# Math 4B: Differential Equations

## Lecture 24: Homogeneous Linear Systems

- Homogeneous Linear Systems with Constant Coefficients,
- Direction Fields, Basic Solutions,
- Phase Planes & More!

# Uncoupled ODEs

**1.** Let's start with a basic example:

$$\begin{aligned}\frac{dx_1}{dt} &= r_1 x_1 \\ \frac{dx_2}{dt} &= \qquad\quad r_2 x_2\end{aligned} \qquad \text{or} \qquad \mathbf{x}'(t) = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \mathbf{x}(t)$$

Solution: This really amounts to two simple ODEs:

$$\frac{dx_1}{dt} = r_1 x_1 \qquad \text{and} \qquad \frac{dx_2}{dt} = r_2 x_2,$$

which have solutions $x_1(t) = a_1 e^{r_1 t}$ and $x_2(t) = a_2 e^{r_2 t}$. Thus the solution to our linear system is

$$\mathbf{x}(t) = \begin{pmatrix} a_1 e^{r_1 t} \\ a_2 e^{r_2 t} \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{r_1 t} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{r_2 t}.$$

# Again with Eigenvectors!

**2.** Now suppose that $P(t) = A$ is again constant, but now we have eigenvalues and eigenvectors:

$$A\boldsymbol{\xi}_1 = \lambda_1 \boldsymbol{\xi}_1 \qquad \text{and} \qquad A\boldsymbol{\xi}_2 = \lambda_2 \boldsymbol{\xi}_2$$

(where $\boldsymbol{\xi}_1 \neq \mathbf{0}$ and $\boldsymbol{\xi}_2 \neq \mathbf{0}$).

Solution: If we write $\mathbf{x}(t) = a_1(t)\boldsymbol{\xi}_1 + a_2(t)\boldsymbol{\xi}_2$, then our ODE becomes

$$a_1'(t)\boldsymbol{\xi}_1 + a_2'(t)\boldsymbol{\xi}_2 = \lambda_1 a_1(t)\boldsymbol{\xi}_1 + \lambda_2 a_2(t)\boldsymbol{\xi}_2.$$

This is essentially decoupled:

$$a_1'(t) = \lambda_1 a_1(t) \qquad \text{and} \qquad a_2'(t) = \lambda_2 a_2(t),$$

so we get solutions $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{\xi}_1 + c_2 e^{\lambda_2 t}\boldsymbol{\xi}_2$.

By our theorems, this encompasses all solutions.

# An Example

**3.** Find the general solution to

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}(t).$$

Solution: Here's the plan:

- We find the eigenvalues $\lambda_1$ and $\lambda_2$
- ...and the corresponding eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.
- Then the general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{\xi}_1 + c_2 e^{\lambda_2 t}\boldsymbol{\xi}_2$.

# Details

**Eigenvalues:** The eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ are the roots of

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \left(\lambda^2 - 4\lambda + 4\right) - 1 = \lambda^2 - 4\lambda + 3$$

Thus $\lambda_1 = 1$ and $\lambda_2 = 3$.

**Eigenvectors:**

$$\mathrm{Null}\,(A - 1I) = \mathrm{Null}\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix} = \mathrm{Null}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathrm{Null}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{Null}\,(A - 3I) = \mathrm{Null}\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} = \mathrm{Null}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \mathrm{Null}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

So $\boldsymbol{\xi}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. **Solution:** $\boxed{\mathbf{x}(t) = c_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

# Summary & Comments

The general solution of $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

Notice that this can be written as

$$\mathbf{x}(t) = e^t \left( c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \right),$$

so the direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will dominate if $c_2 \neq 0$.

# Sophisticated Version:

If we write $\mathbf{x}(0) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = S\mathbf{c}$, then $\mathbf{c} = S^{-1}\mathbf{x}_0$. Then

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda_1 t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= S e^{Dt} S^{-1} \mathbf{x}_0 \end{aligned}$$

where we "exponentiate" a diagonal matrix by simply exponentiating the diagonal entries:

$$e^{Dt} = e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

# Let's Graph This!

To graph solutions of

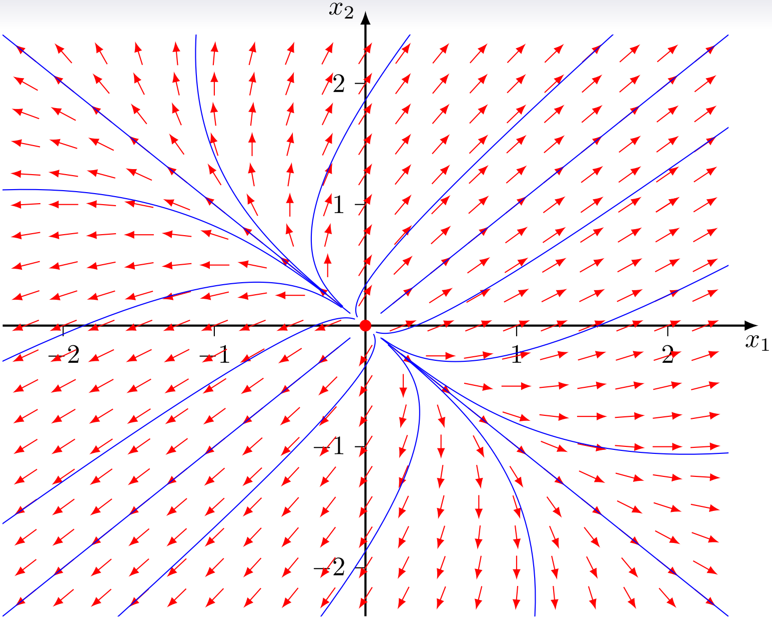
$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} \qquad \text{or} \qquad \begin{aligned} x_1' &= 2x_1 + 1x_2 \\ x_2' &= 1x_1 + 2x_2 \end{aligned}$$

we visualize things in the $x_1x_2$-plane (the ***phase plane***).

