Problem 1.1. Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ be independent. Calculate $\mathbb{E}(\min\{X, Y\}|X < Y)$.

$$E(\min\{X,Y\} \mid X<Y) = E(\min(\text{Exp}(\lambda), \text{Exp}(\mu)) \mid X<Y)$$

$$= E(\text{Exp}(\lambda, \mu)) = \frac{1}{\lambda+\mu}$$

Problem 1.2. Let $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$. Using the fact that the moment generating function of X is given by

$$m(t) = \frac{\lambda}{\lambda - t}, \qquad \text{for } t < \lambda,$$

calculate $\mathbb{E}(X^3)$.

$$E(X^3) = \frac{d^3}{dt^3} M_t(0) = \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)\right)\right)$$

$$= \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{1}{\lambda-t}\right)\lambda\right)\right)$$

$$= \frac{d}{dt}\left(\frac{d}{dt}\left(\left(\frac{-(0-1)}{(\lambda-t)^2}\right)\cdot\lambda\right)\right)$$

$$= \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\lambda}{(\lambda-t)^2}\right)\right)$$

$$= \frac{6\lambda}{(\lambda-t)^4} \Leftarrow \text{wolfram}$$

$$t = 0 \Rightarrow \frac{6\lambda}{\lambda^4} = \frac{6}{\lambda^3}$$

Problem 1.3. A bird repeatedly leaves and returns to the same island; each trip away from the island is called a sojourn away from the island. We can model the length of the bird's sojourns as independent exponential random variables with a mean of 330 days (i.e., each time the bird leaves the island, the time that it takes to return is an Exp $\left(\frac{1}{330}\right)$ random variable).

(a) Over the course of its life, the bird makes 10 sojourns away from (and back to) the island. Let T denote the total time that the bird spends away from the island over the course of its life. What is the probability distribution of T?

(b) Calculate the probability that the total length of the 10 sojourns is at most 4000 days.[1]

a) Sum of EXP $\sim$ Gamma (n, λ) = Gamma $(10, \frac{1}{330})$

$$= \frac{\left(\frac{1}{330}\right)^{10}}{9!} x^9 e^{-\frac{1}{330}x}, \quad x \geq 0$$

b) $$\int_0^{4000} \frac{\left(\frac{1}{330}\right)^{10}}{9!} x^9 e^{-\frac{1}{330}x}\, dx = 0.768$$

Problem 1.4. Recall that $\mathbb{N} \doteq \{1, \ldots\}$ denotes the set of non-negative integers. Let X be a discrete random variable taking values in $\mathbb{N}$. Show that X has the memoryless property if and only if X follows a geometric distribution. [2]

Prop 2.14: X has memoryless propety iff $X \sim \exp(\lambda)$ for some $\lambda > 0$.

which:

$$P(X > s+t \mid X > t) = P(X > s)$$

$\because$ And Geo dist is in family of Exp dist

$\therefore$ Geo $\Rightarrow$ $P(X \geq x) = P(1-P)^x$

Plug in $P(X \geq s+t \mid X > t) = \dfrac{P(X \geq s+t,\, X > t)}{P(X > t)}$

$$= \frac{P(X \geq s+t)}{P(X > t)} = \frac{(1-P)^{(s+t)}}{(1-P)^t} = (1-P)^s = P(X > s)$$

Geometric has memoryless Property

Geo $\rightarrow$ Memoryless CDF write out: $F(n) = 1 - q^n$

$P(X > 2 \mid X > 1) = P(X > 2)$

$\Downarrow$

Bayes $\dfrac{P(X > 2)}{P(X > 1)}$

$P(X > 2) = P(X > 1)^2$

$P(X > n) = (P(X > 1))^{n/m} = q^n$

$q = P(X > 1)$

Problem 1.5. A scientist is interested in two different types of particles; type A and type B. The time that it takes for a particle of type A to decay can be modeled as an exponential distribution with a mean of 75 minutes, and the time that it takes for a particle of type B to decay can be modeled as an exponential distribution with a mean of 50 minutes.

Suppose that a container holds 10 particles; 7 of type A, and 3 of type B. Assume that the rate at which each of the particles decays is independent of all of the other particles in the container.

