Math 174E
Lecture 12

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References

Hull
Chapter 13.1, 13.2

One-Step Binomial Model 5/11

Computing V_0, Δ_0:

Find V_0, Δ_0 s.t.: $V_T = (V_0 - \Delta_0 S_0)e^{rT} + \Delta_0 S_T = (S_T - K)^+$ (*)

Two possible scenarios: $S_T^u = 22$ or $S_T^d = 18$

$\Rightarrow$ (*) yields two equations:

$$\text{(i)} \quad (V_0 - \Delta_0 20)e^{(0.04)3/12} + \Delta_0 22 = 1$$

$$\text{(ii)} \quad (V_0 - \Delta_0 20)e^{(0.04)3/12} + \Delta_0 18 = 0$$

Solution: $V_0 = 0.545$ $\quad \Delta_0 = 1/4$

One-Step Binomial Model 6/11

Interpreting V_0, Δ_0:

- $V_0 =$ **value** of the **replicating portfolio** of the call option
- $\Delta_0 =$ **replication strategy** (also called **delta strategy**)

0 $\qquad\qquad$ $T = 3/12$

		$S_T = 22$	$S_T = 18$
buy $\Delta_0 = 1/4$ shares: (long stock)	$+5$	$+5.5$	$+4.5$
borrow @ $r = 4\%$: (short cash)	-4.455	$(-4.455)e^{(0.04)3/12} = -4.5$	-4.5
	$V_0 = 0.545$	$V_T = 1$	$V_T = 0$

$\underbrace{V_0 = 0.545}_{\text{portfolio value}}$

portfolio value
$=$
call price C_0

$\underbrace{V_T = 1 \qquad V_T = 0}_{\text{portfolio generates same payoff as call}}$

portfolio generates same payoff as call

One-Step Binomial Model 7/11

Observe following **equivalence**:

long call	$=$	long replicating portfolio
(buy 1 call)		(buy $\Delta_0 = 1/4$ shares + borrow \$4.455)
short call	$=$	short replicating portfolio
(sell 1 call)		(short sell $\Delta_0 = 1/4$ shares + invest \$4.455)

One-Step Binomial Model 8/11

Perfect hedging strategy for the seller:

- seller is **short** 1 call option
- seller should take a **long position** in the **replicating portfolio** in order to be *perfectly hedged* against the financial risk of paying the payoff to the option holder at maturity T
  - indeed, at maturity T her short position in the call is cancelled out by her long position in the replicating portfolio
- also note that the premium $C_0 = \$0.545$ the seller receives from selling the call option is *exactly* the amount needed to set up the long position in the replicating portfolio at time 0

- Δ_0 is the seller's **hedging strategy** (**delta hedging**)
  - call option: Δ_0 is positive
    - selling a call: long position in Δ_0 shares
  - put option: Δ_0 is negative (see Example 13.2 below)
    - selling a put: short position in Δ_0 shares

One-Step Binomial Model 9/11

Arbitrage-free price of one call option: \$0.545

Arbitrage opportunity if call price is *higher*: $C_0 = 0.6 > 0.545$

	$t = 0$	$t = T$	
		$S_T = 18$	$S_T = 22$
sell call (short)	-0.6	0	-1
buy replicating strategy (long)			
$\rightarrow$ buy $\Delta_0 = 1/4$ shares	$+5$	$+4.5$	$+5.5$
$\rightarrow$ borrow money	-4.4	-4.44	
net value	0	$+0.06$	

Note: The arbitrage gain at time T is exactly

$$(0.6 - 0.545) \cdot e^{0.04 \cdot 3/12} = 0.06$$

One-Step Binomial Model 10/11

Arbitrage-free price of one call option: \$0.545

Arbitrage opportunity if call price is *lower*: $C_0 = 0.5 < 0.545$

	$t = 0$	$t = T$	
		$S_T = 18$	$S_T = 22$
buy call (long)	$+0.5$	0	$+1$
sell replicating strategy (short)			
$\rightarrow$ short sell $\Delta_0 = 1/4$ shares	-5	-4.5	-5.5
$\rightarrow$ invest money	$+4.5$	$+4.55$	
net value	0	$+0.05$	

Note: The arbitrage gain at time T is exactly

$$(0.545 - 0.5) \cdot e^{0.04 \cdot 3/12} = 0.05$$

One-Step Binomial Model 11/11

Example 13.2

In the above considered one-step binomial model of Example 13.1 the price $P_0(K, T)$ of a **European put option** written on the same stock with strike price $K = \$19$ and maturity $T = 3/12$ is

$$P_0(K, T) = 0.445.$$

The replicating strategy is

$$\Delta_0 = -\frac{1}{4}.$$

One-Step Binomial Model: Notation

- T = maturity (in years)
- r = risk-free interest rate p.a. (continuously compounded)
- only one time step: today ($t = 0$) and maturity ($t = T$)

