:xpert solutions / Mathematical Statistics with Applications

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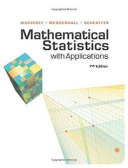
Exercise 23

Chapter 14, Section 14.5, Page 731



Mathematical Statistics with Applications

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Solution

Verified

Step 1

We are testing these hypothesis:

$$H_0 : p_1 - p_2 = 0$$
$$H_1 : p_1 - p_2 \neq 0$$

We know, from chapter 10.3, that the test statistic is given by

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

Now, we have to observe the χ^2 test.Remember that $\hat{p}$ is defined as

$$\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$$

The table with the values is given by

	Treated	Not treated	Sum
Improved	$n_{11} = n_1\hat{p}_1$	$n_{12} = n_2\hat{p}_2$	$n_{11} + n_{12}$
Not improved	$n_{21} = n_1\hat{q}_1$	$n_{22} = n_2\hat{q}_2$	$n_{21} + n_{22}$
Sum	$n_{11} + n_{21} = n_1$	$n_{12} + n_{22} = n_2$	$n_1 + n_2 = n$

Step 2

Now we have to find the expectd value.
Let's take n_{11}:

$$\hat{E}(n_{11}) = \frac{(n_{11}+n_{21})(n_{11}+n_{12})}{n_1+n_2}$$
$$= \frac{(n_{11}+n_{21})(y_1+y_2)}{n_1+n_2}$$
$$= \frac{(n_{11}+n_{21})(n_1\hat{p}_1+n_2\hat{p}_2)}{}$$

$$= n_1\hat{p}$$

Using the same thing on every element of the table, we can get a table with expected values:

	Treated	Not treated
Improved	$n_1\hat{p}$	$n_2\hat{p}$
Not improved	$n_1\hat{q}$	$n_2\hat{q}$

Step 3

Thus, the χ^2 statistic is given by

$$\chi^2 = \frac{(n_{11} - \hat{E}(n_{11}))^2}{\hat{E}(n_{11})} + \frac{(n_{12} - \hat{E}(n_{12}))^2}{\hat{E}(n_{12})} + \\ + \frac{(n_{21} - \hat{E}(n_{21}))^2}{\hat{E}(n_{21})} + \frac{(n_{22} - \hat{E}(n_{22}))^2}{\hat{E}(n_{22})}$$

$$= \frac{(n_1\hat{p}_1 - n_1\hat{p})^2}{n_1\hat{p}} + \frac{(n_2\hat{p}_2 - n_2\hat{p})^2}{n_2\hat{p}} + \\ + \frac{(n_1\hat{q}_1 - n_1\hat{q})^2}{n_1\hat{q}} + \frac{(n_2\hat{q}_2 - n_2\hat{q})^2}{n_2\hat{q}}$$

$$= \frac{n_1^2(\hat{p}_1 - \hat{p})^2}{n_1\hat{p}} + \frac{n_1^2(\hat{q}_1 - \hat{q})^2}{n_1\hat{q}} + \\ + \frac{n_2^2(\hat{p}_2 - \hat{p})^2}{n_2\hat{p}} + \frac{n_2^2(\hat{q}_1 - \hat{q})^2}{n_2\hat{q}}$$

$$= \frac{n_1(\hat{p}_1 - \hat{p})^2}{\hat{p}} + \frac{n_1(\hat{q}_1 - \hat{q})^2}{\hat{q}} + \\ + \frac{n_2(\hat{p}_2 - \hat{p})^2}{\hat{p}} + \frac{n_2(\hat{q}_1 - \hat{q})^2}{\hat{q}}$$

$$= \frac{n_1(\hat{p}_1 - \hat{p})^2}{\hat{p}} \frac{n_1(1 - \hat{p}_1 - 1 + \hat{p})^2}{\hat{q}} + \\ + \frac{n_2(\hat{p}_2 - \hat{p})^2}{\hat{p}} + \frac{n_2(1 - \hat{p}_1 - 1 + \hat{p})^2}{\hat{q}}$$

$$= \frac{n_1(\hat{p}_1 - \hat{p})^2 + n_2(\hat{p}_2 - \hat{p})^2}{\hat{p}\hat{q}}$$

Step 4

Substituing $\hat{p}$, we get:

$$\begin{aligned}\chi^2 &= \frac{n_1(\hat{p}_1-\hat{p})^2+n_2(\hat{p}_2-\hat{p})^2}{\hat{p}\hat{q}}\\ &= \frac{n_1n_2(\hat{p}_1-\hat{p})^2}{\hat{p}\hat{q}(n_1+n_2)}\\ &= \left(\frac{\hat{p}_1-\hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}\right)^2\\ &= Z^2\end{aligned}$$

This means that the tests are equivalent.

Result

Using the definitions af χ^2 and Z statistics, we managed to show that $\chi^2 = Z^2$, thus the tests are equivalent.

Exercise 22c

Exercise 24a