PRACTICE MIDTERM SOLUTIONS

PSTAT 160A - FALL 2021
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These problems are to help you prepare for the midterm. Solutions will be posted, but please try to work through each of the problems before reading the solution.

The midterm itself will have fewer questions than this.

Problems

Problem 1. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of numbers in $(0,1)$ such that $np_n \to \lambda \in (0,1)$ as $n \to \infty$. For each $n \in \mathbb{N}$, let $X_n \sim \text{Binomial}(n, p_n)$, and let $X \sim \text{Poisson}(\lambda)$. Show that $X_n \xrightarrow{d} X$.

(a) Calculate the moment generating function (m.g.f.) of X_n.[1]
(b) Calculate the m.g.f. of X.
(c) Show that $X_n \xrightarrow{d} X$.[2]

Solution 1.

(a) The m.g.f. of X_n is given by, for $t \in \mathbb{R}$,

$$\begin{aligned} m_n(t) &\doteq \mathbb{E}[e^{tX_n}] \\ &= \sum_{k=0}^{n} \binom{n}{k} e^{tk} p_n^k (1-p_n)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (e^t p_n)^k (1-p_n)^{n-k} \\ &= (e^t p_n + 1 - p_n)^n \end{aligned}$$

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[1] *Hint:* use the binomial theorem, which says that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, to find a simple expression for this m.g.f.
[2] *Hint:* use the fact that if $a_n \to a$, then $\left(1 + \frac{a_n}{n}\right)^n \to e^a$.

(b) The m.g.f. of X is given by, for $t \in \mathbb{R}$,

$$\begin{aligned} m(t) &\doteq \mathbb{E}[e^{tX}] \\ &= \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda}\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t\lambda)^k}{k!} \\ &= e^{-\lambda} e^{e^t\lambda} \\ &= e^{(e^t-1)\lambda}. \end{aligned}$$

(c) Note that $(e^t - 1)np_n \to (e^t - 1)\lambda$, so

$$m_n(t) = \big(1 + (e^t - 1)p_n\big)^n = \left(1 + \frac{(e^t-1)np_n}{n}\right)^n \to e^{(e^t-1)\lambda} = m(t).$$

It then follows from the m.g.f. convergence theorem that $X_n \xrightarrow{d} X$.

Problem 2. Let $\{\sigma_n\}$ be a sequence such that $\sigma_n > 0$ for each n, and such that $\sigma_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, let $X_n \sim \mathcal{N}(0, \sigma_n)$, and let X be a random variable such that $\mathbb{P}(X = 0) = 1$.

(a) Show that $X_n \xrightarrow{\mathbb{P}} 0$.[3]

(b) Using the fact that the m.g.f. of the $\mathcal{N}(0, 1)$ distribution is given by, for $t \in \mathbb{R}$,

$$\tilde{m}(t) \doteq e^{\frac{t^2}{2}},$$

calculate the m.g.f. of X_n.

(c) Calculate the m.g.f. of X.

(d) Show that $X_n \xrightarrow{d} 0$.

Solution 2.

(a) For each $\epsilon > 0$, Chebyshev's inequality says that

$$\mathbb{P}(|X_n| > \epsilon) \leq \frac{\sigma_n^2}{\epsilon^2} \to 0,$$

since $\sigma_n \to 0$. Therefore $X_n \xrightarrow{\mathbb{P}} 0$.

(b) Note that if $Z \sim \mathcal{N}(0, 1)$, then $X_n \stackrel{d}{=} \sigma_n Z$, so

$$m_n(t) \doteq \mathbb{E}[e^{tX_n}] = \mathbb{E}[e^{t\sigma_n Z}] = e^{\frac{(t\sigma_n)^2}{2}}.$$

(c) The m.g.f. of X is given by

$$m(t) \doteq \mathbb{E}[e^{tX}] = e^{t\cdot 0} \cdot 1 = 1.$$

(d) For each $t \in \mathbb{R}$,

$$m_n(t) = e^{\frac{(t\sigma_n)^2}{2}} \to e^0 = 1 = m(t),$$

so it follows that $X_n \xrightarrow{d} X$.

