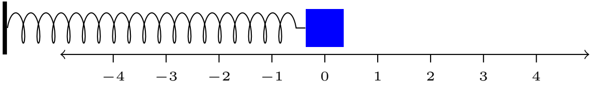
Math 4B: Differential Equations

Lecture 14: Repeated Roots

- Repeated Roots,
- Reduction of Order,
- Some examples & More!

Once More to Mass & Spring

A mass on a spring:



Again we'll focus on the homogeneous case (no external forcing):

$$mx'' + \gamma x' + kx = 0.$$

We saw to solve this we got $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where r_1, r_2 are roots of the ***characteristic equation***

$$mr^2 + \gamma r + k = 0 \qquad \Longrightarrow \qquad r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

So there are three cases:

- $\gamma^2 - 4mk > 0 \implies$ 2 distinct real roots. Done this!
- $\gamma^2 - 4mk < 0 \implies$ 2 complex (non-real) roots. Last time!
- $\gamma^2 - 4mk = 0 \implies$ 1 repeated real root. Today!

Repeated Roots

So today we're assuming that $\gamma^2 - 4mk = 0$, so the only root of the characteristic polynomial

$$mr^2 + \gamma r + k = 0$$

is $r_1 = r_2 = -\frac{\gamma}{2m}$.

Question: Is $\{e^{r_1 t}, e^{r_2 t}\}$ a fundamental set of solutions?

Of course not! These are the same function, so the Wronskian is always zero.

Question 2: Can we find a second function to go with $x_1 = e^{rt}$ (where $r = r_1 = r_2$)?

Trick / Technique: Rather than look at a constant multiple of x_1, we'll look at a ***function*** multiple. That is, we guess that $x_2(t) = v(t) \cdot x_1(t)$.

Trying the Technique

So we're going to substitute $x_2(t) = v(t)x_1(t)$ into

$$mx'' + \gamma x' + kx = 0$$

where $x_1(t) = e^{rt}$ is one solution when $r = \dfrac{-\gamma}{2m}$.

Let's Plug In: $x_2' = v'x_1 + vx_1'$ and $x_2'' = v''x_1 + 2v'x_1' + vx_1''$, so we want

$$\begin{aligned} mx_2'' + \gamma x_2' + kx_2 &= m\big(v''x_1 + 2v'x_1' + vx_1''\big) + \gamma\big(v'x_1 + vx_1'\big) + kvx_1 \\ &= v''mx_1 + v'\big(2mx_1' + \gamma x_1\big) + v\big(mx_1'' + \gamma x_1' + kx_1\big). \end{aligned}$$

Notice that

- $mx_1'' + \gamma x_1' + kx_1 = 0$
- $x_1 = e^{rt}$ implies $x_1' = re^{rt} = -\frac{\gamma}{2m}x_1$, so $2mx_1' = -\gamma x_1$.
- Thus $2mx_1' + \gamma x_1 = 0$ as well.

So $v''mx_1 = 0$, which means $v'' = 0$. Hence $v(t) = at + b$.

A Fundamental Set

So we have seen that if $x_1 = e^{rt}$ is one solution of

$$mx'' + \gamma x' + kx = 0 \qquad \text{with } \gamma^2 - 4mk = 0$$

then $(at+b)e^{rt}$ is also a solution. So we'll take $x_2(t) = te^{rt}$ as our second solution.

Repeated Roots Theorem

Suppose the differential equation $ay'' + by' + cy = 0$ satisfies $b^2 - 4ac = 0$. Then the general solution is

$$y = c_1 e^{rt} + c_2 te^{rt} = \big(c_1 + c_2 t\big)e^{rt},$$

where $r = -\dfrac{b}{2a}$.

Why? Check the Wronskian:

$$W[e^{rt}, te^{rt}] = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & (rt+1)e^{rt} \end{vmatrix} = rte^{2rt} + e^{2rt} - rte^{2rt} = e^{2rt} \neq 0.$$

Example 1

1. Find the general solution to

$$y'' + 6y' + 9y = 0$$

then find the particular solution satisfying the initial conditions $y(0) = 4$, $y'(0) = 1$.

