Math 174E
Lecture 17

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References

Hull
Chapter 15.4, 15.5, 15.6, 15.11
Chapter 19
Chapter 20

Black-Scholes-Merton Model Assumption

The **assumptions** used to derive the Black-Scholes-Merton pricing formula:

1. The stock price process follows a geometric Brownian motion with constant parameters μ and σ.
2. The short selling of securities with full use of proceeds is permitted.
3. There are no transaction costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest, r, is constant and the same for all maturities (and for borrowing an lending)

Log-Returns

Crucial assumption of Black–Scholes–Merton model:

- given a time series (e.g., of one year) of daily stock prices $S_0, S_{\Delta t}, S_{2\Delta t}, \ldots$ with $\Delta t = 1/252$, the daily **log-returns**

$$\log\left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right) \qquad (i = 1, \ldots, 252)$$

  in the Black-Scholes-Merton model are *assumed* to be **independent** and $\mathcal{N}((\mu - \sigma^2/2)\Delta t, \sigma^2 \Delta t)$ **normally distributed** (independent of point in time = stationarity)
- this is a direct consequence of the properties of the **Brownian motion** (recall Definition 14.3, Lecture 14)

However:

- in practice, log-returns computed from *historical stock prices* are typically **not** independent and normally distributed
- normal distribution underestimates occurence of extreme price moves (historical log-return data exhibits "heavier tails")

Volatility

- volatility σ of a stock is a measure of the uncertainty about the returns provided by the stock ($\sigma =$ standard deviation of the log-returns)
- stocks typically have a volatility between 15% and 60% p.a.
- σ is the **most important parameter** in the Black-Scholes-Merton model and for the pricing of derivatives
- in contrast, the expected return μ does not play any role in the pricing of derivatives (thanks to the risk-neutral valuation approach)
- in fact, a stock's expected return is very difficult to estimate (from historical stock prices)
- **Capital Asset Pricing Model (CAPM)** deals with the question of how to determine the expected return of a risky asset (see Math 179)

Estimating Volatility from Historical Data

- suppose stock price is observed daily
- $S_i =$ stock price at the end of i-th day ($i = 0, 1, \ldots, n$)
- compute daily log-returns

$$x_i = \log\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for } i = 1, \ldots, n$$

- empirical estimate for the standard deviation $\hat{\sigma}_N^{\text{daily}}$ of the daily log-returns

$$\hat{\sigma}_n^{\text{daily}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x}_n^{\text{daily}})^2} \quad \text{where} \quad \bar{x}_n^{\text{daily}} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

  is the empirical mean of the daily log-returns (sometimes assumed to be zero when estimating volatility)
- empirical estimate $\hat{\sigma}^{\text{annual}}$ for the annual volatility σ is then obtained via

$$\hat{\sigma}^{\text{annual}} = \sqrt{252} \cdot \hat{\sigma}_n^{\text{daily}} \quad \text{(assuming 252 trading days)}$$

Sensitivities: “Greeks” 1/2

Black-Scholes-Merton **call option pricing formula** from Theorem 15.2 only depends on following parameters:

1. current stock price s
2. time-to-maturity $\tau = T - t$
3. volatility σ
4. risk-free rate r
5. strike price K

$$C_t(K, T) = C^{\mathrm{BS}}(s, \tau, \sigma, r, K)$$

Sensitivities: “Greeks” 2/2

Sensitivities: (here ϕ denotes the $\mathcal{N}(0,1)$-density)

- Delta:
$$\Delta = \frac{\partial}{\partial s} C^{\mathrm{BS}} = \Phi\left(d_+(s,\tau)\right)$$
- Gamma:
$$\Gamma = \frac{\partial}{\partial s}\Delta = \frac{\partial^2}{\partial s^2} C^{\mathrm{BS}} = \phi\left(d_+(s,\tau)\right)\frac{1}{s\sigma\sqrt{\tau}}$$
- Theta:
$$\Theta = \frac{\partial}{\partial \tau} C^{\mathrm{BS}} = \frac{s\sigma}{2\sqrt{\tau}}\phi\left(d_+(s,\tau)\right) + rKe^{-r\tau}\Phi\left(d_-(s,\tau)\right)$$
- Rho:
$$\rho = \frac{\partial}{\partial r} C^{\mathrm{BS}} = K\tau e^{-r\tau}\Phi\left(d_-(s,\tau)\right)$$
- Vega:
$$\mathcal{V} = \frac{\partial}{\partial \sigma} C^{\mathrm{BS}} = s\sqrt{\tau}\phi\left(d_+(s,\tau)\right)$$

Same “greeks” with similar formulas can also be computed for the put option.

