Math 174E
Lecture 15

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References

Hull
Chapter 14.2, 14.3, 14.4, 14.6, 14.7

Brownian motion with drift and diffusion coefficient

Also called **generalized Wiener process**:

Definition 14.5

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. For $x_0, \mu \in \mathbb{R}$ and $\sigma > 0$ the process $(X_t)_{t\geq 0}$ defined as

$$X_t = x_0 + \mu t + \sigma B_t \quad (t \geq 0)$$

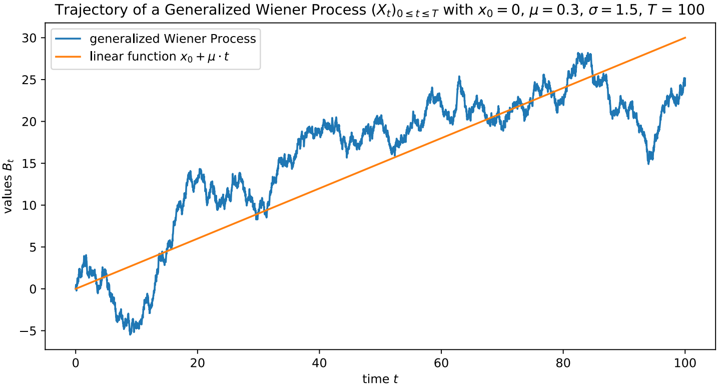
is called **Brownian motion with drift parameter** μ and **variance parameter** σ^2.

In particular:

- $X_t \sim \mathcal{N}(x_0 + \mu t, \sigma^2 t)$ for all $t \geq 0$.
- stationary and independent increments with $X_t - X_s \stackrel{d}{=} X_{t-s} - X_0 \sim \mathcal{N}(\mu(t-s), \sigma^2(t-s))$ and $X_t - X_s$ is independent of X_s (for $0 < s < t$)
- standard Brownian motion: $\mu = 0$, $\sigma^2 = 1$, $x_0 = 0$.

Illustration: Generalized Wiener Process

$X_t = 0.3 \cdot t + 1.5 \cdot B_t$ $\quad\quad$ $\mathbb{E}[X_t] = 0.3t$



Simulating Generalized Wiener Process

Consider simulating a generalized Wiener process on $[0, T]$ with parameters x_0, μ, σ:

- grid of discrete time points $0 = t_0 < t_1 < \ldots < t_n = T$
- by stationary and independent increments, with $X_{t_0} = x_0$,

  $$X_{t_i} = X_{t_{i-1}} + \mu \cdot (t_i - t_{i-1}) + \sigma \cdot \left(B_{t_i} - B_{t_{i-1}}\right) \quad (i = 1, 2, \ldots, n)$$

  where $B_{t_i} - B_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$ independent of $B_{t_{i-1}}$
- recursive representation: $Z_1, \ldots, Z_n$ i.i.d. $\sim \mathcal{N}(0, 1)$

  $$X_{t_i} = X_{t_{i-1}} + \mu \cdot (t_i - t_{i-1}) + \sigma \sqrt{t_i - t_{i-1}} \cdot Z_i \quad (i = 1, 2, \ldots, n)$$

- generates the samples $X_{t_0}, X_{t_1}, \ldots, X_{t_n}$ on the discrete time grid
- typically equally spaced time points: $t_i = i \cdot \frac{T}{n}$ and hence $t_i - t_{i-1} = T/n$

Differential Form

Generalized Wiener process:

$$X_t = x_0 + \mu t + \sigma B_t \qquad (t \geq 0)$$

- in a small time interval $[t, t + \Delta t]$ the change in the value of X is given by

$$\underbrace{X_{t+\Delta t} - X_t}_{\Delta X} = \mu \cdot \underbrace{(t + \Delta t - t)}_{\Delta t} + \sigma \underbrace{(B_{t+\Delta t} - B_t)}_{\Delta B \sim \mathcal{N}(0,\Delta t)}$$

$$\Leftrightarrow \qquad \Delta X = \mu \cdot \Delta t + \sigma \cdot \Delta B \sim \mathcal{N}(\mu \Delta t, \sigma^2 \Delta t)$$

- in the limit as $\Delta t \to 0$ we formally write

$$dX_t = \mu \, dt + \sigma \, dB_t \qquad \textbf{(differential form)}$$

- Interpretation:
  - $dX_t \approx$ changes of X_t in an *infinitesimal* small time interval dt
  - deterministic linear term μdt (drift term)
  - random perturbation $\sigma dB_t \sim \mathcal{N}(0, \sigma^2 dt)$ (diffusion term)

