ASSIGNMENT 5 - SOLUTIONS

PSTAT 160B - SPRING 2022
DUE DATE: FRIDAY, MAY 27 AT 11:59PM

Instructions for the homework: Solve all of the homework problems, and submit them on GradeScope. Your reasoning has to be comprehensible and complete.

Homework Problems

Problem 5.1. Let $\{W_t\}$ be an SBM, and denote the first hitting time of state $a \in \mathbb{R}$ by T_a. Calculate the following:

(a) $\mathbb{P}(W_3 \geq 2)$.
(b) $\mathbb{P}(W_3 \geq 2 | W_1 = 1.5)$.
(c) $\mathbb{E}[W_{17} | W_5 = 3]$.

Solution 5.1.

(a) We have

$$\mathbb{P}(W_3 \geq 2) = 0.1241065.$$

(b) We have

$$\mathbb{P}(W_3 \geq 2 | W_1 = 1.5) = \mathbb{P}(W_3 - W_1 \geq 0.5 | W_1 = 1.5) = \mathbb{P}(W_3 - W_1 \geq 0.5) = \mathbb{P}(W_2 \geq 0.5) = 0.3618368.$$

(c) Note that

$$\mathbb{E}[W_{17} | W_5 = 3] = \mathbb{E}[W_{17} - W_5 + 3 | W_5 = 3] = \mathbb{E}[W_{17} - W_5 | W_5 = 3] + 3 = \mathbb{E}[W_{17} - W_5] + 3 = 3,$$

where we have used the independent increments property.

Problem 5.2. Fix $\alpha > 0$ and let $\{W_t\}$ be an SBM. Define the process $\{\hat{W}_t\}$ by

$$\hat{W}_t \doteq \frac{1}{\sqrt{\alpha}} W_{\alpha t}.$$

Show that $\{\hat{W}_t\}$ is an SBM.

Solution 5.2. Note that if $f : [0, \infty) \to \mathbb{R}$ is a continuous function, then

$$f_\alpha(t) \doteq \frac{1}{\sqrt{\alpha}} f(\alpha t), \quad t \geq 0,$$

is continuous as well. To see this, fix $t \geq 0$ and let $t_n \to t$; note that $\alpha t_n \to \alpha t$, so, since f is continuous at αt,

$$\lim_{n\to\infty} f_\alpha(t_n) = \lim_{n\to\infty} \frac{1}{\sqrt{\alpha}} f(\alpha t_n) = \frac{1}{\sqrt{\alpha}} \lim_{n\to\infty} f(\alpha t_n) = \frac{1}{\sqrt{\alpha}} f(\alpha t)$$

From this it follows that

$$\mathbb{P}(\hat{W}_t \text{ is continuous at all } t \geq 0) = 1.$$

Furthermore, if $t_2 - t_1 = u \geq 0$, then

$$\hat{W}_{t_2} - \hat{W}_{t_1} = \frac{1}{\sqrt{\alpha}}(W_{\alpha t_2} - W_{\alpha t_1}) \stackrel{d}{=} \frac{1}{\sqrt{\alpha}} W_{\alpha(t_2-t_1)} = \frac{1}{\sqrt{\alpha}} W_{\alpha u},$$

which shows that $\{\hat{W}_t\}$ has stationary increments. Also, if $s_1 < s_2 \leq t_1 < t_2$, then

$$\hat{W}_{t_2} - \hat{W}_{t_1} = \frac{1}{\sqrt{\alpha}}(W_{\alpha t_2} - W_{\alpha t_1}),$$

and

$$\hat{W}_{s_2} - \hat{W}_{s_1} = \frac{1}{\sqrt{\alpha}}(W_{\alpha s_2} - W_{\alpha s_1}).$$

Since $\{W_t\}$ has independent increments, it follows that $\hat{W}_{t_2} - \hat{W}_{t_1}$ is independent of $\hat{W}_{s_2} - \hat{W}_{s_1}$, meaning that $\{\hat{W}_t\}$ has independent increments. Additionally, for each $t \geq 0$, $W_{\alpha t} \sim \mathcal{N}(0, \alpha t)$, so

$$\hat{W}_t = \frac{1}{\sqrt{\alpha}} W_{\alpha t} \sim \mathcal{N}(0, t).$$

It follows that $\{\hat{W}_t\}$ is an SBM.

