Math 4B: Differential Equations

Lecture 29: Boundary Value Problems

- Boundary Value Problems,
- Homogeneity,
- Eigenvalues, Eigenfunctions, & More!

Today's Plan

Up until now, we've been talking about ***initial value problems*** like

$$\begin{aligned} y'' + p(x)y' + q(x)y &= g(x) \\ y(x_0) &= y_0 \\ y'(x_0) &= y_0'. \end{aligned}$$

Today we'll be considering a different kind of problem, a ***boundary value problem*** on an interval $\alpha < x < \beta$:

$$\begin{aligned} y'' + p(x)y' + q(x)y &= g(x) \\ y(\alpha) &= y_0 \\ y(\beta) &= y_1 \end{aligned}$$

Notice:

- Existence / uniqueness for the IVP holds with relatively minor conditions.
- The BVP can have a unique solution, but might have none or infinitely many. This behavior is like solutions to $A\mathbf{x} = \mathbf{b}$.

Example 1

1. Solve the boundary value problem

$$\begin{aligned} y'' + 3y &= 0 \\ y(0) &= 1 \\ y(\pi) &= 0. \end{aligned}$$

Solution: We already know how to solve the ODE. The equation $r^2 + 3 = 0$ has roots $r = \pm i\sqrt{3}$, so the ODE has solution

$$y(x) = c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x).$$

From $y(0) = 1$, we get $1 = c_1 \cdot 1 + c_2 \cdot 0 = c_1$, or $\boxed{c_1 = 1}$. From $y(\pi) = 0$, we get

$$0 = \cos(\sqrt{3}\,\pi) + c_2 \sin(\sqrt{3}\,\pi) \qquad \Longrightarrow \qquad c_2 = -\frac{\cos(\sqrt{3}\,\pi)}{\sin(\sqrt{3}\,\pi)} = -\cot(\sqrt{3}\,\pi).$$

Thus $\boxed{y(x) = \cos(\sqrt{3}\,x) - \cot(\sqrt{3}\,\pi)\,\sin(\sqrt{3}\,x)}$. **Note:** Unique solution!

Example 2

2. Solve the boundary value problem

$$\begin{aligned} y'' + 9y &= 0 \\ y(0) &= 1 \\ y(\pi) &= a. \end{aligned}$$

Solution: Again we know the solution of the ODE is

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x).$$

The initial conditions require that

$$1 = c_1 \cdot 1 + c_2 \cdot 0 \qquad \text{and} \qquad a = c_1 \cdot (-1) + c_2 \cdot 0.$$

Thus $c_1 = 1$ and $c_1 = -a$, while c_2 can be anything. That is,

- If $a \neq -1$, there are **no solutions**.
- If $a = -1$, there are **infinitely many solutions**.

Homogeneous Boundary Value Problems

Corresponding to a boundary value problem

$$\begin{aligned} y'' + p(x)y' + q(x)y &= g(x) \\ y(\alpha) &= y_0 \\ y(\beta) &= y_1 \end{aligned}$$

we identify the corresponding ***homogeneous*** BVP:

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(\alpha) &= 0 \\ y(\beta) &= 0. \end{aligned}$$

A Few Observations:

- The ***trivial solution*** $y = 0$ is always a solution to this BVP, no matter what $p(x)$ and $q(x)$ are.
- We very rarely care explicitly about the trivial solution.
- The question is: Are there other ***non***-trivial solutions to this BVP?

Example 3

3. Solve the homogeneous boundary value problem

$$\begin{aligned} y'' + 3y &= 0 \\ y(0) &= 0 \\ y(\pi) &= 0. \end{aligned}$$

Solution: Again we know the solution of the ODE is

$$y(x) = c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x).$$

The initial conditions require that

$$\begin{aligned} y(0) = 0 \quad &\implies \quad 0 = c_1 \cdot 1 + c_2 \cdot 0 && \implies \quad c_1 = 0 \\ y(\pi) = 0 \quad &\implies \quad 0 = 0 \cdot \cos(\sqrt{3}\,\pi) + c_2 \sin(\sqrt{3}\,\pi) && \implies \quad c_2 = 0. \end{aligned}$$

Thus the trivial solution $y(x) = 0$ is the **unique solution**.

Example 4

4. Solve the homogeneous boundary value problem

$$\begin{aligned} y'' + 9y &= 0 \\ y(0) &= 0 \\ y(\pi) &= 0. \end{aligned}$$

Solution: Again we know the solution of the ODE is

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x).$$

The initial conditions require that

$$\begin{aligned} y(0) = 0 &\quad \Longrightarrow \quad 0 = c_1 \cdot 1 + c_2 \cdot 0 &\quad \Longrightarrow \quad c_1 = 0 \\ y(\pi) = 0 &\quad \Longrightarrow \quad 0 = 0 \cdot (-1) + c_2 \cdot 0 &\quad \Longrightarrow \quad c_1 = 0. \end{aligned}$$

Thus there are **infinitely many solutions** $y(x) = c_2 \sin(3x)$.

