Math 4B: Differential Equations

Lecture 02: Solutions & Terminology

- Solutions of DEs
- Some New Terminology,
- & More!

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Solutions & Terminology

Differential equation of order n

We will say that a differential equation is order n if it can be written

$$F\big(t, y, y', y'', y^{(3)}, \ldots, y^{(n)}\big) = 0.$$

In this course we will also assume that this can be written as

$$y^{(n)} = G\big(t, y, y', y'', y^{(3)}, \ldots, y^{(n-1)}\big). \qquad (*)$$

A ***solution*** to equation $(*)$ on the interval $\alpha < t < \beta$ is a function $y = \phi(t)$ such that $\phi', \phi'', \ldots, \phi^{(n)}$ are all defined and

$$\phi^{(n)} = G\big(t, \phi, \phi', \phi'', \phi^{(3)}, \ldots, \phi^{(n-1)}\big)$$

for all t in the interval $\alpha < t < \beta$.

Linear ODEs

Linear ODEs of order n

We will say that a differential equation or order n is ***linear*** if it can be written

$$F\big(t, y, y', y'', y^{(3)}, \ldots, y^{(n)}\big) = 0,$$

where F is linear in y and its derivatives. This means linear first-order ODEs are those of the form

$$a_1(t)\frac{dy}{dt} + a_0(t)y = g(t).$$

Similarly, linear second-order ODEs are those of the form

$$a_2(t)\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = g(t).$$

Homogeneous ODEs

Homogeneous ODEs of order n

We will say that a linear differential equation or order n is ***homogeneous*** if it is of the form

$$a_n(t)y^{(n)}+a_{n-1}(t)y^{(n-1)}+\cdots+a_2(t)y''+a_1(t)y'+a_0(t)y=0,$$

where each a_k is a function of the independent variable t. This means homogeneous linear first-order ODEs are those of the form

$$a_1(t)\frac{dy}{dt}+a_0(t)y=0.$$

Similarly, homogeneous linear second-order ODEs are those of the form

$$a_2(t)\frac{d^2y}{dt^2}+a_1(t)\frac{dy}{dt}+a_0(t)y=0.$$

Solutions of ODEs and IVPs

- What can we say about solutions of ODEs?
- Can we find solutions of ODEs?
- In the Direction Fields slides, we've seen that there are ***lots*** of solutions.
- An ODE like $\frac{dy}{dt} = 2(y-1)$ together with an ***initial condition*** like $y(0) = 5$ is called an ***initial value problem*** (or IVP).
- We might expect that there is exactly one solution to the IVP given above. (This is two statements: There's at least one solution. There are not two different solutions.)
- This isn't true in general – that IVPs always have a unique solution – but we'll see some conditions this quarter. If we are modeling physical phenomena, we expect solutions.

Examples of Solutions

Question: Which of the following ODEs has $y = \sin(t)$ as a solution?

(A) $(y')^2 + y^2 = 1$ (B) $y' = y$ (C) $y'' = y$ (D) $y'' = -y$

Answer: Both $\boxed{\text{A}}$ and $\boxed{\text{D}}$:

$$y = \sin(t) \qquad y' = \cos(t) \qquad y'' = -\sin(t)$$

So...

(A) says $\big(+ \cos(t)\big)^2 + \big(\sin(t)\big)^2 = 1$ True!

(B) says $\cos(t) = \sin(t)$ False!

(C) says $-\sin(t) = \sin(t)$ False!

(D) says $-\sin(t) = -\big(\sin(t)\big)$ True!

Examples of Solutions II

Question: Which of the following functions is a solution of the IVP

$$\begin{cases} y' = \frac{1}{2}y - t \\ y(0) = 6 \end{cases}$$

(A) $y = 2t + 6$ (B) $y = 2t + 6 - 3e^{t/2}$

(C) $y = 6e^{t/2}$ (D) $y = 2t + 4 + 2e^{t/2}$

(A) $y' = 2$ but $\frac{1}{2}y - t = \big(t + 3\big) - t = 3 \neq 2$

(B) $y(0) = 0 + 6 - 3 = 3 \neq 6$

(C) $y' = 3e^{t/2}$ but $\frac{1}{2}y - t = 3e^{t/2} - t$

(D) satisfies both!

Answer: $\boxed{\text{D}}$