Model for the Stock Price Process 1/5

Dynamics of the risk-free asset:

- risk-free asset = bank account (or risk-free zero-coupon bond) earning the risk-free interest rate $r > 0$ (continuous compounding)
- r = risk-free asset's **rate of return** (per annum)
- S_t^0 = value of the risk-free asset at time $t \geq 0$ (in years)
- $S_0^0 = x$ initial value (capital) today at time $t = 0$

$f(x) = e^{rx}$
$f'(x) = r \cdot e^{rx}$
$= r \cdot f(x)$

$$S_t^0 = S_0^0 \cdot e^{r \cdot t} = x \cdot e^{r \cdot t} \qquad (t \geq 0)$$

- note that $(S_t^0)_{t \geq 0}$ as a function in time t is solving a **(linear) ordinary differential equation (ODE)**

$$\frac{dS_t^0}{dt} = r \cdot S_t^0 \quad \Leftrightarrow \quad \frac{dS_t^0}{S_t^0} = r\,dt \quad \Leftrightarrow \quad dS_t^0 = r \cdot S_t^0\,dt$$

with initial condition $S_0^0 = x$

Model for the Stock Price Process 2/5

Interpretation of the ODE:

$$\frac{dS_t^0}{dt} = r \cdot S_t^0 \quad \Leftrightarrow \quad \frac{dS_t^0}{S_t^0} = r\,dt \quad \Leftrightarrow \quad dS_t^0 = r \cdot S_t^0\,dt$$

- the ODE is deterministic (no uncertainty/randomness)
- dS_t^0 = change of the **risk-free asset's price** (= value) in an *infinitesimal* small time interval dt
- $dS_t^0 = r \cdot S_t^0\,dt$: changes in the price during dt are proportional to S_t^0 with rate of return $r \cdot dt$
- $\frac{dS_t^0}{S_t^0} = r\,dt$: risk-free asset's return during dt is $r \cdot dt$
- in a small time interval $[t, t + \Delta t]$ the change in the risk-free asset's value is given by

$$\underbrace{S_{t+\Delta t}^0 - S_t^0}_{\Delta S^0} = r \cdot S_t^0 \cdot \underbrace{(t + \Delta t - t)}_{\Delta t} \quad \Leftrightarrow \quad \Delta S^0 = r \cdot S^0 \cdot \Delta t$$

Model for the Stock Price Process 3/5

Dynamics of the risky asset:

- risky asset = stock
- μ = risky asset's (expected) return (per annum)
- S_t = price (value) of the risky stock at time $t \geq 0$
- $S_0 = s$ initial share price today at time $t = 0$
- we use a **stochastic (linear) differential equation (SDE)** to model the dynamics of $(S_t)_{t\geq 0}$

$$\frac{dS_t}{S_t} = \mu\, dt \underbrace{+ \sigma\, dB_t}_{\text{randomness (noise)}}$$

$$\Leftrightarrow \quad dS_t = \mu \cdot S_t\, dt + \sigma \cdot S_t\, dB_t = S_t \cdot \underbrace{(\mu\, dt + \sigma\, dB_t)}_{\text{random return}}$$

with initial condition $S_0 = s$

Model for the Stock Price Process 4/5

Interpretation of the SDE:

$$dS_t = S_t \cdot (\mu\, dt + \sigma\, dB_t) \quad \Leftrightarrow \quad \frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t$$

- the SDE is stochastic (uncertainty)
- dS_t = change of the **stock price** in an *infinitesimal* small time interval dt
- $\frac{dS_t}{S_t}$ = risky stock's return during dt
  - deterministic part $\mu\, dt$ (= expected rate of return)
  - random fluctuations $\sigma\, dB_t \sim \mathcal{N}(0, \sigma^2 dt)$
  - σ = volatility (standard deviation) of the annual returns (captures the unpredictable variability of the stock return)
- in a small time interval $[t, t + \Delta t]$ the change in the stock price is given by

$$\underbrace{S_{t+\Delta t} - S_t}_{\Delta S} = \mu \cdot S_t \cdot \underbrace{(t + \Delta t - t)}_{\Delta t} + \sigma \cdot S_t \cdot \underbrace{(B_{t+\Delta t} - B_t)}_{\Delta B \sim \mathcal{N}(0, \Delta t)}$$

$$\Leftrightarrow \qquad \Delta S = \mu \cdot S \cdot \Delta t + \sigma \cdot S \cdot \Delta B$$

Model for the Stock Price Process 5/5

Question:
What is the solution to the stochastic differential equation

$$dS_t = S_t\,\mu\,dt + S_t\,\sigma\,dB_t \quad ?$$

In other words, how does the stochastic process $(S_t)_{t\geq 0}$ modeling the stock price look like, i.e., what is

$$S_t = ?$$

To answer this question we will need **Itô's formula** ("Itô calculus").

