Math 174E
Lecture 16

Moritz Voss

September 7, 2022

Chapter 15: The Black–Scholes–Merton Model

Hull

Chapter 15.1, 15.2, 15.3, 15.7, 15.8, 15.9

Introduction 1/3

- Black–Scholes–Merton (or Black–Scholes) model was introduced in the early 1970s by Fischer Black, Myron Scholes, and Robert Merton
- major breakthrough in the pricing of European stock options
- model has had a huge influence on the way traders price and hedge derivatives
- in 1997 the importance of the model was recognized when Robert Merton and Myron Scholes were awarded the **Nobel price for economics** (Fischer Black died in 1995)

Introduction 2/3

Mathematically:

- *simplest* continuous-time model for option pricing
- stock price process $(S_t)_{t\in[0,T]}$ is modeled by a continuous-time stochastic process: the **geometric Brownian motion**
- allows for an *explicit formula* for computing arbitrage-free prices of European call and put options: **Black-Scholes(-Merton) formula**
- allows for an *explicit hedging strategy* for the option seller: **delta-hedging strategy**

Introduction 3/3

Similar to the binomial-tree model (Chapter 13):

- arbitrage-free prices of European options are derived by a replication argument (Merton's approach)
- replicating portfolio changes *continuously* through time: **Black-Scholes(-Merton) partial differential equation (PDE)**
- Black-Scholes-Merton model is **complete**: every European option's payoff is perfectly replicable and has a unique arbitrage-free price
- **risk-neutral valuation**

Some History

Fischer Black (1938–1995), Myron Scholes (1941), Robert Merton (1944).

Source of pictures: Wikipedia.

Black–Scholes–Merton model 1/2

"Real (Physical) World": Stock price process $(S_t)_{t\geq 0}$ is modeled by a **geometric Brownian motion** (recall Lecture 15, slides 18f.)

$$S_t = S_0 \cdot e^{R_t} \quad \text{with} \quad R_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t \quad (t \geq 0) \tag{1}$$

satisfying the stochastic differential equation (SDE)

$$dS_t = S_t \mu \, dt + S_t \sigma \, dB_t = S_t \left(\mu \, dt + \sigma \, dB_t\right)$$

where $(B_t)_{t\geq 0}$ denotes the (standard) Brownian motion.

Two model parameters:

- $\mu =$ expected return (annualized)
  (depends on riskiness, higher than risk-free rate r)
- $\sigma =$ volatility (standard deviation) of the return (annualized)

Black–Scholes–Merton model 2/2

Interpretation:

- S_t = price at time $t \geq 0$ (years)
- $S_t \sim \text{Lognormal}\left(\log(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right)$
- $R_t = \log\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ represents the **log-return** (= continuously compounded return) on $[0, t]$
- μ = **expected rate of return** (per annum) in the sense that (see Lemma 14.13)

$$\mathbb{E}[S_t] = \mathbb{E}[S_0 \cdot e^{R_t}] = S_0 \cdot e^{\mu \cdot t}$$

- σ = **volatility** (per annum) or standard deviation of the **log-returns**

Risk-Neutral Modeling and Valuation of Derivatives 1/3

"Risk-Neutral World": Stock price process $(S_t)_{t\geq 0}$ is modeled by a geometric Brownian motion

$$S_t = S_0 \cdot e^{R_t} \quad \text{with} \quad R_t = \left(r - \frac{1}{2}\sigma^2 \right) t + \sigma B_t \quad (t \geq 0) \quad (2)$$

satisfying the stochastic differential equation (SDE)

$$dS_t = S_t r \, dt + S_t \sigma \, dB_t = S_t \left(r \, dt + \sigma \, dB_t \right)$$

where $(B_t)_{t\geq 0}$ denotes the (standard) Brownian motion and $r > 0$ denotes the risk-free interest rate.

