Math 4B: Differential Equations

Lecture 04: Separable ODEs

- Separable First-order ODEs,
- Those equations from last time,
- & More!

Basic Example

Two of our basic examples of ODEs was

$$\frac{dy}{dt} = ky \qquad \text{and} \qquad \frac{dy}{dt} = p(t)y$$

where k was a constant and $p(t)$ was a function of t. We found the answers, but the question today is...

HOW?

The idea is today's topic: separation of variables.

Separation of Variables

The idea:

Move all the "y"s to one side and the "t"s to the other:

$$\frac{1}{y}\frac{dy}{dt} = k \qquad \text{or} \qquad \frac{1}{y}\,dy = k\,dt.$$

Which one is better?

We integrate and both are the same:

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k\,dt \qquad \text{or} \qquad \int \frac{1}{y}\,dy = \int k\,dt.$$

So $\ln|y| = kt + C_1 \implies |y| = e^{C_1}e^{kt} = C_2 e^{kt} \implies \boxed{y = Ce^{kt}}$.

The General Idea

Last Time we dealt with solving the general first order linear ODE

$$y' + p(t)y = q(t),$$

where $p(t)$ and $q(t)$ are arbitrary functions of t.

Today we'll look at separable first order ODEs; that is, ones that can be written as

$$M(y)y' = N(t) \qquad \text{or, equivalently} \qquad M(y)\,dy = N(t)\,dt.$$

Idea: Just integrate (both are the same):

$$\int M(y)\,\frac{dy}{dt}\,dt = \int N(t)\,dt \qquad \text{or} \qquad \int M(y)\,dy = \int N(t)\,dt.$$

Example 1

1. Find the most general solution to the initial value problem (IVP)

$$\begin{cases} y\dfrac{dy}{dt} = 4t \\ y(0) = 3. \end{cases}$$

Solution: The equation arrives separated, so integrate:

$$\int y\frac{dy}{dt}\,dt = \int 4t\,dt \qquad \text{or} \qquad \int y\,dy = \int 4t\,dt$$

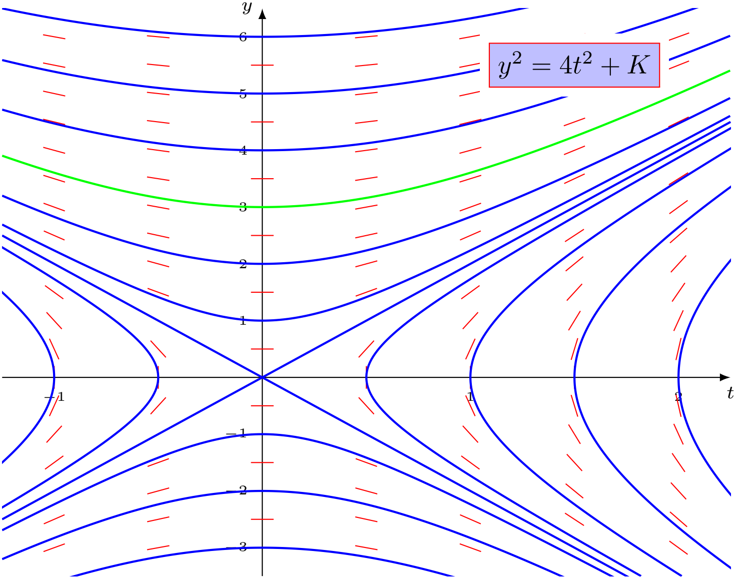
Then

$$\frac{1}{2}y^2 = 2t^2 + C \qquad \Longrightarrow \qquad y = \pm\sqrt{4t^2 + 2C}.$$

When $t = 0$, $y = 3$ means $3 = +\sqrt{4(0)^2 + 2C}$, so $2C = 9$.

Thus $\boxed{y = \sqrt{4t^2 + 9}}$.

Here's a picture of the direction field and a few solutions:



Example 2

2. Solve the ODE

$$\frac{dy}{dt} = te^{t^2+y}.$$

That is, find the general solution to this ODE.

Solution: We separate to get

$$e^{-y}\frac{dy}{dt} = te^{t^2} \qquad \text{or} \qquad e^{-y}\,dy = te^{t^2}\,dt.$$

We integrate to get

$$\int e^{-y}\frac{dy}{dt}\,dt = \int te^{t^2}\,dt \qquad \text{or} \qquad \int e^{-y}\,dy = \int te^{t^2}\,dt.$$

Then $-e^{-y} = \frac{1}{2}e^{t^2} + C$, so

$$e^{-y} = K - \frac{1}{2}e^{t^2} \qquad \Longrightarrow \qquad \boxed{y = -\ln\left(K - e^{t^2}/2\right)}.$$

Example 3

3. Solve the ODE

$$\frac{dy}{dx} = \frac{x}{1+y^2}$$

That is, find the general solution to this ODE.

