# Math 4B: Differential Equations

## Lecture 01: Welcome to Math 4B!

- Differential Equations,
- Definitions, Classification,
- Direction Fields & More!

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# Today's Outline

- What is a differential equation? What are they good for?
- Understanding differential equations? What do they tell us?
- Direction fields. How can we understand a solution to a differential equation we can't solve?

# Differential Equations

A **differential equation** is simply an equation involving (wait for it!) derivatives.

## Examples & Applications

**1.** $y' = f(x)$

**2.** $y' = ky$

The rate of growth (or decay) is proportional to the current value. Common in…

- interest
- population growth
- radioactive decay

All have solution $y = y_0 e^{kt}$

# Examples & Applications (cont'd)

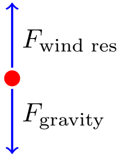
**3.** A falling object

$$F = ma \qquad \Longrightarrow \qquad m\frac{dv}{dt} = ma = F.$$

Is $F$ constant?

If there's wind resistance, the equation becomes

$$\begin{aligned} m\frac{dv}{dt} &= F_{\text{gravity}} - F_{\text{wind resistance}} \\ &= mg - \gamma v \qquad\qquad \gamma = \text{“gamma,” a constant} \end{aligned}$$



# Examples & Applications (cont'd)

**4.** Population with predation

Suppose we have the population of wild burritos in Isla Vista. Absent a natural predator, the population grows according to the model

$$\frac{dp}{dt} = rp,$$

where $p(t)$ is the number of burritos at time $t$ days. What is $r$?

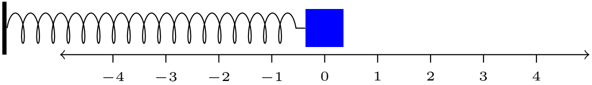
The feral UCSB students catch and eat the burritos at a rate of $k$ burritos/day. How will this impact the model?

$$\frac{dp}{dt} = rp - k.$$

Note that $r$ and $k$ are parameters based on our situation: the rate of growth of burritos and the voracity of the UCSB student.

# Examples & Applications (cont'd)

**5.** A mass on a spring:




Hooke's law says that the spring force is $F = kx$.

Thus (disregarding friction) $F = ma = -kx$ or

$$m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad mx'' = -kx$$

If friction is again proportional to velocity, then $F_{\text{friction}} = -\gamma v$ (is $\gamma > 0$ or $< 0$?). Thus

$$ma = F = F_{\text{spring}} + F_{\text{friction}} \qquad \Longrightarrow \qquad mx'' = -kx - \gamma x'.$$

# Examples & Applications (cont'd)

**6.** The price of a derivative (stock) is modeled by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

the Black-Scholes equation . Here

- $t$ = time
- $S$ = stock price
- $V(t, S)$ = value of European put or call
- $r$ = risk-free interest rate
- $\sigma$ = volatility of the stock

We won't study this kind of differential equation this quarter.

# Properties of Differential Equations

We generally classify differential equations using three different criteria:

**1.** Ordinary derivatives versus partial derivatives

- We'll study ODEs (Ordinary Differential Equations)
- PDEs (Partial Differential Equations) are more complicated

**2.** Order of an ODE

- The ***order*** of an ODE is the highest derivative involved.
- Examples of first-order ODEs:

$$y' = f(x) \qquad y' = ky \qquad m\frac{dv}{dt} = mg - \gamma v \qquad \frac{dp}{dt} = rp - k$$

- Examples of second-order ODEs:

$$mx'' = -kx - \gamma x' \qquad \sqrt{\frac{d^2y}{dx^2} + y\frac{dy}{dx}} = \left(\frac{dy}{dx}\right)^2 - y^3$$

# Properties (continued)

We generally classify differential equations using three different criteria:

**3.** Linear versus Non-linear

- By this we mean linear or non-linear in the dependent variable
- Examples of linear ODEs:

$$y' = f(x) \qquad y' = ky \qquad m\frac{dv}{dt} = mg - \gamma v \qquad mx'' = -kx - \gamma x'$$

- Examples of non-linear ODEs:

$$y' = y^2 \qquad y'' + yy' = t^2 \qquad \sqrt{\frac{d^2y}{dx^2} + y\frac{dy}{dx}} = \left(\frac{dy}{dx}\right)^2 - y^3$$

