Lecture 7: 08/15/22

Example 4.8:

Market data:

bond principal	maturity (years)	annual coupon	price
100	0.25	0	99.6
100	0.5	0	99.0
100	1	0	97.8
100	1.5	4	102.5
100	2	5	105.0

Implied annual n-year zero rates (cont. comp.):

Notation: $r_0(n)$ = today's n-year spot rate

$n = 0.25$: $$100 \cdot e^{-r_0(0.25) \cdot 0.25} = 99.6$$

$$\Leftrightarrow \quad r_0(0.25) = \log\left(\frac{100}{99.6}\right) \cdot \frac{1}{0.25} = 0.01603$$

$\Rightarrow$ 3-month spot rate (p.a.): $r_0(0.25) = 1.603\ \%$ p.a.

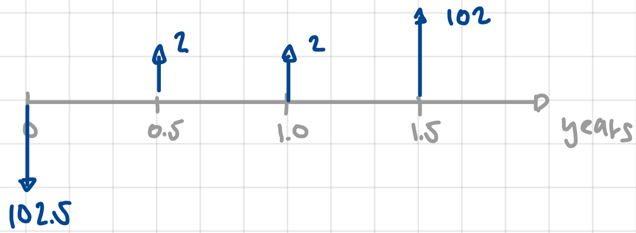
$n = 0.5$: $$100\, e^{-r_0(0.5) \cdot 0.5} = 99.0 \quad \Leftrightarrow \quad r_0(0.5) = \log\left(\frac{100}{99}\right) \cdot \frac{1}{0.5} = 0.02010$$

$\Rightarrow$ 6-month spot rate (p.a.): $r_0(0.5) = 2.010\ \%$ p.a.

$n = 1$: $$100\, e^{-r_0(1) \cdot 1} = 97.8 \quad \Leftrightarrow \quad r_0(1) = \ldots = 0.02225$$

$\Rightarrow$ 1-year spot rate (p.a.): $r_0(1) = 2.225\ \%$ p.a.

$n = 1.5$:

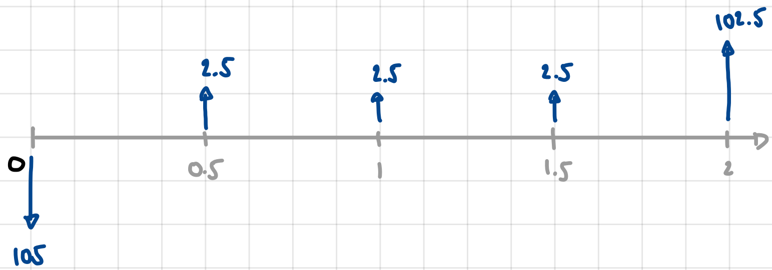


$$102.5 = 2 \cdot e^{-r_0(0.5) \cdot 0.5} + 2 \cdot e^{-r_0(1) \cdot 1} + 102\, e^{-r_0(1.5) \cdot 1.5}$$

$\Leftrightarrow \quad \dots \quad \Leftrightarrow \quad r_0(1.5) = 0.02284$

$\Rightarrow$ 1.5-year spot rate : $\quad r_0(1.5) = 2.284\ \%$ p.a.

$n = 2$:



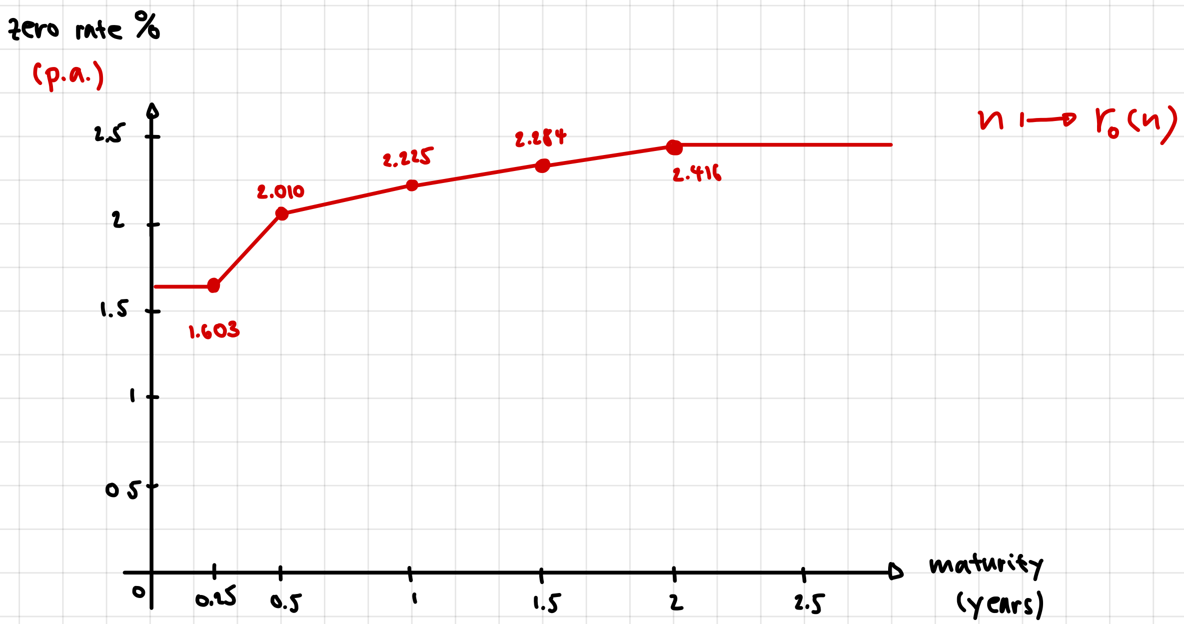
$$105 = 2.5\, e^{-r_0(0.5) \cdot 0.5} + 2.5\, e^{-r_0(1) \cdot 1} + 2.5\, e^{-r_0(1.5) \cdot 1.5} + 102.5\, e^{-r_0(2)\, 2}$$

