#2.1

(a) $f_Y(y) = \frac{12}{5}\int_0^1 2x - x^2 - xy\,dx$

$= \frac{12}{5}\left[x^2 - \frac{1}{3}x^3 - \frac{1}{2}x^2y \Big|_0^1 \right]$

$= \frac{12}{5}\left[1 - \frac{1}{3} - \frac{1}{2}y \right]$

$= \frac{24}{15} - \frac{12}{10}y = \frac{2(-3y+4)}{5}$

$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{12}{5}x(2-x-y)}{\frac{2(-3y+4)}{5}} = \frac{\cancel{12}^{6}}{\cancel{5}}x(2-x-y)\cdot\frac{\cancel{5}}{\cancel{2}(-3y+4)}$

$= \frac{6x(2-x-y)}{-3y+4}$

(b) $P\left[X > \frac{1}{2} \,\middle|\, Y = \frac{2}{3}\right] = \int_{\frac{1}{2}}^{1} \frac{6x(\frac{4}{3}-x)}{2}\,dx + \int_1^\infty 0\,dx = \frac{5}{8}$

(c) $E\left[X \,\middle|\, Y = \frac{2}{3}\right] = \int_0^1 x \cdot \frac{6x(2-x-\frac{2}{3})}{4 - 3\cdot\frac{2}{3}}\,dx$

$= \int_0^1 4x^2 - 3x^3\,dx$

$= \frac{4}{3}x^3 - \frac{3}{4}x^4 \Big|_0^1$

$= \frac{7}{12}$

#2.2

(a) $$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad (\lambda>0)$$

$$P(A=x) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!}\cdot \binom{n}{x} p^{x}\cdot(1-p)^{n-x}$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{e^{-\lambda}}\cancel{\lambda^{x}}}{x!}\cdot\frac{n!}{x!(n-x)!}\cdot\cancel{p^{x}}\cdot(1-p)^{n-x}$$

$$= e^{-\lambda}\cdot\lambda^{x}\cdot p^{x}\cdot\sum_{n=0}^{\infty}\frac{\lambda^{n-x}}{x!}\cdot\frac{n!}{x!(n-x)!}\cdot(1-p)^{n-x}$$

$$= e^{-\lambda}\cdot(\lambda p)^{x}\sum_{n=x}^{\infty}\frac{\lambda^{n-x}\cdot(1-p)^{n-x}}{\cancel{n!}}\cdot\frac{\cancel{n!}}{x!(n-x)!}$$

$$= e^{-\lambda}(\lambda p)^{x}\sum_{n=x}^{\infty}\frac{[\lambda(1-p)]^{n-x}}{\cancel{x!}(n-x)!}$$

$$= \frac{e^{-\lambda}(\lambda p)^{x}}{x!}\underbrace{\sum_{i=0}^{\infty}\frac{[\lambda(1-p)]^{i}}{i!}}_{\to e^{\lambda(1-p)}}$$

$$= \frac{e^{-\lambda p}(\lambda p)^{x}}{x!}$$

$$= \frac{e^{-\frac{1}{2}\lambda}(\frac{1}{2}\lambda)^{x}}{x!}$$

(b) $$P(B|A) = \frac{P(A|B)\cdot P(B)}{P(A)} \quad (\text{independent})$$

$$= \frac{\frac{e^{-\frac{1}{2}\lambda}(\frac{1}{2}\lambda)^{x}}{x!}\cdot\frac{e^{-\lambda}\lambda^{x}}{x!}}{\frac{e^{-\lambda p}(\lambda p)^{x}}{x!}}$$

$$= \frac{e^{-\lambda(1-p)}\lambda^{n-x}(1-p)^{n-x}}{(n-x)!} = \frac{e^{-\lambda\frac{1}{2}}\lambda^{n-x}(\frac{1}{2})^{n-x}}{(n-x)!}$$

(c) $A \sim$ possion $(\frac{1}{2})$

$$Var(A) = \frac{\lambda}{2}$$

#2.3

(a)

$$f_X(x) = \begin{cases} \frac{1}{600} e^{-\frac{x}{600}} & x \geq 200 \\ 0 & x < 200 \end{cases}$$

$$E(p) = \int_{200}^{\infty} x \cdot \frac{1}{600} e^{-\frac{x}{600}} \, dx$$

$$= \frac{1}{600} \int_{200}^{\infty} x \cdot e^{-\frac{x}{600}} \, dx$$

$$= 800 e^{-\frac{1}{3}}$$

(b)

$$Var(p) = \int_{200}^{\infty} x^2 \cdot f_X(x) \, dx - [E(p)]^2$$

$$= \frac{1}{600} \int_{200}^{\infty} x^2 \cdot e^{-\frac{x}{600}} \, dx - (800 e^{-\frac{1}{3}})^2$$

$$= 1000000 e^{-\frac{1}{3}} - (800 e^{-\frac{1}{3}})^2$$

$$\approx 387944.35$$

$$SD \approx \sqrt{Var(p)}$$

$$\approx 622.85$$

#2.4

$i=0: \quad m_0 = (N-0)\cdot 0 = 0$

$i=1: \quad m_i = 1 + \frac{1}{2} m_{i-1} + \frac{1}{2} m_{i+1}$

$$= 1 + \tfrac{1}{2}[N-(i-1)](i-1) + \tfrac{1}{2}[N-(i+1)](i+1)$$

$$= 1 + \tfrac{1}{2}[N-i+1](i-1) + \tfrac{1}{2}[N-i-1](i+1)$$

$$= 1 + (\tfrac{1}{2}N - \tfrac{1}{2}i + \tfrac{1}{2})(i-1) + (\tfrac{1}{2}N - \tfrac{1}{2}i - \tfrac{1}{2})(i+1)$$

