Lecture 9: 08/18/22

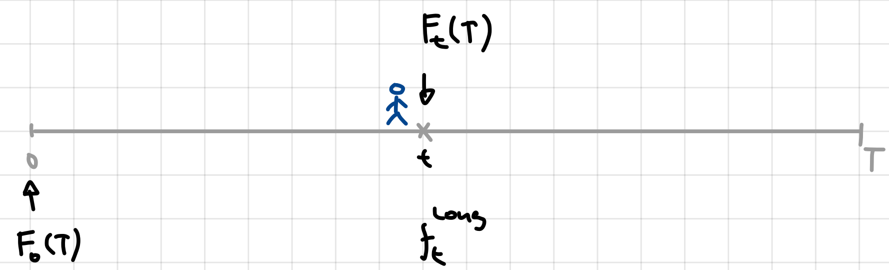
Proof of Lemma 5.8:

Claim: $$f_t^{long} = (F_t(T) - F_0(T))\, e^{-r(T-t)} \qquad t \in [0,T]$$

Check: $t=0$ $\quad f_0^{long} = (F_0(T) - F_0(T))\, e^{-rT} = 0$ ✓

$t=T$ $\quad f_T^{long} = F_T(T) - F_0(T) = S_T - F_0(T)$ ✓

Illustration:



Consider following (hypothetical) portfolio at time t:

- enter long position in forward contract w/ $F_0(T)$ ← has to have value (*) at t
- enter short position in forward contract w/ $F_t(T)$ ← value 0 at time t

Portfolio's payoff at time T:

$$(S_T - F_0(T)) + (F_t(T) - S_T) = F_t(T) - F_0(T)$$ known to us at time t

$\Rightarrow$ value of portfolio at time t: $(F_t(T) - F_0(T))\, e^{-r(T-t)}$ (*)

Proof of lemma 5.10:

Claim: It must hold $F_0(T)\, e^{r_f T} = S_0\, e^{rT}$ (interest rate parity)

$$\left(\Leftrightarrow \quad F_0(T) = S_0\, e^{(r-r_f)T} \right)$$

Idea: Suppose today ($t=0$) we have 1 EUR (one unit of foreign currency)

Q: What will be the value of that asset (foreign currency) in USD at time T (from today's perspective)?

Two investment possibilities (from today's perspective)

	(1)	(2)
$t=0$:	(i) sell 1 EUR today in spot market and receive $1 \cdot S_0$ USD (ii) invest $1 \cdot S_0$ USD on US bank account w/ risk-free rate r until T	(i) invest 1 EUR on European bank account w/ interest rate r_f until T (ii) enter short position in T-forward contract w/ forward price $F_0(T)$ USD per one EUR
$t=T$:	$1 \cdot S_0 \cdot e^{rT}$ USD on US bank account	$1 \cdot e^{r_f T}$ EUR on bank account sell in forward contract and receive $1 \cdot e^{r_f T} \cdot F_0(T)$ USD

$\Rightarrow$ $1 \cdot S_0 \cdot e^{rT}$ USD $= 1 \cdot e^{r_f T} \cdot F_0(T)$ USD, i.e., both possibilities must lead to the same USD amount at T; otherwise there is an arbitrage opportunity

$$\Rightarrow \quad F_0(T) = S_0\, e^{(r - r_f)T}$$

Arbitrage opportunities:

$$F_0(T) > S_0\, e^{(r-r_f)T}$$

$$\left(\Leftrightarrow \quad F_0(T)\, e^{r_f T} > S_0\, e^{rT}\right)$$

"buy asset in spot market and sell asset in forward market"

$$F_0(T) < S_0\, e^{(r-r_f)T}$$

$$\left(\Leftrightarrow \quad F_0(T)\, e^{r_f T} < S_0\, e^{rT}\right)$$

"short sell asset in spot market and buy back / close out short position in forward market"

	$F_0(T) > S_0\, e^{(r-r_f)T}$	$F_0(T) < S_0\, e^{(r-r_f)T}$
$t=0$:	borrow S_0 USD	borrow 1 EUR
	buy 1 EUR	sell 1 EUR and receive S_0 USD
	invest 1 EUR @ r_f	invest S_0 USD @ r
	sell $1 \cdot e^{r_f T}$ EUR forward @ $F_0(T)$	buy $1 \cdot e^{r_f T}$ EUR forward @ $F_0(T)$
$t=T$:	owe $S_0 e^{rT}$ USD	own $S_0 e^{rT}$ USD
	receive $F_0\, e^{r_f T}$ USD	pay $F_0(T) \cdot 1 \cdot e^{r_f T}$ USD
		pay back $1 \cdot e^{r_f T}$ EUR