Problem 2.1. Customers arrive at a coffee shop according to a Poisson process with a rate of $\lambda = 10$ customers per hour. Each customer orders exactly one beverage; fifty percent of the customers order a coffee, 30 percent order green tea, and 20 percent order herbal tea.

(a) Calculate the probability that between time $t = 5$ hours and time $t = 8$ hours that exactly 8 customers order green tea.

(b) Calculate the probability that between time $t = 5$ hours and time $t = 8$ hours that exactly 8 customers order green tea and that exactly 5 customers order herbal tea.

(c) At time $t = 4$ hours, 35 customers have visited the cafe. Given this, what is the probability that exactly 20 of the customers ordered coffee?

a) $P(N_8^g - N_5^g = 8) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$

$= e^{-(0.3\times 10)\cdot(3)} \frac{[(0.3\times 10)\cdot(3)]^8}{8!}$

$= 0.131756$

b). $P(N_8^G - N_5 = 8,\ N_8^H - N_5 = 5)$

$= P(N_3 = 8)\ P(N_3 = 5)$

$= \left(e^{-(0.3\times 10)\cdot(3)} \frac{[(0.3\times 10)\cdot(3)]^8}{8!}\right) \cdot \left(e^{-(0.2\times 10)\cdot(3)} \frac{[(0.2\times 10)\cdot(3)]^5}{5!}\right)$

c). $P(N_4^c - N \overset{\text{coffee}}{=} 20 \mid N_4^c - N_0 \overset{\text{total}}{=} 35)$

$= P(N_4 = 20 \mid N_4 = 35)$

$= \binom{35}{20} P_1^{20} P_2^{15}$ $\qquad \binom{k}{j} P_1^{j} P_2^{k-j}$

$= \binom{35}{20} (0.5)^{20} (0.5)^{15}$

$= \binom{35}{20} (0.5)^{35}$

Problem 2.2. Let $\{N_t^1\}$ be a Poisson process with rate $\lambda_1 = 3$, and let $\{N_t^2\}$ be a Poisson process with rate $\lambda_2 = 5$. Assume that $\{N_t^1\}$ and $\{N_t^2\}$ are independent.

(a) Compute $\mathbb{P}(N_{0.5}^1 = 1, N_{0.5}^2 = 2)$.
(b) Compute $\mathbb{P}(N_2^1 + N_2^2 = 15)$.

a). $$P(N_{0.5}^1 = 1, N_{0.5}^2 = 2) = P(N_{0.5}^1 = 1)\, P(N_{0.5}^2 = 2)$$
$$= e^{-\frac{1}{2}\cdot 3} \frac{(\frac{1}{2}\cdot 3)^1}{1!} \cdot e^{-\frac{1}{2}\cdot 5} \frac{(\frac{1}{2}\cdot 5)^2}{2!}$$

b). $$P(N_2^1 + N_2^2 = 15) = e^{-(3+5)\cdot 2} \frac{(8\cdot 2)^{15}}{15!}$$

Problem 2.3. Subatomic particles of type α enter a chamber according to a Poisson process with a rate of $\lambda_\alpha = 10$ per second, and subatomic particles of type β enter the chamber according to a Poisson process with a rate of $\lambda_\beta = 15$ per second. Additionally, the two types of particles arrive independently of one another. Let α_t denote the number of type α particles by time t, and let β_t denote the number of type β particles by time t

(a) The 10^{th} particle of type α arrives at time $t = 1.5$. Find the probability that the 20^{th} particle of type α arrives within 0.5 seconds after the 10^{th} particle.
(b) Find the probability that, in total, exactly 20 particles enter within the first second.
(c) Denote the total number of particles that have entered the chamber at time t by N_t. Compute $\mathbb{E}[N_3|\alpha_2 = 25]$.
(d) Each time a particle of type β enters the chamber, it has a 0.001% chance of being detected. How long, on average, will it take for the first particle of type β to be detected?