$$= \cancel{1} + \underline{\tfrac{1}{2}Ni} - \tfrac{1}{2}i^2 \cancel{+ \tfrac{1}{2}i} \cancel{- \tfrac{1}{2}N} \cancel{+ \tfrac{1}{2}i} \cancel{- \tfrac{1}{2}} + \underline{\tfrac{1}{2}Ni} - \cancel{\tfrac{1}{2}i^2} - \tfrac{1}{2} \cancel{+ \tfrac{1}{2}N} \cancel{- \tfrac{1}{2}i} \cancel{- \tfrac{1}{2}}$$

$$= Ni - i^2$$

$$= (N-i)\,i$$

$i=N: \quad M_N = (N-N)\cdot N = 0\cdot N = 0.$

which all satisify the system. so prove.

#2.5

(a) $f_X(x) = \int_0^1 1 \, dy = y\Big|_0^1 = 1$

$f_Y(y) = \int_0^1 1 \, dx = x\Big|_0^1 = 1$

(b) $0 < W < \sqrt{2}$

$0 < \frac{W}{\sqrt{2}} < 1$

$$F_U(u) = P(U \le u) = P\left(\frac{W}{\sqrt{2}} \le u\right) = P(W \le u \cdot \sqrt{2})$$

$$F_W(\sqrt{2}u) = \begin{cases} \int_{-\infty}^{\sqrt{2}u} 0 \, du \\ \int_{-\infty}^{0} 0 \, du + \int_0^{\sqrt{2}u} \frac{1}{\sqrt{2}} du \\ \int_{-\infty}^{0} 0 \, du + \int_0^1 \frac{1}{\sqrt{2}} du + 0 \end{cases} = \begin{cases} 0 & u < 1 \\ u & 0 \le u \le 1 \\ 1 & u > 1 \end{cases}$$

$$f_w = F'_W = \begin{cases} 0 & \text{o.w} \\ 1 & 0 \le u \le 1 \end{cases}$$

(c) Same step as previous.

$$F_V(\sqrt{2}v) = \begin{cases} 0 & v < 1 \\ v & 0 \le v \le 1 \\ 1 & u > 1 \end{cases} \qquad f_V = F'_V = \begin{cases} 0 & \text{o w} \\ 1 & 0 \le v \le 1 \end{cases}$$

(d) $U \sim$ uniform $(0,1)$ and $V \sim$ uniform $(0,1)$, (U, V) are dependent uniformly distributed on the diagonal of the unit square. Since (X, Y) is independent uniformly distributed on the unit square. So is NOT uniquely determine the joint distribution.

#2.6

(a)

$$F_{X_1 X_2}(X_1 X_2) = P(X_1 \le X,\ S_2 \le S)$$
$$= P(X_1 \le X,\ X_1 + X_2 \le S)$$
$$= P(X_1 \le X,\ X_2 \le S - X_1)$$
$$= \int_0^X \int_0^{S-X_1} f_{X_1 X_2}(X_1 X_2)\, dx_2\, dx_1$$
$$= \int_0^X \int_0^{S-X_1} \lambda^2 \cdot e^{-\lambda(X_1+X_2)}\, dx_2\, dx_1$$
$$= \lambda^2 \int_0^X \int_0^{S-X_1} e^{-\lambda(X_1+X_2)}\, dX_2\, dX_1$$
$$= 1 - e^{-\lambda X} - \lambda X_1 e^{-\lambda S}$$

(b)

$$f_{X_1 S_2}(X, S_2) = \frac{d}{dX_1}\left[\frac{d}{dS}\left(1 - e^{-\lambda X_1} - \lambda X_1 e^{-\lambda S}\right)\right]$$
$$= \frac{d}{dX_1}\left(\lambda X_1 e^{-\lambda S}\right)$$
$$= \lambda^2 e^{-\lambda S_2}$$

(c)

$$f_{S_2}(S_2) = \int_0^{S_2} \lambda^2 e^{-\lambda S}\, dx_1 = \lambda^2 e^{-\lambda S_2} X \Big|_0^{S_2} = \lambda^2 S_2 e^{-\lambda S_2}$$

$$f_{X_1|S}(X|S_2) = \frac{f_{X_1 S_2}(X_1 S_2)}{f_{S_2}(S_2)} = \frac{\cancel{\lambda^2 e^{-\lambda S}}}{\cancel{\lambda^2} S \cancel{e^{-\lambda S}}} = \frac{1}{S_2}$$

$$f_{X_1|S}(X|S_2) = \begin{cases} \frac{1}{S_2} & [0, S] \\ 0 & \text{o.w.} \end{cases}$$

(d)

$$E[X_1|S_2] = \int_0^S X_1 \frac{1}{S_2}\, dX_1 = \frac{1}{2} X_1^2 \cdot \frac{1}{S_2}\Big|_0^S = \frac{S_2}{2}$$

(e) since $S_2 = X_1 + X_2$ and X_1 & X_2 are same thing somehow.

So we can intuitive see the $\left.\begin{array}{l} S_2 = X_1 + X_2 \\ X_1 = X_2 \end{array}\right\} \Rightarrow X_1 = \frac{S_2}{2}$,

$E[X_1 | S_n] = \frac{S_n}{n}$.