Given the second-order IVP:

$$y'' = p(t)y' + q(t)y + r(t),\ a \le t \le b,$$
$$y(a) = \alpha_1, y'(a) = \alpha_2$$

Let $p(t), q(t), r(t)$ be continuous functions on $[a, b]$.

Convert to a system of first-order equations and then use a theorem from class to prove a unique solution exists.

$Y_1 \doteq y \ , \quad Y_2 \doteq y'$

$\Rightarrow \quad Y_1' = y' = Y_2$

$\Rightarrow \quad Y_2' = y'' = P(t)y' + q(t)y + r(t)$

$\Rightarrow$ In 1st order equation

$$\begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} Y_2 \\ Y_2' \end{pmatrix} \Rightarrow \begin{pmatrix} Y_1(a) \\ Y_2(a) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$\Rightarrow \quad y' = P(t)y' + q(t)y + r(t)$

$y' = \dfrac{q(t)y + r(t)}{1 - P(t)}$ where $a \le t \le b$

By Fundamental thm of Calc.

There is $Y_3 < Y_3 + \varepsilon < Y_4$ when $Y_3 < Y_4$,

w/ $\dfrac{f(t, Y_4) - f(t, Y_3)}{Y_4 - Y_3} = \dfrac{\partial}{\partial y} f(t, \varepsilon) = P(t)\alpha_2 + q(t)\alpha_1 + r(t)$

So, $|f(t, Y_4) - f(t, Y_3)| = |Y_4 - Y_3|\,|P(t)\cdot\alpha_2 + q(t)\cdot\alpha_1 + r(t)|$

Since $P(t)$, $q(t)$, $r(t)$ are Continuous fn on $[a, b]$

There is some γ_1 γ_2 γ_3 exists S.T.

$P(t_1) \geq P(t)$, $q(t_2) \geq q(t)$, $r(t_3) \geq r(t)$, $a \le t \le b$

Let $k = \max(P(t_1), q(t_2), r(t_3))$

So $|P(t)\alpha_2 + q(t)\alpha_1 + r(t)|\,|y_4 - y_3| \leq k\,|y_4 - y_3|$

By Thm 5.4, there is unique solution