Problem 3.1. (10 points) Application of MGFs: historically insurance companies have used the *exponential transform principle* to set premia. Let X be the random variable denoting the amount of loss. The pricing question is how much to charge for insuring X. Charging the expected loss, i.e. $\mathbb{E}[X]$ is often too low and does not adequately account for possibility of very large losses that incur additional expenses (such as the risk of the insurer being bankrupt). The exponential principle sets the premium to be charged to

$$P(a) \doteq \frac{1}{a}\log\left(\mathbb{E}\left[e^{aX}\right]\right),$$

where a is a parameter picked by the insurance firm. Suppose that $X \sim \text{Exp}(1/400)$, so that the expected value of X is \$400. $E(X) = \lambda = 400$

(a) (3 points) Let $a = 0.001$. Evaluate the resulting premium $P(a)$ and compare it to the expected loss.

(b) (3 points) Compute $P(a)$ for $a = 0.002$ and $a = 10^{-4}$. How does the choice of a affect the premium $P(a)$?

(c) (4 points) Show that

$$\lim_{a \downarrow 0} P(a) = \mathbb{E}[X].$$

Thus, actuaries interpret a as a type of "risk loading" that affects how much extra to charge above the expected loss.

(a)

$$E[e^{ax}] = \int_0^\infty e^{ax} \cdot \frac{1}{400} e^{-\frac{1}{400}x}\,dx$$

$$= \frac{1}{400}\int_0^\infty e^{ax} \cdot e^{-\frac{1}{400}x}\,dx$$

$$= \frac{1}{400}\int_0^\infty e^{(a-\frac{1}{400})x}\,dx$$

$$= \frac{1}{\cancel{400}}\left(\frac{\cancel{400}}{-1+400a}\right)\left[e^{(-\frac{1}{400}+a)x}\Big|_0^\infty\right]$$

$$= \frac{1}{400a-1}(-1)$$

$$= -\frac{1}{400a-1}$$

$$= \frac{1}{1-400a}$$

$$P(a) = P(0.001) = \frac{1}{0.001}\log\left(\frac{1}{1-400\cdot 0.001}\right)$$

$$= \frac{1}{0.001}\cdot \ln\left(\frac{1}{0.6}\right)$$

$$\approx 510.8256$$

$$P(a) > E(x)$$

(b)

$$P(a) = P(0.002) = \frac{1}{0.002}\ln\left(\frac{1}{1-400\cdot 0.002}\right)$$

$$\approx 804.7189$$

$$P(a) = P(10^{-4}) = \frac{1}{0.0001}\ln\left(\frac{1}{1-400\cdot 0.0001}\right)$$

$$\approx 408.2199$$

When $a \uparrow$, $P(a) \uparrow$. positive relationship.

(c)

$$\lim_{a\to 0} P(a) = \frac{1}{a}\cdot \ln\left(\frac{1}{1-400a}\right)$$

$$= -\frac{1}{a}\cdot \ln(1-400a)$$

$$= -\lim_{a\to 0}\frac{\ln(1-400a)}{a}$$

L'H

$$= -\lim_{a\to 0}\frac{[\ln(1-400a)]'}{a'}$$

$$= -\lim_{a\to 0}\frac{-\frac{400}{1-400a}}{1}$$

$$= -\frac{-400}{1}$$

$$= 400$$

$$= E(X)$$

Problem 3.2. (10 points) A car insurance company has 2000 policy holders. The expected claim paid to a policy holder during a year is \$900 with a standard deviation of \$1000.

What premium should the company charge to each policy holder to assure that, approximately, with probability 0.999, the total premium income will cover the cost of all the claims?

Use the Central Limit Theorem to answer this question. You can use that $\Phi(3.0903) = 0.999$, where Φ denotes the c.d.f. of the standard normal distribution.