$\Leftrightarrow \quad r_0(2) = 0\ 02416$

$\Rightarrow$ 2-year spot rate (p.a.) : $\quad r_0(2) = 2.416\ \%$ (p.a.)

Example 4.10:

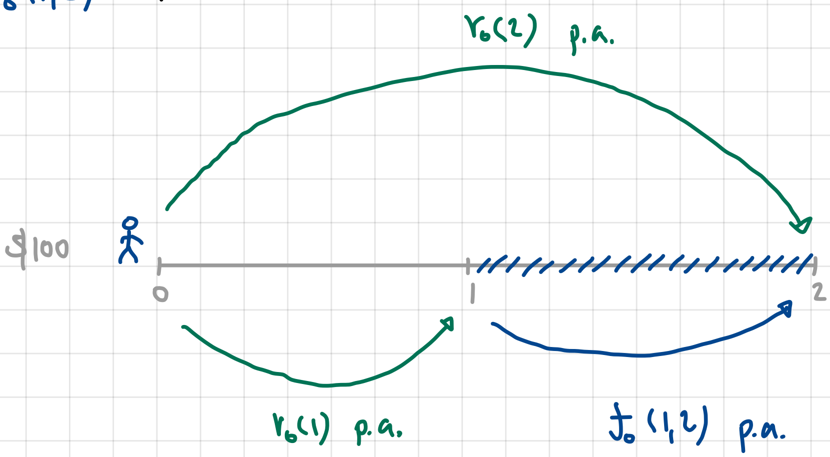
Zero curve: (= term structure of interests)



Interpolated linearly between given maturities, and assume curve to be flat before maturity 0.25 and after maturity 2.

Determination of forward rates:

$n=2$: $\quad f_0(1,2) = \,?$



It must hold true: From today's perspective

$$100 \cdot e^{r_0(2)\cdot 2} \overset{\overset{(i)\quad (ii)}{<\quad >}}{\underset{\uparrow}{=}} \left(100 \cdot e^{r_0(1)\cdot 1}\right) e^{f_0(1,2)\cdot 1}$$

"no arbitrage"

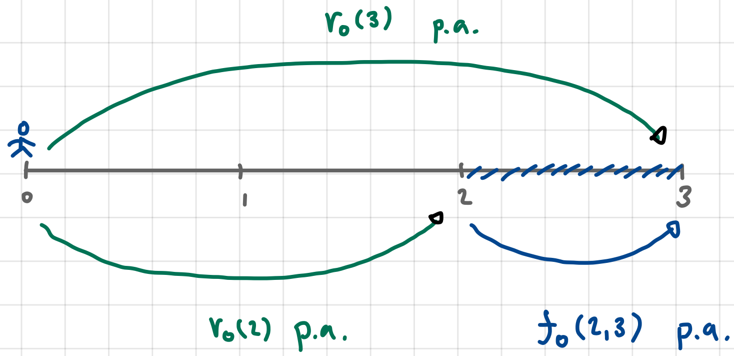
$$\Leftrightarrow \quad r_0(2)\cdot 2 = r_0(1)\cdot 1 + f_0(1,2)\cdot 1$$

$$\Leftrightarrow \quad f_0(1,2) = \frac{r_0(2)\cdot 2 - r_0(1)\cdot 1}{2-1}$$

(i): borrow @ $r_0(2)$ for two years and invest @ $r_0(1)$ for year 1 and @ $f_0(1,2)$ for year 2

(ii): ~~borrow~~ invest @ $r_0(2)$ for two years and ~~invest~~ borrow @ $r_0(1)$ for year 1 and @ $f_0(1,2)$ for year 2

$n=3$: $f_0(2,3)$



It must hold. From today's perspective

$$\$100\, e^{r_0(3)\cdot 3} = \left(\$100\, e^{r_0(2)\cdot 2}\right) e^{f_0(2,3)\cdot 1}$$

"no arbitrage"

General formula :

$$f_0(T_1, T_2) = \frac{r_0(T_2)\cdot T_2 - r_0(T_1)\cdot T_1}{T_2 - T_1}$$