ASSIGNMENT 4 - KEY

PSTAT 160B - SUMMER 2022

Instructions for the homework: Solve all of the homework problems, and submit them on GradeScope. Your reasoning has to be comprehensible and complete.

Homework Problems

Problem 5.1. Let $\{W_t\}$ be an SBM, and denote the first hitting time of state $a \in \mathbb{R}$ by T_a. Calculate the following:

(a) $\mathbb{P}(W_3 \geq 2)$.

(b) $\mathbb{P}(W_3 \geq 2 | W_1 = 1.5)$.

(c) $\mathbb{E}[W_{17} | W_5 = 3]$.

Solution 5.1.

(a) We have

$$\mathbb{P}(W_3 \geq 2) = 0.1241065.$$

(b) We have

$$\mathbb{P}(W_3 \geq 2 | W_1 = 1.5) = \mathbb{P}(W_3 - W_1 \geq 0.5 | W_1 = 1.5) = \mathbb{P}(W_3 - W_1 \geq 0.5) = \mathbb{P}(W_2 \geq 0.5) = 0.3618368.$$

(c) Note that

$$\mathbb{E}[W_{17} | W_5 = 3] = \mathbb{E}[W_{17} - W_5 + 3 | W_5 = 3] = \mathbb{E}[W_{17} - W_5 | W_5 = 3] + 3 = \mathbb{E}[W_{17} - W_5] + 3 = 3,$$

where we have used the independent increments property.

Problem 5.2. Fix $\alpha > 0$ and let $\{W_t\}$ be an SBM. Define the process $\{\hat{W}_t\}$ by

$$\hat{W}_t \doteq \frac{1}{\sqrt{\alpha}} W_{\alpha t}.$$

Show that $\{\hat{W}_t\}$ is an SBM.

Solution 5.2. Note that if $f : [0, \infty) \to \mathbb{R}$ is a continuous function, then

$$f_\alpha(t) \doteq \frac{1}{\sqrt{\alpha}} f(\alpha t), \quad t \geq 0,$$

is continuous as well. To see this, fix $t \geq 0$ and let $t_n \to t$; note that $\alpha t_n \to \alpha t$, so, since f is continuous at αt,

$$\lim_{n\to\infty} f_\alpha(t_n) = \lim_{n\to\infty} \frac{1}{\sqrt{\alpha}} f(\alpha t_n) = \frac{1}{\sqrt{\alpha}} \lim_{n\to\infty} f(\alpha t_n) = \frac{1}{\sqrt{\alpha}} f(\alpha t)$$

From this it follows that

$$\mathbb{P}(\hat{W}_t \text{ is continuous at all } t \geq 0) = 1.$$

Furthermore, if $t_2 - t_1 = u \geq 0$, then

$$\hat{W}_{t_2} - \hat{W}_{t_1} = \frac{1}{\sqrt{\alpha}}(W_{\alpha t_2} - W_{\alpha t_1}) \overset{d}{=} \frac{1}{\sqrt{\alpha}} W_{\alpha(t_2 - t_1)} = \frac{1}{\sqrt{\alpha}} W_{\alpha u},$$

which shows that $\{\hat{W}_t\}$ has stationary increments. Also, if $s_1 < s_2 \leq t_1 < t_2$, then

$$\hat{W}_{t_2} - \hat{W}_{t_1} = \frac{1}{\sqrt{\alpha}}(W_{\alpha t_2} - W_{\alpha t_1}),$$

and

$$\hat{W}_{s_2} - \hat{W}_{s_1} = \frac{1}{\sqrt{\alpha}}(W_{\alpha s_2} - W_{\alpha s_1}).$$

Since $\{W_t\}$ has independent increments, it follows that $\hat{W}_{t_2} - \hat{W}_{t_1}$ is independent of $\hat{W}_{s_2} - \hat{W}_{s_1}$, meaning that $\{\hat{W}_t\}$ has independent increments. Additionally, for each $t \geq 0$, $W_{\alpha t} \sim \mathcal{N}(0, \alpha t)$, so

$$\hat{W}_t = \frac{1}{\sqrt{\alpha}} W_{\alpha t} \sim \mathcal{N}(0, t).$$

It follows that $\{\hat{W}_t\}$ is an SBM.