a). $P(N_2^\alpha \geq 20 \mid N_{1.5}^\alpha = 10) = 1 - P(N_{0.5}^\alpha \leq 10)$

$$= 1 - e^{-\lambda t} \sum_{r=0}^{k-1} \frac{(\lambda t)^r}{r!}$$

$$= 1 - e^{-10\cdot\frac{1}{2}} \sum_{r=0}^{9} \frac{(10\cdot\frac{1}{2})^9}{9!}$$

$$= 0.0318$$

$\alpha_s | \alpha_t = k$

$E[\alpha_2 | \alpha_3] \to$ Binomial $\Rightarrow \frac{s}{t}\cdot k$

b). $$P(N_1 = 20) = e^{-(\lambda_1+\lambda_2)\cdot 1} \frac{[(\lambda_1+\lambda_2)\cdot 1]^{20}}{20!} = 0.052$$

c). $$E[N_3 | \alpha_2 = 25] = E[\alpha_3 + \beta_3 | \alpha_2 = 25] = E[\alpha_3 | \alpha_2 = 25] + E[\beta_3]$$

$$= E[\alpha_3 - \alpha_2 + \alpha_2 | \alpha_2 = 25] + 3\cdot 15$$

$$= E[\alpha_2 | \alpha_2 = 25] + E[\alpha_1] + 3\cdot 15$$

$$= 25 + 10 + 45 = 80$$

d). $$\lambda_\beta^{\text{Detected}} = 0.00001 \times 15 = 0.000015$$

$$E[\lambda_\beta^{\text{Detected}} = 1] = \frac{1}{0.000015} = 6666.\bar{6} \text{ Second} \Rightarrow 1.85 \text{ hrs}$$

Problem 2.4. Customers arrive at a shop according to a Poisson process with rate $\lambda = 10$ per hour. The amount of money that each customer spends in the store can be modeled as an exponentially distributed random variable with a mean of 5 dollars. Additionally, we can assume that the amount of money that each customer spends is independent of the spending of the other customers.

(a) Compute the probability that the first two customers combined spend more than 12 dollars.[1]

(b) The shop opens at 8am and closes at 5pm. Compute the expected amount of money that customers will spend at the store over the course of the day.

(c) Compute the variance of the amount of money that customers will spend at the store over the course of the day.

(d) Compute the probability that, over the course of the day, every customer spends at least 2 dollars.

$\lambda = \frac{1}{5}$ $\quad x = 12$

a). $P(N_1 + N_2 \geq 12) \Rightarrow S \sim \text{Gamma}(2, \frac{1}{5}) = \int_{12}^{\infty} \frac{\beta^n}{(n-1)!} x^{n-1} e^{-\beta x}$

$= \int_{12}^{\infty} \frac{(\frac{1}{5})^2}{1!} x^1 e^{-\frac{1}{5}x} = 0.3084$

b). Let N_9 denote # of Customer Entered Shop.

$N_9 \sim \text{Poiss}(9 \cdot 10) \Rightarrow E[N_9] = 9 \cdot 10 = 90$

Let x denote # of A Customer Spent in a day

$E[x^{total}] \sim \text{Exp}(5) \Rightarrow E[X] = \frac{1}{1/5} = 5$

Amount money spend on a course of a day $= 90 \cdot 5 = 450$

c). $Var(X) = 10 \cdot \frac{2}{(\frac{1}{5})^2} \cdot 9 = 4500$

d). By thm 2.8 independence of M and N:

Let N_9 denote amount of people came by to shop.

$\Rightarrow P(\text{Min}(x_1 \cdots x_n) \geq 2,\ N_9 = j) = P(\text{Min}(x_1 \cdots x_n) \geq 2)\ P(N_9 = j)$

$\Downarrow$

people visited

$N_9 \sim \text{Poiss}(9 \cdot 10)$

$\Rightarrow \boxed{P(M > x, N = i) = \frac{\lambda_i}{\lambda} e^{-\lambda x}}$, $x \geq 0$, $i \in \{1, 2 \dots n\}$.