- S_0 = current stock price (today at time $t = 0$)
- S_T = stock price at maturity T, takes only two values
  - $S_0 \cdot u$ = one-step price upward move ($u > 1$)
  - $S_0 \cdot d$ = one-step price downward move ($d < 1$)
- $u - 1$ = percentage increase
- $1 - d$ = percentage decrease

- f_0 = (stock) option price (today at time $t = 0$)
- f_T = (stock) option payoff at maturity T, takes also only two values
  - f^u = (stock) option's payoff if stock price moved up
  - f^d = (stock) option's payoff if stock price moved down

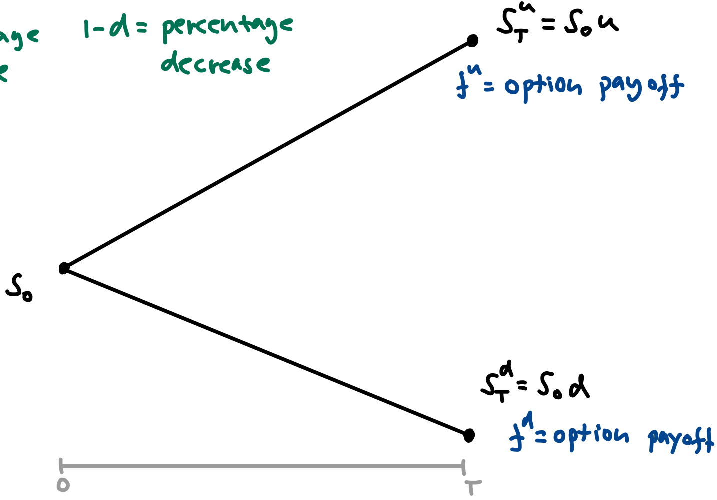
One-Step Binomial Model: Illustration

$u > 1$

$u - 1$ = percentage increase

$d < 1$

$1 - d$ = percentage decrease



European option with payoff : $f_T = h(S_T)$

One-Step Binomial Model: Replication Argument 1/4

Replication argument from above:

$$(V_0 - \Delta_0 S_0)e^{rT} + \Delta_0 S_0 u = f^u \qquad \Big| \cdot e^{-rT}$$

$$(V_0 - \Delta_0 S_0)e^{rT} + \Delta_0 S_0 d = f^d \qquad \Big| \cdot e^{-rT}$$

$$\Leftrightarrow \qquad V_0 + \Delta_0 (e^{-rT} S_0 u - S_0) = e^{-rT} f^u$$

$$V_0 + \Delta_0 (e^{-rT} S_0 d - S_0) = e^{-rT} f^d$$

Subtracting both equations and solving for Δ_0:

$$\boxed{\Delta_0 = \frac{f^u - f^d}{S_0 u - S_0 d}}$$

One-Step Binomial Model: Replication Argument 2/4

Introduce auxiliary variable $p^* \in (0,1)$:

$$V_0 + \Delta_0 (e^{-rT} S_0 u - S_0) = e^{-rT} f^u \quad \Big| \cdot p^*$$

$$V_0 + \Delta_0 (e^{-rT} S_0 d - S_0) = e^{-rT} f^d \quad \Big| \cdot (1-p^*)$$

Adding up both equations:

$$V_0 + \Delta_0 \underbrace{\left[e^{-rT} (S_0 u \cdot p^* + S_0 d \cdot (1-p^*)) - S_0 \right]}_{(*)} = e^{-rT} (f^u \cdot p^* + f^d \cdot (1-p^*))$$

One-Step Binomial Model: Replication Argument 3/4

choose p^* s.t. $(*) = 0$:

$$e^{-rT}\left(S_0 u \cdot p^* + S_0 d \cdot (1-p^*)\right) - S_0 = 0$$

$$\Leftrightarrow \quad S_0 e^{rT} = S_0 u\, p^* + S_0 d\, (1-p^*)$$

$$\Leftrightarrow \quad \boxed{p^* = \frac{e^{rT} - d}{u - d}}$$

Hence, we obtain for V_0 the formula:

$$\boxed{V_0 = e^{-rT}\left(f^u p^* + f^d (1-p^*)\right)}$$

One-Step Binomial Model: Replication Argument 4/4

Interpretation:

(i) Arbitrage-free option premium at $t=0$:

$$V_0 = \mathbb{E}^*\left[e^{-rT} f_T\right] = e^{-rT}\left(f^u p^* + f^d (1-p^*)\right)$$

(ii) Probability p^* is chosen such that

$$S_0 e^{rT} = S_0 u\, p^* + S_0 d\,(1-p^*) = \mathbb{E}^*[S_T]$$

$$\Leftrightarrow \quad \mathbb{E}^*\left[\frac{S_T}{S_0}\right] = e^{rT}$$

expected return of S_T under p^* is just the risk-free rate r