ASSIGNMENT 3 - SOLUTIONS

PSTAT 160B - SPRING 2022
DUE DATE: ___

Instructions for the homework: Solve all of the homework problems, and submit them on GradeScope. Your reasoning has to be comprehensible and complete.

Homework Problems

Problem 3.1. Consider a continuous-time Markov chain $\{X_t\}$ on $\mathcal{S} = \{1, 2, 3, 4\}$ with generator matrix

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 2 & -5 & 2 \\ 1 & 0 & 2 & -3 \end{pmatrix}$$

(a) Write down the transition matrix of the associated embedded DTMC.
(b) If the chain is currently in state 3, how long, on average will it take before moving to a new state?
(c) If the chain is in state 3 at time $t = 0$, what is the probability that it remains there until at least time $t = 2$?
(d) If the chain is currently in state 3 and its next move is to state 4, how long, on average, would you expect the chain to have stayed in state 3 before making that jump?

Solution 3.1.

(a) The transition matrix of the associated embedded DTMC is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/5 & 2/5 & 0 & 2/5 \\ 1/3 & 0 & 2/3 & 0 \end{pmatrix}$$

(b) Denote the holding time of state 3 by τ_3. Then, $\tau_3 \sim \text{Exp}(5)$, so $\mathbb{E}(\tau_3) = 1/5$.
(c) Using the fact that $\tau_3 \sim \text{Exp}(5)$, we can see that $\mathbb{P}[\tau_3 \geq 2 | X_0 = 3] = e^{-5 \cdot 2} = e^{-10}$.
(d) We would like to calculate $\mathbb{E}[\tau_3 | X_{\tau_3} = 4]$. Recall, from our construction of CTMCs via exponential random variables, that

$$\mathbb{E}[\tau_3 | X_{\tau_3} = 4] = \mathbb{E}[M | V = 4],$$

where $E_1 \sim \text{Exp}(1), E_2 \sim \text{Exp}(2)$, and $E_4 \sim \text{Exp}(2)$ are independent and where

$$M \doteq \min\{E_1, E_2, E_4\},$$

and

$$V = \begin{cases} 1, & \min\{E_1, E_2, E_4\} = E_1 \\ 2, & \min\{E_1, E_2, E_4\} = E_2 \\ 4, & \min\{E_1, E_2, E_4\} = E_4. \end{cases}$$

Since M and V are independent, it follows that

$$\mathbb{E}[\tau_3 | X_{\tau_3} = 4] = \mathbb{E}[\tau_3] = \frac{1}{5}.$$

Problem 3.2. Customers arrive at a bank according to a Poisson process with a rate of 30 customers per hour. The bank has a single line for the customers, and has three tellers. Each time a customer reaches the front of the line, if there is a teller who is not currently servicing someone, the customer immediately begins being serviced by that teller. If all of the tellers are busy servicing other customers, the customer at the front of the line waits until one of the tellers finishes servicing their customer.

The servicing times of the tellers are independent of each other and of the number of customers in line, and are exponentially distributed with a mean servicing time of 5 minutes.

Finally, if a customer arrives and there are already four customers in line (in addition to the three that are being serviced), they do not join the queue and instead immediately leave the bank.

(a) Model this system as a CTMC by specifying its transition rate diagram and generator matrix.
(b) Suppose that the queue currently has four people in it and that each of the three tellers are busy servicing a customer. What is the probability that two new customers arrive and immediately depart (due to the fact that the queue is full) before any of the tellers finish servicing their customers?[1]

Solution 3.2.

(a) Consider the state space $\mathcal{S} \doteq \{(0,0),(1,0),(2,0),(3,0),(3,1),(3,2),(3,3),(3,4)\}$, where $(x, y) \in \mathcal{S}$ corresponds to x tellers servicing customers and y customers waiting in line. The transition diagram is given below.