At a point $(x_1, x_2)$, we can find $x_1'$ and $x_2'$ and thus the "direction" of tangent vectors to solutions. We call this set of tangent vectors a ***direction field***.

On direction fields, we can sketch solution curves which we will call ***phase portraits***.

For example, when $(x_1, x_2) = (2, 1)$, we get $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.



$x_2$
$x_1$
2
1
−1
−2
−2
−1
1
2

# Another Example

**4.** Find the general solution to

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix} \mathbf{x}(t).$$

Solution: Remember the plan:

- We find the eigenvalues $\lambda_1$ and $\lambda_2$
- ...and the corresponding eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.
- Then the general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \boldsymbol{\xi}_1 + c_2 e^{\lambda_2 t} \boldsymbol{\xi}_2$.

# Details

**Eigenvalues:** The eigenvalues of $A = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix}$ are the roots of

$$0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -4 \\ 4 & -5-\lambda \end{vmatrix} = \left(\lambda^2 - 25\right) + 16 = \lambda^2 - 9.$$

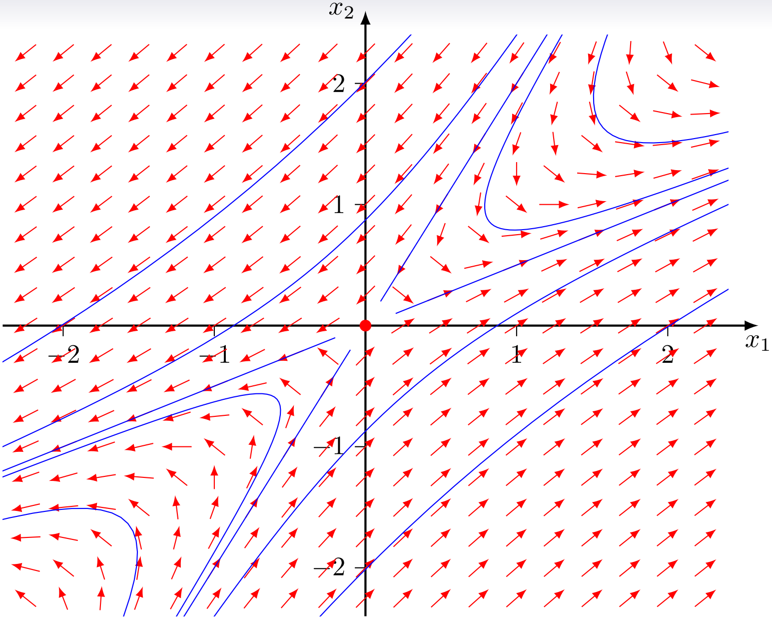
Thus $\lambda_1 = 3$ and $\lambda_2 = -3$.

**Eigenvectors:**

$$\operatorname{Null}(A - 3I) = \operatorname{Null}\begin{pmatrix} 5-3 & -4 \\ 4 & -5-3 \end{pmatrix} = \operatorname{Null}\begin{pmatrix} 2 & -4 \\ 4 & -8 \end{pmatrix} = \operatorname{Null}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

$$\operatorname{Null}(A - (-3)I) = \operatorname{Null}\begin{pmatrix} 5+3 & -4 \\ 4 & -5+3 \end{pmatrix} = \operatorname{Null}\begin{pmatrix} 8 & -4 \\ 4 & -2 \end{pmatrix} = \operatorname{Null}\begin{pmatrix} 1 & -0.5 \\ 0 & 0 \end{pmatrix}$$

So $\boldsymbol{\xi}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. **Solution:** $\boxed{\mathbf{x}(t) = c_1 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$.



$x_2$
$x_1$
2
1
−1
−2
−2
−1
1
2

# General Plan

**Linear Homogeneous Systems with Constant Coefficients**

Suppose $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A$ is an $n \times n$ matrix with $n$ linearly independent eigenvectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly repeated). Then the general solution of this ODE is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{\xi}_1 + c_2 e^{\lambda_2 t}\boldsymbol{\xi}_2 + \cdots + c_n e^{\lambda_n t}\boldsymbol{\xi}_n.$$

# Sophisticated Version

**Linear Homogeneous Systems with Constant Coefficients**

Suppose $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A$ is an $n \times n$ matrix with $n$ linearly independent eigenvectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly repeated). Then the general solution of this ODE is

$$\mathbf{x}(t) = Se^{Dt}S^{-1}\mathbf{x}(0),$$

where

$$S = \begin{pmatrix} | & | & & | \\ \boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 & \cdots & \boldsymbol{\xi}_n \\ | & | & & | \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix},$$

and thus $A = SDS^{-1}$.