# Falling Objects & Burritos

## Remember from earlier:

$$m\frac{dv}{dt} = mg - \gamma v \qquad\qquad \frac{dp}{dt} = rp - k$$

A Falling Object | Population with Predation

These can both be written as

$$\frac{dy}{dt} = a(y - b)$$

where $a$ and $b$ are parameters:

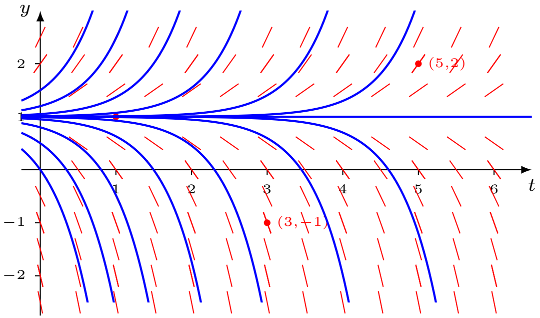
$$\frac{dv}{dt} = -\frac{\gamma}{m}\left(v - \frac{mg}{\gamma}\right) \qquad\qquad \frac{dp}{dt} = r\left(p - \frac{k}{r}\right)$$

# What can we say about $y$?

Consider

$$\frac{dy}{dt} = 2(y-1).$$

What can we say about $y$?
If we pick a random point, say $(t, y) = (1, 1)$, then we can tell the slope of the curve: $y' = 2(1-1) = 0$.

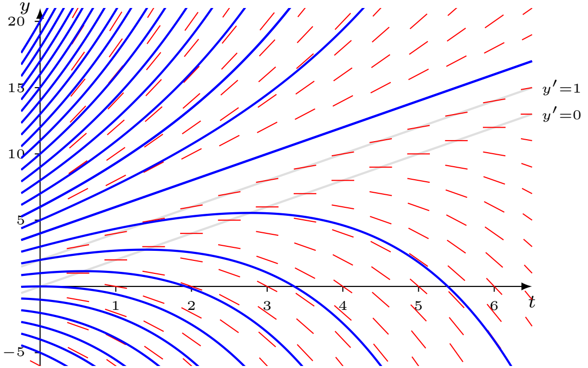


# A More Complicated Direction Field

Try to make a similar direction field for

$$\frac{dy}{dt} = \frac{1}{2}y - t$$

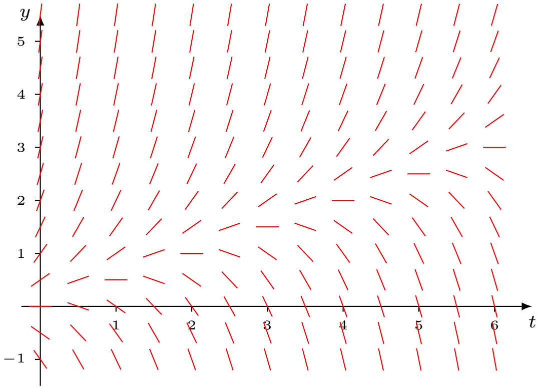
Notice: $y' = 0$ when $\frac{1}{2}y - t = 0$ or $y = 2t$.



# Examples of Solutions

Question: Which of the following ODEs matches this direction field?

(A) $y' = 2y + t$ (B) $y' = 2y - t$ (C) $y' = y - 2t^2$ (D) $y' = y + 2t^2$




Answer: $\boxed{\text{B}}$

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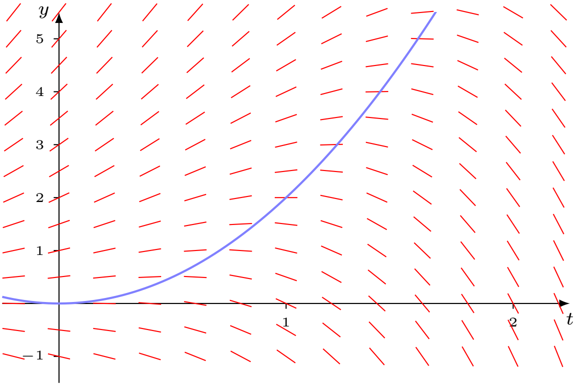


Answer: C