MIDTERM EXAM - VERSION A - SOLUTIONS - PSTAT 160B - SPRING 2022

Name:____________________________________

UCSB Perm Number:______________________________

Instructions: You must show all of your work and justify your answers to receive any credit. You may justify your answer by either writing brief sentences explaining your reasoning or annotating your math work with brief explanations (e.g., "by independence...").

If your work is illegible, or if it is unclear what your answer is or which work/answers you intend to be graded, you may not receive credit. For questions where it is appropriate, I suggest circling, boxing, or writing a sentence indicating your answer. **You may write your solutions either directly on the exam, or in a blue book.**

If you aren't sure how complete a problem or finish a calculation, explain clearly where and why you are stuck, and what you believe the correct approach to the problem is.

It may be helpful or necessary to use previous parts of problems to solve later parts. For example, part (a) might be useful for solving part (b). In such a situation, if you are unable to solve (a), and you believe that the answer to (a) is needed to solve (b), you may use (a) to solve (b).

The midterm has 5 problems (most have multiple parts). Please make sure your exam has all 5 problems.

Academic Integrity: *Please don't cheat!* If you are caught cheating you will receive 0 points for the midterm, and, according to university rules, the incident will be reported to the College of Letters and Sciences.

You may not discuss any of the questions on this exam with anyone besides the instructor until your graded exams have been returned. Doing so is a violation of the university's academic integrity policy.

You are allowed to have one page of notes (front and back) during this exam. As stated previously, your note sheet may not contain any worked out examples, homework problems, or practice problems. If your note sheet has worked out problems please bring it to the instructor now.

Notation: You do not need to convert numerical answers to decimals. For example, if the answer to a question was $7e^{-5}$ or $e^{-4}\frac{4^2}{2!}$, it would be ok to leave your answer in that form.

Please sign your name in the space provided below to verify that you have read and agree to the following: "I pledge that my solutions are solely from my own individual work, and that I have followed all UCSB academic integrity policies when taking this test. Additionally, I have read the **Instructions** and **Academic Integrity** sections above."

Signature: ______________________________

FIGURE 1. A kangaroo rat (Dipodomys phillipsii).

Problem 1. (16 points)

Kangaroo rats (Dipodomys phillipsii) arrive at a burrow (i.e., a home) according to a Poisson process with a rate of 2 kangaroo rats per hour. Each time a Kangaroo rat arrives at a burrow, it brings seeds with it. The weight of the seeds brought by each kangaroo rat follows an exponential distribution with a mean of 100 grams.

Additionally, the weight of the seeds brought by each kangaroo rat is independent of the weight of the seeds brought by the other kangaroo rats and of the number of kangaroo rats that have arrived at the burrow.

(a) **(8 points)** Calculate the expected value of the total weight of the seeds brought to the burrow over a period of 4 hours.

(b) **(8 points)** Calculate the probability that, over the course of 3 hours, every kangaroo rat that arrives at the burrow brings at least 50 grams of seeds with it.

Solution 1.

(a) Let N_t denote the number of kangaroo rats that have arrived by time t and let X_i denote the weight of the seeds brought by the i-th kangaroo rat. Then the total weight of the seeds brought over four hours is given by

$$T \doteq \sum_{i=1}^{N_4} X_i.$$

From our results regarding compound Poisson processes, it follows that

$$\mathbb{E}(T) = \mathbb{E}(N_4)\mathbb{E}(X_i) = 8 \cdot 100 = 800 \text{ grams}.$$

(b) Let L_t denote the number of kangaroo rats that have brought less than 50 grams of seeds by time t. The probability that a kangaroo rat brings less than 50 trams of seeds is given by

$$\mathbb{P}(X_i < 50) = 1 - e^{-\frac{50}{100}} = 1 - e^{-\frac{1}{2}},$$

so, from our results regarding split Poisson processes, it follows that $\{L_t\}$ is a Poisson process with rate of $2\left(1 - e^{-\frac{1}{2}}\right)$ rats with less than 50 grams of seeds per hour. Thus, the probability is given by

$$\mathbb{P}(L_3 = 0) = \exp\left(-3 \cdot 2\left(1 - e^{-\frac{1}{2}}\right)\right) \frac{\left(3 \cdot 2\left(1 - e^{-\frac{1}{2}}\right)\right)^0}{0!} = \exp\left(-6\left(1 - e^{-\frac{1}{2}}\right)\right) \approx 0.0943.$$

Problem 2. (10 points)

The lifetimes of desert tortoises (Gopherus agassizii) follow an exponential distribution with a mean of 60 years. Two desert tortoises share a burrow (home). Suppose that the lifetimes of the two tortoises are independent of each other.

(a) **(4 points)** What is the expected lifetime of the tortoise with the shorter lifetime?

(b) **(6 points)** What is the expected lifetime of the tortoise with the longer lifetime?

Solution 2.

