$\mathbb{P}(|X_n - X| > \varepsilon)$

- Chebyshev's ineq. is usually useful to prove convergence in probability.

Ex: (WLLN) $X_1, \dots, X_n$ iid mean μ, SD σ

$$\mathbb{P}(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n} \to 0.$$

Proving Chebyshev:

$$\mathbb{P}(X - \mu > \varepsilon) \leq \mathbb{P}((X-\mu)^2 > \varepsilon^2)$$
$$\leq \frac{\mathbb{E}((X-\mu)^2)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}$$

Idea: apply Markov's ineq. to $(X-\mu)^2$.

Convergence in dist. (MGF)

Q12: If $n >>> 0$, then $\mathbb{P}(X_n = 1) \approx \frac{1}{2}$

$\mathbb{P}(X_n = -1) \approx 1/2$

This suggests that $X_n \xrightarrow{d} X$, where

$$\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$$

$$m_n(t) = \left(\frac{1}{2} + \frac{1}{n+1}\right)e^t + \left(\frac{1}{2} - \frac{1}{n+1}\right)e^{-t}$$
$$= \underline{e^t + e^{-t}} + \left(\underline{e^t - e^{-t}}\right)$$

$$\frac{\quad}{2} \cdot \left(\frac{\quad}{n+1} \right)$$

$\to e^t \cdot \frac{1}{2} + e^{-t} \cdot \frac{1}{2}$ (mgf of X, w/ $\mathbb{P}(X=1) = \frac{1}{2}$, $\mathbb{P}(X=-1) = \frac{1}{2}$.

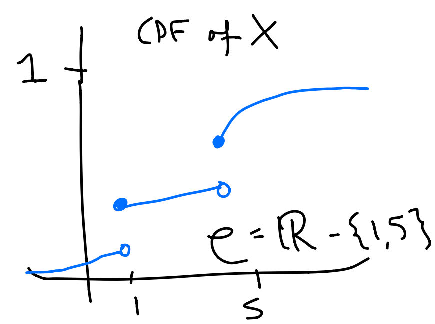
In general: say Y is a discrete RV w/ $\mathbb{P}(Y = a_1) = p_1$, $\mathbb{P}(Y = a_2) = p_2, \dots, \mathbb{P}(Y_n = a_n) = p_n$.

$$\mathbb{E}[e^{tY}] = e^{ta_1} \cdot p_1 + e^{ta_2} \cdot p_2 + \cdots + e^{ta_n} \cdot p_n$$

Convergence in Dist (CDF)

F_n = CDF of X_n

F = CDF of X



We say $X_n \xrightarrow{d} X$ if for all $x \in \mathcal{C} = \{y : F \text{ is cont. at } y\}$, we have $F_n(x) \to F(x)$.

$$F(y) = \mathbb{P}(X \le y) = \begin{cases} 0, & y < -1 \\ \frac{1}{2}, & -1 \le y < 1 \\ 1, & 1 \le y \end{cases}$$

$$F_n(y) = \begin{cases} 0, & y < -1 \\ \frac{1}{2} - \frac{1}{n+1}, & -1 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

(i) $y < -1$

$$F_n(y) = 0 = F(y), \qquad F_n(y) \to F(y)$$

(ii) $-1 < y < 1$

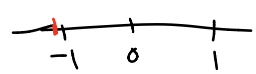
$$F_n(y) = \frac{1}{2} - \frac{1}{n+1} \to \frac{1}{2} = F(y).$$

(iii) $y > 1$

$$F_n(y) = 1 = F(y), \text{ so } F_n(y) \to F(y).$$

$$X_n \xrightarrow{d} X.$$

$$\mathbb{P}(X_n = -1) = \frac{1}{2} - \frac{1}{n+1}$$
$$\mathbb{P}(X_n = 1) = \frac{1}{2} + \frac{1}{n+1}$$



$$\mathbb{P}(X_n \le -2) = 0$$
$$\mathbb{P}(X_n \le -1.00001) = 0$$

Whenever $y < -1$, $\mathbb{P}(X_n \le y) = 0$.

