| notation | $exp(\lambda)$ |
|---|---|
| cdf | $1-e^{-\lambda x}$ for $x \geq 0$ |
| pdf | $\lambda e^{-\lambda x}$ for $x \geq 0$ |
| expectation | $\frac{1}{\lambda}$ |
| variance | $\frac{1}{\lambda^2}$ |
| mgf | $\frac{\lambda}{\lambda - t}$ |
| ind. sum | $\sum_{i=1}^{k} X_i \sim Gamma(k, \lambda)$ |
| minimum | $\sim exp\left(\sum_{i=1}^{k} \lambda_i\right)$ Geo as well |

$$\mathbb{E}(X) = \int_0^1 X^*(p)dp$$

$$\mathbb{E}(X) = \int_{-\infty}^{0} F_X(t)\,dt + \int_0^{\infty} (1 - F_X(t))\,dt$$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X x dx$$

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X x dx$$

$$\mathbb{E}(aX+b) = a\mathbb{E}(X) + b$$

| notation | $Bin(n,p)$ |
|---|---|
| cdf | $\sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$ |
| pmf | $\binom{n}{i} p^i (1-p)^{n-i}$ |
| expectation | $np$ |
| variance | $np(1-p)$ |
| mgf | $(1-p+pe^t)^n$ |

| notation | $Poisson(\lambda)$ |
|---|---|
| cdf | $e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$ |
| pmf | $\frac{\lambda^k}{k!} \cdot e^{-\lambda}$ for $k \in \mathbb{N}$ |
| expectation | $\lambda$ |
| variance | $\lambda$ |
| mgf | $\exp(\lambda(e^t - 1))$ |
| ind. sum | $\sum_{i=1}^{n} X_i \sim Poisson\left(\sum_{i=1}^{n} \lambda_i\right)$ |

$$\mathbb{P}(X > s+t | X > s) = \frac{\mathbb{P}(X > s+t \text{ and } X > s)}{\mathbb{P}(X > s)} = \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t)$$

Gamma

$$f_Y(x) = \frac{\beta^n}{(n-1)!} x^{n-1} e^{-\beta x}, \quad x \geq 0.$$

$$F_Y(x) = 1 - e^{-\lambda x} \sum_{r=0}^{n-1} \frac{(\lambda x)^r}{r!}, \quad z \geq 0.$$

$$m_Y(t) = \left(\frac{\beta}{\beta - t}\right)^n, \quad t < \beta.$$

*tive random variable $X$ is* ***memoryless*** *if*

$$\mathbb{P}(X > s+t | X > s) = \mathbb{P}(X > t), \quad s, t \geq 0.$$

*, let $X_i \sim Exp(\lambda_i)$, where $\lambda_i > 0$. Let*

$$M = \min\{X_1, X_2, \ldots, X_n\},$$

$$N \doteq \sum_{i=1}^{n} i 1_{\{M = X_i\}}$$

$$\mathbb{P}(M > x, N = i) = \frac{\lambda_i}{\lambda} e^{-\lambda x}, \quad N \perp\!\!\!\perp M$$

$$\mathbb{P}(M > x, N = i) = \mathbb{P}(M > x)\mathbb{P}(N = i)$$

*Let $\{X_n\}$ be a sequence of iid $Exp(\lambda)$ random variables, and define*

$$S_0 \doteq 0, \quad S_n \doteq \sum_{i=1}^{n} X_i$$

*The counting process $\{N_t\}_{t \geq 0}$ defined by*

$$N_t \doteq \max\{n \geq 0 : S_n \leq t\}$$

*$\{S_n\}$ as the* ***arrival times****.*

*$\{X_n\}$ as the* ***inter-arrival times****,*

$$\mathbb{E}[X] = \frac{d}{dt} m_X(t)\Big|_{t=0} = \frac{1}{\lambda},$$

$$\mathbb{E}[X^2] = \frac{d^2}{dt^2} m_X(t)\Big|_{t=0} = \frac{2}{\lambda^2}.$$

$$\frac{d^3}{dt^3} m(t) = \frac{6\lambda}{(\lambda - t)^4}, \quad t < \lambda,$$

$$\mathbb{E}(\min\{X, Y\} | X < Y) = \mathbb{E}(M | N = 1) = \mathbb{E}(M) = \frac{1}{\lambda + \mu}.$$

