ASSIGNMENT 1

PSTAT 160B - SUMMER 2022

The first part of this assignment sheet contains exercises for the sections, which will not be turned in. The second part consists of homework problems which have to be turned in on the due date.

Instructions for the homework: Solve all of the homework problems, and submit them on GradeScope. Your reasoning has to be comprehensible and complete.

Assignment 1 Homework Problems

Problem 1.1. Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ be independent. Calculate

$$\mathbb{E}(\min\{X, Y\} | X < Y).$$

Let $M \doteq \min\{X, Y\}$ and

$$N \doteq \begin{cases} 1 & X < Y \\ 0 & Y \leq X \end{cases}.$$

From our proposition in class, we know that M and N are independent and that $M \sim \text{Exp}(\lambda + \mu)$. Thus,

$$\mathbb{E}(\min\{X, Y\} | X < Y) = \mathbb{E}(M | N = 1) = \mathbb{E}(M) = \frac{1}{\lambda + \mu}.$$

Problem 1.2. Let $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$. Using the fact that the moment generating function of X is given by

$$m(t) = \frac{\lambda}{\lambda - t}, \quad \text{for } t < \lambda,$$

calculate $\mathbb{E}(X^3)$.

Solution 1.1. We begin by calculating the third derivative of the moment generating function;

$$\frac{d^3}{dt^3} m(t) = \frac{6\lambda}{(\lambda - t)^4}, \quad t < \lambda,$$

from which we see that

$$\mathbb{E}(X^3) = \frac{d^3}{dt^3} m(t)\bigg|_{t=0} = \frac{6}{\lambda^3}.$$

Problem 1.3. A bird repeatedly leaves and returns to the same island; each trip away from the island is called a sojourn away from the island. We can model the length of the bird's sojourns as independent exponential random variables with a mean of 330 days (i.e., each time the bird leaves the island, the time that it takes to return is an Exp $\left(\frac{1}{330}\right)$ random variable).

(a) Over the course of its life, the bird makes 10 sojourns away from (and back to) the island. Let T denote the total time that the bird spends away from the island over the course of its life. What is the probability distribution of T?

(b) Calculate the probability that the total length of the 10 sojourns is at most 4000 days.[1]

Solution 1.2.

(a) Let X_i denote the length of the i^{th} sojourn away from the island. Then

$$T = \sum_{i=1}^{10} X_i,$$

where $X_i \overset{iid}{\sim} \text{Exp}\left(\frac{1}{300}\right)$, so $T \sim \text{Gamma}\left(10, \frac{1}{330}\right)$.

(b) This probability is given by

$$\mathbb{P}(T \leq 4000) = \int_0^{4000} \left(\frac{1}{330}\right)^{10} \frac{x^9}{9!} e^{-\frac{x}{330}} dx \approx 0.7680.$$

Problem 1.4. Recall that $\mathbb{N} \doteq \{1, \dots\}$ denotes the set of non-negative integers. Let X be a discrete random variable taking values in $\mathbb{N}$. Show that X has the memoryless property if and only if X follows a geometric distribution. [2]

Solution 1.3. Suppose that $X \sim \text{Geom}(p)$ for some $p \in (0, 1)$. Then, for each $n \in \mathbb{N}$,

$$\mathbb{P}(X > n) = \sum_{m=n+1}^{\infty} \mathbb{P}(X = m) = \sum_{m=n+1}^{\infty} (1-p)^{m-1} p = p(1-p)^n \sum_{m=n+1}^{\infty} (1-p)^{m-1}(1-p)^{-n}$$

$$= p(1-p)^n \sum_{m=n+1}^{\infty} (1-p)^{m-(n+1)} = p(1-p)^n \sum_{m=0}^{\infty} (1-p)^m = p(1-p)^n \frac{1}{1-(1-p)} = (1-p)^n.$$

This says that, for each $m, n \in \mathbb{N}$,

$$\mathbb{P}(X > n+m | X > n) = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > n)} = \frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m = \mathbb{P}(X > m),$$

so X has the memoryless property.

[1] Hint: you may use a numerical integral calculator for this.

[2] Hint: recall that X follows a geometric distribution with parameter $p \in (0, 1)$ if its probability mass function is given by $p(n) = (1-p)^{n-1} p$, for $n \in \mathbb{N}$.