(a) Let T_1 and T_2 denote the lifetimes of the two tortoises. Then, $\min\{T_1, T_2\}$ is an Exponential$\left(\frac{1}{30}\right)$ random variable, so $\mathbb{E}(\min\{T_1, T_2\}) = 30$.

(b) Note that $\max\{T_1, T_2\} + \min\{T_1, T_2\} = T_1 + T_2$, so

$$\mathbb{E}(\max\{T_1, T_2\}) = \mathbb{E}(T_1) + \mathbb{E}(T_2) - \mathbb{E}(\min\{T_1, T_2\}) = 60 + 60 - 30 = 90.$$

Problem 3. (16 points) Throughout the day, birds land on a cactus according to a Poisson process with a rate of 2 birds per hour. Each time a bird lands, there is a 60% chance that it is a cactus wren (Campylorhynchus brunneicapillus), a 30% chance that it is a curve-billed thrasher (Toxostoma curvirostre), and a 10% chance that it is a Gila woodpecker (Melanerpes uropygialis).

Additionally, the species of each bird is independent of the species of the other birds.

(a) **(8 points)** Calculate the probability that between 10am and 11am, exactly two cactus wrens land on the cactus and exactly one Gila woodpecker lands on the cactus.

(b) **(8 points)** Let C denote the number of curve-billed thrashers that land on the cactus between 2pm and 4pm. What is the probability distribution of C?

Solution 3.

(a) Let C_t, T_t, and G_t, respectively, denote the number of cactus wrens, curve-billed thrashers, and Gila woodpeckers that have landed on the cactus by time t. Then, from our results regarding split Poisson processes, $\{C_t\}$ is a Poisson process with rate of $2 \cdot 0.6 = 1.2$ birds per hour, and $\{G_t\}$ is a Poisson process with rate of $2 \cdot 0.1 = 0.2$ birds per hour, and the two processes are independent. Thus,

$$\mathbb{P}(C_1 = 2, G_1 = 1) = \mathbb{P}(C_1 = 2)\mathbb{P}(G_1 = 1) = e^{-1.2}\frac{(1.2)^2}{2!} \cdot e^{-0.2}\frac{(0.2)^1}{1!} \approx 0.0355.$$

(b) Note that $\{T_t\}$ is a Poisson process with rate of $2 \cdot 0.3 = 0.6$ birds per hour. Thus $C \overset{d}{=} T_2$ is a Poisson(1.2) random variable.

Problem 4. (20 points)

Two Gila monsters (Heloderma suspectum, a large lizard) share a burrow. One of the Gila monsters is male and the other is female.

Each time the male Gila monster returns to the burrow, the time that he spends at the burrow follows an exponential distribution with a mean of 8 hours. Each time the male Gila monster leaves the burrow, the time that it takes him to return follows an exponential distribution with a mean of 4 hours.

Each time the female Gila monster returns to the burrow, the time that she spends at the burrow follows an exponential distribution with a mean of 6 hours. Each time the female Gila monster leaves the burrow, the time that it takes her to return follows an exponential distribution with a mean of 3 hours.

Each time a Gila monster returns to the burrow (or leaves the burrow), the length of time that it stays at the burrow (or is away from the burrow) is independent of the whether the other Gila monster is there, and independent of the time that each Gila monster has spent at and away form the burrow.

(a) **(12 points)** Model this system as a continuous-time Markov chain by specifying its state space and generator matrix.

(b) **(8 points)** Currently, neither Gila monster is in the burrow. What is the probability that the male Gila monster returns to the burrow before the female Gila monster?

Solution 4.

(a) Let $\mathcal{S} = \{0, M, F, B\}$, where 0 is the state corresponding to neither Gila monster being in the burrow, M is the state corresponding to only the male Gila monster being in the burrow, F is the state corresponding to only the female Gila monster being in the burrow, and B is the state corresponding to both the Gila monsters being in the burrow. Then we can model the system as a CTMC on $\mathcal{S}$ with generator matrix

$$Q = \begin{pmatrix} -\frac{7}{12} & \frac{1}{4} & \frac{1}{3} & 0 \\ \frac{1}{8} & -\frac{11}{24} & 0 & \frac{1}{3} \\ \frac{1}{6} & 0 & -\frac{5}{12} & \frac{1}{4} \\ 0 & \frac{1}{6} & \frac{1}{8} & -\frac{7}{24} \end{pmatrix}$$

(b) Let $\{X_t\}$ denote the state of the system at time t, and let τ_0 denote the holding time of state 0. Then,

$$\mathbb{P}(X_{\tau_0} = M | X_0 = 0) = \frac{Q_{0,M}}{|Q_{0,0}|} = \frac{\frac{1}{4}}{\frac{7}{12}} = \frac{3}{7}.$$

Problem 5. (1 point)

If you have any comments or if there is anything it might be helpful for me to know about the course or the exam please let me know here.

Feel free to leave this blank if you have no comments; you will still get the point for this question.