Math 174E
Lecture 7

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References

Hull

Chapters 4.7, 4.8, 4.9, 4.10, 4.12

Determining Zero Rates

n-year **zero rates** (spot rates) can be computed from given Treasury bill and Treasury bond prices (**Treasury zero rates**).

Example 4.8

Consider following market data on the prices of 5 Treasury bonds:

Bond principal ($)	Time to maturity (years)	Annual coupon ($)	Bond price ($)
100	0.25	0	99.6
100	0.50	0	99.0
100	1.00	0	97.8
100	1.50	4	102.5
100	2.00	5	105.0

Coupon payments are semiannually. Compute the 0.25-, 0.5-, 1-, 1.5-, and 2-year annual zero rates which are consistent with the market data above (**bootstrap method**).

Source of table: Hull, Chapter 4.7, Table 4.3, page 86.

Zero Curve

Definition 4.9

A chart showing the annual zero rates (spot rates) as a function of maturity is called the **zero curve** (or zero-coupon yield curve).

Example 4.10

Draw the zero curve for the data from Example 4.8:

Maturity (years)	Annual zero rate (% cont. comp.)
0.25	1.603
0.50	2.010
1.00	2.225
1.50	2.284
2.00	2.416

Interpolate linearly between the given maturities.

Source of table: Hull, Chapter 4.7, Table 4.4, page 87.

Slope of Zero Curve

- upward sloping:
  - the longer the maturity, the higher the yield (p.a.)
  - normal market
  - possibly explanation (among others): market is anticipating a rise in short-term interest rates
- downward sloping:
  - short-term interest rates higher than long-term interest rates
  - inverted market
  - possibly explanation (among others): market is anticipating falling short term interest rates
- flat or hump-shaped

Other factors influencing shape: supply and demand, volatility (risk premium), ...

Duration 1/3

- the **duration** of a bond is a measure of how long the holder of the bond has to wait before receiving the present value of the cash payments
- a zero-coupon bond that lasts n years has a duration of n years
- however, a coupon-bearing bond lasting n years has a duration of less than n years, because the holder receives some of the cash payments prior to year n

Duration 2/3

Suppose that a bond provides the holder with cash flows c_i at time t_i $(1 \leq i \leq n)$. The bond price B and the bond yield y (continuously compounded) are related by (recall Definition 4.6)

$$B = \sum_{i=1}^{n} c_i \cdot e^{-y \cdot t_i}.$$

Definition 4.11

The **duration of the bond** D is defined as

$$D = \frac{\sum_{i=1}^{n} t_i \cdot c_i \cdot e^{-y \cdot t_i}}{B} = \sum_{i=1}^{n} t_i \cdot \left(\frac{c_i \cdot e^{-y \cdot t_i}}{B} \right).$$

Observe that the duration is a *weighted average* of the times when payments are made, with the weight applied to time t_i being equal to the ratio of the present value of the cash flow at time t_i to the bond price (the present value of all cash flows).

Duration 3/3

Application of duration:
When a small change Δy in the bond's yield is considered, it is approximately true that

$$\Delta B \approx \frac{dB}{dy} \cdot \Delta y \qquad (1^{\text{st}} \text{ order approximation}).$$

Since

$$\frac{dB}{dy} = -\sum_{i=1}^{n} t_i \cdot c_i \cdot e^{-y \cdot t_i}$$

we obtain the key **duration relationship**

$$\Delta B \approx -B \cdot D \cdot \Delta y.$$

This is an approximate relationship between percentage changes in a bond price and changes in its yield. Note that there is a *negative* relationship between B and y (see Assignment 3 for an example).

Forward Rates 1/2

Definition 4.12

Forward interest rates are the rates of interest *implied* by current zero rates (spot rates) for periods of time in the future.

Today's **forward rate** for the ***n*-th year**
= interest rate **today** for borrowing or lending/investing for a **one-year period** in $\boldsymbol{n-1}$ **years**
= interest rate today for the period from year $\boldsymbol{n-1}$ to year $\boldsymbol{n}$

Notation:

- $r_0(n)$ = today's n-year spot rate (p.a.)
- $f_0(n-1, n)$ = today's n-th year forward rate (p.a.) for the period from $n-1$ to n (alternatively: $n-1$ year forward rate)

Forward Rates 2/2

Example 4.13

Today's spot rates and corresponding **implied** forward rates:

year (n)	spot rate $r_0(n)$ p.a.	forward rate $f_0(n-1,n)$ p.a.
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

Source of table: Hull, Chapter 4.8, Table 4.6, page 90.

General formula:

$$f_0(T_1, T_2) = \frac{r_0(T_2)\cdot T_2 - r_0(T_1)\cdot T_1}{T_2 - T_1} = r_0(T_2) + (r_0(T_2) - r_0(T_1))\cdot\frac{T_1}{T_2 - T_1}$$

with general maturities T_1 and T_2 (measured in years).

Forward Rate Agreement 1/3

Definition 4.14

A **forward rate agreement (FRA)** is an over-the-counter contract to *fix the interest rate* that will apply to either borrowing or lending a certain principal amount L during a specified future time period from T_1 to T_2.

- most FRAs are based on LIBOR
- it is an agreement where interest at a predetermined **fixed rate** R_K is exchanged for the **floating market interest rate** R_M
- when an FRA is first negotiated the specified fixed interest rate R_K usually equals the **forward rate** R_F, so that the contract has **zero value** when initiated
- FRAs are written on a given underlying **principal** amount L
- FRAs can be used to **hedge interest rate risk** by *locking in* a fixed interest rate R_K

Forward Rate Agreement 2/3

The rates R_K, R_M, R_F are typically expressed with *simple compounding* and a compounding frequency which corresponds to $T_2 - T_1$.

