Math 174E
Lecture 14

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References

Hull
Chapter 13.7, 13.8, 13.9

Increasing the Number of Steps

- one or two stock price movements during the life of an option are not very realistic
- when binomial trees are used in practice one can use a much higher numbers of steps (or, equivalently, a smaller step size $\Delta t = \frac{T}{N}$)
  - N steps $=$ $N+1$ terminal stock price values S_T
  - 2^N possible trajectories
- risk-neutral valuation approach for computing arbitrage-free stock option prices as in the one- and two-step trees still applies and holds true

Binomial tree formulas: $S_0, r, T, \Delta t, \sigma$ are given parameters

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad p^* = \frac{e^{r\Delta t} - d}{u - d}$$

Mathematical Comments 1/4

Binomial tree model = **Cox-Ross-Rubinstein model (1979)**

- simplest multi-period financial market model
- *discrete time* model for the stock price $(S_n)_{n=0,\dots,N}$
- split the time interval $[0, T]$ into N equidistant time steps of size $\Delta t = \frac{T}{N}$

$$S_n = S_0 \cdot \prod_{i=1}^{n} R_i, \qquad R_i = \begin{cases} u & \text{prob. } p^* \\ d & \text{prob. } 1 - p^* \end{cases} \quad \textbf{i.i.d.} \tag{1}$$

  for all $n = 1, \dots, N$
- we have

$$\mathbb{P}^*[S_n = S_0 \cdot u^j \cdot d^{n-j}] = \binom{n}{j} \cdot (p^*)^j \cdot (1 - p^*)^{n-j}$$

  for all $n = 1, \dots, N$ and $j = 1, \dots, n$ (**binomial distribution**)

Mathematical Comments 2/4

- risk-neutral European call option pricing formula

$$
\begin{aligned}
C_0(K,T) &= \mathbb{E}^*[e^{-rT}(S_N-K)^+] \\
&= e^{-rT}\sum_{j=0}^{N}(S_0u^jd^{N-j}-K)^+\cdot\mathbb{P}^*[S_N=S_0u^jd^{N-j}] \\
&= e^{-rT}\sum_{j=0}^{N}(S_0u^jd^{N-j}-K)^+\cdot\binom{N}{j}\cdot(p^*)^j\cdot(1-p^*)^{N-j} \\
&= e^{-rT}\sum_{j=0}^{N}(S_0u^jd^{N-j}-K)^+\cdot\frac{N!}{(N-j)!\,j!}\cdot(p^*)^j\cdot(1-p^*)^{N-j}
\end{aligned}
$$

(compare with formula in Theorem 13.3 for $N=1$ and in Theorem 13.5 for $N=2$)

- similar for European put options and any other European option with payoff function $h(S_N)$ (e.g., power option, binary option etc.)

Mathematical Comments 3/4

- note that we obtain from (1)

$$\log\left(\frac{S_n}{S_0}\right) = \sum_{i=1}^{n} \underbrace{\log(R_i)}_{=L_i}$$

  where

$$L_i = \begin{cases} \log(u) > 0 & \text{prob. } p^* \\ \log(d) < 0 & \text{prob. } 1 - p^* \end{cases} \qquad \text{i.i.d.}$$

- hence, we can write the evolution of the stock price process $(S_n)_{n=0,\dots,N}$ as

$$S_n = S_0 e^{\sum_{i=1}^{n} L_i}$$

- a process of the form $X_n = \sum_{i=1}^{n} L_i$ with i.i.d. increments $L_1, L_2, \dots, L_n$ which take only two values (positive and negative, respectively) is called a **simple random walk**

Mathematical Comments 4/4

- under suitable technical assumptions one can show via the **Central Limit Theorem** that

$$\text{distribution of } \log\left(\frac{S_N}{S_0}\right) \xrightarrow[N\uparrow\infty]{} \mathcal{N}\left(\left(r-\frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

- here $\mathcal{N}\big((r-\sigma^2/2)T, \sigma^2 T\big)$ denotes the **normal distribution** with **mean** parameter $(r-\sigma^2/2)T$ and **variance** parameter $\sigma^2 T$

- this leads to the continuous time **Black-Scholes-Merton model (1973)** in the limit as the step size $\Delta t = \frac{T}{N}$ converges to 0 when the number of steps N goes to $+\infty$
- in particular, an **option price** computed in the binomial tree model converge to the **Black-Scholes-Merton price** as the number of time steps N goes to $+\infty$ (see Appendix of Chapter 13 in Hull for a non-rigorous proof)

Chapter 14: Wiener Processes and Itô's Lemma

Hull
Chapter 14.1, 14.2

Preliminaries

Definition 14.1

A **stochastic process** (random process) is a collection of random variables $(S_t)_{t \in I}$ with index set I, taking values in a state space $\mathcal{S}$.

Typically: index t represents time!

