Math 4B: Differential Equations

Lecture 22: Linear Algebra

- All of Linear Algebra,
- In 50 Minutes or Less,
- (& More?)

Basics

You should remember the basic objects of linear algebra:

- matrices, and what $m \times n$ and $n \times n$ matrices are
  (Remember: $m = \#$ of rows, $n = \#$ of columns)

- Special matrices: the zero matrix $\mathbf{0}$ and the $n \times n$ identity matrix I_n (or just I)

- Vectors = column vectors = $n \times 1$ matrices: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

  Row vectors = $1 \times n$ matrices $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$

Operations

You should be familiar with the basic operations:

- Addition: You can add $m \times n$ matrices entry by entry:

$$A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

- Scalar Multiplication: You can multiply an $m \times n$ matrix by a constant (scalar) c by multiplying each entry:

$$cA = c(a_{ij}) = (ca_{ij})$$

- Multiplication: If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$ and BA makes no sense unless $p = m$.

  The ij entry of $C = AB$ is $c = \displaystyle\sum_{k=1}^{n} a_{ik} b_{kj}$

- It all plays nicely: $c(A + B) = cA + cB$, $\quad A(BC) = (AB)C$, $A(B + C) = AB + AC$, $\quad$ but $\boxed{BA \neq AB}$

Slightly Sophisticated

- The product of a matrix A and a vector $\mathbf{x}$ gives a linear combination of the columns of A:

$$A\mathbf{x} = \begin{pmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a_n} \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

- The matrix product AB can be thought of as p products of A with the columns of B:

$$AB = A\begin{pmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b}_2 & \cdots & \mathbf{b}_p \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ A\mathbf{b_1} & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \\ | & | & & | \end{pmatrix}.$$

More Operations

You should also remember...

- The ***transpose*** of a matrix:

$$A = (a_{ij}) \implies A^T = (a_{ji}) \quad \text{if } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

- The ***conjugate*** of a matrix:

$$A = (a_{ij}) \implies \overline{A} = (\overline{a_{ij}}) \quad \text{if } A = \begin{pmatrix} 1 & 2i \\ 4 & 5+6i \end{pmatrix}, \text{ then } \overline{A} = \begin{pmatrix} 1 & -2i \\ 4 & 5-6i \end{pmatrix}.$$

- The ***adjoint*** (or conjugate transpose) of a matrix:

$$A = (a_{ij}) \implies A^* = (\overline{a_{ji}}) \quad \text{if } A = \begin{pmatrix} 1 & 2i \\ 4 & 5+6i \end{pmatrix}, \text{ then } A^* = \begin{pmatrix} 1 & 4 \\ -2i & 5-6i \end{pmatrix}.$$

Vector Products

We have a couple of ways to take two vectors to produce a scalar:

- The ***dot product*** $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$.

- The ***inner product*** or ***scalar product*** is
$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \overline{y_i} = \mathbf{x}^T \overline{\mathbf{y}}.$$

- Notice that:
$$(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}, \qquad (\alpha \mathbf{x}, \mathbf{y}) = \alpha(\mathbf{x}, \mathbf{y}), \qquad (\mathbf{x}, \alpha \mathbf{y}) = \overline{\alpha}\,(\mathbf{x}, \mathbf{y}),$$
$$(\mathbf{x}, \mathbf{y} + \mathbf{z}) = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z}) \qquad \text{and} \qquad (\mathbf{x} + \mathbf{z}, \mathbf{y}) = (\mathbf{x}, \mathbf{y}) + (\mathbf{z}, \mathbf{y}).$$
$$(A\mathbf{x}, \mathbf{y}) = (A\mathbf{x})^T \overline{\mathbf{y}} = \mathbf{x}^T (A^T \overline{\mathbf{y}}) = \mathbf{x}^T \overline{A^* \mathbf{y}} = (\mathbf{x}, A^* \mathbf{y}).$$

- We define the ***length*** or ***magnitude*** of $\mathbf{x}$ by $\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$.
- We say two vectors $\mathbf{x}$ and $\mathbf{y}$ are ***orthogonal*** or ***perpendicular*** if $(\mathbf{x}, \mathbf{y}) = 0$.

Systems of Equations

Here is a system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 3x_2 \phantom{{}+x_3+x_4} &= -1 \\ 3x_1 + 3x_2 - x_3 - x_4 &= -1 \end{aligned}$$

We can write this as

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 3 & 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix} \qquad \text{or} \qquad A\mathbf{x} = \mathbf{b}.$$

Questions:

- How can we solve this?
- How many solutions do we get?
- Does it matter if the right-hand side is zero ("homogeneous") or not ("nonhomogeneous")?