Problem 5.3. Let $\{W_t^1\}, \ldots, \{W_t^d\}$ be independent SBMs. The $\mathbb{R}^d$-valued process $\{\boldsymbol{W}_t\}$ defined as

$$\boldsymbol{W}_t \doteq \begin{pmatrix} W_t^1 & \ldots W_t^d . \end{pmatrix}$$

What is the probability distribution of $\boldsymbol{W}_t$? Note that, for each $t \geq 0$, $\boldsymbol{W}_t$ is an $\mathbb{R}^d$-valued random variable.

Solution 5.3. For $a_1, \ldots, a_d \in \mathbb{R}^d$,

$$\sum_{i=1}^{d} a_i W_t^i \sim \mathcal{N}\left(0, t\sum_{i=1}^{d} a_i^2\right),$$

so $\boldsymbol{W}_t$ follows a multivariate normal distribution with mean vector

$$\boldsymbol{\mu} \doteq \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix},$$

and covariance matrix

$$\Sigma = \operatorname{diag}(t, \ldots, t).$$

Problem 5.4. Let $X, X_1, X_2, \ldots, X_d$ be a collection of iid $\mathcal{N}(\mu, \sigma^2)$ random variables.

(a) Let $\mathbb{X} = \begin{pmatrix} X & X & \ldots & X \end{pmatrix} \in \mathbb{R}^d$. Determine the probability distribution of $\mathbb{X}$.

(b) Let $\mathbb{Y} = \begin{pmatrix} X_1 & X_2 & \ldots & X_d \end{pmatrix} \in \mathbb{R}^d$. Determine the probability distribution of $\mathbb{Y}$.

Solution 5.4.

(a) Here $\boldsymbol{X} \sim \mathcal{N}_d(\boldsymbol{\mu}, \Sigma^2)$, where

$$\boldsymbol{\mu} \doteq \begin{pmatrix} \mu & \mu & \dots & \mu \end{pmatrix},$$

and

$$\Sigma_{i,j} = \sigma^2, \quad 1 \le i, j \le d.$$

(b) Here $\boldsymbol{X} \sim \mathcal{N}_d(\boldsymbol{\mu}, \Sigma^2)$, where

$$\boldsymbol{\mu} \doteq \begin{pmatrix} \mu & \mu & \dots & \mu \end{pmatrix},$$

and

$$\Sigma_{i,j} = \begin{cases} \sigma^2 & i = j \\ 0 & i \neq j \end{cases}$$

Problem 5.5. Let $\{W_t\}$ be an SBM. For $s < t$, what is the probability distribution of the $\mathbb{R}^2$-valued random variable (W_s, W_t)?

Solution 5.5. Note that for $a_1, a_2 \in \mathbb{R}$, since $W_t - W_s$ and W_s are independent,

$$a_1 W_s + a_2 W_t = a_2(W_t - W_s) + (a_1 + a_2) W_s \sim \mathcal{N}(0, a_2^2(t-s) + (a_1 + a_2)^2 s).$$

Thus (W_s, W_t) is multivariate normal, with mean vector $\boldsymbol{\mu} = (0, 0)$, and covariance matrix

$$\Sigma = \begin{pmatrix} s & \min\{s,t\} \\ \min\{s,t\} & t \end{pmatrix}.$$

Problem 5.6. Let $\{W_t\}$ be an SBM. Define the process $\{B_t\}$ on the time interval $[0, 1]$ by

$$B_t \doteq W_t - tW_1.$$

(a) What is the probability distribution of B_t?

(b) Briefly explain why $\mathbb{P}(B_1 = 0) = 1$.

(c) At what time is the variance of the process maximized?

Solution 5.6.

(a) Note that, since $W_1 - W_t$ and W_t are independent,

$$B_t = W_t - tW_1 = -t(W_1 - W_t) + (1 - t)W_t \sim \mathcal{N}(0, t - t^2).$$

(b) We have

$$B_1 = W_1 - 1W_1 = W_1 - W_1 = 0.$$

(c) From (a), we know that $\mathrm{Var}(B_t) = t - t^2$, so the variance is maximized at $t = \frac{1}{2}$.

Problem 5.7. Let $\{X_n\}$ be a sequence of iid random variables such that

$$\mathbb{P}(X_n \ge 0) = 1, \quad \mathbb{E}(X_n) = 1.$$

Let $M_n \doteq \prod_{i=1}^{n} X_i$. Show that $\{M_n\}$ is a martingale with respect to $\{X_n\}$.

