The Poisson process.

Let

$$I_1, I_2, \ldots, I_m, \ldots$$

be a sequence of independent identically distributed continuous random variables. Let $T_0 = 0$ and, for each positive integer m, let

$$T_m = \sum_{i=0}^{m} I_i.$$

Evidently,

$$0 = T_0 < T_1 < \cdots < T_m < \cdots.$$

For nonnegative integers m, n with $m \le n$ let

$$T_{m,n} = \sum_{m<i\le n} I_i;$$

note that

$$T_n = T_m + T_{m,n};$$

and that

$$T_{m,n} \text{ and } T_{m-n} \text{ have the same distribution, which is to say that } f_{T_{m,n}} = f_{T_{m-n}}.$$

Let $F : \mathbf{R} \to [0,1]$ be such that

$$F(t) = P(I_m \le t) \quad \text{whenever } t \in \mathbf{R} \text{ and } m = 1, 2, \ldots$$

and let

$$f = F'.$$

Let $f_1 = f$ and, for each $m = 2, 3, \ldots$ let

$$f_m = \underbrace{f * \cdots * f}_{m \text{ times}}.$$

For each $m = 1, 2, \ldots$ let

$$F_m(t) = \int_{-\infty}^{t} f_m(\tau)\, d\tau \quad \text{whenever } t \in \mathbf{R}.$$

Theorem. We have

(1) $$F_{T_m} = F_m, \quad \text{and} \quad f_{T_m} = f_m \quad m = 1, 2, \ldots.$$

Proof. This was shown earlier. $\square$

Theorem. Suppose k and $m_1, \ldots, m_k$ are positive integers,

$$m_k > \cdots > m_1,$$

$t_l, \ldots, t_k$ are positive real numbers and

$$t_k > \cdots > t_1.$$

Then

(2) $$f_{T_{m_k}, \ldots, T_{m_1}}(t_k, \ldots, t_1) = f_{m_k - m_{k-1}}(t_k - t_{k-1}) \cdots f_2(t_2 - t_1) f_1(t_1).$$

Proof. This is a good exercise in conditioning.

The key point is the following. Suppose $j = 1, \ldots, k$. For any $u \in \mathbf{R}$ we have

$$\begin{aligned} &P(T_{m_j} \le u | T_{m_{j-1}} = t_{j-1}, \ldots, T_{m_1} = t_1) \\ &\qquad = P(T_{m_{j-1}} + T_{m_{j-1},m_j} < u | T_{m_{j-1}} = t_{j-1}, \ldots, T_{m_1} = t_1) \\ &\qquad = P(t_{j-1} + T_{m_{j-1},m_j} < u | T_{m_{j-1}} = t_{j-1}, \ldots, T_{m_1} = t_1) \\ &\qquad = P(T_{m_{j-1},m_j} < u - t_{j-1}) \\ &\qquad = P(T_{m_j - m_{j-1}} < u - t_{j-1}) \end{aligned}$$

which implies that

$$f_{T_{m_j}|T_{m_{j-1}},\ldots,T_1}(t_j|t_{j-1}, \ldots, t_1) = f_{T_{m_j - m_{j-1}}}(t_j - t_{j-1}).$$

It follows that

$$\begin{aligned} &f_{T_{m_k},\ldots,T_{m_1}}(t_k, \ldots, t_1) \\ &\qquad = f_{T_{m_k}|T_{m_{k-1}},\ldots,T_{m_1}}(t_k|t_{k-1}, \ldots, t_1) \cdots f_{T_{m_2}|T_{m_1}}(t_2|t_1) f_{T_{m_1}}(t_1) \\ &\qquad = f_{T_{m_k - m_{k-1}}}(t_k - t_{k-1}) \cdots f_{T_{m_2 - m_1}}(t_2 - t_1) f_{T_1}(t_1). \end{aligned}$$

□

For each $t \in (0, \infty)$ we let

$$N_t$$

be the nonnegative integer random variable such that

$$\{N_t = n\} = \{T_n \le t < T_{n+1}\} \quad \text{for any nonnegative integer } n.$$

Theorem. For each $t \in (0, \infty)$ we have

$$P(N_t = n) = \int_0^t \left(\int_t^\infty f(t_{n+1} - t_n)\, dt_{n+1} \right) f_n(t_n)\, dt_n. \tag{3}$$

Proof. We have

$$\begin{aligned} P(N_t = n) &= P(T_n \le t, T_{n+1} > t) \\ &= \int \int_{t_n \le t < t_{n+1}} f_{T_{n+1},T_n}(t_{n+1}, t_n)\, dt_{n+1} dt_n. \end{aligned}$$

Now apply (3). □

Theorem. Suppose s and t are positive real numbers and m and n are nonnegative integers. Then

$$\begin{aligned} &P(N_{s+t} = m + n, N_s = m) = \\ &\quad \int_0^s \left(\int_s^{s+t} \left(\int_{t_{m+1}}^{s+t} \left(\int_{s+t}^\infty g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) dt_{m+n+1} \right) dt_{m+n} \right) dt_{m+1} \right) dt_m \end{aligned} \tag{4}$$

where we have set

$$g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) = f(t_{m+n+1} - t_{m+n}) f_{n-1}(t_{m+n} - t_{m+1}) f(t_{m+1} - t_m) f_m(t_m)$$

for $0 < t_m < t_{m+1} < t_{m_n} < t_{m+n+1}$.

