PSTAT 122 Exam I 5/5/2022

1.(a) $\bar{y}_{..} = \frac{1}{4}(47+48+29+42) = 41.5$

$SS_{Treatments} = 5[(47-41.5)^2 + (48-41.5)^2 + (29-41.5)^2 + (42-41.5)^2] = 1145$

$SSE = 4(6.5+5+6.5+8) = 104$

$SS_{Tot} = 104 + 1145 = 1249$

Source	SS	df	MS
Between Brands	1145	3	381.67
Error	104	16	6.5
Total	1249	19	

1b) $H_0: \mu_A = \mu_B = \mu_C = \mu_D$ $\quad H_1:$ not H_0

Reject H_0 if $F > F_{3,16,0.05} = 3.24$

$$F = \frac{MS_{Treatments}}{MSE} = \frac{381.67}{6.5} = 58.72 > 3.24$$

$\Longrightarrow$ Reject H_0, the four tire brands do not have the same average miles of wear.

(c) $y_{ij} = \mu_i + \varepsilon_{ij}$, $\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$ $\quad i=1,2,3,4$, $j=1,2,3,4,5$

$\hat{\mu}_1 = \bar{y}_{1.} = 47$, $\hat{\mu}_2 = \bar{y}_{2.} = 48$, $\hat{\mu}_3 = \bar{y}_{3.} = 29$, $\hat{\mu}_4 = \bar{y}_{4.} = 42$

$\hat{\sigma}^2 = MSE = 6.5$

or $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$ $\quad \varepsilon_{ij} \sim N(0, \sigma^2)$ $\quad i=1,2,3,4$, $j=1,2,3,4,5$

$\hat{\mu} = \bar{y}_{\cdot\cdot} = 41.5 \qquad \hat{\sigma}^2 = MSE = 6.5$

$\hat{\tau}_1 = \bar{y}_{1\cdot} - \bar{y}_{\cdot\cdot} = 47 - 41.5 = 5.5, \quad \hat{\tau}_2 = \bar{y}_{2\cdot} - \bar{y}_{\cdot\cdot} = 48 - 41.5 = 6.5$

$\hat{\tau}_3 = \bar{y}_{3\cdot} - \bar{y}_{\cdot\cdot} = 29 - 41.5 = -12.5, \quad \hat{\tau}_4 = \bar{y}_{4\cdot} - \bar{y}_{\cdot\cdot} = 42 - 41.5 = 0.5$

(d) $H_0: \mu_C = \frac{1}{3}(\mu_A + \mu_B + \mu_D) \Leftrightarrow H_0: W = \frac{1}{3}\mu_A + \frac{1}{3}\mu_B + \frac{1}{3}\mu_D - \mu_C = 0$

$H_1: W \neq 0$

$$SS_W = \frac{\left[\frac{1}{3}(47+48+42) - 29\right]^2}{\frac{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + 1^2 + \left(\frac{1}{3}\right)^2}{5}}$$

$$= \frac{\left(\frac{1}{3}(47+48+42) - 29\right)^2}{\frac{4}{15}} = 1041.67$$

Reject H_0 if $F > F_{1,16,0.05} = 4.49$

$$F = \frac{SS_W}{MSE} = \frac{1041.67}{6.5} = 160.26 > 4.49$$

$\Longrightarrow$ Reject H_0.

(f) $C_1 = \mu_A - \mu_B \qquad SS_{C_1} = \frac{(47-48)^2}{\frac{1^2}{5} + \frac{1^2}{5}} = \frac{1}{\frac{2}{5}} = 2.5$

$C_2 = \mu_A + \mu_B - 2\mu_D \quad SS_{C_2} = \frac{(47+48-2\times(42))^2}{\frac{1^2}{5} + \frac{1^2}{5} + \frac{2^2}{5}} = \frac{11^2}{\frac{6}{5}} = 100.83$

Reject $H_0: \mu_A = \mu_B = \mu_D$ if $F > F_{2,16,0.05} = 3.63$

$$F = \frac{(SS_{C_1} + SS_{C_2})/2}{MSE} = \frac{(2.5 + 100.83)/2}{6.5} = 7.95 > 3.63$$

$\Longrightarrow$ Reject $H_0 \Longrightarrow$ Brands ABD are not all the same.

(e) 95% CI for $\mu_A - \mu_B$

$$\frac{\sum_{i=1}^{a} c_i \bar{Y}_{i\cdot} - \sum_{i=1}^{a} c_i \mu_i}{S\sqrt{\sum_{i=1}^{a} \frac{c_i^2}{n}}} \sim t_{N-a}$$

$\Longrightarrow$ $100(1-\alpha)\%$ CI for $\sum_{i=1}^{a} c_i \mu_i$:

$$\sum_{i=1}^{a} c_i \bar{y}_{i\cdot} \mp t_{N-a,\alpha/2}\, S\sqrt{\sum_{i=1}^{a} \frac{c_i^2}{n}}$$

$$\Longrightarrow (\bar{y}_{1\cdot} - \bar{y}_{2\cdot}) \mp t_{16,0.025}\sqrt{6.5}\sqrt{\frac{1^2}{5} + \frac{1^2}{5}}$$

$$\Longrightarrow (47 - 48) \mp 2.12\sqrt{6.5}\sqrt{\frac{2}{5}}$$

$$\Longrightarrow -1 \mp 3.418 \Longrightarrow (-4.418, 2.418)$$

2(a) $\bar{y}_{..} = (495.5 + 482.5 + 487.2)/3 = 488.4$

$SS_{Treatments} = 4\left[(495.5 - 488.4)^2 + (482.5 - 488.4)^2 + (487.2 - 488.4)^2\right]$

$= 346.64 \qquad df = 2 \qquad MS_{Treatments} = \frac{346.64}{2} = 173.32$

$SSE = 3(10.85 + 24.7 + 12.37) = 143.76 \qquad df = 9$

$SS_{Tot} = 490.4 \qquad df = 11$

$\Longrightarrow MSE = \frac{143.76}{9} = 15.973$

$H_0: \sigma_\tau^2 = 0$ vs $H_1: \sigma_\tau^2 > 0$

Reject H_0 if $F > F_{2,9,0.05} = 4.26$

$F = \frac{MS_{Treatments}}{MSE} = \frac{173.32}{15.973} = 10.851 > 4.26$

$\Longrightarrow$ Reject $H_0 \Longrightarrow$ significant variation

(b) $y_{ij} = \mu + \tau_i + \varepsilon_{ij} \qquad i = 1,2,3 \quad j = 1,2,3,4$

$\tau_1, \tau_2, \tau_3 \overset{iid}{\sim} N(0, \sigma_\tau^2)$, $\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$

$\hat{\mu} = \bar{y}_{..} = 488.4 \longrightarrow \mu$

$\hat{\sigma}^2 = MSE = 15.973 \longrightarrow \sigma^2$

$\hat{\sigma}_\tau^2 = \frac{1}{n}(MS_{Treatments} - MSE) = \frac{1}{4}(173.32 - 15.973) = 39.337$

$\longrightarrow \sigma_\tau^2$

(c) 95% CI for μ:

$$\bar{y}_{..} \mp t_{N-a,\alpha/2} \sqrt{\frac{MS_{\text{Treatments}}}{N}} \qquad t_{9,0.025} =$$

$$\Rightarrow 488.4 \mp 2.262 \sqrt{\frac{173.32}{12}}$$

$$\Rightarrow 488.4 \mp 8.597$$

$$\Rightarrow (479.803, 496.997)$$