Solution 5.7. Note that, since the $\{X_n\}$ are independent and non-negative,

$$\mathbb{E}[|M_n|] = \mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} \mathbb{E}[X_i] = 1 < \infty.$$

Additionally,

$$\begin{aligned}
\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] &= \mathbb{E}\left[X_{n+1}\left(\prod_{i=1}^{n} X_i\right)|X_0, \ldots, X_n\right] \\
&= \left(\prod_{i=1}^{n} X_i\right)\mathbb{E}[X_{n+1}|X_0, \ldots, X_n] \\
&= M_n \mathbb{E}[X_{n+1}] \\
&= M_n,
\end{aligned}$$

so $\{M_n\}$ is a martingale with respect to $\{X_n\}$.

Problem 5.8. Let $\{W_t\}$ be an SBM.

(a) Using Itô's lemma, show that

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}.$$

(b) Using Itô's lemma, show that

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds.$$

Solution 5.8.

(a) See the lecture notes; we use Itô's formula to evaluate

$$f(W_t) = \frac{W_t^2}{2}.$$

(b) Let $f(x) \doteq \frac{x^3}{3}$. Then, Itô's formula tells us that

$$\begin{aligned}
d\left(\frac{W_t^3}{3}\right) &= df(W_t) \\
&= f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt \\
&= W_t^2 dW_t + W_t dt.
\end{aligned}$$

Thus,

$$\frac{W_t^3}{3} = \int_0^t W_s^2 dW_s + \int_0^t W_s ds,$$

so rearranging yields

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds.$$

Problem 5.9. Let $\{W_t\}$ be an SBM. Consider a process $\{X_t\}$ satisfying the SDE

$$
\begin{aligned}
dX_t &= \alpha dW_t + \beta dt \\
X_0 &= x_0,
\end{aligned}
$$

where $\alpha, \beta, x_0 > 0$. Let $Y_t \doteq \exp(\gamma X_t)$, where $\gamma > 0$. By applying Itô's formula, find the SDE solved by $\{Y_t\}$. That is, "calculate" dY_t.

Solution 5.9. Let $f(x) = \exp(\gamma x)$, so that $f'(x) = \gamma \exp(\gamma x)$, and $f''(x) = \gamma^2 \exp(\gamma x)$. Then, Itô's formula says that, with $Y_t \doteq f(X_t)$, that, with $b(x) \doteq \beta$, and $\sigma(x) \doteq \alpha$,

$$
\begin{aligned}
d(Y_t) &= f'(X_t)b(X_t)dt + f'(X_t)\sigma(X_t)dW_t + \frac{1}{2}f''(X_t)\sigma^2(X_t)dt \\
&= \beta\gamma \exp(\gamma X_t)dt + \alpha\gamma \exp(\gamma X_t)dW_t + \frac{1}{2}\alpha^2\gamma^2 \exp(\gamma X_t)dt \\
&= \beta\gamma Y_t dt + \alpha\gamma Y_t dW_t + \frac{1}{2}\alpha^2\gamma^2 Y_t dt \\
&= \left(\beta\gamma + \frac{1}{2}\alpha^2\gamma^2\right) Y_t dt + \alpha\gamma Y_t dW_t.
\end{aligned}
$$

Problem 5.10. Let $\{W_t\}$ be an SBM. Solve the SDE

$$
\begin{aligned}
dX_t &= 3X_t^{\frac{2}{3}} dW_t + 3X_t^{\frac{1}{3}} dt \\
X_0 &= 0.
\end{aligned}
$$

Solution 5.10. See the Practice Questions.

Optional Problems (these use a version of Itô's formula we will see Tuesday)

Problem 5.11. Recall that if $\{W_t\}$ is an SBM, and $\{Y_t\}$ is a process for which the Itô integral

$$I_t \doteq \int_0^t Y_s dW_s,$$

is defined, then $\{I_t\}$ is a martingale. Using this and Itô's formula, show that the process $\{X_t\}$ defined by

$$X_t \doteq \exp\left(\frac{t}{2}\right)\cos(W_t), \quad t \geq 0,$$

is a martingale.

Hint: apply Itô's formula to the function $f(t,x) \doteq \exp\left(\frac{t}{2}\right)\cos(x)$.

Problem 5.12. Let $\{W_t\}$ be an SBM. Show that the process $\{X_t\}$ defined by

$$X_t = \mu + (x_0 - \mu)\exp(-rt) + \sigma\int_0^t \exp(-r(t-s))dW_s,$$

satisfies the SDE

$$\begin{aligned} dX_t &= -r(X_t - \mu)dt + \sigma W_t \\ X_0 &= x_0. \end{aligned}$$

Hint: apply Itô's formula to the function $f(t,x) \doteq \exp(rt)x$.