Solution: We separate to get

$$(1+y^2)\frac{dy}{dx} = x \qquad \text{or} \qquad (1+y^2)\,dy = x\,dx.$$

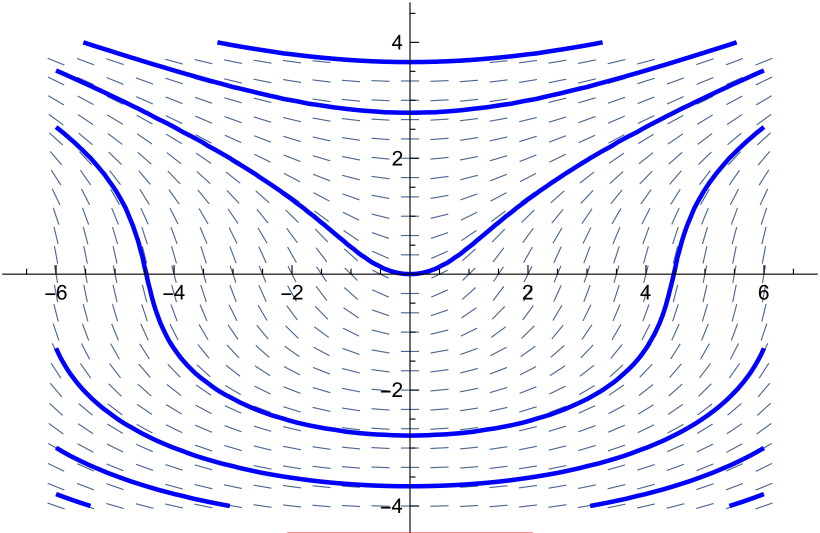
We integrate to get

$$\int (1+y^2)\frac{dy}{dx}\,dx = \int x\,dx \qquad \text{or} \qquad \int (1+y^2)\,dy = \int x\,dx.$$

Then

$$y + \frac{1}{3}y^3 = \frac{1}{2}x^2 + C \qquad \Longrightarrow \qquad \boxed{y = ???}.$$

Here's a picture of the direction field and some solutions:



$$\boxed{y + \frac{1}{3}y^3 = \frac{1}{2}x^2 + C}$$

Example 4

4. Find the most general solution to the ODE

$$\frac{dy}{dt} = \frac{1}{4}y(5-y).$$

How does the behavior of the solution change based on the initial condition $y(0)$?

Solution: The equation is not hard to separate:

$$\frac{1}{y(5-y)}\frac{dy}{dt} = \frac{1}{4} \qquad \text{or} \qquad \frac{1}{y(5-y)}\,dy = \frac{1}{4}\,dt.$$

As usual, we integrate:

$$\int \frac{1}{y(5-y)}\frac{dy}{dt}\,dt = \int \frac{1}{4}\,dt \qquad \text{or} \qquad \int \frac{1}{y(5-y)}\,dy = \int \frac{1}{4}\,dt.$$

We use partial fractions to write $\dfrac{1}{y(5-y)} = \dfrac{1}{5}\left(\dfrac{1}{y} + \dfrac{1}{5-y}\right)$.

Example 4 (cont'd)

Integrating, we get

$$\frac{1}{5}\left(\ln|y| - \ln|5-y|\right) = \frac{1}{4}t + C_1 \qquad \Longrightarrow \qquad \frac{y}{5-y} = Ke^{5t/4}.$$

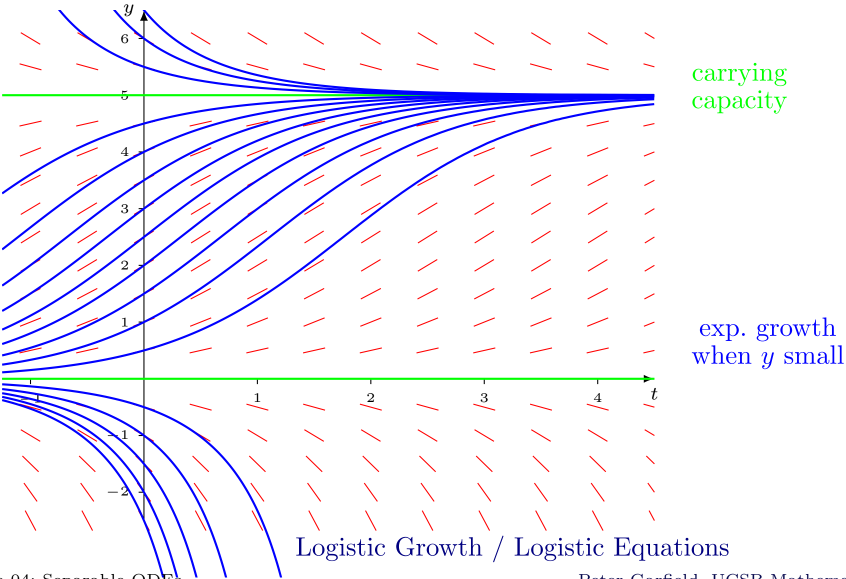
Thus

$$\boxed{y = \frac{5Ke^{5t/4}}{1+Ke^{5t/4}}}$$

or, equivalently,

$$\boxed{y = \frac{5}{1+Ce^{-5t/4}}}.$$

Here's a picture of the direction field and a few solutions:



Logistic Growth / Logistic Equations