**Theorem 2.26.** *A stochastic process $\{N_t\}_{t \geq 0}$ is a Poisson process with rate $\lambda > 0$ if and only if it has stationary and independent increments and, for each $t \geq 0$, $N_t \sim Poisson(\lambda t)$.*

$$\mathbb{P}(N_t = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k \in \mathbb{N}_0. \text{ PP formula}$$

pp:

$$\mathbb{P}(S_1 \leq t) = \mathbb{P}(N_t \geq 1) = 1 - \mathbb{P}(N_t = 0) = 1 - e^{-\lambda t} \frac{(\lambda t)^0}{0!} = 1 - e^{-\lambda t},$$

$$N_t^s \doteq N_{t+s} - N_s, \quad t \geq 0. \text{ Shifted PP}$$

*Note that $N_t^s$ denotes the number of events that occur in the $t$ time-units after time $s$.*

Superposition PP

$$N_t \doteq \sum_{i=1}^{n} N_t^i, \quad \lambda \doteq \sum_{i=1}^{n} \lambda_i. \qquad \mathbb{P}(N_{t_2} - N_{t_1} = i, N_{t_4} - N_{t_3} = j) = \mathbb{P}(N_{t_2} - N_{t_1} = i)\mathbb{P}(N_{t_4} - N_{t_3} = j)$$

$$\mathbb{P}(N_{t_2} - N_{t_1} = k) = \mathbb{P}(N_{u+t_1} - N_{t_1} = k) = \mathbb{P}(N_u^{t_1} = k) = e^{-\lambda u} \frac{(\lambda u)^k}{k!},$$

$$\mathbb{P}(N_{t_1} = k_1) = e^{-\lambda t_1} \frac{(\lambda t_1)^{k_1}}{k_1!} = e^{-\lambda t_1} \frac{(\lambda(t_1 - t_0))^{k_1 - k_0}}{(k_1 - k_0)!}$$

$P(N_1 = 3, N_{2.5} = 4) = P(N_{2.5} - N_1 = 4 - 3) = P(N_{1.5} = 1) = $ PP formula

split pp

$$N_t^1 | N_t = k \sim \text{Binomial}(k, p_1),$$

$$\mathbb{P}(N_t^1 = j | N_t = k) = \binom{k}{j} p_1^j (1 - p_1)^{k-j} = \binom{k}{j} p_1^j p_2^{k-j} = \frac{k!}{j!(k-j)!} p_1^j p_2^{k-j}.$$

$$\mathbb{P}(N_t^1 = k_1, N_t^2 = k_2) = \mathbb{P}(N_t^1 = k_1 | N_t = k_1 + k_2)\mathbb{P}(N_t = k_1 + k_2)$$

Bin × PP

$$= \binom{k_1 + k_2}{k_1} p_1^{k_1} p_2^{k_2} e^{-\lambda t} \frac{(\lambda t)^{k_1 + k_2}}{(k_1 + k_2)!}$$

Non-homo pp

$$\lambda(t) = t^2. \quad \Lambda(t) = \int_0^t \lambda(s) ds = \int_0^t s^2 ds = \frac{t^3}{3}, \quad \text{so } N_t \sim Poisson\left(\frac{t^3}{3}\right).$$

$$p_t(x) \doteq \mathbb{P}(N_t = x) = e^{-\frac{t^3}{3}} \frac{\left(\frac{t^3}{3} \cdot t\right)^x}{x!} = e^{-\frac{t^3}{3}} \frac{t^{4x}}{x! 3^x} \qquad \mathbb{E}(N_t) = \frac{t^3}{3}.$$

compound pp

*Suppose that $\{Z_t\}$ is a CPP of the form $Z_t = \sum_{n=1}^{N_t} X_n$*

*$\{N_t\} \sim PP(\lambda)$, $\mathbb{E}(X_n) = \mu$, and $\mathbb{E}(X_n^2) = \theta^2$.*

$$\mathbb{E}(Z_t) = \lambda \mu t, \quad Var(Z_t) = \lambda \theta^2 t.$$

spitial pp

$$\mathbb{P}(N_B = k | N_A = n) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\text{where } p = \frac{|B|}{|A|}.$$

CTMC

$$\mathbb{P}(X_{t+s} = y | X_s = x, X_u = x_u \text{ for all } u \in [0, s)) = \mathbb{P}(X_{t+s} = y | X_s = x).$$

$$\mathbb{P}(X_{t+s} = y | X_s = x) = \mathbb{P}(X_t = y | X_0 = x) \quad \text{Time Homo Always}$$

$$P_{x,y}(t) = \mathbb{P}(X_t = y | X_0 = x). \quad \text{Transition fn}$$

*That is $P_{x,y}(t)$ describes the probability that if the chain is currently in state $x$, that after $t \geq 0$ time units, it will be in state $y$. Note that for each $t \geq 0$, $P(t)$ can be interpreted as an $|\mathscr{S}| \times |\mathscr{S}|$ matrix.*

holding time

*The* ***holding time*** *$T_x$ of state $x$ is the amount of time that the chain spends in state $x$ before it jumps to another state.*