Problem 5.3. Let $\{W_t^1\}, \ldots, \{W_t^d\}$ be independent SBMs. The $\mathbb{R}^d$-valued process $\{\boldsymbol{W}_t\}$ defined as

$$\boldsymbol{W}_t \doteq \begin{pmatrix} W_t^1 & \ldots W_t^d . \end{pmatrix}$$

What is the probability distribution of $\boldsymbol{W}_t$? Note that, for each $t \geq 0$, $\boldsymbol{W}_t$ is an $\mathbb{R}^d$-valued random variable.

Solution 5.3. For $a_1, \ldots, a_d \in \mathbb{R}^d$,

$$\sum_{i=1}^{d} a_i W_t^i \sim \mathcal{N}\left(0, t \sum_{i=1}^{d} a_i^2\right),$$

so $\boldsymbol{W}_t$ follows a multivariate normal distribution with mean vector

$$\boldsymbol{\mu} \doteq \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix},$$

and covariance matrix

$$\Sigma = \operatorname{diag}(t, \ldots, t).$$

Problem 5.4. Let $\{W_t\}$ be an SBM. For $s < t$, what is the probability distribution of the $\mathbb{R}^2$-valued random variable (W_s, W_t)?

Solution 5.4. Note that for $a_1, a_2 \in \mathbb{R}$, since $W_t - W_s$ and W_s are independent,

$$a_1 W_s + a_2 W_t = a_2(W_t - W_s) + (a_1 + a_2) W_s \sim \mathcal{N}(0, a_2^2(t-s) + (a_1 + a_2)^2 s).$$

Problem 5.5. Let $\{W_t\}$ be an SBM. Define the process $\{B_t\}$ on the time interval $[0, 1]$ by

$$B_t \doteq W_t - t W_1.$$

(a) What is the probability distribution of B_t?
(b) Briefly explain why $\mathbb{P}(B_1 = 0) = 1$.
(c) At what time is the variance of the process maximized?

Solution 5.5.

(a) Note that, since $W_1 - W_t$ and W_t are independent,
$$B_t = W_t - tW_1 = -t(W_1 - W_t) + (1-t)W_t \sim \mathcal{N}(0, t - t^2).$$
(b) We have
$$B_1 = W_1 - 1W_1 = W_1 - W_1 = 0.$$
(c) From (a), we know that $\mathrm{Var}(B_t) = t - t^2$, so the variance is maximized at $t = \frac{1}{2}$.

Problem 5.6. Recall that the density of an SBM at time $t \geq 0$ is given by
$$f(x,t) \doteq \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$
Show that the density satisfies the following partial differential heat equation:
$$\frac{\partial}{\partial t} f = \frac{1}{2}\frac{\partial^2}{\partial x^2} f.$$
In particular, show that with
$$g(y,s) \doteq \frac{\partial}{\partial t} f(y,t)\bigg|_{t=s},$$
and
$$h(y,s) \doteq \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x,s)\bigg|_{x=y},$$
we have, for all $x \in \mathbb{R}$ and $s \geq 0$ that $g(y,s) = h(y,s)$.

Solution 5.6. Note that
$$g(x,t) = \frac{x^2 - t}{2\sqrt{2\pi} t^{5/2}} \exp\left(\frac{x^2}{2t}\right),$$
and
$$h(x,t) = \frac{1}{2} \cdot \frac{x^2 - t}{\sqrt{2\pi} t^{5/2}} \exp\left(\frac{x^2}{2t}\right) = \frac{x^2 - t}{2\sqrt{2\pi} t^{5/2}} \exp\left(\frac{x^2}{2t}\right),$$
so $g(x,t) = h(x,t)$.