Simulating the Stock Price Process

Idea: Discretization of the SDE **(Euler method)**

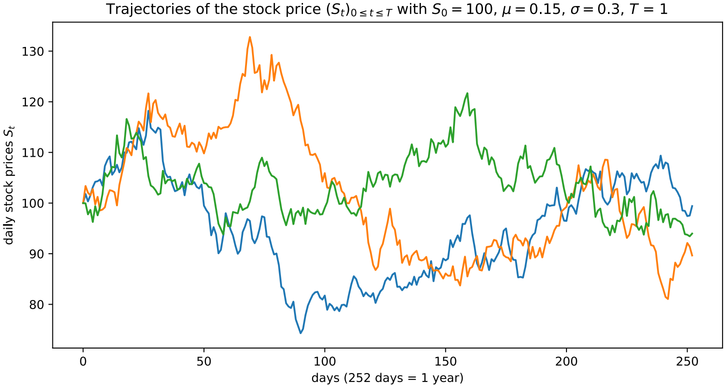
$$dS_t = \mu \cdot S_t \, dt \, + \, \sigma \cdot S_t \, dB_t, \quad S_0 = s$$

- grid of discrete time points $0 = t_0 < \ldots < t_n = T$
- set $S_0 = s$ and compute recursively for $i = 1, \ldots, n$:

$$S_{t_i} = S_{t_{i-1}} + S_{t_{i-1}} \cdot \mu \cdot (t_i - t_{i-1}) + S_{t_{i-1}} \cdot \sigma \cdot \sqrt{t_i - t_{i-1}} \cdot Z_i$$

  where $Z_1, \ldots, Z_n$ i.i.d. $\sim \mathcal{N}(0, 1)$
- generates the stock prices $S_{t_0}, S_{t_1}, \ldots, S_{t_n}$ on the discrete grid
- typically equally spaced time points: $t_i = i \cdot \frac{T}{n}$ and hence $t_i - t_{i-1} = T/n$

Illustration



Possible evolution of the stock price $(S_t)_{0 \le t \le 1}$ with volatility $\sigma = 30\%$ per annum and expected return $\mu = 15\%$ per annum. Initial price is $S_0 = 100$.

Itô Processes and Itô's Lemma 1/4

KIYOSI ITÔ (1915 – 2008) was a Japanese mathematician. He pioneered the theory of *stochastic integration* and *stochastic differential equations*, also known as the **Itô calculus**. Its basic concept is the **Itô integral**, and among the most important results is a change of variable formula known as **Itô's lemma**.

Source of pictures: Wikipedia.

Itô Processes and Itô's Lemma 2/4

Very informal definition:

Definition 14.6

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Let $(H_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ be stochastic processes (satisfying suitable technical assumptions).

A continuous-time stochastic process with dynamics (differential form)

$$dX_t = K_t\, dt + H_t\, dB_t, \quad X_0 = x$$

is called **Itô process**.

K_t: *drift* coefficient, H_t: *diffusion* coefficient.

Note: We can also have that $K_t = K(t, X_t)$ and $H_t = H(t, X_t)$ are (deterministic) functions of t and X_t.

Itô Processes and Itô's Lemma 3/4

Example 14.7

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

1. The standard Brownian motion $(B_t)_{t\geq 0}$ is an Itô process with $K_t = 0$ and $H_t = 1$.
2. The Brownian motion with drift parameter $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$ (generalized Wiener process from Definition 14.5) with dynamics

$$dX_t = \mu\, dt + \sigma\, dB_t$$

   is an Itô process with $K_t = \mu$ and $H_t = \sigma$.
3. The stock price process (from above) with dynamics

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t$$

   is an Itô process with $K_t = \mu S_t$ and $H_t = \sigma S_t$.

