Problem 3.1. Consider a continuous-time Markov chain on $\mathcal{S} = \{1, 2, 3, 4\}$ with generator matrix

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 2 & -5 & 2 \\ 1 & 0 & 2 & -3 \end{pmatrix}$$

(a) Write down the transition matrix of the associated embedded DTMC.
(b) If the chain is currently in state 3, how long, on average will it take before moving to a new state?
(c) If the chain is in state 3 at time $t = 0$, what is the probability that it remains there until at least time $t = 2$?
(d) If the chain is currently in state 3 and its next move is to state 4, how long, on average, would you expect the chain to have stayed in state 3 before making that jump?

a)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & 0 & \frac{2}{5} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{pmatrix} \end{array}$$

b) Find $E(X_2 \neq 3 \mid X_0 = 3) \Rightarrow E(\text{inter-Arrival}) \sim EXP(1+2+2) \Rightarrow \frac{1}{5}$

c) $P(\min(\text{Stay state } 3) \geq 2 \mid X_0 = 3) = P(\min(\cdots) \geq 2)$

$= e^{-\lambda_{x \neq y} \cdot 2}$

d) $E(\text{Time inter Arrival} \mid \underline{X_0 = 3, X_{next} = 4})$

$= E(EXP(5)) = \frac{1}{5}$

↳ No info given Inter-Arrival Time

Problem 3.2. Customers arrive at a bank according to a Poisson process with a rate of 30 customers per hour. The bank has a single line for the customers, and has three tellers. Each time a customer reaches the front of the line, if there is a teller who is not currently servicing someone, the customer immediately begins being serviced by that teller. If all of the tellers are busy servicing other customers, the customer at the front of the line waits until one of the tellers finishes servicing their customer.

The servicing times of the tellers are independent of each other and of the number of customers in line, and are exponentially distributed with a mean servicing time of 5 minutes.

Finally, if a customer arrives and there are already four customers in line (in addition to the three that are being serviced), they do not join the queue and instead immediately leave the bank.

(a) Model this system as a CTMC by specifying its transition rate diagram and generator matrix.

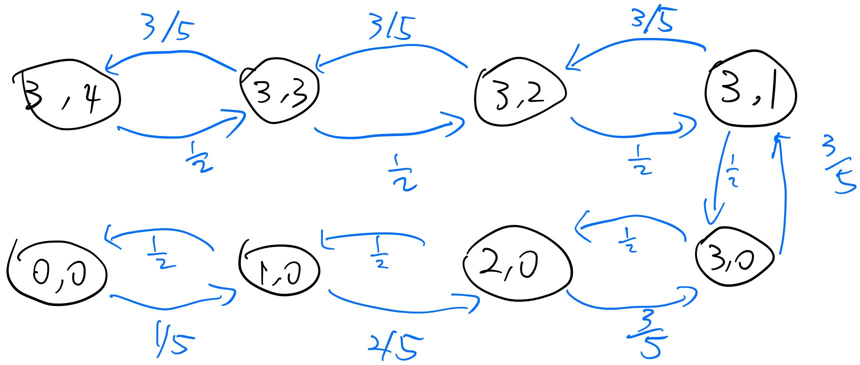
30/hour = 1/2 min

$\lambda = \frac{1}{2}$ $\longrightarrow$

$\longleftarrow$ $\lambda = \frac{1}{5}$

no more than 4 Customer in line

a) Let Y denote Customer in Queue, X denote # of teller avialiable



$$Q = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & -\frac{7}{10} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & -\frac{9}{10} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{5} & -\frac{11}{10} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{5} & -\frac{11}{10} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{5} & -\frac{11}{10} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & -\frac{11}{10} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & -\frac{3}{5} \end{pmatrix}$$

b). $P(X_1 + X_2 \leq Y) = E(P[X_1 + X_2 \leq Y \mid X_1 + X_2])$

$= E(e^{-\frac{3}{5}(X_1+X_2)})$

$= E(e^{-\frac{3}{5}X_1}) E(e^{-\frac{3}{5}X_2}) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - (-\frac{3}{5})}\right)\left(\frac{\frac{1}{2}}{\frac{1}{2} - (-\frac{3}{5})}\right)$

Problem 3.3. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

(a) Diagonalize A.
(b) Compute $\exp(A)$.

a) $A = UDU^{-1}$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\det(A) = \begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = 0$