Summary of Examples

Examples **1.** and **2.** are nonhomogeneous boundary value problems and **3.** and **4.** are the corresponding homogeneous boundary value problems.

When the nonhomogeneous BVP has a unique solution (like **1.**), the corresponding homogeneous BVP has only the trivial solution (like **3.**).

When the nonhomogeneous BVP has no solutions (like **2.**) or infinitely many solutions, the corresponding homogeneous BVP has infinitely many solutions (like **4.**).

For the rest of today, we'll focus on this last case: homogeneous boundary value problems.

Eigenvalues

Consider the homogeneous boundary value problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ y(\pi) &= 0. \end{aligned}$$

We've seen this has

- **no non-trivial solutions** if $\lambda = 3$
- **infinitely many solutions** if $\lambda = 9$

We're going to call λ an ***eigenvalue*** if there are non-trivial solutions, and we'll call these non-trivial solutions ***eigenfunctions***.

Question: What are the eigenvalues / eigenfunctions for this BVP?

Case I: $\lambda = 0$

If $\lambda = 0$, then the BVP is

$$\begin{aligned} y'' &= 0 \\ y(0) &= 0 \\ y(\pi) &= 0. \end{aligned}$$

The solutions to this ODE are $y(x) = c_1 x + c_2$. The initial conditions tell us that

$$\begin{aligned} y(0) = 0 &\quad\Longrightarrow\quad 0 = c_1 \cdot 0 + c_2 &\quad\Longrightarrow\quad c_2 = 0 \\ y(\pi) = 0 &\quad\Longrightarrow\quad 0 = c_1 \cdot \pi + 0 &\quad\Longrightarrow\quad c_1 = 0. \end{aligned}$$

Thus $\lambda = 0$ is ***not an eigenvalue*** for this BVP

Case II: $\lambda < 0$

If $\lambda = -\mu^2 < 0$, then the BVP is

$$\begin{aligned} y'' - \mu^2 y &= 0 \\ y(0) &= 0 \\ y(\pi) &= 0. \end{aligned}$$

The solutions to this ODE are $y(x) = k_1 e^{\mu x} + k_2 e^{-\mu x}$, although we'll use $y(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$. The initial conditions tell us that

$$\begin{aligned} y(0) = 0 \quad &\implies \quad 0 = c_1 \cdot 1 + c_2 \cdot 0 &&\implies \quad c_1 = 0 \\ y(\pi) = 0 \quad &\implies \quad 0 = 0 \cdot \cosh(\pi) + c_2 \cdot \frac{e^{\pi} - e^{-\pi}}{2} &&\implies \quad c_2 = 0. \end{aligned}$$

Thus there are ***no negative eigenvalues***.

Case III: $\lambda > 0$

If $\lambda = \mu^2 > 0$, then the BVP is

$$\begin{aligned} y'' + \mu^2 y &= 0 \\ y(0) &= 0 \\ y(\pi) &= 0. \end{aligned}$$

The solutions to this ODE are $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$. The initial conditions tell us that

$$\begin{aligned} y(0) = 0 \quad &\Longrightarrow \quad 0 = c_1 \cdot 1 + c_2 \cdot 0 &&\Longrightarrow \quad c_1 = 0 \\ y(\pi) = 0 \quad &\Longrightarrow \quad 0 = 0 \cdot \cos(\mu\pi) + c_2 \cdot \sin(\mu\pi) &&\Longrightarrow \quad ??? \end{aligned}$$

We have two cases:

- If μ is ***not*** an integer, then $\sin(\mu\pi) \neq 0$ and $c_2 = 0$.
- If μ is an integer, then $\sin(\mu\pi) = 0$ and c_2 is arbitrary. Thus $\boxed{\lambda_n = n^2 \text{ is an eigenvalue}}$ for each positive integer n, with $\boxed{\text{eigenfunction } y_n(x) = \sin(nx)}$.

Boundary Value Problem on $[0, L]$

Suppose we generalize this to the interval $[0, L]$:

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ y(L) &= 0. \end{aligned}$$

Again if $\lambda = 0$ or $\lambda < 0$, there are no non-trivial solutions. That is, there are no eigenvalues $\lambda \leq 0$.

If $\lambda = \mu^2$ (with $\mu > 0$), then again we get

$$\begin{array}{lcl cl} y(0) = 0 & \implies & 0 = c_1 \cdot 1 + c_2 \cdot 0 & \implies & c_1 = 0 \\ y(L) = 0 & \implies & 0 = 0 \cdot \cos(\mu L) + c_2 \cdot \sin(\mu L) & \implies & ??? \end{array}$$

In this case we get $\sin(\mu L) = 0$ if $\mu L = n\pi$ for some integer $n > 0$. Thus the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } n = 1, 2, 3, \ldots$$