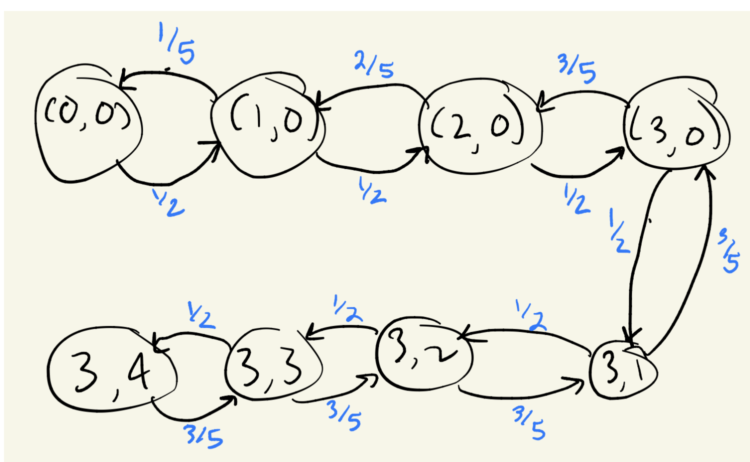

FIGURE 1. Here, the rates are all in minutes.

[1] **Hint:** if you use the law of total expectation and you recall the moment generating function of the exponential distribution, you can solve this problem without evaluating any integrals.

The generator matrix is given by

$$Q = \begin{pmatrix} -1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/5 & -7/10 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/5 & -9/10 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/5 & -11/10 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/5 & -11/10 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/5 & -11/10 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3/5 & -11/10 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/5 & -3/5 \end{pmatrix}$$

(b) Let X_1 denote the arrival time of the first new customer, and let X_2 denote the inter-arrival time of the second new customer. Let Y denote the time that it takes for one of the tellers to finish servicing a customer. Then, $X_1, X_2 \overset{iid}{\sim} \text{Exp}(1/2)$, and $Y \sim \text{Exp}(3/5)$. Note also that Y is independent of X_1 and X_2. Then, the probability that the two customers arrive before any of the tellers finish servicing their customer is given by

$$\begin{aligned} \mathbb{P}(X_1 + X_2 \leq Y) &= \mathbb{E}\left(\mathbb{P}[X_1 + X_2 \leq Y | X_1 + X_2]\right) \\ &= \mathbb{E}\left(e^{-\frac{3}{5}(X_1+X_2)}\right) \\ &= \mathbb{E}\left(e^{-\frac{3}{5}X_1}\right)\mathbb{E}\left(e^{-\frac{3}{5}X_2}\right) \\ &= \left(\frac{\frac{1}{2}}{\frac{1}{2} - (-\frac{3}{5})}\right)\left(\frac{\frac{1}{2}}{\frac{1}{2} - (-\frac{3}{5})}\right) \end{aligned}$$

Problem 3.3. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

(a) Diagonalize A.
(b) Compute $\exp(A)$.

Solution 3.3.

(a) The eigenpairs of A are given by $(\lambda_1, v_1), (\lambda_2, v_2)$, where $\lambda_1 = 3, \lambda_2 = -1$, and

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

It follows that the diagonalization of A is given by $A = UDU^{-1}$, where

$$U = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}, \quad D = \text{diag}(3, -1), \quad U^{-1} = \begin{pmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{pmatrix}.$$

(b) Using the diagonalization from part (a), we have

$$\exp(A) = U\exp(D)U^{-1} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^3 & 0 \\ 0 & e^{-1} \end{pmatrix}\begin{pmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e^{-1} + e^3) & e^3 - e^{-1} \\ \frac{1}{4}(e^3 - e^{-1}) & \frac{1}{2}(e^{-1} + e^3) \end{pmatrix}$$

Problem 3.4. Consider the matrix

$$A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Without diagonalizing A, compute $\exp(A)$. [2]

Solution 3.4. Begin by noting that A is a projection matrix, as

$$A^2 = AA = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} = A.$$

Inductively, we can see that if $A^k = A$ for some $k \in \mathbb{N}$, then

$$A^{k+1} = AA^k = AA = A,$$

so it follows that $A^k = A$ for all $k \in \mathbb{N}$. Thus,

$$\begin{aligned}\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} = I + \sum_{k=1}^{\infty} \frac{A}{k!} = I + \left(\sum_{k=1}^{\infty} \frac{1}{k!}\right) A &= I + (e-1)A \\ &= \begin{pmatrix} \frac{1}{3}(1+2e) & \frac{1}{3}(e-1) & \frac{1}{3}(e-1) \\ \frac{1}{3}(e-1) & \frac{1}{3}(1+2e) & \frac{1}{3}(1-e) \\ \frac{1}{3}(e-1) & \frac{1}{3}(1-e) & \frac{1}{3}(1+2e). \end{pmatrix}\end{aligned}$$

