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1. Let x have density function given by

$$f_X(x) = \begin{cases} \frac{1}{15}(2x+6) & -2 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

a) Find distribution function $F_X(x)$ of X.

$$\int_{-2}^{x} \frac{1}{15}(2t+6)\,dt = \frac{1}{15}\int_{-2}^{x}(2t+6)\,dt = \frac{1}{15}\left(t^2+6t\right)\Big|_{-2}^{x}$$

$$= \frac{1}{15}\left[x^2+6x-(4-12)\right] = \boxed{\frac{1}{15}\left[x^2+6x+8\right] \quad \text{for } -2 < x < 1.}$$

b) Find PDF where $Y = 2X+1$.

increasing $\quad y = h(x) = 2x+1 \quad -2 < x < 1 \iff -2 < \frac{y-1}{2} < 1 \iff -3 < y < 3$

$x = \frac{y-1}{2} = h^{-1}(x)$

$$\frac{dh^{-1}(y)}{du} = \frac{1}{2}.$$

$$f_Y(y) = f_X\left(\frac{y-1}{2}\right)\left|\frac{1}{2}\right|$$

$$= \frac{1}{15}\left(2\left(\frac{y-1}{2}\right)+6\right)\cdot\frac{1}{2}$$

$$= \frac{1}{30}(y-1+6)$$

$$= \frac{1}{30}(y+5)$$

$$= \begin{cases} \frac{1}{30}(y+5) & -3 < y < 3 \\ 0 & \text{o.w.} \end{cases}$$

c). Last.

D. X_1 X_2 X_3 be sample of size 3 from population drawn from PDF $f_X(x)$. $X_{(1)} = \min(X_1, X_2, X_3)$ Find $P(X_{(1)} > 0)$

$$P(X_{(1)} > 0) = 1 - P(X_{(1)} \leq 1)$$

$$= 1 - [F_X(1)]^1$$

$$= 1 - \left[\frac{1}{15} [x^2 + 6x + 8]_{x=1} \right]^1$$

$$= 1 - [\tfrac{1}{15}(1 + 6 + 8)]$$

$$\boxed{= 1 - 1 = 0.}$$

2. X be density function given by

$$f_X(x) = \begin{cases} \frac{1}{2^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/2} & x>0 \\ 0 & \text{elsewhere} \end{cases}$$

a. Show that MGF for X is $M_X(t) = (1-2t)^{-\alpha}$.

Gamma dist.

$\beta = \frac{1}{2}$

$$M_X(t) = E(e^{xt})$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \frac{1}{2^{\alpha}\Gamma(\alpha)} x^{(\alpha-1)} e^{(-x/2)}\,dx$$

$$= \frac{1}{2^{\alpha}\Gamma(\alpha)} \int_0^{\infty} e^{tx} x^{(\alpha-1)} e^{(-x/2)}\,dx$$

$$\boxed{\begin{aligned} tx - \tfrac{x}{2} &= u \\ x(t-\tfrac{1}{2}) &= u \end{aligned}}$$

$$= \frac{1}{2^{\alpha}\Gamma(\alpha)} \int_0^{\infty} x^{(\alpha-1)} e^{tx - x/2}\,dx$$

Let $u = x(t-\frac{1}{2}) \Rightarrow -u = x(\frac{1}{2}-t)$

$$\frac{1}{2^{\alpha}\Gamma(\alpha)} \frac{\int_0^{\infty} \left(\frac{u}{t-\frac{1}{2}}\right)^{\alpha-1} e^{-u}\,du}{\Gamma(\alpha)}$$

$$= \frac{(\frac{1}{2})^{\alpha}\Gamma(\alpha)}{\Gamma(\alpha)(\frac{1}{2}-t)^{\alpha}}$$

$$= \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\alpha}$$

$$= \left(\frac{\frac{1}{2}-t}{\frac{1}{2}}\right)^{-\alpha}$$

$= \left(1 - \frac{t}{\frac{1}{2}}\right)^{-\alpha}$

$\boxed{= (1-2t)^{-\alpha}}$

b). Mean of Gamma distribution is β which is 2α.

$\frac{d}{dt}(1-2t)^{-\alpha} = 2\alpha(1-2t)^{-\alpha-1}$

plug in 0.

$= 2\alpha(1)^{-\alpha-1} = 2\alpha$

Variance : $\frac{d}{dt}\left(2\alpha(1-2t)^{-\alpha-1}\right) = 4\alpha(\alpha+1)(1-2t)^{-\alpha-2}$

plug in 0 : $4\alpha(\alpha+1)(1)^{-\alpha-2} = 4\alpha(\alpha+1)$

c). it follows Gamma distribution

3. Let X_1 X_2 X_3 be a random sample of size 4 from $N(0,1)$

$\bar{X} = \frac{1}{4}(X_1+X_2+X_3+X_4)$

Find c s.t.

a). $P(\bar{x} > c) = 0.025$

$\bar{X} \sim N(0,1)$ $\quad P(\bar{x} > c) = P\left(\frac{\bar{x}-0}{\sqrt{1/4}} > \frac{c-0}{\sqrt{1/4}}\right) = P(Z > 2c) = 0.025$

$= P(Z > 1.96) = 0.025$

$2c = 1.96 \Rightarrow \boxed{c = 0.98}$

b). $P[\underbrace{(x_1-\bar{x})^2}_{A_1} + \underbrace{(x_2-\bar{x})^2}_{A_2} + \underbrace{(x_3-\bar{x})^2}_{A_3} + \underbrace{(x_4-\bar{x})^2}_{A_4} > c] = 0.05$

$A_1^2 + A_2^2 + A_3^2 + A_4^2 \sim \chi^2_4$

$P(\chi^2_4 > c) = 0.05$

$P(\chi^2_4 > 9.48773) = 0.05$

$\boxed{c = 9.48773}$

c). $P\left(\frac{X_1^2}{X_2^2+X_3^2} > c\right) = 0.05$

$\frac{X_1^2}{(X_2^2+X_3^2)/2} \sim F_{1,2} \Rightarrow P\left(\frac{2X_1^2}{(X_2^2+X_3^2)} > 2c\right) = 0.05$

$P(F_{1,2} > 18.51) = 0.05$

$2c = 18.51$ $\boxed{c = 9.225}$

d). $\frac{X_1}{\sqrt{X_2/1}} \sim t_1 \Rightarrow \frac{X_1}{|X_2|} \sim t_1$

t-table

$P\left(\frac{X}{|X_2|} > c\right) = P(t_1 > c) = 0.05$

$P(t_1 > 6.314) = 0.05$

$\boxed{c = 6.314}$

E). $P\left(\frac{X_1^2}{X_1^2 + X_2^2} < c\right) = 0.05$

$P\left(\frac{X_1^2 + X_2^2}{X_1^2} > \frac{1}{c}\right) = 0.05$

$P\left(1 + \frac{X_2^2}{X_1^2} > \frac{1}{c}\right) = 0.05$

$P\left(\frac{X_2^2}{X_1^2} > \frac{1}{c} - 1\right) = 0.05$

$P\left(\frac{X_2^2}{X_1^2} > \frac{1}{c} - 1\right) = 0.05$ chi-dist Df: 2

$P\left(\frac{X_2^2}{X_1^2} > 10.5966\right) = 0.05$

$\frac{1}{c} - 1 = 10.5966$

$\boxed{c = \frac{1}{11.5966}}$