Problem 3. The *first Borel-Cantelli lemma* says that if $\{A_n\}_{n=1}^{\infty}$ is a collection of events such that

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty,$$

[3] *Hint:* use Chebyshev's inequality.

then $\mathbb{P}$(infinitely many A_n occur) $= 0$. The *second Borel-Cantelli lemma* says that if $\{A_n\}_{n=1}^{\infty}$ is a collection of **independent** events such that

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty,$$

then $\mathbb{P}$(infinitely many A_n occur) $= 1$. In this problem you may use both of the Borel-Cantelli lemmas without proof.

(a) I have a keyboard with 50 keys on it. Each time Helen walks across the keyboard, she presses 5 of the keys. The 5 keys she presses each time are chosen uniformly at random, are independent of each other, and may be repeated. Additionally, the 5 keys she presses each time she walks across the keyboard are chosen independently of the 5 keys she presses each of the other times she walks across the keyboard.

Further, assume that she walks across the keyboard repeatedly, infinitely many times. Let A denote the event "Helen types "Meow!" on infinitely many of her trips across the keyboard." Show that $\mathbb{P}(A) = 1$.

(b) Now, suppose that each time Helen crosses the keyboard, I replace it with a new keyboard that has an additional key. I start out with a keyboard that has 51 keys, and after she walks across it the first time, I replace it with a keyboard that has 52 keys. In general, after she walks across the keyboard the n-th time, I replace it with a new keyboard that has $50 + (n + 1)$ keys.

Again, assume that she walks across the keyboard repeatedly, infinitely many times. Let B denote the event "Helen types "Meow!" on infinitely many of her trips across the keyboard". Show that $\mathbb{P}(B) = 0$.

Solution 3.

(a) Let A_n denote the event "Helen types"Meow!" on her n-th trip across the keyboard." The events $\{A_n\}$ are mutually independent, and A is simply the event that infinitely many of the A_n occur. Furthermore, for each $n \in \mathbb{N}$,

$$\mathbb{P}(A_n) = \left(\frac{1}{50}\right)^5,$$

so

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty.$$

Therefore, the second Borel-Cantelli lemma says that $\mathbb{P}(A) = 1$.

(b) Let B_n denote the event "Helen types "Meow!" on her n-th trip across the keyboard." Then, for each $n \in \mathbb{N}$,

$$\mathbb{P}(B_n) = \left(\frac{1}{50+n}\right)^5 < \left(\frac{1}{n}\right)^5 = \frac{1}{n^5},$$

so

$$\sum_{n=1}^{\infty} \mathbb{P}(B_n) < \sum_{n=1}^{\infty} \frac{1}{n^5} < \infty.$$

Note that B is simply the event that infinitely many of the B_n occur, so it follows from the first Borel-Cantelli lemma that $\mathbb{P}(B) = 0$. That is, Helen will type "Meow!" at most a finite number of times.

Problem 4. Let $X \sim \text{Unif}(1, 2)$ and suppose that when $X = x$, $Y \sim \text{Exp}(x)$.

(a) Write down the joint probability density function of (X, Y)

(b) Calculate explicitly $\mathbb{E}[Y|X=x]$, for $x \in [1,2]$.
(c) Using part (b), calculate $\mathbb{E}[Y]$.

Solution 4.