Black-Scholes-Merton PDE and Replication

- one can show (see Math 179) that the call price given by the Black-Scholes-Merton formula $C^{\mathrm{BS}}(s, T-t, \sigma, r, K) = v(t, s)$ viewed as a function in time t and stock price s satisfies following partial differential equation (PDE) (**Black-Scholes-Merton PDE**)

$$\frac{\partial}{\partial t} v(t,s) + rs \frac{\partial}{\partial s} v(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} v(t,s) = rv(t,s), \ (t,s) \in [0,T) \times (0,+\infty),$$
$$v(T,s) = (s-K)^+, \ s \in (0,+\infty).$$

- the PDE can be derived from a **replication argument** where $v(t,s)$ denotes the **value of the replicating portfolio** (derivation requires **Itô's Lemma**; see Math 179)
- in the replication portfolio the share holdings are prescribed by the **delta hedge**

$$\Delta = \frac{\partial}{\partial s} C^{BS}(s,t) = \# \text{ of shares to hold at time } t \text{ if stock price is } s$$

(continuously rebalanced trading strategy!)

Implied Volatility 1/5

- in the Black-Scholes-Merton model the **volatility** σ is assumed to be a **constant**
- in practice this assumption is **not** satisfied

Indeed: Two observations can be made

1. Empirical estimates (slide 6 above) for historical volatility varies.
2. In the Black-Scholes-Merton formula $C^{\mathrm{BS}}(s, \tau, \sigma, r, K)$ the volatility σ does not depend on strike K and maturity T. However, quoted market prices of European call and put options on exchanges reveal that σ depends on K and T by computing the so-called **implied volatility**!

Implied Volatility 2/5

Definition 15.4

The **implied volatility** σ^{implied} of a call option is the *volatility parameter* for which the Black-Scholes-Merton price equals the market price, i.e.,

$$C_0^{\text{market}}(K, T) = C^{\text{BS}}(S_0, T, \sigma^{\text{implied}}, r, K).$$

- traders and brokers often quote implied volatilities of an option *rather than the option price*
- in turns out that for market prices of call options on the **same stock** with *different* strike K and time-to-maturity $\tau = T - 0$ lead to *different* implied volatilities:

$$\sigma^{\text{implied}} = \sigma^{\text{implied}}(K, T)$$

Implied Volatility 3/5

Example 15.5

Suppose the market price today of a European call option on a non-dividend paying stock is $C_0^{\text{market}}(K, T) = \$1.875$ where $S_0 = 21,\ K = 20,\ r = 0.1,\ T = 0.25.$

The corresponding implied volatility is $\sigma^{\text{implied}} = 0.2355$ (23.5% per annum) because

$$C^{\text{BS}}(S_0, T, \sigma^{\text{implied}}, r, K) = 1.875$$

Remark: Using **put-call parity** one can show that the implied volatility is the *same for European put and call options* which are written on the same stock with same K and T.

Implied Volatility 4/5

One can show the following (see Math 179):

- vega (same for calls and puts!)

$$\mathcal{V} = \frac{\partial}{\partial \sigma} C_0^{\text{BSM}} = s\sqrt{\tau}\phi\left(d_+(s,\tau)\right) > 0$$

- C_0^{BSM} is an increasing function in σ
- we have

$$\lim_{\sigma \to 0} C_0^{\text{BSM}} = (S_0 - e^{-rT}K)^+$$
$$\lim_{\sigma \to \infty} C_0^{\text{BSM}} = S_0$$

- hence we obtain

$$(S_0 - e^{-rT}K)^+ < C_0^{\text{BSM}} < S_0$$

Implied Volatility 5/5

As a consequence:

- for *any* arbitrage-free market price $C_0^{\text{market}}(K, T)$ satisfying

  $$(S_0 - e^{-rT}K)^+ < C_0^{\text{market}}(K, T) < S_0$$

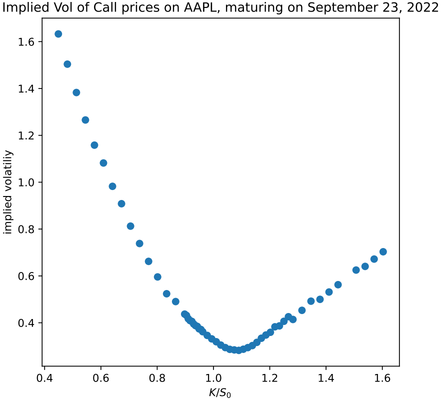
  there exists a **unique** $\sigma^{\text{implied}}(K, T) \in (0, \infty)$ such that

  $$C_0^{\text{market}}(K, T) = C_0^{\text{BSM}}(S_0, T, \sigma^{\text{implied}}(K, T), r, K)$$

  (by virtue of the inverse function theorem)
- iterative search procedure must be used to find $\sigma^{\text{implied}}(K, T)$ (**Newton's method**)

Volatility Smile 1/3

Plotting $\sigma^{\text{implied}}(K/S_0, T)$ as a function in K/S_0 ("moneyness") for fixed T is referred to as the **volatility smile**



Implied volatility of call options on Apple maturing on September 23, 2022, with $S_0 = 155.96$ (as of September 7, 2022, 9:00 p.m.)