$\Rightarrow \lambda_1 = -1,\ \lambda_2 = 3$

$D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$

b) $\mathrm{EXP}(A) = \mathcal{U}\,\mathrm{EXP}(D)\,\mathcal{U}^{-1}$

To find $\mathcal{U}$

$\Rightarrow \begin{pmatrix} 1-(-1) & 4 \\ 1 & 1-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow x_1 + 2x_2 = 0 \qquad x_1 = -2x_2 \Rightarrow \frac{x_1}{x_2} = \frac{-2}{1} \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\begin{pmatrix} 1-(3) & 4 \\ 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \cdots \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow \mathcal{U} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow \mathcal{U}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{-1}{4} & \frac{-1}{2} \end{pmatrix}$

$\mathrm{EXP}(A) = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \mathrm{EXP}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{-1}{4} & -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 \\ 0 & e^{3} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}$

$$A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

$$\mathrm{Exp}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$\mathrm{Tr}(A) = \frac{2}{3} \cdot 3 = 2$ Rank = 2 (by wolfRam)

Since $\mathrm{Tr}(A) = \mathrm{Rank}(A) \Rightarrow A$ can be idempotent

if $A \cdot A = A \Rightarrow A$ is idempotent

$\Rightarrow A^2 = A$ by wolfRam $\Rightarrow A^k = A$

Since $\frac{A^0}{0!} = 1 = I \Rightarrow \sum_{k=1}^{\infty} \frac{A^k}{k!} + I = \sum_{k=1}^{\infty} \frac{A}{k!} + I$

$$= A \sum_{k=1}^{\infty} \frac{1^k}{k!} + I$$

$$= A \cdot (e-1)^1 + I$$

Problem 3.5. Consider a CTMC $\{X_t\}$ on $\mathcal{S} = \{1, 2, 3, 4\}$ with generator matrix

$$Q = \begin{pmatrix} -3 & 0 & 1 & 2 \\ 2 & -4 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 2 & 2 & 1 & -5 \end{pmatrix}.$$

Note that Q can be diagonalized as $Q = UDU^{-1}$, where

$$U = \begin{pmatrix} -3 & -2 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ -1 & -2 & -3 & 1 \\ 5 & 3 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 0 & -1/4 & 0 & 1/4 \\ -1/5 & 2/5 & 0 & -1/5 \\ 1/4 & -1/8 & -1/4 & 1/8 \\ 7/20 & 7/40 & 1/4 & 9/40 \end{pmatrix}$$

Let $P(t)$ denote the transition function of $\{X_t\}$. Does $P(t)$ converge as $t \to \infty$? If so, what is the limit?

$P(t) = U \exp(Dt) U^{-1}$ as $P(t) = \exp(Qt)$

$$\exp(Dt) = \begin{pmatrix} e^{-6t} & 0 & 0 & 0 \\ 0 & e^{-5t} & 0 & 0 \\ 0 & 0 & e^{-4t} & 0 \\ 0 & 0 & 0 & e^{0} \end{pmatrix} \quad \text{As } t \to \infty \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By wolfRam, $U \exp(Dt) U^{-1}$ as $t \to \infty$

$$P(t) = \frac{1}{40} \begin{pmatrix} 14 & 7 & 10 & 9 \\ 14 & 7 & 10 & 9 \\ 14 & 7 & 10 & 9 \\ 14 & 7 & 10 & 9 \end{pmatrix} \Rightarrow P(t) \text{ Converges}$$

Problem 3.6. Your friend proposes a game of chance. You roll a fair 6-sided die repeatedly until it lands on 1, at which point the game ends. Each time the roll does not land on 1, you pay your friend a random amount. The amount that you pay after each roll (including the roll on which you roll 1) is exponentially distributed with a mean of 1 dollar, and is independent of what you have paid on all of the others rolls (and the number of rolls).

(a) How much do you expect to pay in total after playing the game once?

a) Roll die $\sim$ Geom $(\overset{\lambda}{\frac{1}{6}})$ $\quad E[\text{Geom}(\frac{1}{6})] = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6$

(b) After playing the game once, what is the probability that you pay more than 10 dollars?

$P(\text{min Pay} > 10) = e^{-\lambda x} = e^{-\frac{1}{6}\cdot 10}$

Problem 3.7. A squirrel in your yard is either in a tree or on the ground; the only time it switches positions is when a pedestrian walks by. If the squirrel is currently on the ground and a pedestrian walks by, there is a 30% chance that it stays on the ground and a 70% chance that it moves to the tree. If the squirrel is currently in a tree and a pedestrian walks by, there is a 90% chance that it stays in the tree and a 10% chance that it moves to the ground. Assume that the time that it takes for the squirrel to move from the ground to the tree (and from the tree to the ground) is negligible.