*Let $T_x$ denote the holding time of state $x$.*

*Then there is some $q_x > 0$ such that*

$$T_x \sim Exp(q_x).$$

*Let $\{X_t\}$ be a CTMC with transition function $P$. Then for each $s, t \geq 0$,*

$$P(s+t) = P(s)P(t).$$

*That is, for each $x, y \in \mathscr{S}$ and $s, t \geq 0$,*

$$P_{x,y}(s+t) = [P(s)P(t)]_{x,y} = \sum_{z \in \mathscr{S}} P_{x,z}(s) P_{z,y}(t).$$

$$P_{x,y} \doteq \mathbb{P}(X_{T_x} = y | X_0 = x).$$

$$q_{x,y} \doteq q_x P_{x,y},$$

$$q_{x,x} \doteq -\sum_{y \neq x} q_{x,y}.$$

$$Q_{x,y} = q_{x,y}, \quad x, y \in \mathscr{S}.$$

$$q_x \doteq |q_{x,x}| = \sum_{y \neq x} q_{x,y},$$

generator matrix Q

rate from x to y

pp transition fn

prove pp is CTMC

*Let $\{N_t\} \sim PP(\lambda)$. Then $\{N_t\}$ is a CMTC on $\mathscr{S} = \{0, 1, 2, \ldots\}$ with transition function*

$$P_{x,y}(t) = e^{-\lambda t} \frac{(\lambda t)^{y-x}}{(y-x)!}, \quad x, y \in \mathscr{S}, t \geq 0.$$

forward/backward eq

$$P'(t) = QP(t), \quad t \geq 0,$$

$$P'(t) = P(t)Q, \quad t \geq 0,$$

$$P(0) = I,$$

component wise

$$P'_{x,y}(t) = \sum_{z \in \mathscr{S}} Q_{x,z} P_{z,y}(t), \quad x, y \in \mathscr{S}, t \geq 0$$

$$P'_{x,y}(t) = \sum_{z \in \mathscr{S}} P_{x,z}(t) Q_{z,y}, \quad x, y \in \mathscr{S}, t \geq 0.$$

*Let $A$ be a $d \times d$ matrix.*

*The* ***matrix exponential*** *of $A$, which is the $d \times d$ matrix denoted as $e^A$ or $\exp(A)$, is defined as*

$$\exp(A) \doteq \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

*Then the transition function for $\{X_t\}$ is given by*

$$P(t) = \exp(Qt).$$

embeded chain

*The transition matrix of the embedded chain associated with $\{X_t\}$ is given by*

$$P_{x,y} = \begin{cases} \frac{q_{x,y}}{\sum_{z \neq x} q_{x,z}}, & x \neq y \\ 0 & x = y. \end{cases}$$

$$T_1 \doteq \inf\{t \geq 0 : X_t \neq X_0\},$$

*ransition. Similarly, let*

$$T_2 \doteq \inf\{t > T_1 : X_t \neq X_{T_1}\},$$

$$T_n \doteq \inf\{t > T_{n-1} : X_t \neq X_{T_{n-1}}\}$$

q is element of Q matrix

*A $d \times d$ matrix $A$ is said to be* ***diagonalizable***

$$A = UDU^{-1}$$

*Observe that if $D = diag(\lambda_1, \ldots, \lambda_d)$*

$$\exp(D) = diag(e^{\lambda_1}, \ldots, e^{\lambda_d}).$$

$$A = UDU^{-1},$$

$$\exp(A) = U \exp(D) U^{-1}$$

*Let $\{E_m\} \overset{iid}{\sim} Exp(\lambda)$ be independent of $X \sim Geom(p)$, and define*

$$S_m \doteq \sum_{k=1}^{m} E_k.$$

*Then $S_X \sim Exp(\lambda p)$.*

$$P \doteq I + \frac{Q}{\lambda}.$$

$$P(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} P^n, \quad t \geq 0.$$

***subordinated to a Poisson process.***

$$Q \doteq \lambda(P - I),$$

a DTMC w/
transition matrix P
can simulate CTMC
using rate lambda

method of chose
M large enough

**Proposition 3.32.** *Let $\{X_t\}$ be a CTMC on $\mathscr{S} = \{1, \ldots, d\}$ with generator matrix $Q$. Let*

$$\lambda \doteq \max_{1 \le x \le d} |q_{x,x}|,$$

*and define*

$$P \doteq I + \frac{Q}{\lambda}.$$

*For each $M \in \mathbb{N}$, let*

$$\hat{P}[M](t) = \sum_{n=0}^{M} e^{-\lambda t} \frac{(\lambda t)^n}{n!} P^n, \quad t \ge 0.$$

