Math 4B: Differential Equations

Lecture 26: Repeated Eigenvalues

- What if we ***don't*** have enough eigenvalues?
- The repeated eigenvalues case,
- Generalized eigenvectors & More!

General Plan

Linear Homogeneous Systems with Constant Coefficients

Suppose $\mathbf{x}'(t) = A\mathbf{x}(t)$ where A is an $n \times n$ matrix with n linearly independent eigenvectors $\xi_1, \ldots, \xi_n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly repeated). Then the general solution of this ODE is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2 + \cdots + c_n e^{\lambda_n t}\xi_n.$$

Sophisticated Version

Linear Homogeneous Systems with Constant Coefficients

Suppose $\mathbf{x}'(t) = A\mathbf{x}(t)$ where A is an $n \times n$ matrix with n linearly independent eigenvectors $\xi_1, \ldots, \xi_n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly repeated). Then the general solution of this ODE is

$$\mathbf{x}(t) = Se^{Dt}S^{-1}\mathbf{x}(0),$$

where

$$S = \begin{pmatrix} | & | & & | \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ | & | & & | \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix},$$

and thus $A = SDS^{-1}$.

Repeated Roots

Question: What happens if there are repeated roots and not enough eigenvectors?

- If there are enough eigenvectors (if there is a **basis** of eigenvectors), then all is ***exactly*** as before.
- If A is a 2×2 matrix with one repeated eigenvalue λ, then a basis of eigenvectors means two linearly independent eigenvectors for λ. This means
$$A\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$$
for ***every*** vector $\boldsymbol{\xi}$ in $\mathbf{R}^2$. This means $A = \lambda I$.
- What if there are not enough eigenvectors?

Today's Problem

So what if there are not enough eigenvectors?
Remember from Linear Algebra: Sometimes with a repeated eigenvalue we have

$$\text{the geometric multiplicity} \; < \; \text{the algebraic multiplicity},$$

where

- the geometric multiplicity of λ is the number of linearly independent eigenvectors for λ, and
- the algebraic multiplicity of λ is the number of times λ is a root of the characteristic polynomial.

What do we do here?

1. Find the general solution to the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{x}.$$

Our Example

1. Find the general solution to the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{x}.$$

Solution: The **eigenvalues** here are the solutions to

$$0 = \det\left(A - \lambda I\right) = \begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Thus $\lambda_1 = \lambda_2 = 3$.
The **eigenvectors** are linearly independent vectors in

$$\mathrm{Null}\left(A - 3I\right) = \mathrm{Null}\begin{pmatrix} 4-3 & -1 \\ 1 & 2-3 \end{pmatrix} = \mathrm{Null}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \mathrm{Null}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Thus there is only one linearly independent eigenvector: $\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

What can we do?

Repeated Roots: Attempt #1

We might guess that a fundamental set of solutions of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{x}$$

are

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}_2(t) = \boldsymbol{\xi}\, te^{3t}$$

for some constant vector $\boldsymbol{\xi}$. This would mean

$$\left(\boldsymbol{\xi}\, te^{3t}\right)' = A\left(\boldsymbol{\xi}\, te^{3t}\right) \qquad \Longrightarrow \qquad \boldsymbol{\xi}\left(3te^{3t} + e^{3t}\right) = (A\boldsymbol{\xi})te^{3t}.$$

Dividing by e^{3t}, this would mean

$$\boldsymbol{\xi}\left(3t + 1\right) = (A\boldsymbol{\xi})t$$

for all t. This only works for all t if $\boldsymbol{\xi} = \mathbf{0}$.

Guess: Maybe try $\mathbf{x}_2(t) = \boldsymbol{\xi}\, te^{3t} + \boldsymbol{\eta}\, e^{3t}$ instead?