(a) The joint pdf of (X,Y) is given by
$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = xe^{-xy}, \quad x \in [1,2], y \geq 0.$$
(b) We have, for $x \in [1,2]$,
$$\mathbb{E}[Y|X=x] = \int_0^\infty y f_{Y|X}(y|x)dy = \int_0^\infty yxe^{-xy}dy = \frac{1}{x}.$$
(c) Part (b) tells us that $\mathbb{E}[Y|X] = \frac{1}{X}$, so we can calculate
$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}\left[\frac{1}{X}\right] = \int_1^2 \frac{1}{x}dx = \log(2).$$

Problem 5. Consider jointly continuous random variables X and Y with joint probability density function
$$f_{X,Y}(x,y) \doteq \begin{cases} cx^2, & 0 \leq y \leq 1-x^2, x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$
(a) Find the constant c.
(b) Determine the marginal probability density functions of X and Y.
(c) Determine whether X and Y are independent.
(d) Calculate $\mathbb{E}[Y|X=x]$.

Solution 5.

(1) We have
$$c \doteq \left(\int_0^1 \int_0^{1-x^2} x^2 dydx\right)^{-1} = \frac{15}{2}.$$
(2) The marginal density of X is given by, for $x \in [0,1]$,
$$f_X(x) = \frac{15}{2}\int_0^{1-x^2} x^2 dy = \frac{15}{2}(x^2 - x^4),$$
and the marginal density of Y, for $y \in [0,1]$, is
$$f_Y(y) = \frac{15}{2}\int_0^{\sqrt{1-y}} x^2 dx = \frac{5}{2}(1-y)^{\frac{3}{2}}.$$
(3) Note that, in general,
$$f_{X,Y}(x,y) \neq f_X(x) f_Y(y),$$
so X and Y are not independent.
(4) The density of Y given $X=x$ is, for $0 \leq y \leq 1-x^2$ and $x \in [0,1]$,
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x^2}{x^2-x^4} = \frac{1}{1-x^2},$$
so
$$\mathbb{E}[Y|X=x] = \int_0^{1-x^2} y\frac{1}{1-x^2}dy = \frac{1-x^2}{2}.$$

Problem 6. Consider jointly continuous random variables X and Y with joint probability density function

$$f_{X,Y}(x,y) \doteq \begin{cases} cx^2y & 0 \le x \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant c.
(b) Compute $\mathrm{Cov}(X,Y)$.

Solution 6.

(a) Note that

$$\int_0^1 \int_0^y x^2 y dx dy = \frac{1}{15},$$

so $c = 15$.

(b) The marginal density of X is

$$f_X(x) = \int_x^1 f_{X,Y}(x,y)dy = 15x^2 \int_x^1 y dy = \frac{15}{2}(x^2 - x^4), \quad x \in [0,1],$$

and the marginal density of Y is

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)dy = 5y^4, \quad y \in [0,1].$$

Thus

$$\mathbb{E}[X] = \frac{5}{8}, \quad \mathbb{E}[Y] = \frac{5}{16},$$

and

$$\mathbb{E}[XY] = \int_0^1 \int_0^y xyf_{X,Y}(x,y)dxdy = 15 \int_0^1 \int_0^y x^3y^2 dxdy = \frac{15}{28},$$

so

$$\mathrm{Cov}(X,Y) = \frac{15}{28} - \frac{5}{8} \cdot \frac{5}{16} = \frac{305}{896}.$$

Problem 7. Let $\mu, \lambda > 0$. You flip a fair two-sided coin. If it lands on heads, then $X \sim \text{Poisson}(\lambda)$. If it lands on tails, then $X \sim \text{Poisson}(\mu)$. Calculate $\mathbb{E}[X]$.

Solution 7. Let

$$Y = \begin{cases} 1, & \text{coin lands on heads} \\ 0, & \text{coin lands on tails,} \end{cases}$$

so that $Y \sim \text{Bernoulli}(0.5)$. The conditional pmf of X given $Y = y$ is

$$p_{X|Y}(x|y) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & y = 1 \\ \frac{e^{-\mu}\mu^x}{x!}, & y = 0, \end{cases}$$

so

$$\mathbb{E}[X|Y=1] = \lambda, \quad \mathbb{E}[X|Y=0] = \mu.$$

Therefore, the law of total expectation says that

$$\mathbb{E}[X] = \mathbb{E}[X|Y=1]\mathbb{P}(Y=1) + \mathbb{E}[X|Y=0]\mathbb{P}(Y=0) = \frac{\lambda + \mu}{2}.$$

Problem 8. Let $X \sim \mathcal{N}(0,1)$ and recall that the moment generating function of the $\mathcal{N}(0,1)$ distribution is given by, for $t \in \mathbb{R}$, $m(t) \doteq e^{\frac{t^2}{2}}$. Using this, calculate $\mathbb{E}[X^n]$ for $n \in \{1,2,3,4\}$.