$E(X) = n \cdot \mu$

$= 2000 \cdot 900$

$= 1800000$

$SD(X) = \sqrt{n} \cdot \sigma$

$= \sqrt{2000} \cdot 1000$

$$\frac{2000X - 1800000}{\sqrt{2000} \cdot 1000} \geq 3.0903$$

$$2000X - 1800000 \geq 3.0903 \left(\sqrt{2000} \cdot 1000 \right)$$

$$X \geq 969.101$$

At least 969.101

Problem 3.3. (10 points) Consider the following game of chance. First, a number U is chosen uniformly at random from the interval $[1, 10]$. Next, an integer X is chosen according to the Poisson distribution with parameter U. The player receives a reward of \$X.

What would be the fair price to charge for this game? That is, how much should it cost to play so that the expected net gain is zero?

$U \sim \text{Uniform}(1, 10)$

$f_X(x) = \frac{1}{9}$

$E(U) = \frac{1+10}{2} = \frac{11}{2}$

$X \sim \text{Poission}(U)$

$E[X] = E[X|U]$

$E[X|U] = U$

$$E[X] = E[E[X|U]]$$
$$= E[U]$$
$$= \frac{11}{2}$$

Problem 3.4. (10 points) Suppose that X has the m.g.f. given by

$$m_X(t) = \frac{1}{2} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{5t}.$$

(a) (5 points) Find the expected value and variance of X using its m.g.f.

(b) (5 points) Find the p.m.f. of X. Using the p.m.f., verify that you answer to part (a).

a) $M_X'(t) = 0 + \frac{1}{3}e^{-4t} \cdot (-4) + \frac{1}{6}e^{5t} \cdot 5$

$E(X) = M_X'(0) = -\frac{4}{3} + \frac{5}{6} = -\frac{\cancel{6}\,1}{\cancel{18}\,2} = -\frac{1}{2}$

$M_X''(t) = \frac{1}{3}e^{-4(t)} \cdot (-4)(-4) + \frac{5}{6}e^{5t} \cdot 5$

$E(X^2) = M_X''(0) = \frac{16}{3} + \frac{25}{6} = \frac{57}{6}$

$Var(X) = E(X^2) - [E(X)]^2$

$= \frac{57}{6} - \frac{1}{4}$

$= 9.5 - 0.25$

$= 9.25$

(b) $M_X(t) = \sum e^{tx} \cdot f(x)$

$= \frac{1}{2} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{5t}$

$= \frac{1}{2} + \sum_{k=1}^{2} \frac{1}{k^2+k} \cdot e^{tx}$

$x=0 \quad f(x) = \frac{1}{2}$

$x=-4 \quad f(x) = \frac{1}{3}$

$x=5 \quad f(x) = \frac{1}{6}$

$E(X) = \sum X \cdot p(x) = 0 \cdot (\frac{1}{2}) + (-4)(\frac{1}{3}) + 5(\frac{1}{6}) = -\frac{1}{2}$

$E(X^2) = \sum x^2 \cdot p(x) = 0 + \frac{16}{3} + \frac{25}{6} = \frac{19}{2}$

$Var(X) = E(X^2) - [E(X)]^2$

$= \frac{19}{2} - \frac{1}{4}$

$= 9.25$

Problem 3.5. (10 points) Let $X \sim \text{Poisson}(1)$ and $Y \sim \text{Unif}(0, 2)$ be independent.

(a) (5 points) Use Markov's inequality to bound $\mathbb{P}[X + Y \geq 7]$. $\leq \frac{E(X+Y)}{7}$

(b) (5 points) Use Chebyshev's inequality to bound $\mathbb{P}[X + Y \geq 7]$.