Repeated Roots: Attempt #2

So we're going to see if $\mathbf{x}_2(t) = \boldsymbol{\xi}\, te^{3t} + \boldsymbol{\eta}\, e^{3t}$ will work as a solution to

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{x}.$$

Plugging in, we find

$$\boldsymbol{\xi}\left(3te^{3t} + e^{3t}\right) + 3\boldsymbol{\eta}\, e^{3t} = (A\boldsymbol{\xi})\, te^{3t} + (A\boldsymbol{\eta})\, e^{3t}.$$

From this we see that

$$A\boldsymbol{\xi} = 3\boldsymbol{\xi} \qquad \text{and} \qquad A\boldsymbol{\eta} = 3\boldsymbol{\eta} + \boldsymbol{\xi} \text{ or } (A - 3I)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Which means $\boldsymbol{\xi}$ is an eigenvector; we'll take $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, from which we find

$$\left[\begin{array}{c|c} A - 3I & \boldsymbol{\xi} \end{array}\right] = \left[\begin{array}{cc|c} 4-3 & -1 & 1 \\ 1 & 2-3 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

From this we get $\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as one possible solution.

Our Example (Conclusion)

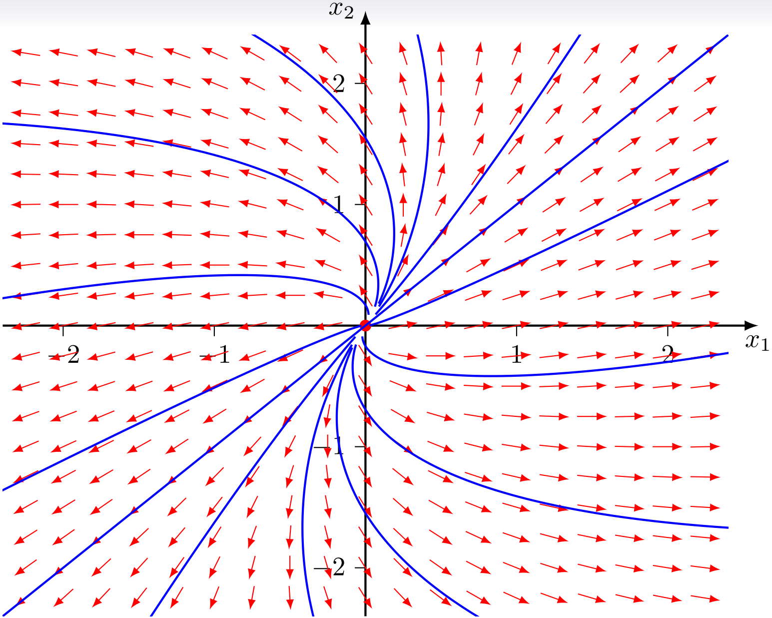
Thus the general solution of our system is

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\\ &= c_1\boldsymbol{\xi}\, e^{3t} + c_2\left(\boldsymbol{\xi}\, te^{3t} + \boldsymbol{\eta}e^{3t}\right)\\ &= c_1\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + c_2\left[\begin{pmatrix}1\\1\end{pmatrix}te^{3t} + \begin{pmatrix}1\\0\end{pmatrix}e^{3t}\right].\end{aligned}$$

General Approach for Double Eigenvalues:

- Find repeated eigenvalue λ with one eigenvector $\boldsymbol{\xi}$.
- $\mathbf{x}_1(t) = \boldsymbol{\xi}\, e^{\lambda t}$
- Solve $(A - \lambda I)\boldsymbol{\eta} = \boldsymbol{\xi}$ for $\boldsymbol{\eta}$.

  Note: Even though $A - \lambda I$ is singular, there is ***always*** a solution to this system.
- $\mathbf{x}_2(t) = \boldsymbol{\xi}\, te^{\lambda t} + \boldsymbol{\eta}\, e^{\lambda t}$
- Conclusion: $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ is a fundamental set of solutions.


x_2
x_1
-2
-1
1
2
2
1
-1
-2

Example 2

2. Find the general solution to the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix} \mathbf{x}.$$

Eigenvalues: These are the roots of

$$0 = \det\left(A - \lambda I\right) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

Thus $\lambda = -2$ is a repeated eigenvalue.