Proof. Since

$$P(N_{s+t} = m+n, N_s = m) = P(T_m \le s < T_{m+1}, T_{m+n} \le s+t < T_{m+n+1})$$

the desired probability is the integral over

$$\{(t_{m+n+1}, t_{m+n}, t_{m+1}, t_m) : 0 < t_m \le s < t_{m+1}, t_{m+n} \le s+t < t_{m+n+1} < \infty\}$$

of the joint density

$$f_{T_{m+n+1}, T_{m+n}, T_{m+1}, T_m}(t_{m+n+1}, t_{m+n}, t_{m+1}, t_m).$$

Now apply (3). □

The Poisson process. Now suppose $0 < \lambda < \infty$ and

$$P(I_m > t) = e^{-\lambda t} \quad \text{whenever } 0 < t < \infty \text{ and } m = 1, 2, \ldots.$$

That is, I_m, $m = 1, 2, \ldots$ is exponentially distributed with parameter λ.

Theorem. For any $m = 1, 2, \ldots$ we have

(5) $$f_m(t) = \begin{cases} 0 & \text{if } t < 0, \\ \lambda^m e^{-\lambda t} \frac{t^{m-1}}{(m-1)!} & \text{if } t > 0. \end{cases}$$

Proof. We induct on n. (4) holds by definition if $m = 1$.

Suppose (5) holds for some positive integer k and $t > 0$ then

$$f_{k+1}(t) = f * f_k(t) = \int_0^t e^{-\lambda(t-\tau)} e^{-\lambda\tau} \frac{\tau^{m-1}}{(m-1)!}\, d\tau = e^{-\lambda t} \int_0^t \frac{\tau^{m-1}}{(m-1)!}\, d\tau = e^{-\lambda t} \int_0^t \frac{t^m}{m!}.$$

□

Theorem. For any $t \in (0, \infty)$ and any nonnegative integer n we have

(6) $$P(N_t = n) = e^{-\lambda} \frac{(\lambda t)^n}{n!}.$$

Remark. Thus N_t has the Poisson distribution with parameter λt.

Proof. Suppose n is a nonnegative integer. By (3) and (5)

$$P(N_t = n) = \int_0^t \left(\int_t^\infty e{-}\lambda(t_{n+1} - t_n)\, dt_{n+1} \right) \lambda^n e^{-\lambda t_n} \frac{t_n^{n-1}}{(n-1)!}\, dt_n = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

□

Theorem. Suppose $s, t \in (0, \infty)$ and m, n are nonnegative integers. Then

(7) $$P(N_{s+t} = m+n, N_s = m) = P(N_s = m) P(N_t = n).$$

Proof. This will follow from (5) and (4).

Suppose $n > 1$. If g is as in (4) we have from (5) that

$$g(t_{m+n+1}, t_{m+n}, t_{m+1}, t_m) = \lambda^{m+n+1} e^{-\lambda t_{m+n+1}} \frac{(t_{m+n} - t_{m+1})^{n-2}}{(n-2)!} \frac{t_m^{m-1}}{(m-1)!}$$

whenever $0 < t_m < t_{m+1} < t_{m+n+1} < t_{m+n+1} < \infty$. Then

$$\begin{aligned} &\int_0^s \left(\int_s^{s+t} \left(\int_{t_{m+1}}^{s+t} \left(\int_{s+t}^{\infty} g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) dt_{m+n+1} \right) dt_{m+n} \right) dt_{m+1} \right) dt_m \\ &= \lambda^{m+n+1} \left(\int_{s+t}^{\infty} e^{-\lambda t_{m+n+1}} \, dt_{m+n+1} \right) \\ &\qquad \left(\int_s^{s+t} \left(\int_{t_{m+1}}^{s+t} \frac{(t_{m+n} - t_{m+1})^{n-2}}{(n-2)!} \, dt_{m+n} \right) dt_{m+1} \right) \left(\int_0^s \frac{t_m^{m-1}}{(m-1)!} \, dt_m \right) \\ &= \lambda^{m+n+1} \frac{e^{-\lambda(s+t)}}{\lambda} \frac{t^n}{n!} \frac{s^m}{m!}. \end{aligned}$$

We leave it to the reader to use similar techniques to handle the case $n = 0$ or $n = 1$.

Corollary. Suppose s and t are positive real numbers. Then $N_{s+t} - N_s$ is independent of N_s and has the Poisson distribution with parameter λt.

Proof. We have

$$(1) \qquad P(N_{s+t} - N_s = n, N_s = m) = P(N_{s+t} = m + n, , N_s = m) = P(N_s = m)P(N_t = n)$$

for any nonnegative integers m and n. Summing over m in (1) we infer that $N_{s+t} - N_s$ has the same distribution as N_t. Substituting $P(N_t = n) = (N_{s+t} - N_s = n)$ in the right hand side of (1) we infer that $N_{s+t} - N_s$ is independent of N_s. $\square$

Remark. Suppose m is an integer not less than 2 and $0 < t_1 < t_2 < \cdots < t_m < \infty$. Applying the preceding Corollary repeatedly we infer that

$$N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_m} - N_{t_{m-1}}$$

are independent with Poisson distributions with parameters

$$\lambda t_1, \lambda(t_2 - t_1), \ldots, \lambda(t_m - t_{m-1}),$$

respectively.

We say the Poisson process has **independent increments**.