Suppose that pedestrians walk by your yard according to a Poisson process with a rate of 2 per hour, and let X_t denote the location of the squirrel at time t (i.e., in the tree or on the ground).

Is $\{X_t\}$ a CTMC? If so, what is its generator matrix?

X_t is CTMC. From description we obtain transition Matrix

$$A = \begin{array}{cc} & \begin{array}{cc} G & T \end{array} \\ & \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix} \begin{array}{c} G \\ T \end{array} \end{array} \Rightarrow Q = \lambda(A - I) = 2\left(A - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \boxed{\begin{pmatrix} -1.4 & 1.4 \\ 0.2 & -0.2 \end{pmatrix}}$$

Problem 3.8. Consider a CTMC $\{X_t\}$ on $\mathcal{S} = \{1, 2, 3\}$ with generator matrix

$$Q = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix},$$

and denote the transition function of $\{X_t\}$ by $\{P(t), t \geq 0\}$.

Obtained by R

(a) Let $\{N_t\}$ be a Poisson process with rate $\lambda = 3$. Find the smallest value $M \in \mathbb{N}_0$ such that

$$\mathbb{P}(N_2 > M) \leq 0.01.$$

```{r}
1-ppois(q: 12, lambda: 6)
```

```
[1] 0.008827484
```

$M = \min(M_i),\ i \in \mathbb{N}$

$= 12$

(b) Using Poisson subordination and part (a), find an estimate $\hat{P}(2)$ such that

$$\max_{x,y\in\mathcal{S}} |\hat{P}(2) - P(2)| \leq 0.01.$$

```python
import numpy as np
import scipy.linalg

from scipy.stats import poisson
from numpy.linalg import matrix_power
def EstimatePHat(Q, epsilon, t):
    lam = max(abs(np.diag(Q)))
    d = len(Q)
    I = I = np.identity(d)
    P = I + np.multiply(Q,1/lam)
    M = int(poisson.ppf(1 - epsilon, lam*t))
    PHat = np.zeros([d,d])
    for n in range(0,M+1):
        PHat += np.multiply(matrix_power(P, n),poisson.pmf(n,lam*t))
    return(PHat)
Q = np.array([[-3,2,1],[1,-2,1],[2,1,-3]])
I = np.identity(3)
lam = max(abs(np.diag(Q)))
P = I + np.multiply(Q,1/lam)
EstimatePHat(Q,0.01,2)
```

	0	1	2
0	0.309804	0.433659	0.247709
1	0.309469	0.433994	0.247709
2	0.310140	0.432988	0.248045

$x,y\in\mathcal{S}$

(c) Without explicitly computing the transition function of $\{X_t\}$, compute the limiting distribution of $\{X_t\}$.[3]

Since X_t is irreducible, $\alpha = \pi$

and $\pi Q = 0$

Wolfram gives $\pi = (5/16,\ 1/4,\ \frac{7}{16})$

(d) If the chain starts at $X_0 = 1$, in the long term, what proportion of time will it spend in state 1?

$\frac{5}{16}$ as 2 States

(e) If the chain starts at $X_0 = 2$, in the long term, what proportion of time will it spend in state 1?

$5/16$

Problem 3.9. Suppose that $\{X_t\}$ is an irreducible CTMC on $\mathcal{S} = \{1, 2, 3\}$ with stationary distribution $\pi = \begin{pmatrix} 0.3 & 0.5 & 0.2 \end{pmatrix}$.

(a) Compute $\mathbb{E}[X_t^2 | X_0 \sim \pi]$. Recall that we write

$$\mathbb{E}[\cdot | X_0 \sim \pi]$$

$$\begin{aligned} E[X_t^2 | X_0 \sim \pi] &= 1^2 \cdot \pi_1 + 2^2 \pi_2 + 3^2 \pi_3 \\ &= 0.3 + 4 \cdot 0.5 + 9 \cdot 0.2 \\ &= 0.3 + 2 + 1.8 = 4.1 \end{aligned}$$

(b) Note that we have not specified the transition function or generator matrix of $\{X_t\}$. Does the quantity

$$\lim_{t \to \infty} \mathbb{E}[X_t^2 | X_0 \sim \pi].$$

depend on the transition function or generator matrix of the chain? Why or why not?

No, it does not depend on Q. as it's irreducible chain

$\pi = \alpha$

(c) Does the quantity Does the quantity

$$\lim_{t \to \infty} \mathbb{E}[X_t^2 | X_0 = 1].$$

depend on the transition function or generator matrix of the chain? Why or why not?