$E(X) = 1$

$E(Y) = 1$

$Z = X + Y.$

$E(Z) = E(X+Y)$

$= E(X) + E(Y)$

$\operatorname{Var}(Z) = \operatorname{Var}(X+Y)$

$= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X, Y)$

$= 1 + \frac{\cancel{5}\,\cancel{4}\,1}{\cancel{12}\,3} + 0$

$= \frac{4}{3}$

(a)

$$P[X+Y \geq 7] \leq \frac{E(X+Y)}{7}$$

$$\leq \frac{E(X)+E(Y)}{7}$$

$$\leq \frac{2}{7}$$

(b)

$$P[X+Y \geq 7] = P[Z \geq 7]$$

$$\leq P[(Z - E(Z))^2 \geq 49]$$

$$\leq \frac{E(Z-\mu)^2}{(7-2)^2}$$

$$\leq \frac{\frac{4}{3}}{25}$$

$$\leq \frac{4}{75}$$

Problem 3.6. (10 points) Let $X \sim \text{Unif}(0,1)$ and [redacted]. $Y \sim \text{Unif}(0,1)$

(a) (5 points) Compute the p.d.f. f_Z of $Z \doteq X + Y$.

(b) (5 points) Let f_X and f_Y denote the p.d.f.'s of X and Y, respectively. The *convolution* of f_X and f_Y is the function $f_X \star f_Y$ defined as

$$(f_X \star f_Y)(z) \doteq \int_{-\infty}^{\infty} f_X(z-x) f_Y(x) dx.$$

Compute $f_X \star f_Y$ and observe that it is equal to f_Z.[1]

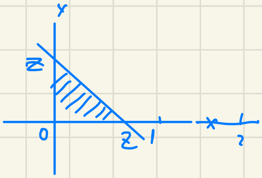
$0 < x < 1$

$-1 < -x < 0$

$-1+z < z-x < z$

$0 < z-x < 1$

(a) $f_Z = 1$



$F_Z(z) = P(X+Y \le z)$

$= P(Y \le z-x)$

$$\int_0^z \int_0^{z-x} 1 \, dy \, dx$$

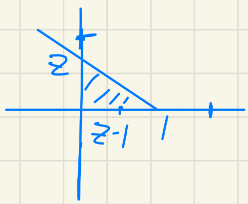
$$= \int_0^z y \Big|_0^{z-x} dx$$

$$= \int_0^z [z-x] \, dx$$

$$= zx - \tfrac{1}{2}x^2 \Big|_0^z$$

$$= z^2 - \tfrac{1}{2}z^2$$

$f_Z(z) = F_Z'(z) = 2z - z = z$ $\quad [0,1]$



$$\int_{z-1}^{1} \int_{1}^{z-x} 1 \, dy \, dx$$

$$= -z^2 + 3z - \frac{-z^2+2z}{2} - 2$$

$f_Z(z) = F_Z'(z) = -z+2$

$$f_Z(z) = \begin{cases} z & [0,1] \\ -z+2 & [1,2] \\ 0 & \text{o.w} \end{cases}$$

(b) $f_X * f_Y = \int_0^1 f_X(z-x) \cdot 1 \, dx$

$0 \le z \le 1 \quad f_Z(z) = \int_0^z dy = z$

$1 \le z \le 2 \quad f_Z(z) = \int_{z-1}^1 dy = 2-z$

$$f_Z(z) = \begin{cases} z & [0,1] \\ 2-z & [1,2] \\ 0 & \text{o.w} \end{cases}$$

Problem 3.7. (10 points) We say that two random variables X and Y are bivariate normal (or jointly normal) if for all $a, b \in \mathbb{R}$, the random variable $aX + bY$ has a normal distribution.[2]

If X and Y are jointly normal, we write $(X, Y) \sim \mathcal{N}(\mu, \Sigma)$, where

$$\mu \doteq \begin{pmatrix} \mathbb{E}[X] \\ \mathbb{E}[Y] \end{pmatrix} \in \mathbb{R}^2, \qquad \Sigma \doteq \begin{pmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}(Y) \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

The two-dimensional vector μ is called the mean of (X, Y) and the 2×2 matrix Σ is called the covariance matrix of (X, Y).[3]

(a) (2 points) Suppose that $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are independent. Show that (X, Y) follows a bivariate normal distribution. Be sure to specify the mean and covariance matrix of (X, Y).[4]