Solution Approaches

1. Row reduction or Gaussian elimination
2. If we have n equations in n unknowns (so A is an $n \times n$ matrix), we can solve $A\mathbf{x} = \mathbf{b}$ by taking the inverse of A:

   $$A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x} \qquad \text{and so} \qquad \mathbf{x} = A^{-1}\mathbf{b}.$$

   Two comments:

   - This needs A to be ***invertible*** or ***non-singular***. We often test this by taking the ***determinant*** $\det(A)$: a matrix A is invertible (non-singular) exactly when $\det(A) \neq 0$.
   - Often we invert matrices (and find determinants!) using row reduction.
3. If A is non-singular, this shows that $A\mathbf{x} = \mathbf{b}$ has a unique solution.
4. If A is singular, then $A\mathbf{x} = \mathbf{b}$ will have either 0 or infinitely many solutions.

Linear (In)Dependence

Remember that a set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is ***linearly independent*** if the only linear combination of them that gives $\mathbf{0}$ is the zero combination:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \qquad \Longrightarrow \qquad x_1 = x_2 = \cdots = x_n = 0.$$

Viewed as a linear system, this says that the only solution to

$$\begin{pmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{or} \qquad A\mathbf{x} = \mathbf{0}$$

is the zero vector. That is, the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linearly independent if and only if the matrix

$$\begin{pmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{pmatrix}$$

is non-singular.

Calculus with Matrices

We can define matrices that are functions $A(t)$ (this includes vectors!). We can then do calculus:

- $\dfrac{d}{dt}\big(A+B\big) = \dfrac{dA}{dt} + \dfrac{dB}{dt}$
- $\dfrac{d}{dt}\big(AB\big) = \dfrac{dA}{dt}B + A\dfrac{dB}{dt}$
- If $A(t) = \big(a_{ij}(t)\big)$, then $\displaystyle\int_a^b A(t)\,dt = \left(\int_a^b a_{ij}(t)\,dt\right)$.
- Example: If $A(t) = \begin{pmatrix} \sin(t) & t \\ e^t & 1 \end{pmatrix}$, then

$$\frac{dA}{dt} = \begin{pmatrix} \cos(t) & 1 \\ e^t & 0 \end{pmatrix} \qquad \text{and} \qquad \int_0^\pi A(t)\,dt = \begin{pmatrix} 2 & \pi^2/2 \\ e^\pi - 1 & \pi \end{pmatrix}.$$

Eigenvalues and eigenvectors

Remember that an $n \times n$ matrix A has ***eigenvalue*** λ and ***eigenvector*** $\mathbf{x}$ if $A\mathbf{x} = \lambda\mathbf{x}$ (where $\mathbf{x} \neq \mathbf{0}$).

Some facts about eigenvalues and eigenvectors:

- Counting with multiplicity, A has n eigenvalues (some may be complex). (The eigenvalues of A are exactly the roots of the ***characteristic polynomial*** $\det(A - \lambda I)$.) We call this multiplicity the ***algebraic multiplicity***.
- Given an eigenvalue λ of A, the ***eigenspace*** E_λ is the subspace of all eigenvectors (plus $\mathbf{0}$) corresponding to λ. We call $\dim(E_\lambda)$ the ***geometric multiplicity*** of λ. It is the number of linearly independent eigenvectors for λ.
- For each λ, $1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity.
- Eigenvectors from different eigenvalues are linearly independent.

The Spectral Theorem

The Spectral Theorem

If A is an $n \times n$ Hermitian matrix (so $A^* = A$), then A is orthogonally diagonalizable. This means

- The eigenvalues of A are all real.
- There is an orthonormal basis of eigenvectors of A.

Note: If A is a real matrix, then "$A^* = A$" just says that A is symmetric.

Example: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ can be written $A = SDS^{-1}$.

Try it!

Examples: Eigenvalues

To find the eigenvalues of $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, we take the determinant

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\ &= (-\lambda)^3 + 1 + 1 - \big(-\lambda - \lambda - \lambda \big) \\ &= -\lambda^3 + 3\lambda + 2. \end{aligned}$$

It turns out this factors to $\det(A - \lambda I) = -(\lambda + 1)^2(\lambda - 2)$, so the eigenvalues are

- $\lambda = -1$ (algebraic multiplicity 2), and
- $\lambda = 2$.

We'll write $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 2$.

Examples: Eigenvectors

To find the eigenvectors of A, we look at the eigenspaces E_{-1} and E_2.

Here $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, so

$$E_{-1} = \mathrm{Null}\left(A - (-1)I\right) = \mathrm{Null}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \mathrm{Null}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} = \mathrm{Span}\left\{\mathbf{v}_1, \mathbf{v}_2\right\} \text{ and}$$

$$E_2 = \mathrm{Null}\left(A - 2I\right) = \mathrm{Null}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$= \mathrm{Null}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \mathrm{Span}\left\{\mathbf{v}_3\right\}.$$

Example: The Spectral Theorem

Example: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has eigenvalues $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 2$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This can be written $A = SDS^{-1}$, where $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ and S has columns given by the corresponding eigenvectors of A:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}.$$

Another Example

Consider $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. This has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 4$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

Thus $B = SDS^{-1}$, or

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

or $B = QDQ^T$, where Q is orthogonal and $Q^{-1} = Q^T$:

$$Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$