Solution 8. We have

$$\begin{cases} m^{(1)}(t) \doteq \frac{d}{dt} m(t) = te^{\frac{t^2}{2}}, \\ m^{(2)}(t) \doteq \frac{d}{dt} m^{(1)}(t) = e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}}, \\ m^{(3)}(t) \doteq \frac{d}{dt} m^{(2)}(t) = 3te^{\frac{t^2}{2}} + t^3 e^{\frac{t^2}{2}}, \\ m^{(4)}(t) \doteq \frac{d}{dt} m^{(3)}(t) = 3e^{\frac{t^2}{2}} + 6t^2 e^{\frac{t^2}{2}} + t^4 e^{\frac{t^2}{2}}, \end{cases}$$

so

$$\begin{cases} \mathbb{E}[X] = m^{(1)}(0) = 0, \\ \mathbb{E}[X^2] = m^{(2)}(0) = 1, \\ \mathbb{E}[X^3] = m^{(3)}(0) = 0, \\ \mathbb{E}[X^4] = m^{(4)}(0) = 3. \end{cases}$$

Problem 9. This problem deals with the memorylessness property of exponentially distributed random variables. Let $X \sim \text{Exp}(\lambda)$.

(a) Show that for all $s, t > 0$, $\mathbb{P}[X > t + s | X > t] = \mathbb{P}(X > s)$.

(b) For $t > 0$, calculate $\mathbb{E}[X | X > t]$.

Solution 9.

(a) We have

$$\mathbb{P}[X > t + s | X > t] = \frac{\mathbb{P}(X > t + s, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > t + s)}{\mathbb{P}(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}(X > s).$$

(b) We have

$$\mathbb{E}[X | X > t] = \frac{\mathbb{E}[X 1_{(t,\infty)}(X)]}{\mathbb{P}(X > t)} = \frac{\lambda \int_t^\infty x e^{-\lambda x} dx}{e^{-\lambda t}} = \frac{\lambda e^{-\lambda t}(\lambda t + 1)}{\lambda^2 e^{-\lambda t}} = t + \frac{1}{\lambda}.$$

This shows that $\mathbb{E}[X | X > t] = t + \mathbb{E}[X]$.

Problem 10. Let $\{S_n\}_{n=0}^\infty$ be a simple random walk with

$$p \doteq \mathbb{P}(S_{n+1} - S_n = 1) = \frac{2}{3}, \quad q \doteq 1 - p = \mathbb{P}(S_{n+1} - S_n = -1) = \frac{1}{3}.$$

Calculate $\mathbb{P}_0[S_8 = 4, S_{14} = 6]$.

Solution 10. We have

$$\begin{aligned} \mathbb{P}_0[S_8 = 4, S_{14} = 6] &= \mathbb{P}_0[S_{14} = 6 | S_8 = 4] \mathbb{P}_0[S_8 = 4] \\ &= \mathbb{P}_0[S_{14} - S_8 = 2 | S_8 = 4] \mathbb{P}_0[S_8 = 4] \\ &= \mathbb{P}_0[S_6 = 2] \mathbb{P}_0[S_8 = 4] \\ &= \binom{6}{\frac{1}{2}(6+2)} \left(\frac{2}{3}\right)^{\frac{1}{2}(6+2)} \left(\frac{1}{3}\right)^{\frac{1}{2}(6-2)} \cdot \binom{8}{\frac{1}{2}(8+4)} \left(\frac{2}{3}\right)^{\frac{1}{2}(8+4)} \left(\frac{1}{3}\right)^{\frac{1}{2}(8-4)} \\ &= \frac{80}{243} \cdot \frac{1792}{6561} \\ &= \frac{143360}{1594323}. \end{aligned}$$