Observe:

- compared to the dynamics of the stock price process $(S_t)_{t\geq 0}$ in the "real world" in equation (1), the expected return μ is replaced by the risk-free rate r
- the volatility σ is the same

Risk-Neutral Modeling and Valuation of Derivatives 2/3

Interpretation: (compare with slide 8 above)

- S_t = price at time $t \geq 0$ (years)
- $S_t \sim \text{Lognormal}\left(\log(S_0) + \left(r - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right)$
- $R_t = \log\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}((r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ log-return on $[0, t]$
- the dynamics of the stock price in (2) are "risk-neutral" in the sense that (again Lemma 14.13)

$$\mathbb{E}^*[S_t] = \mathbb{E}^*[S_0 \cdot e^{R_t}] = S_0 \cdot e^{r \cdot t}$$

- that is, the expected return in the "risk-neutral world" is just the risk-free rate $r > 0$ (same approach as in the binomial tree model; compare with Lecture 13, slide 6)
- $\mathbb{E}^*$ = expected value computed in the "risk-neutral world" (it just means that we assume that S_t is given by (2))

Risk-Neutral Modeling and Valuation of Derivatives 3/3

Principal of risk-neutral valuation:

- risk-neutral stock price dynamics in (2) are used to compute **arbitrage-free** prices of (European-type) financial derivatives, namely by **computing expected values of discounted future payoffs** at maturity T written on the stock:

$$\text{arbitrage-free price at time } 0 = \mathbb{E}^*[e^{-rT}h(S_T)]$$

  where h represents the payoff function (e.g., call, put, binary option, power option, ...)
- observe that the stock price's "real world" expected return μ is **not** needed for arbitrage-free option pricing (same approach as in the binomial tree model; compare with Lecture 13, slide 8)

Risk-Neutral Modeling and Valuation of Derivatives 4/4

Simple illustration of risk-neutral valuation (= arbitrage-free):

Example 15.1 (Forward contract on a stock)

A forward contract with maturity T and forward price $F_0(T)$ on a non-dividend paying stock S_T is a derivative depending on the stock.

The payoff of a long position at time T is $S_T - F_0(T)$, and the arbitrage-free forward price $F_0(T)$ is determined in such a way that the value of the contract when initiated (at time 0) is 0.

Using the principal of risk-neutral valuation this means that

$$\mathbb{E}^*[e^{-rT}(S_T - F_0(T))] = 0 \quad \Leftrightarrow \quad F_0(T) = \mathbb{E}^*[S_T] = S_0 e^{rT}$$

In other words: the arbitrage-free forward price must satisfy $F_0(T) = S_0 e^{rT}$, which is consistent with Chapter 5, Lemma 5.2 (Lecture 8)!

Famous Black–Scholes–Merton Pricing Formula 1/2

Theorem 15.2 (Black–Scholes–Merton Call Option Formula)

Under the risk-neutral model dynamics in (2) the **arbitrage-free** price of a **European call option** at time $t \in [0, T]$ is given by

$$
\begin{aligned}
C_t(K, T) &= \mathbb{E}^*[e^{-r(T-t)}(S_T - K)^+ \mid S_t = s] \\
&= s \cdot \Phi\left(d_+(s, T-t)\right) - K \cdot e^{-r(T-t)} \cdot \Phi\left(d_-(s, T-t)\right)
\end{aligned}
$$

where

$$
d_\pm(s, \tau) = \frac{\log\left(s/K\right) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}
$$

and Φ denotes the cumulative distribution function of the $\mathcal{N}(0,1)$ distribution.

Proof: See Lecture Notes.

(Remark: $d_-(s,\tau) = d_+(s,\tau) - \sigma\sqrt{\tau}$)

Famous Black–Scholes–Merton Pricing Formula 2/2

Theorem 15.3 (Black–Scholes–Merton Put Option Formula)

Similarly, under the risk-neutral model dynamics in (2) the **arbitrage-free** price of a **European put option** at time $t \in [0, T]$ is given by

$$\begin{aligned} P_t(K, T) &= \mathbb{E}^*[e^{-r(T-t)}(K - S_T)^+ \,|\, S_t = s] \\ &= Ke^{-r(T-t)} \cdot \Phi\left(-d_-(s, T-t)\right) - s \cdot \Phi\left(-d_+(s, T-t)\right) \end{aligned}$$

where $d_\pm(s, \tau)$ are defined as in Theorem 15.2 and Φ denotes the cumulative distribution function of the $\mathcal{N}(0, 1)$ distribution.

Proof: Similar to the call option in Theorem 15.2.