Math 4B: Differential Equations

Lecture 21: Introduction to Systems

- Systems of ODEs,
- From High Order ODEs to Systems,
- Linearity, Homogeneity & More!

Systems of ODEs

A ***system of simultaneous first order ordinary differential equations*** has the general form

$$
\begin{aligned}
x_1'(t) &= F_1(t, x_1, x_2, \ldots, x_n) \\
x_2'(t) &= F_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
x_n'(t) &= F_n(t, x_1, x_2, \ldots, x_n),
\end{aligned}
$$

where each x_k is a function of t. If each F_k is a linear function of $x_1, x_2, \ldots, x_n$, then the system of equations is said to be ***linear***; if not, it is ***nonlinear***.

We could generalize this definition to systems of higher order ordinary differential equations.

An Example

Remember our standard mass on a spring second order ODE:

$$mx'' + \gamma x' + kx = 0.$$

If we add a second variable, v (for velocity), then we can write

$$x' = v \qquad \text{and so} \qquad x'' = v'.$$

Thus we can replace x'' and x' in the second order ODE above and solve for v'. We get

$$\begin{aligned} x' &= v \\ v' &= -\frac{k}{m}\,x - \frac{\gamma}{m}\,v. \end{aligned}$$

This is also written as

$$\begin{aligned} x_1' &= \phantom{-\frac{k}{m}\,x_1 - \frac{\gamma}{m}\,}x_2 \\ x_2' &= -\frac{k}{m}\,x_1 - \frac{\gamma}{m}\,x_2 \end{aligned} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{or} \quad \mathbf{x}'(t) = A\mathbf{x}(t).$$

The Plan

Today:

- Introduction to systems (§7.1)
- Examples
- Theorems

Next Time:

- Linear Algebra Review (§§7.2–7.3)
- Math 4A in 50 minutes

In General:

- Leverage our understanding of linear algebra into an understanding of ODEs and their solutions

nth Order ODEs to Systems

Suppose we have an arbitrary nth Order ODE:

$$y^{(n)}(t) = F(t, y, y', \ldots, y^{(n-1)}). \qquad (*)$$

We define

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \ldots, \quad x_n = y^{(n-1)}.$$

This then becomes a system of n first order ODEs;

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= F(t, x_1, x_2, \ldots, x_n) \end{aligned} \qquad (\dagger)$$

In particular, notice that $(\dagger)$ is linear exactly when $(*)$ is linear.

Solutions

By a ***solution*** to a first order system of ODEs

$$\begin{aligned} x_1'(t) &= F_1(t, x_1, x_2, \dots, x_n) \\ x_2'(t) &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n'(t) &= F_n(t, x_1, x_2, \dots, x_n), \end{aligned} \qquad (**)$$

we mean an interval I: $\alpha < t < \beta$ together with n functions

$$x_1(t) = \phi_1(t), \quad x_2(t) = \phi_2(t), \quad \dots, \quad x_n(t) = \phi_n(t)$$

that are differentiable on I and satisfy the system of ODEs at all values of t in I.

We can also specify initial conditions:

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \dots, \quad x_n(t_0) = x_n^0$$

or a point $(t_0, x_1^0, x_2^0, \dots, x_n^0)$.

Existence/Uniqueness of Solutions

Existence/Uniqueness Theorem

Suppose R is a region in space defined by

$$\Big\{(t, x_1, x_2, \ldots, x_n) \,\Big|\, \alpha < t < \beta,\ \alpha_1 < x_1 < \beta_1,$$
$$\alpha_2 < x_2 < \beta_2,\ \ldots,\ \alpha_n < x_n < \beta_n\Big\}$$

containing the point $(t_0, x_1^0, x_2^0, \ldots, x_n^0)$. Suppose further that $F_1, F_2, \ldots, F_n$ and the n^2 first partial derivatives

$$\frac{\partial F_1}{\partial x_1}, \ldots, \frac{\partial F_1}{\partial x_n}, \ldots, \frac{\partial F_n}{\partial x_1}, \ldots, \frac{\partial F_n}{\partial x_n}$$

are all continuous on R. Then in some interval $t_0 - h < t < t_0 + h$, there exists a unique solution to the system $(**)$ through $(t_0, x_1^0, x_2^0, \ldots, x_n^0)$.

Linear Systems

If each F_k is a linear function in $x_1, x_2, \dots, x_n$, then the system is called a ***linear system*** and can be written

$$\begin{aligned} x_1'(t) &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2'(t) &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n'(t) &= p_{n1}(t)x_1 + p_{n2}(t)x_n + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{‡}$$

or $\mathbf{x}'(t) = P(t)\mathbf{x}(t) + \mathbf{g}(t)$.

If the vector $\mathbf{g}(t)$ is zero (that is, if all the functions $g_1(t)$ through $g_n(t)$ are zero), then this system is ***homogeneous***. If not, the system is ***nonhomogeneous***.

Existence/Uniqueness II

Existence/Uniqueness Theorem for Linear Systems

Suppose I is the interval $\alpha < t < \beta$ containing t_0, and suppose the functions $p_{11}(t)$ through $p_{nn}(t)$ and $g_1(t)$ through $g_n(t)$ are continuous on I. Consider initial value problem formed by the linear system (‡) with the initial conditions

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \dots, \quad x_n(t_0) = x_n^0.$$

Then there is a unique solution

$$x_1(t) = \phi_1(t), \quad x_2(t) = \phi_2(t), \quad \dots, \quad x_n(t) = \phi_n(t)$$

satisfying this IVP. Moreover, this solution is defined (and satisfies the IVP) throughout the interval I.