Volatility Smile 2/3

Volatility Smile:

- implied volatility is relatively low for at-the-money options $K/S_0 \approx 1.0$
- implied volatility becomes progressively higher as an option moves either into the money or out of the money ($K/S_0 > 1$ or $K/S_0 < 1$)
- *Example:* foreign currency options

Volatility Skew:

- implied volatility decreases as strike price increases
- *Example:* equity options

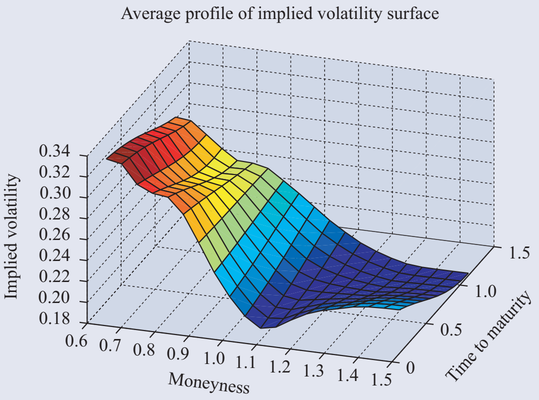
Volatility Smile 3/3

Reason for Volatility Smile/Skew:

- log-normal distribution in the Black-Scholes-Merton model **underestimates** the probability of extreme movements in the price
- traders use different volatilities for pricing options to compensate this (i.e., they use volatility smiles to allow for *"nonlognormality"*)

- plotting $\sigma^{\text{implied}}(K/S_0, T)$ as a function in K/S_0 **and** T is referred to as the **volatility surface**
- volatility surface is used in practice for pricing options

Volatility Surface



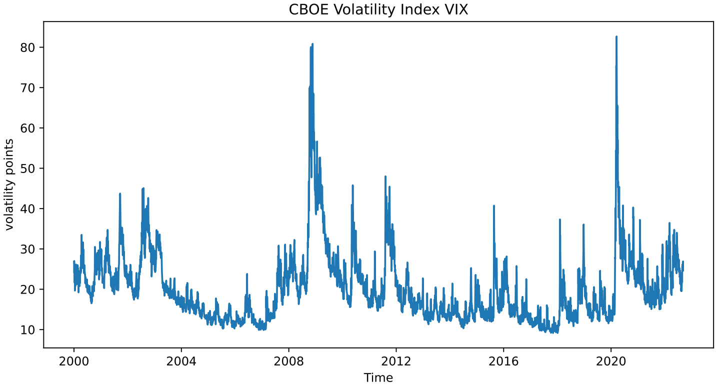
Typical profile of the implied volatility of S&P500 options as a function of time to maturity and moneyness.

Source of figure: Rama Cont & José da Fonseca (2002), Dynamics of implied volatility surfaces, *Quantitative Finance*, 2:1, 45-60, `https://doi.org/10.1088/1469-7688/2/1/304`.

The VIX Index 1/2

- Volatility Index (**VIX Index**) calculated and disseminated by CBOE (Chicago Board Options Exchange)
- measures the 30-day expected volatility of the S&P 500 index
- computed from prices of at- and out-of-the-money put and call options on the S&P 500 Index
- often referred to as the **"fear index"**
- trading in futures on the VIX started in 2004, trading in options on the VIX started in 2006

The VIX Index 2/2



On **March 16, 2020**, the VIX closed at 82.69, the **highest level** since its inception in 1990!

Some Final Comments & Outlook Math 179

- under suitable technical assumptions one can show that the arbitrage-free price of a call option in the **binomial tree model** converges to the **Black-Scholes-Merton** price from Theorem 15.1 (see Lecture 14)
- there are **no** explicit formulas for **American stock options**
  - prices can be computed numerically by either solving the Black-Scholes-Merton PDE with suitable boundary condition or by doing Monte-Carlo simulations and tree-type approximations
- the Black-Scholes-Merton model is also used for pricing **exotic options**:
  - either closed form formulas or pricing via **Monte-Carlo simulation**
- important generalizations of the Black-Scholes-Merton model include **stochastic volatility** models (**Heston model**) and **local volatility models** (calibrated to the **volatility surface**)