$. We get

$$y_p^{(4)} + 4y_p''' + 6y_p'' + 4y_p' + 5y_p = -8\,(A+B)\sin(t) + 8(A-B)\cos(t),$$

so we want $A + B = -4$ and $A - B = 0$. Thus the general solution we want is

$$\boxed{y = c_1\cos(t) + c_2\sin(t) + c_3e^{-2t}\cos(t) + c_4e^{-2t}\sin(t) - 2t\sin(t) - 2t\cos(t)}.$$