Now suppose that X has the memoryless property. Then, for each $n, m \in \mathbb{N}$,

$$\mathbb{P}(X > n+m | X > n) = \mathbb{P}(X > m).$$

Additionally, using the definition of conditional probability,

$$\mathbb{P}(X > n+m | X > n) = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > n)}.$$

Combining these two identities, we see that

$$\mathbb{P}(X > n+m) = \mathbb{P}(X > n)\mathbb{P}(X > m). \tag{1}$$

Let $q \doteq \mathbb{P}(X > 1)$. We now argue inductively that for $n \in \mathbb{N}$, $\mathbb{P}(X > n) = q^n$. Note from (1) that

$$\mathbb{P}(X > 2) = \mathbb{P}(X > 1)\mathbb{P}(X > 1) = q^2.$$

Now assume that $\mathbb{P}(X > n) = q^n$, and, once more using (1), note that

$$\mathbb{P}(X > n+1) = \mathbb{P}(X > n)\mathbb{P}(X > 1) = q^n \cdot q = q^{n+1},$$

so we have seen that for each $n \in \mathbb{N}$, $\mathbb{P}(X > n) = q^n$. It follows that the pmf of X is given by

$$\mathbb{P}(X = n) = \mathbb{P}(X > n-1) - \mathbb{P}(X > n) = q^{n-1} - q^n = q^{n-1}(1-q), \quad n \in \mathbb{N}.$$

Since pmf's uniquely characterize discrete random variables, It follows that $X \sim \text{Geom}(p)$, where $p \doteq 1 - q = \mathbb{P}(X = 1)$.

Problem 1.5. A scientist is interested in two different types of particles; type A and type B. The time that it takes for a particle of type A to decay can be modeled as an exponential distribution with a mean of 75 minutes, and the time that it takes for a particle of type B to decay can be modeled as an exponential distribution with a mean of 50 minutes.

Suppose that a container holds 10 particles; 7 of type A, and 3 of type B. Assume that the rate at which each of the particles decays is independent of all of the other particles in the container.

(a) Calculate the probability that the first particle to decay is of type A.

(b) Calculate the probability of the following event; "it takes at least 30 minutes for any of the particles to decay, and the first particle that decays is of type B".

Solution 1.4.

(a) Let $\lambda_A \doteq \frac{1}{75}$ and $\lambda_B \doteq \frac{1}{50}$. Denote the lifetimes of the particles by $A_1, \ldots, A_7, B_1, \ldots, B_3$ and note that $\{A_i\}_{i=1}^7$ are iid $\text{Exp}(\lambda_A)$ random variables, which are independent of $\{B_i\}_{i=1}^3$, which themselves are iid $\text{Exp}(\lambda_B)$ random variables. If we let

$$M \doteq \min\{A_1, \ldots, A_7, B_1, \ldots, B_3\},$$

and

$$N = \sum_{i=1}^7 A 1_{\{M = A_i\}} + \sum_{j=1}^3 B 1_{\{M = B_j\}},$$

then $N = A$ if and only if the first particle to decay is of type A. Using our corollary regarding the minimum of independent exponentials, we see that, with

$$\lambda \doteq 7\lambda_A + 3\lambda_B = \frac{23}{150},$$

we have

$$\mathbb{P}(N = A) = \sum_{i=1}^7 \mathbb{P}(M = A_i) = \sum_{i=1}^7 \frac{\lambda_A}{\lambda} = \frac{14}{23}.$$

(b) With M and N as above, if we let E be the event "it takes at least 30 minutes for any of the particles to decay, and the first particle that decays is of type B," then

$$\mathbb{P}(E) = \mathbb{P}(M \geq 30, N = B) = e^{-30\lambda}\frac{3\lambda_B}{7\lambda_A + 3\lambda_B} \approx 0.0039.$$

Problem 1.6. Poisson processes can effectively model the arrival of shocks to a system (e.g., disruptions in a financial system, physical phenomena, surges in demand, etc.). Suppose that we model the arrival of shocks to a system as a Poisson process with a rate of $\lambda = 2$ shocks per hour.

(a) The system starts at time $t = 0$. Calculate the probability that exactly three shocks occur by time $t = 1$.
(b) The system experiences 7 shocks over the course of five hours. Given this, calculate the probability that exactly three of the shocks occurred in the first four hours.
(c) Calculate the probability that the system experiences exactly one shock between time $t = 0$ and time $t = 1$ and three shocks between time $t = 3$ and time $t = 6$.

Solution 1.5.

(a) Let N_t denote the number of shocks that have occurred by time t, so that $\{N_t\}$ is a Poisson process with rate $\lambda = 2$. The probability that exactly three shocks occur by time $t = 1$ is given by

$$\mathbb{P}(N_1 = 3) = e^{-2}\frac{2^3}{3!} \approx 0.18,$$

where we have used the fact that $N_1 \sim \text{Poisson}(2 \cdot 1)$.

