Financial Engineering & Risk Management

Option Pricing and the Binomial Model

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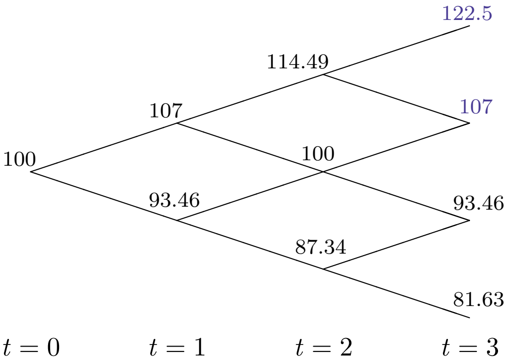
A Brief Overview of Option Pricing

In the next series of modules we'll study:

1. The 1-period binomial model
2. The multi-period binomial model
3. Replicating strategies
4. Pricing European and American options in the binomial lattice
5. The Black-Scholes formula

Stock Price Dynamics in the Binomial Model



- A risk-free asset or cash account also available
  - $\$1$ invested in cash account at $t = 0$ worth R^t dollars at time t

Some Questions

1. How much is an option that pays $\max(0,\ S_3 - 100)$ at $t = 3$ worth?
   (i) do we have enough information to answer this question?
   (ii) should the price depend on the utility functions of the buyer and seller?
   (iii) will the price depend on the true probability, p, of an up-move in each period? Perhaps the price should be

$$\mathsf{E}_0^{\mathbb{P}}[R^{-3} \max(0,\ S_3 - 100)]? \tag{1}$$

2. Suppose now that:
   (i) you stand to lose a lot at date $t = 3$ if the stock is worth 81.63
   (ii) you also stand to earn a lot at date $t = 3$ if the stock is worth 122.49.

   If you don't want this risk exposure could you do anything to eliminate it?

The St. Petersberg Paradox

- Consider the following game
  - a fair coin is tossed repeatedly until first head appears
  - if first head appears on the n^{th} toss, then you receive $\$2^n$
- How much would you be willing to pay in order to play this game?
- The expected payoff is given by

$$\begin{aligned}
\mathsf{E}_0^{\mathbb{P}}[\text{Payoff}] &= \sum_{n=1}^{\infty} 2^n \, \mathsf{P}(1^{st} \text{ head on } n^{th} \text{ toss}) \\
&= \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} \\
&= \infty
\end{aligned}$$

- But would you pay an infinite amount of money to play this game?
  - clear then that (1) does not give correct option price.

The St. Petersberg Paradox

- Daniel Bernouilli resolved this paradox by introducing a utility function, $u(\cdot)$
  - $u(x)$ measures how much utility or benefit you obtains from x units of wealth
  - different people have different utility functions
  - $u(.)$ should be increasing and concave
- Bernouilli introduced the $\log(\cdot)$ utility function so that

$$\mathsf{E}_0^{\mathbb{P}}[u(\mathsf{Payoff})] \;=\; \sum_{n=1}^{\infty} \log(2^n)\,\frac{1}{2^n} \;=\; \log(2) \sum_{n=1}^{\infty} \frac{n}{2^n} \;<\; \infty$$

- So maybe just need to figure out appropriate utility function and use it to compute option price
  - maybe, but who's utility function?
  - in fact we'll see there's a much simpler way.

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The 1-Period Binomial Model

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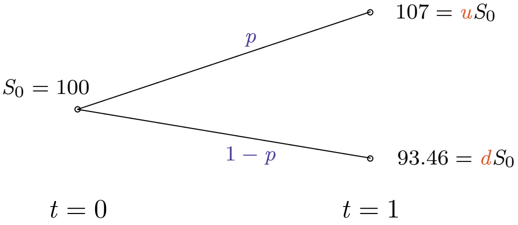
The 1-Period Binomial Model



- Can borrow or lend at gross risk-free rate, R
  - so \$1 in cash account at $t = 0$ is worth \$$R$ at $t = 1$
- Also assume that short-sales are allowed.

The 1-Period Binomial Model

Questions:

1. How much is a call option that pays $\max(S_1 - 107,\ 0)$ at $t = 1$ worth?

2. How much is a call option that pays $\max(S_1 - 92,\ 0)$ at $t = 1$ worth?

Type A and Type B Arbitrage

Earlier definitions of weak and strong arbitrage applied in a deterministic world.

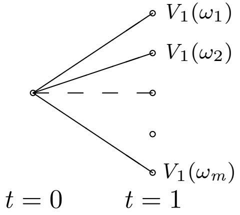
Need more general definitions when we introduce randomness.

Definition. A type A arbitrage is a security or portfolio that produces immediate positive reward at $t = 0$ and has non-negative value at $t = 1$.