(a) Calculate the probability that the first particle to decay is of type A.

(b) Calculate the probability of the following event; "it takes at least 30 minutes for any of the particles to decay, and the first particle that decays is of type B".

a) Collory 2.9 + $\frac{\lambda}{\lambda+\mu}$

Denote $A_i = \min\{A_1 \ldots A_7, B_1 \ldots B_3 \mid \bar{A} < \bar{B}\}$ is type A.

N = 1st to Deecay

$$P(N = A_i) = \sum_{i=1}^{7} \frac{\frac{1}{75}}{7 \cdot \frac{1}{75} + 3 \cdot \frac{1}{50}} = \sum_{i=1}^{7} \frac{\frac{1}{75}}{(23/150)} = \frac{14}{23}$$

b) Thm 2.8 $P(M > x, N = i) = \frac{\lambda_i}{\lambda} e^{-\lambda x}, \ x \geq 0, i \in (1, 2, \ldots n)$

Let $x = 30$ min, $\lambda = \lambda_A + \lambda_B$

$$P(\text{Decay} > 30, N = B_i) = \frac{3 \cdot \frac{1}{50}}{7 \cdot \frac{1}{75} + 3 \frac{1}{50}} e^{-30 \cdot \frac{1}{50}}$$

$$= 0.0038$$

Problem 1.6. Poisson processes can effectively model the arrival of shocks to a system (e.g., disruptions in a financial system, physical phenomena, surges in demand, etc.). Suppose that we model the arrival of shocks to a system as a Poisson process with a rate of $\lambda = 2$ shocks per hour.

(a) The system starts at time $t = 0$. Calculate the probability that exactly three shocks occur by time $t = 1$.

(b) The system experiences 7 shocks over the course of five hours. Given this, calculate the probability that exactly three of the shocks occurred in the first four hours.

(c) Calculate the probability that the system experiences exactly one shock between time $t = 0$ and time $t = 1$ and three shocks between time $t = 3$ and time $t = 6$.

a) $P(N_{t=1} = 3) = e^{-2\cdot 1} \frac{(2\cdot 1)^3}{3!} = \frac{4}{3e^2}$

b). $P(N_{t=4} = 3 \mid N_{t=5} - N_{t=0} = 7)$

$= \frac{P(N_{t=4}=3,\ N_{t=5}=7)}{P(N_{t=5}=7)} = \frac{P(N_{t=4}=3)\, P(N_{t=1}=4)}{P(N_{t=5}=7)}$

$= \frac{e^{-2\cdot 1}\frac{(2\cdot 1)^3}{3!} \quad e^{-2\cdot}\frac{(2\cdot\ )^4}{4!}}{e^{-2\cdot 5}\frac{(2\cdot 5)^7}{7!}} = 0.029$

c). $P(N_{t=1} - N_{t=0} = 1,\ N_{t=6} - N_{t=3} = 3)$

$= P(N_{t=1-0} = 1)\, P(N_{t=3} = 3)$

$= \left(e^{-2\cdot 1}\frac{(2\cdot 1)^1}{1}\right)\left(e^{-2\cdot 3}\frac{(2\cdot 3)^3}{3!}\right)$

Problem 1.7. Let $\{N_t\}$ be a Poisson process with rate $\lambda > 0$. Denote the associated sequence of inter-arrival times by $\{X_n\}$, and the associated sequence of arrival times by $\{S_n\}$.

(a) Calculate $\mathbb{E}(X_7 X_8)$.

(b) Calculate $\mathbb{E}(S_7 S_8)$.

(c) Recall that the correlation between two random variables X and Y is given by

$$\mathrm{Corr}(X,Y) \doteq \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

For $0 \le s < t$, compute $\mathrm{Corr}(N_s, N_t)$.

a) $X \Rightarrow$ individual $\Rightarrow E(X_7 X_8) = E(X_7) E(X_8) \Leftarrow \{X_n\} \overset{iid}{\sim} \mathrm{Exp}(\lambda)$

$\Rightarrow E(X_7) E(X_8) = \frac{1}{\lambda} \cdot \frac{1}{\lambda} = \frac{1}{\lambda^2}$

b) $S_7 = \{X_1 + \cdots + X_7\}$ $\quad S_8 = \{X_1 + \cdots + X_8\}$.