(b) (2 points) Let $Z_1, Z_2 \sim \mathcal{N}(0, 1)$. be independent standard normal random variables. Let $\sigma_X^2, \sigma_Y^2 > 0$, $\rho \in (0, 1)$, and $\mu_X, \mu_Y \in \mathbb{R}$. Define the random variables X and Y by

$$X \doteq \sigma_X Z_1 + \mu_X, \qquad Y \doteq \sigma_Y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_Y.$$

What is the joint probability distribution of (X, Y)?[5] $\int$ (x,y)

(c) (2 points) Let $X \sim \mathcal{N}(0, 1)$ be a standard normal random variable. Let W be a discrete random variable which is independent of X whose probability mass function is given by

$$\mathbb{P}(W = 1) = \frac{1}{2}, \qquad \mathbb{P}(W = -1) = \frac{1}{2}.$$

$WX<1\cdot$

Define the continuous random variable Y by $Y \doteq WX$. What is the probability distribution of Y?[6] $<Y$

$$Y = WX = \begin{cases} \frac{1}{2}\cdot X & W=1 \\ -\frac{1}{2}X & W=-1 \end{cases}$$

(d) (2 points) With X and Y defined as in part (c), calculate $\mathrm{Cov}(X, Y)$.[7]

(e) (2 points) With X and Y defined as in part (c), use the definition of bivariate normal random variables to show that the random variable (X, Y) does not follow a bivariate normal distribution.[8]

(a)

$$M_{aX}(t) = E(e^{t\cdot aX}) = e^{at\mu_x + \frac{\sigma_x^2\cdot(at)^2}{2}}$$

$$M_{bY}(t) = E(e^{t\cdot bY}) = e^{bt\mu_Y + \frac{\sigma_Y^2\cdot(bt)^2}{2}}$$

Since independent.

$$M_{aX+bY}(t) = M_{aX}(t)\cdot M_{bY}(t) = \boxed{e^{(a\mu_x + b\mu_Y)t + (a\sigma_x^2 + b\sigma_Y^2)\frac{t^2}{2}}}$$

μ: $E[aX] = a\mu_x$

$E[bY] = b\mu_Y$

Σ: $\begin{pmatrix} a^2\sigma_x^2 & 0 \\ 0 & Y^2\sigma_Y^2 \end{pmatrix}$

(b)

$$M_{aX+bY}(t) = E[e^{(aX+bY)t}]$$

$$= E[e^{[a(\sigma_x Z_1 + \mu_x) + b(\sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}Z_2) + \mu_Y)]t}]$$

$$= e^{t(a\mu_x + b\mu_Y)} e^{(a\sigma_x + b\sigma_Y)^2\cdot\frac{t^2}{2}}$$

So this is a Normal distribution.

$$\mu = \begin{pmatrix} E(X) \\ E(Y) \end{pmatrix} = \begin{bmatrix} a\mu_x \\ b\mu_Y \end{bmatrix}$$

$$\mathrm{Cov}(X,Y) = E[(X-E(X))(Y-E(Y))]$$

$$= E[(\sigma_x Z_1)(\sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}Z_2))]$$

$$= \sigma_x\sigma_Y E(\rho Z_1^2 + \sqrt{1-\rho^2}Z_1 Z_2)$$

$$= \sigma_x\sigma_Y E(Z_1^2)$$

$$= \sigma_x\sigma_Y\rho$$

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_x\sigma_Y\rho \\ \sigma_x\sigma_Y\rho & \sigma_Y^2 \end{pmatrix}$$

which follow bivariate normal distribution.

(c) $\begin{cases} P(W=1) = \frac{1}{2} \\ P(W=-1) = \frac{1}{2} \end{cases}$

$Y = WX$; $W \sim N(0,1)$

$M_X(t) = e^{\frac{t^2}{2}}$

$M_Y(t) = E(e^{tWX})$

$= E(e^{tWX} | W=1) P(W=1) + E(e^{tWX} | W=-1) P(W=-1)$

$= E(e^{tX})(\frac{1}{2}) + E(e^{-tX})(\frac{1}{2})$

$= e^{\frac{t^2}{2}} \cdot \frac{1}{2} + e^{\frac{(-t)^2}{2}} \cdot \frac{1}{2}$

$= e^{\frac{t^2}{2}}$

So y is also a normol distribution.