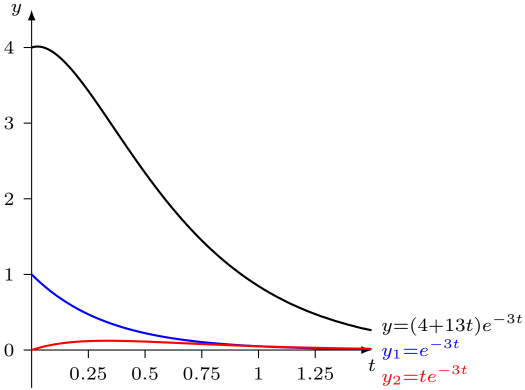
Solution: The repeated root of the characteristic polynomial $r^2 + 6r + 9 = (r+3)^2$ is $r = -3$. Thus the general solution of the ODE is $\boxed{y(t) = c_1 e^{-3t} + c_2 t e^{-3t}}$.

The derivative of $y(t)$ is $y'(t) = \big(-3c_1 + c_2\big)e^{-3t} - 3c_2 t e^{-3t}$, so the initial conditions say

$$y(0) = c_1 = 4$$
$$y'(0) = -3c_1 + c_2 = 1.$$

Thus the particular solution is $\boxed{y = 4e^{-3t} + 13te^{-3t}}$.

Pictures



Example 2

2. Find the general solution to

$$y'' - y' + \frac{1}{4}\,y = 0$$

then find the particular solution satisfying the initial conditions $y(0) = 2$, $y'(0) = 1/2$.

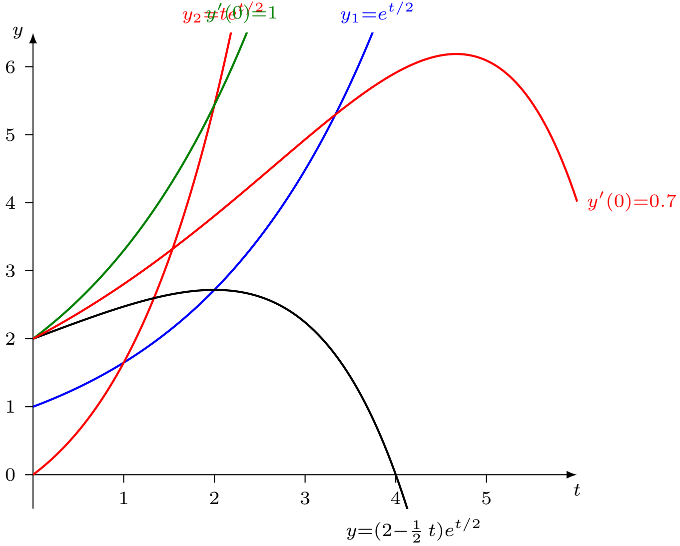
Solution: The repeated root of the characteristic polynomial $r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$ is $r = +\frac{1}{2}$. Thus the general solution of the ODE is $\boxed{y(t) = c_1 e^{t/2} + c_2 t e^{t/2}}$.

The derivative of $y(t)$ is $y'(t) = \left(\frac{1}{2}c_1 + c_2\right)e^{t/2} + \frac{1}{2}c_2 t e^{t/2}$, so the initial conditions say

$$y(0) = c_1 = 2$$
$$y'(0) = \frac{1}{2}c_1 + c_2 = \frac{1}{2}.$$

Thus the particular solution is $\boxed{y = 2e^{t/2} - \frac{1}{2}\,t e^{t/2}}$.

