PSTAT 160B - PRACTICE EXAM I
LAST UPDATED August 16, 2022

SUMMER 2022

Directions: By 2:35pm at the latest, you need to upload photos of your scratch work to the 'Exam I' page on GradeScope. If you have no work and simply guessed for a question, write that on your scratch paper and submit a photo of that. Questions for which you do not submit any work will receive 0 points.

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Cheating or otherwise violating the university's academic integrity policies will result in a failing grade in the course and referral to the office of administrative conduct.

(1) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent $\text{Exp}(\eta)$ random variables, and let $Y \sim \text{Gamma}(n, \eta)$. Recall that the density of an $\text{Exp}(\eta)$ random variable is

$$f(x) = \begin{cases} \eta e^{-\eta x}, & x \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and the density of a $\text{Gamma}(n, \eta)$ random variable is

$$g(x) = \begin{cases} \frac{\eta^n}{(n-1)!} x^{n-1} e^{-\eta x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let $S_n \doteq \sum_{i=1}^{n} X_i$.

Which of the following statements are true (select all that apply)?

(a) $\mathbb{P}(S_n \geq 5) = \int_5^\infty \frac{\eta^n}{(n-1)!} x^{n-1} e^{-\eta x} dx$

(b) $\mathbb{P}(S_n \leq 5) = \int_0^5 \eta e^{-\eta x} dx$

(c) For all $t > 0$, $\mathbb{E}(e^{tY}) = \left(\frac{\eta}{\eta - t}\right)^n$

(d) $\mathbb{E}(Y^2) = \left(\frac{n}{\lambda}\right)^2$

(e) For each $t \geq 0$, $\mathbb{P}(S_n \leq t) = \mathbb{P}(Y \leq t)$.

(f) The process $\{S_n\}_{n=1}^{\infty}$ is a Poisson process with rate η.

(g) None of the above.

a e

$N_t^1 \sim PP(\lambda)$ $N_t^2 \sim PP(\mu)$

(2) Planes arrive at an airport according to a Poisson process with rate λ, and helicopters arrive at the airport according to a Poisson process with rate μ. Additionally, the helicopters and planes arrive independently of each other.

$e^{-\mu} \cdot \frac{(\mu)^3}{3!} \cdot e^{-3\mu} \cdot \frac{(3\mu)^4}{4!}$

$e^{-4\mu}\ \mu^7 \cdot \frac{1}{3!} \cdot \frac{3^4}{4!}$

$6 \cdot 24$ $1\times2\times3$ $1\times2\times3\times4$

$\frac{81}{6 \cdot 24} = \frac{81}{144}$

Which of the following statements are true (select all that apply)?

(a) The total number of aircraft arrivals is a Poisson process with rate $\lambda + \mu$.

$P(N_1=3, N_4=7) = P(N_1=3, N_3=4) = P(N_1=3) \cdot P(N_3=4) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$

(b) The probability that exactly three helicopters have arrived by time $t = 1$ and exactly 7 helicopters have arrived by time $t = 4$ is equal to $\mu^7 e^{-4\mu}\frac{3^6}{6!}$ $\cdot \frac{3^4}{1\times2\times\cancel{3}\times4\times5\times2}$ $16\times$

(c) Given that 10 aircraft have arrived by time $t = 3$, the probability that 8 of the aircraft were planes is equal to $e^{-3\lambda}\frac{(3\lambda)^8}{8!}e^{-3\mu}\frac{(3\mu)^2}{2!}$.

$P(N_3^1=8 \mid N_3=10) = P(N_3^1=8 \mid N_3^1+N_3^2=10) = \frac{P(N_3^1=8,\ N_3^1+N_3^2=10)}{P(N_3=10)}$

$\frac{P(N_3^1=8) \cdot P(N_3^2=2)}{P(N_3=10)}$

$\frac{e^{-3\lambda}\frac{(3\lambda)^8}{8!} \cdot e^{-3\mu}\frac{(3\mu)^2}{2!}}{e^{-3(\mu+\lambda)} \cdot \frac{(3\mu+3\lambda)^{10}}{10!}}$

(d) The probability that it will take more than 3 hours for any aircraft to arrive given that the first aircraft that arrives will be a helicopter is equal to $e^{-3\mu}(\lambda + \mu)^{-1}$.