$= E(S_7) = E(X_1 + \cdots + X_7) \Rightarrow E(X_1) + E(X_2) \cdots + E(X_7)$

if $X_i = X_j \Rightarrow E(X_i X_j) = E(X_i^2) = \frac{2}{\lambda^2}$

$X_i \neq X_j \Rightarrow E(X_i X_j) = E(X_i) E(X_j) = \frac{1}{\lambda} \cdot \frac{1}{\lambda} = \frac{1}{\lambda^2}$

$E(S_7 S_8) = E[(X_1 + \cdots + X_7)(X_1 + \cdots X_8)] = \sum_{i=1}^{7} \sum_{j=1}^{8} E(X_i X_j)$

=> only 7 of 56 of $(X_i X_j)$ has $i = j$ $\{1{:}1, 2{:}2 \cdots 7{:}7\}$.

$\Rightarrow 7 \cdot \frac{2}{\lambda^2} + 49 \cdot \frac{1}{\lambda^2} = \frac{63}{\lambda^2}$

$E(S_7 S_8) = E(S_7(S_7 + X_8))$

$= E(S_7^2) + E(S_7 + X_8)$

$= E(S_7^2) + E(S_7) * E(X_8)$

c). $\mathrm{Corr}(N_s, N_t) = \dfrac{E(N_s N_t) - E(N_s)E(N_t)}{\sqrt{Var(N_s)\,Var(N_t)}}$

$E(N_s N_t) = E[N_s(\lambda(t-s) + N_s)] = \lambda(t-s)E(N_s) + E(N_s^2) = \lambda^2 st + \lambda s$

$\mathrm{Corr}(N_s, N_t) = \dfrac{E(N_s N_t) - E(N_s)E(N_t)}{\sqrt{Var(N_s)\,var(N_t)}} = \boxed{\dfrac{\lambda s}{\sqrt{\lambda s \lambda t}} = \sqrt{\dfrac{s}{t}}}$ ANS

Notes:

$N_s \sim \mathrm{Poisson}(\lambda s)$ / $N_t \sim \mathrm{Poisson}(\lambda_t)$

$Cov(N_s, N_t) = E(N_s N_t) - E(N_s)E(N_t)$

$= E(N_s N_t) - (\lambda_s)(\lambda_t) = \lambda^2 st + \lambda_s - \lambda^2 st = \lambda s$

Assume $s < t$ indep increment for sure

$E(N_s N_t) = E(N_s(N_t - N_s + N_s))$

$= E(N_s(N_t - N_s) + N_s^2)$

$= E[N_s(N_t - N_s)) + E(N_s^2)$

$E(N_s) = \lambda_s$

$Var(N_s) = \lambda_s$

$E(N_s^2) = \lambda_s t (\lambda s)^2$

$t: 0 - t_s$

$= E(N_s)E(N_t - N_s) + \lambda s + (\lambda_s)^2$

$= \lambda_s(\lambda_t - \lambda_s) + \lambda_s + (\lambda_s)^2$

$= \lambda_s \lambda_t + \lambda_s = \lambda^2 st + \lambda_s$

if $s < t$ $\quad Cov(N_s, N_t) = \lambda t$

$Cov(N_s, N_t) = \lambda \min\{s, t\}$

$$Corr(N_s, N_t) = \frac{\lambda \min\{s,t\}}{\sqrt{\lambda s}\sqrt{\lambda t}}$$

$$= \frac{\min\{s,t\}}{\sqrt{st}}$$

Common mistake on Exam

$$E(\min\{X,Y\} \mid X<Y)$$
$$= E(X \mid X<Y)$$
$$= E(X)$$

$X, Y \overset{iid}{\sim} Exp(\lambda)$

$M = \min\{X, Y\}$

$$N = \begin{cases} 1 & X = \min\{X,Y\} \\ 0 & Y = \min\{X,Y\} \end{cases}$$

$M \perp\!\!\!\perp N$

$E(M \mid N=1) = E(M) = \frac{1}{\lambda+\mu}$ $\qquad M \sim Exp(\lambda+\mu)$