Pictures



General Approach

So we have fully solved the ODE

$$mx'' + \gamma x' + kx = 0.$$

We find the roots r_1 and r_2 of

$$mr^2 + \gamma r + k = 0: \qquad r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Three Cases:

Case	Discriminant	Fundamental Set	General Solution
Two real roots $r_1 \neq r_2$	$\gamma^2 > 4mk$	$\left\{e^{r_1 t}, e^{r_2 t}\right\}$	$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
Two non-real roots $\alpha \pm i\beta$	$\gamma^2 < 4mk$	$\left\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\right\}$	$y = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$
One real repeated root r	$\gamma^2 = 4mk$	$\left\{e^{rt}, te^{rt}\right\}$	$y = (c_1 + c_2 t)e^{rt}$

Reduction of Order

This technique works for more general ODEs – ones with non-constant coefficients:

$$y'' + p(t)y' + q(t)y = 0.$$

Suppose $y_1(t)$ is one solution and assume $y_2(t) = v(t)y_1(t)$.
As before: $y_2' = v'y_1 + vy_1'$ and $y_2'' = v''y_1 + 2v'y_1' + vy_1''$, so we want

$$\begin{aligned} y_2'' + py_2' + qy_2 &= \big(v''y_1 + 2v'y_1' + vy_1''\big) + p\big(v'y_1 + vy_1'\big) + qvy_1 \\ &= v''y_1 + v'\big(2y_1' + py_1\big) + v\big(y_1'' + py_1' + qy_1\big). \end{aligned}$$

Since $y_1'' + py_1' + qy_1 = 0$, we get the ODE

$$y_1v'' + (2y_1' + py_1)v' = 0.$$

This is a ***first order equation*** **in v'**.

Example

3. Suppose we know that $y_1 = \sqrt{t}$ is a solution of

$$2t^2y'' + 3ty' - y = 0 \qquad (t > 0).$$

Use reduction of order to find the general solution.

Solution: Set $y_2(t) = v(t)y_1(t) = v(t)\sqrt{t}$, from which we find

$$y_2 = v\,t^{1/2} \qquad y_2' = v'\,t^{1/2} + \frac{1}{2}v\,t^{-1/2} \qquad y_2'' = v''\,t^{1/2} + v'\,t^{-1/2} - \frac{1}{4}v\,t^{-3/2}.$$

Then

$$\begin{aligned} 0 &= 2t^2y_2'' + 3ty_2' - y_2 \\ &= 2t^2\left(v''t^{1/2} + v't^{-1/2} - \frac{1}{4}vt^{-3/2}\right) + 3t\left(v't^{1/2} + \frac{1}{2}vt^{-1/2}\right) - vt^{1/2} \\ &= 2t^{5/2}v'' + 5t^{3/2}v'. \end{aligned}$$

Now: Solve $2t^{5/2}v'' + 5t^{3/2}v' = 0$ or $\boxed{2t^{5/2}u' + 5t^{3/2}u = 0}$ for $u = v'$.

Example (cont'd)

So we'd like to find v, where $2t^{5/2}u' + 5t^{3/2}u = 0$ for $u = v'$. Notice we can re-write this as

$$t^{5/2}u' + \frac{5}{2}\,t^{3/2}u = 0 \qquad \text{or} \qquad t^{5/2}u' + \left(t^{5/2}\right)' u = 0.$$

Thus $t^{5/2}u = C$, or $v' = u = Ct^{-5/2}$. Integrating, we get $v = C't^{-3/2}$ and so

$$y_2(t) = C't^{-3/2}y_1 = C't^{-3/2} \cdot \sqrt{t} = C't^{-1}$$

is also a solution of our ODE. So take $\boxed{y_2 = t^{-1}}$.

Check:

$$2t^2y'' + 3ty' - y = 2t^2(+2t^{-3}) + 3t(-t^{-2}) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0.$$

Wronskian:

$$W[\sqrt{t}, t^{-1}] = \begin{vmatrix} \sqrt{t} & t^{-1} \\ \frac{1}{2\sqrt{t}} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2} \neq 0.$$

General Solution: $\boxed{y = c_1\sqrt{t} + c_2t^{-1}}$