Math 174E
Lecture 6

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References

Hull
Chapters 3.4

Choice of Contract

Two aspects:

1. The choice of the delivery month.
2. The choice of the asset underlying the futures contract.

Principles:

- Choose a delivery month that is as close as possible to, but later than, the end of the life of the hedge.
- When there is no futures contract on the asset being hedged, choose the contract whose futures prices are most closely correlated with the price of the asset being hedged.

  $\rightsquigarrow$ **cross hedging**

Cross Hedging 1/2

- in practice, asset underlying the futures contract is not necessarily the asset whose price is being hedged: **cross hedging**
- *Example:* Airline hedges the price of jet fuel with heating oil futures.

Definition 3.7

The **hedge ratio** is the ratio of the *size of the position in a hedging instrument* (e.g., futures contracts) to the *size of the position being hedged* (exposure).

Cross Hedging 2/2

- When the asset underlying the futures contract is the same as the asset being hedged, it is natural to use a hedge ratio of 1.0 (as, e.g., in Examples 3.5 and 3.6)
  - *Example:* To hedge the purchase of 20,000 barrels of crude oil, use 20 futures contracts on crude oil with a size of 1,000 barrels per contract.

- When cross hedging is used, setting the hedge ratio equal to 1.0 is not always optimal.
- Instead, the hedger chooses a value for the hedge ratio that **minimizes the variance** of the value of the hedged position.

Minimum Variance Hedge Ratio 1/3

For simplicity we ignore the daily settlement of futures contracts.

Notation:

- t_1 = time when the hedge is put in place
- t_2 = time when the hedge is closed out ($t_1 < t_2$)
- S_{t_i} = spot price at time t_i of asset being hedged ($i = 1, 2$)
- F_{t_i} = futures price at time t_i of contract used ($i = 1, 2$)

- $\Delta S = S_{t_2} - S_{t_1}$: change in spot price
- $\Delta F = F_{t_2} - F_{t_1}$: change in futures price

- σ_S = standard deviation of ΔS
- σ_F = standard deviation of ΔF
- ρ = correlation coefficient between ΔS and ΔF

σ_S, σ_F, ρ can be estimated from historical data of ΔS, ΔF (see Assignment 3)

Minimum Variance Hedge Ratio 2/3

Notation: (continued)

- N_A = size of position being hedged (in units of the asset being hedged)
- N_F = size of the hedge (in units of asset underlying the futures contract)
- $h = \frac{N_F}{N_A}$ = hedge ratio
- Q_F = size of one futures contract (in units of asset underlying the futures contract)
- $\frac{h \cdot N_A}{Q_F}$ = number of futures contracts

Minimum Variance Hedge Ratio 3/3

Proposition 3.8

The minimum variance hedge ratio is given by

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

and the optimal number N^* of futures contracts is given by

$$N^* = \frac{h^* \cdot N_A}{Q_F}.$$

Proof: See lecture notes.

Examples:

- For $\rho = 1$ and $\sigma_F = \sigma_S$: $h^* = 1$
- For $\rho = 1$ and $\sigma_F = 2\sigma_S$: $h^* = 0.5$

Numerical Example

Example 3.9

An airline expects to purchase 2 million gallons ($= N_A$) of jet fuel in 1 month and decides to use heating oil futures for hedging.

Suppose that $\sigma_F = 0.0313$, $\sigma_S = 0.0263$, and $\rho = 0.928$. The minimum variance hedge ratio h^* is therefore

$$h^* = 0.928 \cdot \frac{0.0263}{0.0313} = 0.78.$$

Each heating oil futures contract is on 42,000 gallons of heating oil. The optimal number of futures contracts is thus

$$N^* = \frac{0.78 \cdot 2,000,000}{42,000} = 37.1429 \approx 37 \text{ (round to nearest integer)}$$

The size of the hedge is $N_F = 0.78 \cdot 2,000,000 = 1.56$ million gallons of heating oil.

Comments

- the minimum variance hedge ratio h^* from Proposition 3.8 depends on the relationship between changes in the spot price ΔS and changes in the futures price ΔF
- the proof of Proposition 3.8 actually reveals that h^* is the **slope of the best-fit line** from a linear regression of ΔS against ΔF (minimizing the mean squared error)
- this result is intuitively reasonable, since we would expect h^* to be the ratio of the average change in S for a particular change in F

Chapter 4: Interest Rates

Hull
Chapter 4.1, (4.2), 4.3, 4.4, 4.5, 4.6, 4.7, 4.10

Introduction

- an **interest rate** in a particular situation defines the amount of money a borrower promises to pay the lender (time value of money)
- for any given currency, many different types of interest rates are regularly quoted: mortgage rates, deposit rates, prime borrowing rates etc.
- interest rate applicable depends on the **credit risk** (risk of default by the borrower)
- interest rates are often expressed in **basis points**: 1 basis point $= 0.01\% = 0.0001$
- interest rates are an important factor in the **valuation of virtually all financial derivatives**