Excursion: Itô Processes and Itô's Lemma 4/4

Theorem 14.8 (Itô's Formula)

Let

$$dX_t = K_t\,dt + H_t\,dB_t$$

be an Itô process and let $g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. Then, the stochastic process $\boldsymbol{Z_t = g(t, X_t)}$ is again an Itô process with dynamics

$$\begin{aligned} dZ_t &= dg(t, X_t) \\ &= \underbrace{\left(\frac{\partial g}{\partial t}(t, X_t) + \frac{\partial g}{\partial x}(t, X_t) \cdot K_t + \frac{1}{2} \cdot \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot H_t^2\right) dt}_{\text{drift coefficient}} \\ &\qquad + \underbrace{\frac{\partial g}{\partial x}(t, X_t) \cdot H_t}_{\text{diffusion coefficient}}\, dB_t. \end{aligned}$$

Application: Computing the Stock price process

We are now able to compute the stock price process $(S_t)_{t\geq 0}$ with dynamics (SDE)

$$dS_t = \mu \cdot S_t \, dt + \sigma \cdot S_t \, dB_t, \quad S_0 = s \qquad (\star)$$

Example 14.9

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, $S_0 > 0, \mu \in \mathbb{R}$ and $\sigma > 0$.

Show that the process

$$S_t = S_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t} \qquad (t \geq 0)$$

solves the stochastic differential equation in $(\star)$.

Hint: Compute the dynamics of $\log(S_t)$ using the dynamics of S_t in $(\star)$ and Itô's formula.

See Lecture Notes.

Geometric Brownian Motion

Definition 14.10

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. For $S_0, \mu \in \mathbb{R}$ and $\sigma > 0$ the process $(S_t)_{t\geq 0}$ defined as

$$S_t = S_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t} \quad (t \geq 0)$$

is called **geometric Brownian motion**.

The geometric Brownian motion is used as a model for the **stock price** in the **Black-Scholes-Merton model**.

Log-normal distribution

Definition 14.11

A positive random variable X has a **log-normal** distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ if

$$\log(X) \sim \mathcal{N}(\mu, \sigma^2).$$

Notation: $X \sim \text{Lognormal}(\mu, \sigma^2)$.

Note:

- If $X \sim \text{Lognormal}(\mu, \sigma^2)$ then $X = e^Z$ where $Z \sim \mathcal{N}(\mu, \sigma^2)$
- log-normal distribution takes only values in $\mathbb{R}^+$ and has a known density function

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}} \quad (x > 0)$$

Properties of the geometric Brownian motion 1/2

Lemma 14.12

The stock price process $S_t = S_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$ is log-normally distributed, i.e.,

$$S_t \sim \text{Lognormal}\left(\log(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right).$$

Interpretation:

- $S_t =$ price at time $t \geq 0$ (years)
- $R_t = \log\left(\frac{S_t}{S_0}\right) = (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \sim \mathcal{N}((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ represents the **log-return** (= continuously compounded return) on $[0, t]$
- $\sigma =$ **volatility** (per annum) or standard deviation of the **log-returns**
- σ models the **uncertainty** (= riskiness) of the stock price

Properties of the geometric Brownian motion 2/2

Lemma 14.13

The expected value of the stock price process $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$ is given by

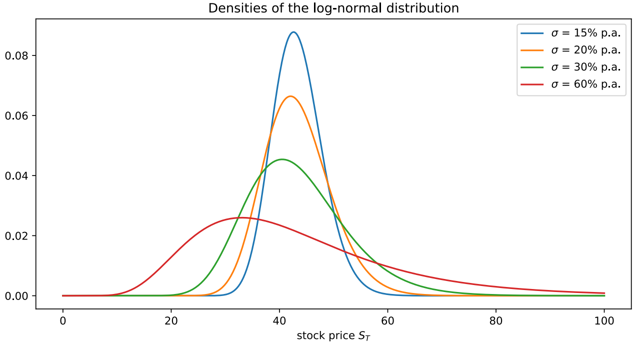
$$\mathbb{E}[S_t] = \mathbb{E}[S_0 \cdot e^{R_t}] = S_0 \cdot e^{\mu \cdot t} \quad (t \geq 0).$$

Proof: See Lecture Notes.

Interpretation:

- $\mu =$ **expected rate of return** (per annum)
  (depends on riskiness, higher than risk-free rate r)

Density of the stock price distribution



Density of the stock price S_T in $T = 1/2$ years (6 months) with initial price $S_0 = \$40$, expected return 16% p.a.; with different volatilities σ p.a. (note $\mathbb{E}[S_T] = S_0 e^{0.16 \cdot T} = \$43.33$)