- $I = \{0, 1, 2, \dots\}$: discrete time
- $I = [0, +\infty)$: continuous time

State space:

- $\mathcal{S} \subseteq \mathbb{N}$: "discrete" random variables
- $\mathcal{S} \subseteq \mathbb{R}$: "continuous" random variables

Most important examples:

- discrete time: **random walk**
- continuous time: **Brownian motion**

Markov Process

A very informal definition:

Definition 14.2

A **Markov process** is a particular type of stochastic process where the future evolution of the process only depends on its current value but not on the past.

Most important examples of Markov processes:

- discrete time: **random walk**
- continuous time: **Brownian motion**

Stock prices are assumed to follow a Markov process

- consistent with the weak form of "market efficiency" (current price of a stock impounds all relevant information)

Brownian Motion

Brownian motion: Continuous-time, continuous-state stochastic process, also called the **Wiener process**.

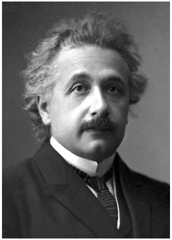
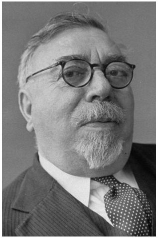
Some history:

- 1827: Botanist Robert Brown observed pollen grains suspended in water, noted the erratic and continuous movement of tiny particles ejected from the grains.
- 1905: Albert Einstein gave a theoretical explanation for this physical phenomenon (atomic nature of matter) and introduced the Brownian motion.
- 1923: American mathematician Norbert Wiener rigorously proved the existence of the Brownian Motion.

- 1900: French mathematician Louis Bachelier is credited with being the first person to introduce a mathematical model of Brownian motion; used it for valuing stock options in his PhD thesis (forefather of financial mathematics).

Some History

Robert Brown (1773 – 1858), Albert Einstein (1879 – 1955),
Norbert Wiener (1894 – 1964), Louis Bachelier (1870 – 1946).

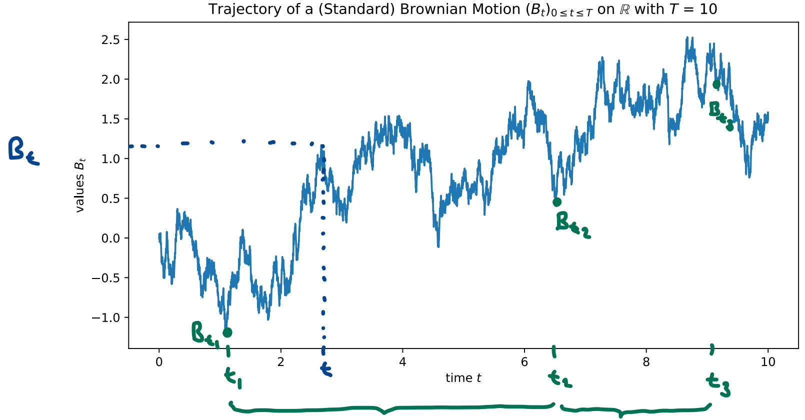
   

Source of pictures: Wikipedia.

Illustration: Brownian Motion on $\mathbb{R}$

$B_t \sim \mathcal{N}(0,t)$: $\mathbb{E}[B_t]=0$, $\mathrm{Var}(B_t)=t$

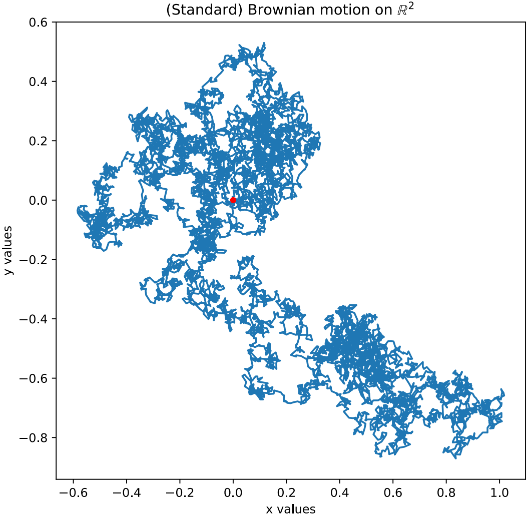
$t \mapsto B_t$ continuous



$B_{t_2} - B_{t_1} \sim \mathcal{N}(0, t_2 - t_1)$ $\quad$ $B_{t_3} - B_{t_2} \sim \mathcal{N}(0, t_3 - t_2)$

independent

Illustration: Brownian Motion on $\mathbb{R}^2$



Standard Brownian Motion

Definition 14.3

A continuous-time stochastic process $(B_t)_{t \geq 0}$ is called a **standard Brownian motion** on $\mathbb{R}$ if it satisfies the following properties:

1. $B_0 = 0$.
2. *(Independent increments)*
   For all $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < \ldots < t_{n-1} < t_n$, the random variables $B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.
3. *(Stationary increments)*
   For all $0 \leq s < t$ the random variable $B_t - B_s$ is **normally distributed** with mean 0 and variance $t - s$, i.e., $B_t - B_s \sim \mathcal{N}(0, t - s)$.
4. *(Continuous paths)*
   The function $t \mapsto B_t$ is continuous.