$\Rightarrow \because i \in \{1, 2, \dots n\}$

$\therefore \text{Min}(X_1 \cdots X_n) \sim \text{EXP}(\frac{n}{5})$

$$\Rightarrow P(\mathrm{Min}(X_1 \cdots X_n) \geq 2) = e^{-\frac{n}{5}\cdot 2}$$

$$P(N_9 = j) = \sum_{j=0}^{\infty} e^{-10\cdot 9} \frac{(9\cdot 10)^j}{j!}$$

$$\Rightarrow P(\mathrm{Min}(X_1 \cdots X_n) \geq 2) \cdot P(N_9 = j) = \left[\sum_{j=0}^{\infty} e^{-10\cdot 9} \frac{(10\cdot 9)^j}{j!}\right] \cdot \left[e^{-\frac{n}{5}\cdot 2}\right]$$

Problem 2.5. You throw darts at a square dartboard, denoted by

$$\square = \{(x, y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\},$$

whose sides all have a length of 2 feet; you throw darts according to a Poisson process with rate $\lambda = 1$ per minute, and each dart you throw hits the square dartboard at a uniformly chosen position. Additionally, all of your throws are independent of one another. Within the square dartboard, there is a circular target with a radius of 1 foot, denoted by

$$\circ = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Denote the total number of darts that you throw at the dartboard by time $t \geq 0$ by $N_t(\square)$.

(a) Let $N_t(\circ)$ denote the number of darts that land in the circular target by time $t \geq 0$. What is the probability mass function of $N_t(\circ)$?

(b) Compute $\mathbb{P}[N_5(\circ) = 4 | N_5(\square) = 6]$ and $\mathbb{P}[N_{20}(\circ) = 4 | N_{20}(\square) = 6]$.

(c) Compute $\mathbb{E}[N_t(\circ) | N_t(\square)]$.

(d) Using (3), compute $\mathbb{E}(N_t(\circ))$.

a). $P\left[\underbrace{N_t(\circ)}_{j} \mid \underbrace{N_t(\square)}_{k}\right] = \binom{k}{j} p^j (1-p)^{k-j} \Leftarrow \text{Binom}(k, p)$

$N_t(\square) \sim PP(t) \Rightarrow e^{-t} \frac{(t)^k}{k!}$

Joint Pmf $= f(\circ) \cdot f(\square) = \left[\binom{k}{j} p^k (1-p)^j\right] \cdot e^{-t} \cdot \frac{(t)^k}{k!}$

b). $P[N_5(\circ) = 4 \mid N_5(\square) = 6]$

$= \binom{6}{4} p^4 (1-p)^2$

$p = \frac{\text{Area}(\circ)}{\text{Area}(\square)} = \frac{\pi r^2}{2 \times 2} = \frac{3.14}{4} = 0.785$

$= 0.263$

$P[N_{20}(\circ) = 4 \mid N_{20}(\square) = 6] = 0.263$ $\because$ k, j Didn't change value.

c). $E[N_t(\circ) \mid N_t(\square)] = \lambda \mu t$ where $\lambda = 1$, $\mu = E[N_t(\circ)] = 0.785 \cdot n$

$\Rightarrow 0.785\, n \cdot t$

d). $E[N_t(\circ)] = E[E[N_t(\circ)] / N_t(\square)] = p \cdot E[N_t(\square)] = pt$

Problem 2.6. A company releases a new product; at first they receive complaints about the product very infrequently, but, as time progresses and more people realize how poorly the product functions. The number of complaints that the company receives can be modeled according a non-homogeneous Poisson process with intensity function $\lambda(t) = e^t$, where $t \geq 0$ denotes the time in days since the product was released.

(a) Find the probability mass function of the number of complaints the company has received within the first month of the product's release.

(b) Calculate expected number of complaints that the store will receive between the first and second month after the product's release.

a). $\Lambda(t) = \int_0^t \lambda(t) = \int_0^t \lambda(t)\,dt = e^t - 1$

$$P(x) = e^{(e^t-1)} \frac{[(e^t-1)\cdot(t)]^x}{x!}, \quad t \in \{28, 30, 31\}$$

x = # of Complicant reeived

b). $E(x) = \int_{30}^{60} e^t\,dt = 1.1420 \times 10^{26}$

Problem 2.7. Thunderstorms arrive according to a Poisson process with a rate of $\lambda = 1$ storm per week. The amount of rain that falls during each thunderstorm follows an Exponential distribution with a mean of 0.5 inches. Find the probability that, within four weeks, at least one rainstorm deposits more than one inch of rain.