So $F_n(y) = 0$ for $y < -1$.

$$\mathbb{P}(X_n \le -1) = \mathbb{P}(X_n = -1) = \frac{1}{2} - \frac{1}{n+1}$$
$$\mathbb{P}(X_n \le -0.3) = \mathbb{P}(X_n = -1) = \frac{1}{2} - \frac{1}{n+1}$$

So $F_n(y) = \frac{1}{2} - \frac{1}{n+1}$, $-1 \le y < 1$

$\mathbb{P}(X_n \le 20) = 1$

$\mathbb{P}(X_n \le 1.001) = 1$

$\mathbb{P}(X_n \le 1) = 1.$

So $F_n(y) = 1$, for $y \geq 1$.

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(3) BC 1: If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then

$\mathbb{P}$(infinitely many A_n happen) $= 0$.

BC 2: If A_n's are indep, then if

$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then

$\mathbb{P}$(infinitely many A_n occur) $= 1$.

(a) Let A = infinitely many Meow!'s are typed.

Let A_n = "types "Meow!" on her n-th trip she types meow.

$$\mathbb{P}(A_n) = \frac{1}{50} \times \frac{1}{50} \times \frac{1}{50} \times \frac{1}{50} \times \frac{1}{50}$$

$$= \left(\frac{1}{50}\right)^5$$

So $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$; so BC2 tells us that, since A_n's are indep,

$\mathbb{P}$(infinitely many A_n's occur) < 1.

(b) B_n = she types "meow!" on trip n.

$$\mathbb{P}(B_n) = \left(\frac{1}{51+n}\right)^5$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{51+n}\right)^5 < \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^5 < \infty.$$

So $\mathbb{P}$(infinitely many B_n's occur) $= 0$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$\leftarrow$ integral test

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \le \int_1^{\infty} \frac{1}{x^2}\,dx = -x^{-1}\Big|_1^{\infty} < \infty.$$

$\boxed{3.14}$ Let $Z \sim N(0,1)$, let $Y = \mu + \sigma Z$.
Then $Y \sim N(\mu, \sigma^2)$.

$m_Z(s) = e^{s^2/2}$

$$\begin{aligned} m_Y(t) = \mathbb{E}[e^{tY}] &= \mathbb{E}\left[e^{t(\mu+\sigma Z)}\right] \\ &= \mathbb{E}[e^{t\mu} e^{t\sigma Z}] \\ &= e^{t\mu}\,\mathbb{E}[e^{t\sigma Z}] \\ &= e^{t\mu} m_Z(t\sigma) \\ &= e^{t\mu} e^{t^2\sigma^2/2} \end{aligned}$$

$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ indep

$$m_{X+Y}(t) = m_X(t)\, m_Y(t) = e^{t\mu_1} e^{t^2\sigma_1^2/2} e^{t\mu_2} e^{t^2\sigma_2^2/2}$$

X,Y indep

$$= e^{t(\mu_1+\mu_2) + t^2(\sigma_1^2+\sigma_2^2)/2}$$

So $X+Y \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$

<u>Ex:</u> SRW w/ $\mathbb{P}(S_n - S_{n-1} = 1) = 0.8$, $\mathbb{P}(S_n - S_{n-1} = -1) = 0.2$

$$\mathbb{P}_0(S_8 = 6) = \binom{8}{7}(0.8)^7(0.2)^1$$

$$\left.\begin{aligned} u - d &= 6 \ \text{"} \\ u + d &= 8 \ \text{"} \end{aligned}\right\} \rightarrow u = 7,\ d = 1$$

$$\mathbb{P}_0(S_n = x) = \begin{cases} \binom{n}{\frac{1}{2}(n+x)} p^{\frac{1}{2}(n+x)} q^{\frac{1}{2}(n-x)} & \text{if } n+x \text{ is even and } |x| \le n \\ 0 & \text{o.w.} \end{cases}$$