In particular: $B_t = B_t - B_0 \sim \mathcal{N}(0, t-0)$

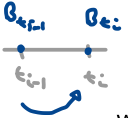
- $B_t \sim \mathcal{N}(0, t)$ with $\mathbb{E}[B_t] = 0$ and $\mathrm{Var}(B_t) = t$ for all $t > 0$

Simulating Brownian Motion

Consider simulating a Brownian motion on $[0, T]$:

- grid of discrete time points
  $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$

- by stationary and independent increments, with $B_{t_0} = B_0 = 0$,



$$B_{t_i} = B_{t_{i-1}} + \left(B_{t_i} - B_{t_{i-1}}\right) \stackrel{d}{=} B_{t_{i-1}} + X_i \quad (i = 1, 2, \ldots, n)$$

  where $X_i \sim \mathcal{N}(0, t_i - t_{i-1})$ independent of $B_{t_{i-1}}$
- recursive representation: $Z_1, \ldots, Z_n$ i.i.d. $\sim \mathcal{N}(0, 1)$

$$B_{t_i} = B_{t_{i-1}} + \sqrt{t_i - t_{i-1}} \cdot Z_i \quad (i = 1, 2, \ldots, n)$$

- generates the samples $B_{t_0}, B_{t_1}, B_{t_2}, \ldots, B_{t_n}$ on the discrete time grid
- typically equally spaced time points: $t_i = i \cdot \frac{T}{n}$ and hence $t_i - t_{i-1} = T/n$

Reminder: Normal Distribution 1/3

Definition 14.4

The **normal distribution (Gaussian distribution)** is a (continuous) probability distribution on $\mathbb{R}$ and is characterized by the *probability density function (PDF)*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \qquad (x \in \mathbb{R})$$

with mean parameter $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 \in \mathbb{R}_+$.

A random variable X is called **normally distributed (Gaussian distributed)** with parameters μ and σ^2 (notation $X \sim \mathcal{N}(\mu, \sigma^2)$) if its *cumulative distribution function (CDF)* is given by

$$F(x) = \mathbb{P}[X \leq x] = \int_{-\infty}^{x} f(z) dz = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\frac{(z-\mu)^2}{\sigma^2}} dz$$

for all $x \in \mathbb{R}$.

Reminder: Normal Distribution 2/3

Important properties and computational rules: $X \sim \mathcal{N}(\mu, \sigma^2)$

- mean and variance:

$$\mathbb{E}[X] = \mu \qquad \mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sigma^2$$

- computing probabilities

$$\mathbb{P}[a \leq X \leq b] = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}\,dx = F(b) - F(a)$$

- computing expected values (function $g : \mathbb{R} \to \mathbb{R}$):

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}\,dx$$

- linear transformation: $Z \sim \mathcal{N}(0, 1)$ **(standard normal)**

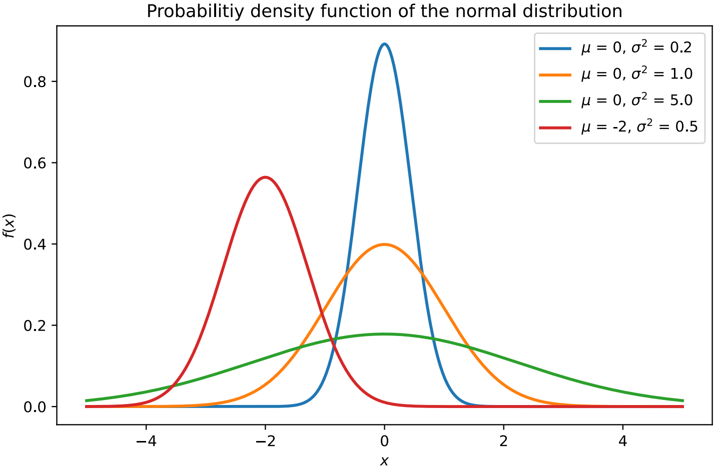
$$X = \mu + \sigma \cdot Z \sim \mathcal{N}(\mu, \sigma^2) \quad \Leftrightarrow \quad Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

- Moment-generating function (MGF):

$$M(u) = \mathbb{E}[e^{u \cdot X}] = e^{\mu \cdot u + \frac{1}{2}\sigma^2 u^2} \quad (u \in \mathbb{R})$$

- $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ independent:
  $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Reminder: Normal Distribution 3/3



Some History

The **normal distribution**, also known as the **Gaussian distribution**, is named after JOHANN CARL FRIEDRICH GAUSS (1777 – 1855), a German mathematician and physicist who made significant contributions to many fields in mathematics and sciences.

Source: Wikipedia