$\underset{\lambda+\mu}{\sim}$ $P(\min \ >3)$

(e) The probability that it will take more than 3 hours for any aircraft to arrive given that the first aircraft that arrives will be a helicopter is equal to $e^{-3\lambda}\lambda^{-1}$.

(f) The probability that it will take more than 3 hours for any aircraft to arrive given that the first aircraft that arrives will be a helicopter is equal to $e^{-3(\mu+\lambda)}$.

(g) None of the above.

$N_t^1(\frac{1}{20}) + N_t^2(\frac{1}{10}) = N_t(\frac{3}{20})$

$\sim$

$\alpha_1 \ldots \alpha_{10} \sim Exp(\frac{1}{20})$ $\beta_1, \ldots \beta_5 \sim Exp(\frac{1}{10})$

(3) 15 students in PSTAT 160B are taking an exam that is too difficult. Since the exam is so difficult, every student eventually gives up and leaves the classroom.

10 of the students are PSTAT majors; the time that it takes for each of these students to leave the classroom follows an exponential distribution with a mean of 20 minutes. The other 5 students are math majors; the time that it takes for each of these students to leave the classroom follows an exponential distribution with a mean of 10 minutes.

Additionally, the time that it takes for each student to leave the classroom is independent of the other students.

Which of the following statements are true (select all that apply)?

$exp(\frac{10}{20} + \frac{5}{10})$

(a) The number of students who have left within the first 15 minutes follows a Poisson(1.5) distribution.

$\{N_t\} \sim PP(\lambda)$

(b) Given that no students leave within the first 30 minutes, the probability that the first student who leaves is a PSTAT major is 1/2. $P(\min\{\alpha_1 \ldots \beta_5\} \mid \min > 30)$

(c) The probability that the first student who leaves is a math major is 1/2.

$\min\{\alpha_1 .. \alpha_{10}\} \mid \min\{\alpha_1 .. \beta_5\} \leq 30$

???

$N_t \sim PP(\frac{3}{30})$

(d) The probability that exactly one student leaves within the first 30 minutes is $30e^{-30}$.

(e) None of the above.

$P(N_{30} = 1) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$

$= e^{-30}(\frac{30}{1!}) = 30e^{-30}$

$N_t = N_t^1 + N_t^2 \sim PP(\lambda_1 + \lambda_2)$

(4) Consider a CTMC $\{X_t\}_{t\geq 0}$ on $\mathcal{S} = \{1,2,3\}$ with generator matrix

$$Q = \begin{pmatrix} ?\,{-4} & 3 & 1 \\ 2 & -5 & 3 \\ 0 & 1 & -1 \end{pmatrix}. \quad = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & \frac{3}{4} & \frac{1}{4} \\ 2 & \frac{2}{5} & 0 & \frac{3}{5} \\ 3 & 0 & 1 & 0 \end{array}$$

Which of the following statements are true (select all that apply)?

(a) The entry denoted by "?" is equal to 4.

(b) Given that $X_4 = 2$, the probability that the next state that $\{X_t\}_{t\geq 0}$ moves to is state 1 is equal to $\frac{2}{3}$.

(c) Given that $X_4 = 2$, the probability that the next state that $\{X_t\}_{t\geq 0}$ moves to is state 1 is equal to $\frac{2}{5}$.

(d) Given that the CTMC is in state 1 at time t_0, we expect that, on average, it will remain in state 1 until time $t_0 + 3$.

why not $t_0 + 1$

(e) Given that the CTMC is in state 1 at time t_0, we expect that, on average, it will remain in state 1 until time $t_0 + 4$.

c

e

(f) None of the above.

(5) Which of the following plots most closely resembles a *typical* trajectory of a Poisson process with rate 5?