Types of rates

- Treasury rates
  - rates an investor earns on Treasury bills and Treasury bonds
  - these instruments are used by governments to borrow in their own currencies (e.g., U.S. Treasury rates are the rates at which US government borrows in U.S. dollars)
  - treasury rates are regarded as a **risk-free rate**
- LIBOR rates
  - London Interbank Offered Rate
  - unsecured short-term borrowing rate between banks (AA-rated)
  - quoted for different currencies and borrowing periods
  - used as a **reference rate** for all lot of derivative transactions throughout the world (e.g, interest rate swaps, see Chapter 7)
  - virtually risk-free, but considered nowadays as a less-than-ideal reference rate for derivatives pricing
  - LIBOR will be phased out by June 30, 2023, and will be replaced by the Secured Overnight Financing Rate (SOFR)
- Overnight rates (**federal funds rate** in the U.S.)
- Repo rates
  - secured borrowing rates (repo = repurchase agreement)

See also `https://www.global-rates.com`.

Measuring Interest Rates

Two factors:

- compounding period (typically expressed in years)
- compounding frequency within the compounding period

Example 4.1

Interest rate $r = 0.1$ per year (i.e., 10% p.a.)

Compounding frequency m	Value of \$100 after one year
annually ($m = 1$)	$100 \cdot (1 + 0.1)^1 = 110$
semiannually ($m = 2$)	$100 \cdot \left(1 + \frac{0.1}{2}\right)^2 = 110.25$
quarterly ($m = 4$)	$100 \cdot \left(1 + \frac{0.1}{4}\right)^4 = 110.38$
monthly ($m = 12$)	$100 \cdot \left(1 + \frac{0.1}{12}\right)^{12} = 110.47$
weekly ($m = 52$)	$100 \cdot \left(1 + \frac{0.1}{52}\right)^{52} = 110.51$
daily ($m = 365$)	$100 \cdot \left(1 + \frac{0.1}{365}\right)^{365} = 110.516$
continuously ($m = \infty$)	$100 \cdot e^{0.1} = 110.517$

Source of table: Hull, Chapter 4.4, Table 4.1, page 82.

Formulas 1/3

General notation:

- interest rate r (per year, per annum, p.a.)
- investment period n (measured in years)
- compounding frequency m per year

Simple compounding: Terminal value (**future value**) of an amount A invested for n years at rate r p.a. compounded m times per year:

$$A \cdot \left(1 + \frac{r}{m}\right)^{m \cdot n}$$

Continuous compounding: Terminal value (**future value**) of an amount A invested for n years at rate r p.a. compounded continuously:

$$A \cdot e^{r \cdot n}$$

Formulas 2/3

Conversion: r_c (annual rate of interest continuously compounded) is *equivalent* to r_m (annual rate of interest compounded m times per year) if and only if

$$\left(1 + \frac{r_m}{m}\right)^m = e^{r_c}$$

and hence

$$r_c = m \cdot \log\left(1 + \frac{r_m}{m}\right) \qquad \text{or} \qquad r_m = m \cdot \left(e^{\frac{r_c}{m}} - 1\right)$$

Formulas 3/3

Simple discounting: Present value today of an amount A received in n years (when interest rate is r p.a. compounded m times per year)

$$A \cdot \left(1 + \frac{r}{m}\right)^{-m \cdot n}$$

Continuous discounting: Present value today of an amount A received in n years (when interest rate is r p.a. continuously compounded)

$$A \cdot e^{-r \cdot n}$$

In this class, interest rates will be measured with **continuous compounding**!

Zero Rates

Definition 4.2

The ***n*-year zero-coupon interest rate** (also called ***n*-year spot rate** or ***n*-year zero rate**) is the *rate of interest earned* on an investment that starts today (at time $t = 0$) and lasts for n years (until $t = n$). All the interest and principal is realized at the end of n years. There are no intermediate payments.

Example 4.3

Suppose a 5-year zero spot rate is quoted as $r = 5\%$ per annum (continuous compounding).

This means that \$100, if invested for 5 years, grows to $100 \cdot e^{0.05 \cdot 5} = \$128.403$.

Similarly, the present value today of \$100 received in 5 years is $100 \cdot e^{-0.05 \cdot 5} = \$77.8801$.

Bonds

- governments and corporations need to raise funds to finance their expenditures and their long-term investments
- one possibility is to issue **bonds**

Definition 4.4

A **bond** is an instrument of *indebtedness* of the bond issuer to the holders. The issuer owes the holders a debt and (depending on the terms of the bond) is obliged to pay them interest (the **coupon**) and the **principal** at a later date, termed the **maturity date**. Interest is usually payable at fixed intervals (semiannual, annual, sometimes monthly).

- most bonds (e.g., U.S. treasury bonds, corporate bonds) pay coupons to the holder periodically (**coupon rate**)
- **zero-coupon bonds** = no coupon payments
- bond's principal (par value, face value) is received at maturity T

Bond Pricing 1/2

- after issuance most bonds are actively traded on financial markets (secondary market)
- the theoretical price of a bond can be calculated as the **present value of all future cash flows** (coupon payments and the principal) that will be received by the holder
- sometimes bond traders use the same discount rate for all the cash flows
- a more accurate approach is to use a different **spot rate** for each cashflow

Bond Pricing 2/2

Example 4.5

Today (at time $t = 0$) suppose that a 2-year bond with a principal of \$100 provides coupons at the rate of 6% per annum semiannually.