(d) $E(W) = 1 \cdot \frac{1}{2} + 1 \cdot (-\frac{1}{2}) = 0$

$cov(X,Y) = cov(X, WX)$

$= E(X^2 W) - E(WX) E(X)$

$= E(W) \cdot E(X) \cdot E(X) - 0$

$cov(X,Y) = 0 - 0 = 0$

$= E(e^X$

$M_X(t) = e^{\frac{t^2}{2}}$

$x + x(-1)$

$x - x = 0$

$t \cdot 0 = 0.$

(e) $M_{X+Y}(t) = E\left[e^{t(X+XW)}\right]$

$= E(e^{t(X+XW)} | W=1) P(W=1) + E(e^{t(X+XW)} | W=-1) P(W=-1)$

$= E(e^{2tX})(\frac{1}{2}) + E(e^{0})(\frac{1}{2})$

$= e^{\frac{(2t)^2}{2}} \cdot (\frac{1}{2}) + \frac{1}{2}$

As we can see, this is not follow a normol distribution form.

Problem 3.8. A natural first guess to how one might define convergence in distribution would be to say that for a sequence of random variables $\{X_n\}$ and a random variable X, the sequence $\{X_n\}$ converges in distribution to X if

$$\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A), \text{ for all events } A \subseteq \mathbb{R}. \tag{1}$$

For the purpose of this question, if (1) holds for all events $A \subseteq \mathbb{R}$, we will say that $\{X_n\}$ converges *setwise* to X. However, as we saw in class, **this is not the definition of convergence in distribution**. This question will illustrate why the proposed definition given in (1) is "incorrect". That is to say, this question will show that, even in very simple situations, the notion of setwise convergence is too restrictive.

$P(\lim_{n\to\infty} X_n = X) = 1$

Recall that for random variables $\{Y_n\}$ and Y, we say that $Y_n \xrightarrow{d} Y$ as $n \to \infty$ if for each $\tilde{y} \in \{y : F_Y \text{ is continuous at } y\}$, we have that

$$\mathbb{P}(Y_n \leq \tilde{y}) \to F_Y(\tilde{y}) \doteq \mathbb{P}(Y \leq \tilde{y}), \tag{2}$$

as $n \to \infty$, where F_Y denotes the cumulative distribution function of Y.

Consider a sequence of random variables $\{X_n\}$ such that $\mathbb{P}\left(X_n = \frac{1}{n}\right) = 1$ for each $n \in \mathbb{N}$ and a random variable X such that $\mathbb{P}(X = 0) = 1$.

(a) (2 points) Denote the cumulative distribution function of X by F_X, and let $\mathcal{C}_X \doteq \{x : F_X \text{ is continuous at } x\}$. Describe the set $\mathcal{C}_X$ by specifying all of the real numbers that belong to $\mathcal{C}_X$.

(b) (4 points) Using the characterization of convergence in distribution given in (2), show that $X_n \xrightarrow{d} X$.

(c) (4 points) Show that X_n does not converge setwise to X. That is, show that there is some event $A \subseteq \mathbb{R}$ such that the convergence in (1) fails to hold.

(a) $F(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$C_x = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$

for any $x \in C_x$. $F_x(x) \xrightarrow{\infty} F(x)$

(b) if $x < 0$, $F_n(x) = 0$ $(\forall n \in X)$ $\to$ $F(x) = 0$ So $F_n(x) \to F(x)$, $x<0$

if $\frac{1}{x} < n$, $F_n(x) = 1$ $\frac{1}{x} \geq n$ $\to$ $F_n(x) \to F(x) = 1$

So $X_n \xrightarrow{d} X$

(c) when $A = \{0\}$ $P(X_n \in A) = 0$ $\not\to$ $P(X \in A) = 1$

So $P(X_n \in A) \not\to P(X \in A) = 1$