PP(5)

Figure 1. (a)

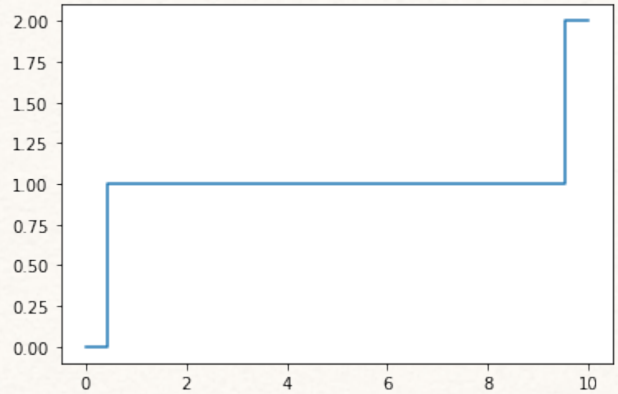


Figure 2. (b)

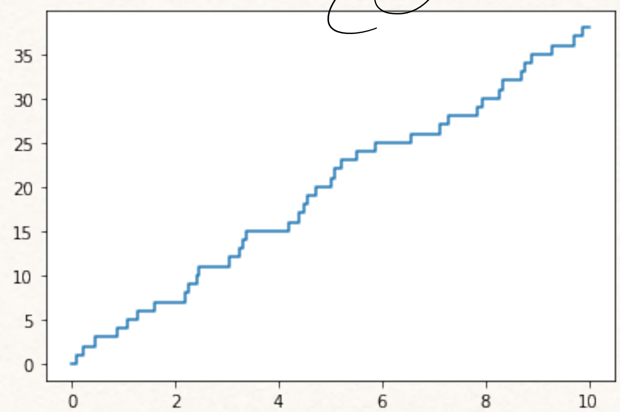


Figure 3. (c)

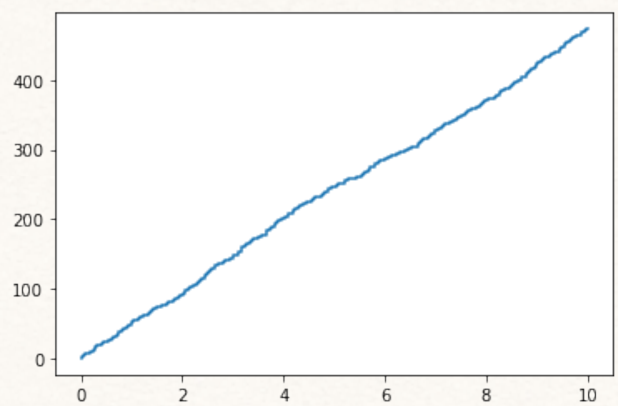
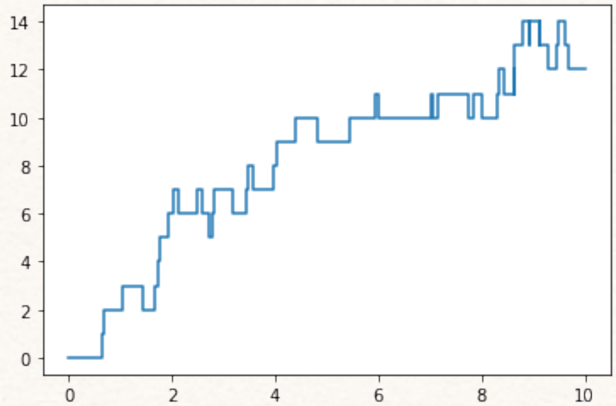


Figure 4. (d)



Q3: 10 pstat: $\alpha_1, \alpha_2, \ldots \alpha_{10} \overset{iid}{\sim} Exp(\frac{1}{20})$ $N_t^1 \sim PP(\frac{1}{20})$

5 math: $\alpha_1 \ldots \alpha_5 \overset{iid}{\sim} Exp(\frac{1}{10})$ $N_t^2 \sim PP(\frac{1}{10})$