(b) We have that

$$\mathbb{P}(N_4 = 3 | N_5 = 7) \overset{1}{=} \frac{\mathbb{P}(N_5 = 7, N_4 = 3)}{\mathbb{P}(N_5 = 7)} \overset{2}{=} \frac{\mathbb{P}(N_5 - N_4 = 4)\mathbb{P}(N_4 = 3)}{\mathbb{P}(N_5 = 7)} \overset{3}{=} \frac{\mathbb{P}(N_1 = 4)\mathbb{P}(N_4 = 3)}{\mathbb{P}(N_5 = 7)},$$

where $\overset{1}{=}$ used the definition of conditional probability, $\overset{2}{=}$ used the independent increments property, and $\overset{3}{=}$ used the stationary increments property. Using the fact that for each $t \geq 0$, $N_t \sim \text{Poisson}(\lambda t)$, it follows that

$$\mathbb{P}(N_4 = 3 | N_5 = 7) = e^{-\lambda}\frac{(\lambda)^4}{4!}e^{-4\lambda}\frac{(4\lambda)^3}{3!}\left(e^{-5\lambda}\frac{(5\lambda)^7}{7!}\right)^{-1}.$$

(c) Using the independent increments property and then the stationary increments property, we have that

$$\begin{aligned}\mathbb{P}(N_1 = 1, N_6 - N_3 = 3) &= \mathbb{P}(N_1 = 1)\mathbb{P}(N_6 - N_3 = 3) \\ &= \mathbb{P}(N_1 = 1)\mathbb{P}(N_3 = 3) = e^{-\lambda}\frac{(\lambda \cdot 1)^1}{1!}e^{-3\lambda}\frac{(\lambda \cdot 3)^3}{3!} \approx 0.024.\end{aligned}$$

Problem 1.7. Let $\{N_t\}$ be a Poisson process with rate $\lambda > 0$. Denote the associated sequence of inter-arrival times by $\{X_n\}$, and the associated sequence of arrival times by $\{S_n\}$.

(a) Calculate $\mathbb{E}(X_7 X_8)$.
(b) Calculate $\mathbb{E}(S_7 S_8)$.

(c) Recall that the correlation between two random variables X and Y is given by

$$\mathrm{Corr}(X,Y) \doteq \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

For $0 \leq s < t$, compute $\mathrm{Corr}(N_s, N_t)$.

Solution 1.6.

(a) Since the inter-arrival times of a Poisson process are independent, we have that

$$\mathbb{E}(X_7X_8) = \mathbb{E}(X_7)\mathbb{E}(X_8) = \left(\frac{1}{\lambda}\right)^2 = \lambda^{-2}.$$

(b) Note that

$$S_n = \sum_{i=1}^{n} X_i,$$

so

$$\mathbb{E}(S_7S_8) = \mathbb{E}\left[\left(\sum_{i=1}^{7} X_i\right)\left(\sum_{i=1}^{8} X_i\right)\right] = \sum_{i=1}^{7}\sum_{j=1}^{8}\mathbb{E}(X_iX_j).$$

Note that there are $7 \cdot 8 = 56$ terms in the sum above; in 7 of those terms we have $i = j$ and in 49 of those terms we have $i \neq j$. If $i = j$, then

$$\mathbb{E}(X_iX_j) = \mathbb{E}(X_i^2) = \frac{2}{\lambda^2},$$

and if $i \neq j$, then

$$\mathbb{E}(X_iX_j) = \mathbb{E}(X_i)\mathbb{E}(X_j) = \frac{1}{\lambda^2}.$$

Together, these observations tell us that

$$\mathbb{E}(S_7S_8) = 7 \cdot \frac{2}{\lambda^2} + 49 \cdot \frac{1}{\lambda^2} = \frac{63}{\lambda^2}.$$

Alternatively, one can note that $S_8 = S_7 + X_8$, which simplifies the calculation somewhat.

(c) Since $N_t \sim \mathrm{Poisson}(\lambda_t)$, we know that $\mathbb{E}(N_t) = \lambda t = \mathrm{Var}(N_t) = \lambda t$. Additionally, using the law of total expectation we have that

$$\mathbb{E}(N_sN_t) = \mathbb{E}\left(\mathbb{E}[N_sN_t|N_s]\right) = \mathbb{E}(N_s\mathbb{E}[N_t|N_s]). \tag{2}$$

Additionally,

$$\mathbb{E}[N_t|N_s] = \mathbb{E}[N_t - N_s + N_s|N_s] = \mathbb{E}[N_t - N_s|N_s] + \mathbb{E}(N_s|N_s) = \mathbb{E}[N_t - N_s] + N_s,$$

which shows that

$$\mathbb{E}[N_t|N_s] = \lambda(t-s) + N_s. \tag{3}$$

Combining (2) and (3), and using the fact that

$$\mathbb{E}(N_s^2) = \mathrm{Var}(N_s) + (\mathbb{E}(N_s))^2 = \lambda s + (\lambda s)^2,$$

we see that

$$\mathbb{E}(N_sN_t) = \mathbb{E}\left[N_s\left(\lambda(t-s) + N_s\right)\right] = \lambda(t-s)\mathbb{E}(N_s) + \mathbb{E}(N_s^2) = \lambda^2 st + \lambda s.$$

Therefore

$$\mathrm{Corr}(N_s, N_t) = \frac{\mathbb{E}(N_sN_t) - \mathbb{E}(N_s)\mathbb{E}(N_t)}{\sqrt{\mathrm{Var}(N_s)\mathrm{Var}(N_t)}} = \frac{\lambda s}{\sqrt{\lambda s \lambda t}} = \sqrt{\frac{s}{t}}$$