Eigenvectors: These are non-zero elements of

$$\operatorname{Null}\left(A - (-2)I\right) = \operatorname{Null}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \operatorname{Null}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Thus $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is one eigenvector for $\lambda = -2$.

Example 2 (continued)

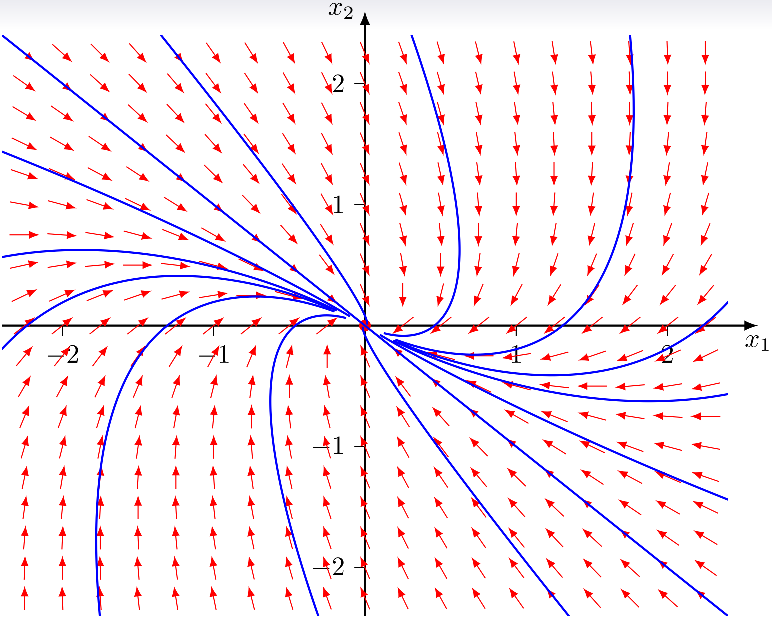
A ***generalized eigenvector*** $\boldsymbol{\eta}$ is a solution to $(A - \lambda I)\boldsymbol{\eta} = \boldsymbol{\xi}$. In this case, we're solving

$$\left[\begin{array}{c|c} A - (-2)I & \boldsymbol{\xi} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

So we can take $\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or many others.

Thus the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) \\ &= c_1 \boldsymbol{\xi}\, e^{-2t} + c_2 \left(\boldsymbol{\xi}\, t e^{-2t} + \boldsymbol{\eta} e^{-2t} \right) \\ &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \right]. \end{aligned}$$


x_2
x_1
2
1
−1
−2
−2
−1
1
2

General Plan

To solve $\mathbf{x}'(t) = A\mathbf{x}(t)\ldots$

- If A has a basis of real eigenvectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ corresponding to the (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$, the general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{\xi}_1 + c_2 e^{\lambda_2 t}\boldsymbol{\xi}_2 + \cdots + c_n e^{\lambda_n t}\boldsymbol{\xi}_n.$$

- If we have a complex (non-real) eigenvalue / eigenvector pair λ and $\boldsymbol{\xi}$, we replace $e^{\lambda t}\boldsymbol{\xi}$ and $e^{\overline{\lambda} t}\overline{\boldsymbol{\xi}}$ with $\operatorname{Re}\left(e^{\lambda t}\boldsymbol{\xi}\right)$ and $\operatorname{Im}\left(e^{\lambda t}\boldsymbol{\xi}\right)$ to get a fundamental set of ***real*** solutions.

- If we have have an eigenvalue λ that is a double root of the characteristic polynomial, but which has only one linearly independent eigenvector $\boldsymbol{\xi}$, then another linearly independent solution is $(\boldsymbol{\xi} t + \boldsymbol{\eta})\, e^{\lambda t}$ where $(A - \lambda I)\boldsymbol{\eta} = \boldsymbol{\xi}$.