$N_t = N_t^1 + N_t^2 \sim PP(\frac{3}{20})$ $\frac{1}{20} + \frac{1}{10} = \frac{3}{20}$

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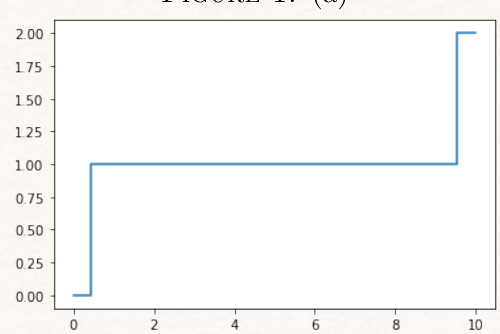

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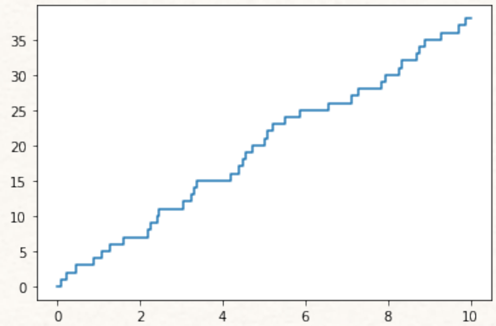

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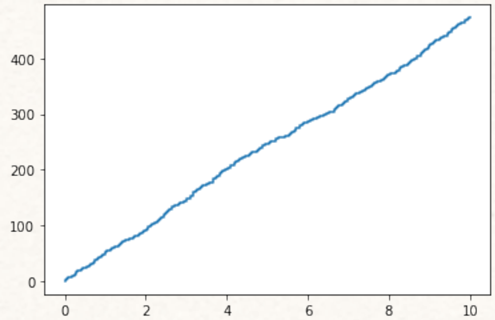

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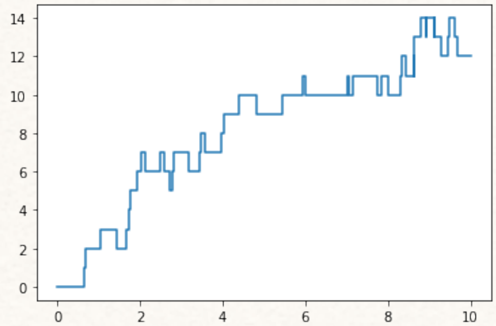

(6) Let $\{N_t\}$ be a Poisson process with rate λ. Consider the process $\{X_t\}$ defined by

$$X_t \doteq \exp(N_t).$$

Which of the following statements are true (select all that apply)?

(a) $\{X_t\}$ has stationary increments.

(b) $\{X_t\}$ has independent increments.

(c) $\{X_t\}$ is a time-homogeneous CTMC on $\mathcal{S} = \{0, e, e^2, \dots\}$.

$e^{\lambda t} \cdot (e^{e} - e^{1})$

(d) $\{X_t\}$ is a time-inhomogeneous CTMC on $\mathcal{S} = \{0, e, e^2, \dots\}$.

(e) For each $t \geq 0$, $\mathbb{E}(X_t) = e^{\lambda t(e-1)}$.

$E(e^{N_t}) = e^{E(N_t)}$ $\quad e^{\lambda t}$

(7) There are three cashiers at a bank. Each cashier takes an Exp(μ) length of time to help a customer, and the service time of each customer is independent of the number of customers and the other service times. Additionally, customers arrive according to a Poisson process with a rate of λ. If a customer arrives and all three cashiers are busy helping other customers, the new customer will wait in line. However, if there are already two customers waiting in line, the new customers will immediately leave.

$q_{i,i+1} = \lambda$

Can this system be modeled as a CTMC? If so, specify the generator matrix.

0 0 0
1 1 1
1
1

0 1 2 3 4 5 6 . 7 . . .

0
1
2
3
4
5 $\quad 3\mu \quad -(3\mu+\lambda) \quad \lambda$
6 $\quad 3\mu \quad \lambda$
7

Why not λ

Why not infinite