Problem 11. Let $\{S_n\}_{n=0}^{\infty}$ be a random walk where $S_0 \doteq 0$, and $S_n \doteq \sum_{i=0}^{n} X_i$ where $X_i \overset{iid}{\sim}$ Poisson(10).

(a) Using Markov's inequality, estimate $\mathbb{P}[S_{47} > 480]$.
(b) Using Chebyshev's inequality, estimate $\mathbb{P}[S_{47} > 480]$.
(c) Using the Central Limit Theorem, estimate $\mathbb{P}[S_{47} > 480]$.

Solution 11.

(a) Markov's inequality yields

$$\mathbb{P}[S_{47} > 480] \leq \frac{\mathbb{E}[S_{47}]}{480} = \frac{\sum_{i=1}^{47} \mathbb{E}[X_i]}{480} = \frac{47}{48}.$$

(b) Note that $\mathbb{E}[S_{47}] = 470$ and $\mathrm{Var}(S_{47}) = 470$, so Chebyshev's inequality yields

$$\mathbb{P}[S_{47} > 480] = \mathbb{P}[S_{47} - 470 > 10] \leq \mathbb{P}[|S_{47} - 470| > 10] \leq \frac{470}{100} = \frac{47}{10}.$$

This bound is not useful since it is larger than 1.

(c) We have

$$\mathbb{P}(S_{47} > 480) = \mathbb{P}\left(\frac{S_{47} - 470}{\sqrt{470}} > \frac{480 - 470}{\sqrt{470}}\right) \approx \mathbb{P}\left(Z > \frac{10}{\sqrt{470}}\right) = 0.322.$$

Problem 12. For each $n \in \mathbb{N}$, let X_n be a discrete random variable with probability mass function

$$\mathbb{P}(X_n = 1) = \frac{1}{2} + \frac{1}{n+1}, \quad \mathbb{P}(X_n = -1) = \frac{1}{2} - \frac{1}{n+1}.$$

(a) Find the distribution of the random variable X such that $X_n \xrightarrow{d} X$. Show, using the characterization of weak convergence via cumulative distribution functions, that $X_n \xrightarrow{d} X$.
(b) In the same setting as above, use moment generating functions to show that $X_n \xrightarrow{d} X$.

Solution 12.

(1) The cdf of X_n is given by

$$F_n(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} - \frac{1}{n+1} & -1 \leq x < 1 \\ 1 & 1 \leq x, \end{cases}$$

and the cdf of X is

$$F_n(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leq x < 1 \\ 1 & 1 \leq x. \end{cases}$$

Note that F is continuous at x for all $x \in \mathbb{R} \setminus \{-1, 1\}$. If $x < -1$, then

$$F_n(x) = 0 \to 0 = F(x),$$

and if $-1 \leq x < 1$, then

$$F_n(x) = \frac{1}{2} - \frac{1}{n+1} \to \frac{1}{2} = F(x),$$

and if $x > 1$, then

$$F_n(x) = 1 \to 1 = F(x),$$

so we have shown that $F_n \xrightarrow{d} F$.

(2) The mgf of X_n is given by

$$m_n(t) = e^t \left(\frac{1}{2} + \frac{1}{n+1} \right) + e^{-t} \left(\frac{1}{2} - \frac{1}{n+1} \right) = \frac{e^t + e^{-t}}{2} + \frac{e^t - e^{-t}}{n+1},$$

and the mgf of X is

$$m(t) = \frac{e^t + e^{-t}}{2},$$

so clearly $m_n(t) \to m(t)$ for all t. Therefore $X_n \xrightarrow{d} X$.