No, B/C $\lim_{t \to \infty} P(X_t = x | X_0 = y) = \pi_0$

$\Rightarrow \sum_{x \in \mathcal{S}} x^2 \pi_x = 1^2 \pi_1 + 2^2 \pi_2 + 3^2 \pi_3$

Problem 3.10. Jobs arrive at a computer server according to a Poisson process with a rate of 5 jobs per hour. The server can complete one job at a time, and the time that it takes for each job to be completed follows an Exponential distribution with a mean of 10 minutes. If the computer server is actively completing a job, then any additional jobs that arrive are added to a queue.

(a) On average, in the long term, how many jobs are there in the queue?

(b) Compute the probability, in the long term, that there at least 3 jobs in the queue.

$5/hr = 5/60 \text{ min} = 1/12 \text{ min}$

a) $\xrightarrow{PP(\frac{1}{12})}$ in

out $\xleftarrow[EXP(\frac{1}{10})]{}$

$$\frac{\text{Job coming in}}{\text{Job done - coming in}} = \frac{\frac{1}{12}}{\frac{1}{10} - \frac{1}{12}} = 5$$

b) $P(\min(X_1 \cdots X_n) \geq 3) = e^{-(\frac{1}{10} - \frac{1}{12}) \cdot 3} = e^{-0.05} \approx 95\%$

Problem 3.11. Let $\boldsymbol{X} = (X_1, \dots, X_d)^T \sim \mathcal{N}_d(\boldsymbol{\mu}, \Sigma)$ for some $\boldsymbol{\mu} \in \mathbb{R}^d$ and $d \times d$ matrix Σ, and let A be a deterministic $n \times d$ matrix. Note that $A\boldsymbol{X}$ is a (random) vector in $\mathbb{R}^n$.

(a) Fix $\boldsymbol{a} \in \mathbb{R}^n$. What is the probability distribution of $\boldsymbol{a}^T A\boldsymbol{X}$?
(b) For $1 \le i \le n$, compute $\mathbb{E}((A\boldsymbol{X})_i)$.
(c) For $1 \le i, j \le n$, compute $\mathrm{Cov}((A\boldsymbol{X})_i, (A\boldsymbol{X})_j)$.
(d) Using (a), (b), and (c), determine the probability distribution of $A\boldsymbol{X}$.

a) we have $AX \sim N(A\mu, A\Sigma A^T)$?

$$\alpha^T AX \sim N(\alpha^T A\mu, \alpha^T A\Sigma A^T \alpha)$$

b) $E((AX)_i) = \sum_{i=0}^{n} A_i \mu$ by notes

c) $$\mathrm{Cov}((AX)_i, (AX)_j) = \sum_{k=1}^{d} \sum_{l=1}^{d} A_{i,k} \mathrm{Cov}(X_k, X_l) A_{j,l}$$
$$= \sum_{k=1}^{d} A_{i,k} \sum_{l=1}^{d} \Sigma_{k,l} A_{jl}$$

d) $AX \sim N(A\mu, A\Sigma A')$

Problem 3.12. Let $X, Y \overset{iid}{\sim} \mathcal{N}(0,1)$, and let W be a random variable with distribution

$$\mathbb{P}(W = 1) = \mathbb{P}(W = -1) = \frac{1}{2},$$

that is independent of X and Y. Define $Z = WX + Y$. Determine whether (X, Z) is jointly normal.

$$X + Z = X + WX + Y$$

$$= (1 + W)X + Y$$

they're not jointly normal

Problem 3.13. Let $\{S_n\}$ be a discrete-time random walk of the form

$$S_n = \sum_{i=1}^{n} X_i,$$

where $\{X_i\}$ are iid random variables with distribution

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

(a) Calculate [4]

$$\mathbb{P}\left(\frac{S_{35}}{\sqrt{50}} \geq 1\right).$$

(b) Using the central limit theorem, estimate the probability from part (a).

b) $P\left(\frac{S_{35}}{\sqrt{35}} \geqslant \frac{\sqrt{35}}{\sqrt{50}}\right) = P\left(z \geqslant \frac{\sqrt{35}}{\sqrt{50}}\right) = 0.0877$

a) $P(S_{35} \geqslant \sqrt{50}) = P(S_{35} \geqslant 7.\cdots) = P(S_{35} \geqslant 8)$

$= P(S_{35} > 8) + P(S_{35} \geqslant 8)$

$= P(S_{35} = 9) + P(S_{35} = 11) \cdots +$

$= P(S_{35} = \sum_{i=1}^{13} 9 + 2i)$

$= \sum_{i=1}^{13} \binom{35}{9+2i} \left(\frac{1}{2}\right)^{35}$