1. A trader who will **borrow** a certain principal amount at the **market rate LIBOR** for a future time period $[T_1, T_2]$ can enter into an FRA where for the specified period $[T_1, T_2]$:
   - she receives LIBOR R_M on the principal amount L
   - she pays the predetermined fixed rate R_K on the principal amount L
   - her payoff (= exchange of interest payments) from the FRA **at time $\boldsymbol{T_2}$** is

     $$L \cdot (R_M - R_K) \cdot (T_2 - T_1)$$

   - she converts uncertain floating LIBOR rate R_M from her **borrowings** into a fixed rate R_K via the FRA
   - she hedges against the risk of *increasing* interest rates
   - this is a **long position** in the FRA

Forward Rate Agreement 3/3

2. A trader who will **lend** a certain principal amount at the **market rate LIBOR** for a future time period $[T_1, T_2]$ can enter into an FRA where for the specified period $[T_1, T_2]$:
   - she pays LIBOR R_M on the principal amount L
   - she receives the predetermined fixed rate R_K on the principal amount L
   - her payoff (= exchange of interest payments) from the FRA **at time $\boldsymbol{T_2}$** is

     $$L \cdot (R_K - R_M) \cdot (T_2 - T_1)$$

   - she converts uncertain floating LIBOR rate R_M from her **lending** into a fixed rate R_K via the FRA
   - she hedges against the risk of *decreasing* interest rates
   - this is a **short position** in the FRA

Forward Rates vs. Future Spot Rates

Typically:

forward interest rates $>$ expected future spot interest rates

(Liquidity preference theory, see Section 4.12 in Hull)

Chapter 5: Determination of Forward and Futures Prices

Hull

Chapter 5.1, 5.2

Introduction

- we examine how **forward prices** (and futures prices) are related to the **spot price** of the underlying asset
- this relationship will determine **arbitrage-free** forward prices (futures prices)
  - *"How has the forward price (futures prices) be determined such that (i) it does not require any upfront cost to enter into a forward contrat (futures contract) and that (ii) there is no arbitrage opportunity neither for the long nor for the short position?"*
- we focus on forward prices because they are easier to analyze than futures prices
  - no daily settlement
  - only one single payment at maturity
- it can be shown that the forward price and futures price of an asset are usually very close when the maturities of the two contracts are the same

Investment vs. Consumption Assets

- **Investment asset**: asset that is held by a significant number of investors/traders solely for investment purposes
  - *Examples:* stocks, bonds, but also some commodities like gold or silver
- **Consumption asset**: asset that is held primarily for consumption (and not for investment purposes)
  - *Examples:* commodities such as copper, crude oil, corn

- important distinction for deriving arbitrage-free forward prices
  - for investment assets, we can use no-arbitrage arguments to determine the forward and futures price from its spot price and other observable market variables
  - for consumption assets, there are some restrictions with using arbitrage arguments

Short Selling

Some of the arbitrage arguments we use in this chapter will involve **short selling**:

Definition 5.1

Short selling ("shorting") involves selling an asset that is not owned.

- short selling is possible for some investment assets (e.g., stocks, bonds), but not all
- sometimes the term "shorting" also loosely refers to "betting on a decline in value/price of an asset"

Procedure of Shorting Shares of a Stock 1/2

- investor instructors a broker to short 500 shares of, say, GameStop
- broker will carry out the instruction by borrowing the shares from someone who owns them (e.g., from another client) and selling them in the stock market in the usual way
- proceeds from selling the shares are transferred to the investor

- at some later stage, the investor will **close out** the position by purchasing 500 shares of GameStop in the market
- these shares are then used to replace the borrowed shares and the short position is closed out

- investor takes a **profit** if the stock price has **declined**, and a **loss** if it has **risen**
- if at any time while the contract is open the broker has to return the borrowed shares and there are no other shares that can be borrowed, *the investor is forced to close out the position, even if not ready to do so*

Procedure of Shorting Shares of a Stock 2/2

- investor with a short position must pay to the broker any income (dividends, interests) that would be received on the securities that have been shorted
- broker transfers this income to the lender

- often a borrowing fee is charged to the investor doing the shorting
- investor with a short position is also required to maintain a **margin account** with the broker (consisting of cash or marketable securities) and **margin calls** occur if the price of the security shorted goes up

- there are regulations on short selling imposed by the SEC (U.S. Securities and Exchange Commission): e.g., restrictions apply if stock price declines by more than 10% during one day; temporary bans on short selling; no "naked short selling" since September 17, 2008 etc.

Short squeeze

> *"In the stock market, a* **short squeeze** *is a rapid increase in the price of a stock owing primarily to an excess of short selling of a stock rather than underlying fundamentals. A short squeeze occurs when there is a lack of supply and an excess of demand for the stock due to short sellers having to buy stocks to cover their short positions."*

Source: Wikipedia.

Examples:

- GameStop short squeeze in January 2021, primarily triggered by the Reddit forum WallStreetBets, led to the share price reaching an all-time intraday high of $483 on January 28, 2021 on the NYSE
- in October 2008, a short squeeze triggered by an attempted takeover by Porsche temporarily drove the shares of Volkswagen from EUR 210.85 to over EUR 1,000 in less than two days, briefly making it the most valuable company in the world