Following are today's market spot rates:

Maturity (years)	spot rate p.a. (in %) cont. comp.
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Today's theoretical price (at time $t = 0$) of the bond can be computed as

$$3 \cdot e^{-0.05 \cdot 0.5} + 3 \cdot e^{-0.058 \cdot 1.0} + 3 \cdot e^{-0.064 \cdot 1.5} + (100 + 3) \cdot e^{-0.068 \cdot 2.0}$$
$$= 98.39.$$

Source of table: Hull, Chapter 4.6, Table 4.2, page 84.

Bond Yield

Definition 4.6

The **bond yield** is the *single discount rate* that makes the present value of all cash flows of the bond equal to its market price.

Example 4.7

Suppose that the theoretical price of the bond from Example 4.5 is also its market price (at time $t = 0$). Then, the annual yield y of the bond must satisfy

$$3 \cdot e^{-y \cdot 0.5} + 3 \cdot e^{-y \cdot 1} + 3 \cdot e^{-y \cdot 1.5} + (100 + 3) \cdot e^{-y \cdot 2.0} = 98.39.$$

This equation can be solved numerically and gives $y = 6.76\%$.

The bond yield is the **rate of return** received from investing in the bond.

Determining Zero Rates

n-year **zero rates** (spot rates) can be computed from given Treasury bill and Treasury bond prices (**Treasury zero rates**).

Example 4.8

Consider following market data on the prices of 5 Treasury bonds:

Bond principal ($)	Time to maturity (years)	Annual coupon ($)	Bond price ($)
100	0.25	0	99.6
100	0.50	0	99.0
100	1.00	0	97.8
100	1.50	4	102.5
100	2.00	5	105.0

Coupon payments are semiannually. Compute the 0.25-, 0.5-, 1-, 1.5-, and 2-year annual zero rates which are consistent with the market data above (**bootstrap method**).

Source of table: Hull, Chapter 4.7, Table 4.3, page 86.

Zero Curve

Definition 4.9

A chart showing the annual zero rates (spot rates) as a function of maturity is called the **zero curve** (or zero-coupon yield curve).

Example 4.10

Draw the zero curve for the data from Example 4.10:

Maturity (years)	Annual zero rate (% cont. comp.)
0.25	1.603
0.50	2.010
1.00	2.225
1.50	2.284
2.00	2.416

Interpolate linearly between the given maturities.

Source of table: Hull, Chapter 4.7, Table 4.4, page 87.

Slope of Zero Curve

- upward sloping:
  - the longer the maturity, the higher the yield (p.a.)
  - normal market
  - possibly explanation (among others): market is anticipating a rise in short-term interest rates
- downward sloping:
  - short-term interest rates higher than long-term interest rates
  - inverted market
  - possibly explanation (among others): market is anticipating falling short term interest rate rates
- flat or hump-shaped

Other factors influencing shape: supply and demand, volatility (risk premium), . . .

Duration 1/3

- the **duration** of a bond is a measure of how long the holder of the bond has to wait before receiving the present value of the cash payments
- a zero-coupon bond that lasts n years has a duration of n years
- however, a coupon-bearing bond lasting n years has a duration of less than n years, because the holder receives some of the cash payments prior to year n

Duration 2/3

Suppose that a bond provides the holder with cash flows c_i at time t_i $(1 \leq i \leq n)$. The bond price B and the bond yield y (continuously compounded) are related by (recall Definition 4.6)

$$B = \sum_{i=1}^{n} c_i \cdot e^{-y \cdot t_i}.$$

Definition 4.11

The **duration of the bond** D is defined as

$$D = \frac{\sum_{i=1}^{n} t_i \cdot c_i \cdot e^{-y \cdot t_i}}{B} = \sum_{i=1}^{n} t_i \cdot \left(\frac{c_i \cdot e^{-y \cdot t_i}}{B} \right).$$

Observe that the duration is a *weighted average* of the times when payments are made, with the weight applied to time t_i being equal to the ratio of the present value of the cash flow at time t_i to the bond price (the present value of all cash flows).

Duration 3/3

Application of duration:

When a small change Δy in the bond's yield is considered, it is approximately true that

$$\Delta B \approx \frac{dB}{dy} \cdot \Delta y \qquad (1^{\text{st}} \text{ order approximation}).$$

Since

$$\Delta B = -\Delta y \cdot \sum_{i=1}^{n} t_i \cdot c_i \cdot e^{-y \cdot t_i}$$

we obtain the key **duration relationship**

$$\Delta B \approx -B \cdot D \cdot \Delta y.$$

This is an approximate relationship between percentage changes in a bond price and changes in its yield. Note that there is a *negative* relationship between B and y (see Assignment 3 for an example).