Problem 3.5. Consider a CTMC $\{X_t\}$ on $\mathcal{S} = \{1, 2, 3, 4\}$ with generator matrix

$$Q = \begin{pmatrix} -3 & 0 & 1 & 2 \\ 2 & -4 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 2 & 2 & 1 & -5 \end{pmatrix}.$$

Note that Q can be diagonalized as $Q = UDU^{-1}$, where

$$U = \begin{pmatrix} -3 & -2 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ -1 & -2 & -3 & 1 \\ 5 & 3 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 0 & -1/4 & 0 & 1/4 \\ -1/5 & 2/5 & 0 & -1/5 \\ 1/4 & -1/8 & -1/4 & 1/8 \\ 7/20 & 7/40 & 1/4 & 9/40 \end{pmatrix}$$

Let $P(t)$ denote the transition function of $\{X_t\}$. Does $P(t)$ converge as $t \to \infty$? If so, what is the limit?

Solution 3.5. The transition function is given by

$$P(t) = \exp(Qt) = U \exp(Dt) U^{-1}.$$

Note that, as $t \to \infty$,

$$\exp(Dt) = \mathrm{diag}(e^{-6t}, e^{-5t}, e^{-4t}, 1) \to \mathrm{diag}(0, 0, 0, 1),$$

so, as $t \to \infty$,

$$P(t) \to U\mathrm{diag}(0, 0, 0, 1)U^{-1} = \begin{pmatrix} 7/20 & 7/40 & 1/4 & 9/40 \\ 7/20 & 7/40 & 1/4 & 9/40 \\ 7/20 & 7/40 & 1/4 & 9/40 \\ 7/20 & 7/40 & 1/4 & 9/40 \end{pmatrix}$$

[2]**Hint:** use the definition of the matrix exponential, and calculate A^k for $k \geq 1$.

Problem 3.6. Your friend proposes a game of chance. You roll a fair 6-sided die repeatedly until it lands on 1, at which point the game ends. Each time the roll does not land on 1, you pay your friend a random amount. The amount that you pay after each roll (including the roll on which you roll 1) is exponentially distributed with a mean of 1 dollar, and is independent of what you have paid on all of the others rolls (and the number of rolls).

(a) How much do you expect to pay in total after playing the game once?
(b) After playing the game once, what is the probability that you pay more than 10 dollars?

Solution 3.6.

(a) let X_i denote the amount that you pay after the i-th roll, so that $X_i \sim \text{Exp}(1)$, and let R denote the number of rolls that it takes for the die to land on 1. Then the total amount paid is given by
$$T = \sum_{i=1}^{R} X_i.$$
Then, since $R \sim \text{Geom}(1/6)$, our result from class tells us that
$$T = \sum_{i=1}^{R} X_i \sim \text{Exp}(1 \cdot 1/6),$$
so $\mathbb{E}(T) = 6$.

(b) From (a), we know that $\mathbb{P}(T \geq 10) = e^{-\frac{10}{6}}$.

Problem 3.7. A squirrel in your yard is either in a tree or on the ground; the only time it switches positions is when a pedestrian walks by. If the squirrel is currently on the ground and a pedestrian walks by, there is a 30% chance that it stays on the ground and a 70% chance that it moves to the tree. If the squirrel is currently in a tree and a pedestrian walks by, there is a 90% chance that it stays in the tree and a 10% chance that it moves to the ground. Assume that the time that it takes for the squirrel to move from the ground to the tree (and from the tree to the ground) is negligible.

Suppose that pedestrians walk by your yard according to a Poisson process with a rate of 2 per hour, and let X_t denote the location of the squirrel at time t (i.e., in the tree or on the ground).

Is $\{X_t\}$ a CTMC? If so, what is its generator matrix?

Solution 3.7. If we let $\{N_t\}$ denote the number of customers who have walked by at time t, and let Y_n denote the position of the squirrel after the n-th person walks by. Note that $\{Y_n\}$ is a DTMC on $\mathcal{S} = \{G, T\}$ with transition matrix
$$P = \begin{pmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{pmatrix}.$$
The position of the squirrel at time t is given by $X_t \doteq Y_{N_t}$. Note that $\{X_t\}$ is a CTMC subordinated to the Poisson process $\{N_t\}$, so its generator matrix is given by
$$Q = 2(P - I) = 2\left(\begin{pmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} -1.4 & 1.4 \\ 0.2 & -0.2 \end{pmatrix}.$$