*Fix $\epsilon > 0$, and let $N^t \sim PP(\lambda t)$. If we choose $M \in \mathbb{N}$ large enough so that*

$$\mathbb{P}(N^t > M) \le \epsilon,$$

*then, for all $s \in [0, t]$,*

$$|P_{x,y}(s) - \hat{P}[M]_{x,y}(s)| \le \epsilon, \quad x, y \in \mathscr{S}.$$

**Definition 3.36.** *Let $\{X_t\}$ be a CTMC on $\mathscr{S}$ with transition function $P$. A probability distribution $\mu$ on $\mathscr{S}$ is the **limiting distribution** of $\{X_n\}$ if, for all $x, y \in \mathscr{S}$,*

$$\lim_{t \to \infty} P_{x,y}(t) = \mu_y.$$

**Definition 3.37.** *ALet $\{X_t\}$ be a CTMC on $\mathscr{S}$ with transition function $P$. A probability distribution $\pi$ on $\mathscr{S}$ is the **stationary distribution** of $\{X_n\}$ if*

$$\pi = \pi P(t), \quad t \ge 0.$$

*Equivalently, $\pi$ is a stationary distribution for $\{X_t\}$ if for all $x \in \mathscr{S}$ and $t \ge 0$,*

$$\pi_x = \sum_{y \in \mathscr{S}} \pi_y P_{y,x}(t).$$

**Definition 3.38.** *Let $\{X_t\}$ be a CTMC on $\mathscr{S}$ with transition function $P$. We say that a collection of states $\mathscr{C} \subseteq \mathscr{S}$ is a **communication class** if for all $x, y \in \mathscr{S}$, there are $s, t \ge 0$ such that*

$$P_{x,y}(s) > 0, \quad P_{y,x}(s) > 0.$$

*A communication class $\mathscr{C}$ is **closed** if for all $y \in \mathscr{S}$ such that there is some $x \in \mathscr{C}$ and $t \ge 0$ such that $P_{x,y}(t) > 0$, we have $y \in \mathscr{C}$. We say that $\{X_t\}$ is **irreducible** if for all $x, y \in \mathscr{S}$, there is some $t > 0$ such that $P_{x,y}(t) > 0$. Equivalently, $\{X_t\}$ is irreducible if and only if $\mathscr{S}$ consists of a single closed communication class.*

**Theorem 3.41.** *Consider a CTMC on $\mathscr{S}$ with generator matrix $Q$. A probability distribution $\pi$ on $\mathscr{S}$ is a stationary distribution for the chain if and only if*

$$\pi Q = \mathbf{0},$$

*where $\mathbf{0} \in \mathbb{R}^{|\mathscr{S}|}$ is a vector of zeroes. Coordinate-wise this says that $\pi$ is a stationary distribution if and only if for each $x \in \mathscr{S}$,*

$$\sum_{y \in \mathscr{S}} \pi_y Q_{y,x} = 0.$$

**Theorem 3.40.** *Let $\{X_t\}$ be a CTMC on a finite state space $\mathscr{S}$ with transition function $P$. If $\{X_t\}$ is irreducible, then there is a unique stationary distribution $\pi$, which is also a limiting distribution. In particular, for each $x, y \in \mathscr{S}$,*

$$\lim_{t \to \infty} P_{x,y}(t) = \pi_y,$$

*or, equivalently,*

$$\lim_{t \to \infty} P(t) = \Pi$$

*where $\Pi$ is an $|\mathscr{S}| \times |\mathscr{S}|$ matrix with entries given by*

$$\Pi_{x,y} = \pi_y, \quad x, y \in \mathscr{S}.$$

*Additionally, as with DTMCs, the unique stationary distribution describes, in the long term, the proportion of time that the chain spends in each state.*

**Definition 2.21.** *Let $\{X_t\}_{t \ge 0}$ be a stochastic process. For $t_1 < t_2$, the quantity $X_{t_2} - X_{t_1}$ is referred to as the increment of $\{X_t\}$ over the interval $(t_1, t_2]$. We say that $\{X_t\}$ has **stationary increments** if for all $s_1 < s_2$ and $t_1 < t_2$ such that $s_2 - s_1 = t_2 - t_1$, we have*

$$X_{t_2} - X_{t_1} \overset{d}{=} X_{s_2} - X_{s_1}.$$

*We saw that $\{X_t\}$ has **independent increments** if for all $t_1 < t_2 \le t_3 < t_4$, the random variables $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ are independent.*