X_i = Amount Rainfall. $\quad N_4$ = # of thunderstorm in 4 wks

$$P(\min(X_1 \cdots X_{N_4}) > 1,\ N_4 = k)$$

$$\Rightarrow \operatorname{Min}(X_1 \cdots X_{N_4}) \sim \operatorname{EXP}(2 \cdot k)$$

$$\Rightarrow P(\operatorname{Min}(X_1 \cdots X_{N_4}) > 1) = e^{-2k \cdot 1}$$

$$\Rightarrow P(N_4 = k) = e^{-1 \cdot 4} \frac{(1 \cdot 4)^k}{k!}$$

$$\Rightarrow P(\operatorname{Min}(X_1 \cdots X_{N_4}) > 1,\ N_4 = k) = \sum_{k=0}^{\infty} \underline{e^{-4} \cdot \frac{4^k}{k!} \cdot e^{-2k}}$$

$$= e^{-4} \sum_{k=0}^{\infty} \frac{(e^{-2} \cdot 4)^k}{k!}$$

$$= e^{-4} \cdot e^{(e^{-2} \cdot 4)}$$

Problem 2.8. Let $\{X_t\}$ be a CTMC on $\mathcal{S} = \{1, 2\}$ with transition function

$$P(t) \doteq \frac{1}{\lambda+\mu} \begin{pmatrix} \mu + \lambda e^{-(\lambda+\mu)t} & \lambda - \lambda e^{-(\lambda+\mu)t} \\ \mu - \mu e^{-(\lambda+\mu)t} & \lambda + \mu e^{-(\lambda+\mu)t} \end{pmatrix},$$

where $\lambda, \mu > 0$.

(a) Calculate $\mathbb{P}[X_{1.7} = 2 | X_{1.2} = 1]$.

(b) Calculate $\mathbb{P}[X_{2.1} = 1, X_{1.9} = 2 | X_{1.1} = 1]$.

a). $P[X_{1.7} = 2 | X_{1.2} = 1] = P(X_{1.7} - X_{1.5} = 2 | X_0 = 1)$

$= P_{1,2}(0.5)$ $\boxed{P_{x,y}(t) = P(X_t = y | X_0 = x)}$

b). $P[X_{2.1} = 1, X_{1.9} = 2 | X_{1.1} = 1]$

$= P(X_{2.1} = 1 | X_{1.9} = 2)\ P(X_{1.9} = 2 | X_{1.1} = 1)$

$= P(X_{0.2} = 1 | X_0 = 2)\ P(X_{0.8} = 2 | X_0 = 1)$

$= (P(0.2))_{2,1} \cdot (P(0.8))_{1,2}$

Problem 2.9. Let $\{X_t\}$ be a CTMC on $\mathcal{S} = \{1, 2\}$ with transition function $P(t)$. Suppose that

$$P(2) = \begin{pmatrix} 0.4 & 0.6 \\ 0.1 & 0.9, \end{pmatrix}$$

Do we have enough information to calculate $\mathbb{P}[X_4 = 2 | X_0 = 1]$? If so, calculate it. If not, explain what additional information would be needed.

Yes, we have Enough info to find Pr.

By definition, $P(X_{t+s} = y | X_s = x) = P(X_t = y | X_0 = x)$

Similarly, $P(X_4 = 2 | X_0 = 1] \Rightarrow P(X_{2+2} = 2 | X_0 = 1)$

$= P(X_4) = P(X_2) \cdot P(X_2) \Rightarrow [P(2) \cdot P(2)]_{1,2}$

$= \begin{bmatrix} 0.22 & 0.78 \\ 0.13 & 0.87 \end{bmatrix}$

$\Rightarrow P(X_4 = 2) =)\ 0.78$