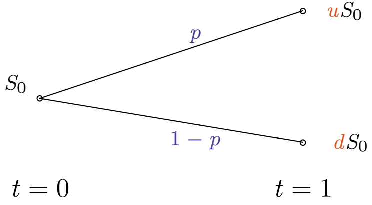
i.e. a security with initial cost, $V_0 < 0$, and time $t = 1$ value $V_1 \geq 0$.

Definition. A type B arbitrage is a security or portfolio that has a non-positive initial cost, has positive probability of yielding a positive payoff at $t = 1$ and zero probability of producing a negative payoff then.

i.e. a security with initial cost, $V_0 \leq 0$, and $V_1 \geq 0$ but $V_1 \neq 0$.



Arbitrage in the 1-Period Binomial Model



- Recall we can borrow or lend at gross risk-free rate, R, per period.
- And short-sales are allowed.

Theorem. There is no arbitrage if and only if $d < R < u$.

Proof: (i) Suppose $R < d < u$. Then borrow S_0 and invest in stock.

(ii) Suppose $d < u < R$. Then short-sell one share of stock and invest proceeds in cash-account.

Both case give a type B arbitrage.

Will soon see other direction, i.e. if $d < R < u$, then there can be no-arbitrage.

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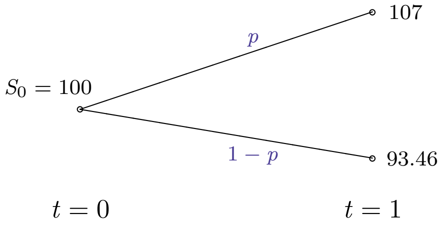


Assume now that $R = 1.01$.

1. How much is a call option that pays $\max(S_1 - 102,\ 0)$ at $t = 1$ worth?
2. How will the price vary as p varies?

To answer these questions, we will construct a replicating portfolio.

The Replicating Portfolio

- Consider buying x shares and investing $\$y$ in cash at $t = 0$
- At $t = 1$ this portfolio is worth:

$$\begin{array}{rl} 107x + 1.01y & \quad \text{when } S = 107 \\ 93.46x + 1.01y & \quad \text{when } S = 93.46 \end{array}$$

- Can we choose x and y so that portfolio equals option payoff at $t = 1$?
- If so, then we must solve

$$\begin{aligned} 107x + 1.01y &= 5 \\ 93.46x + 1.01y &= 0 \end{aligned}$$

The solution is

$$\begin{aligned} x &= 0.3693 \\ y &= -34.1708 \end{aligned}$$

So yes, we can construct a replicating portfolio!

The Replicating Portfolio

Question: What does a negative value of y mean?

Question: What would a negative value of x mean?

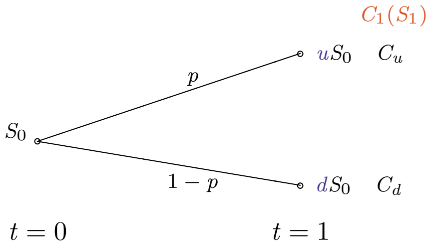
- The cost of this portfolio at $t = 0$ is

$$0.3693 \times 100 - 34.1708 \times 1 \approx 2.76$$

- So the fair value of the option is 2.76
  - indeed 2.76 is the arbitrage-free value of the option.

- So option price does not directly depend on buyer's (or seller's) utility function.

Derivative Security Pricing



- Can use same replicating portfolio argument to find price, C_0, of any derivative security with payoff function, $C_1(S_1)$, at time $t=1$.
- Set up replicating portfolio as before:

$$\begin{aligned} uS_0x + Ry &= C_u \\ dS_0x + Ry &= C_d \end{aligned}$$

- Solve for x and y as before and then must have $C_0 = xS_0 + y$.

Derivative Security Pricing

- Obtain

$$
\begin{aligned}
C_0 &= \frac{1}{R}\left[\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right] \\
&= \frac{1}{R}\left[q C_u + (1-q) C_d\right] \\
&= \frac{1}{R}\mathsf{E}_0^{\mathbb{Q}}[C_1].
\end{aligned} \tag{2}
$$

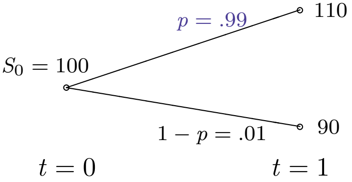
- Note that if there is no-arbitrage then $q > 0$ and $1 - q > 0$
  - we call (2) risk-neutral pricing
  - and $(q, 1-q)$ are the risk-neutral probabilities.

- So we now know how to price any derivative security in this 1-period model.

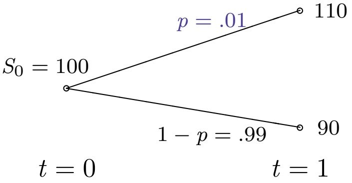
- Can also answer earlier question: "How does the option price depend on p?"
  - but is the answer crazy?!

What's Going On?

- Stock ABC



- Stock XYZ



Question: What is the price of a call option on ABC with strike $K = \$100$?
Question: What is the price of a call option on XYZ with strike $K = \$100$?