Math 4B: Differential Equations

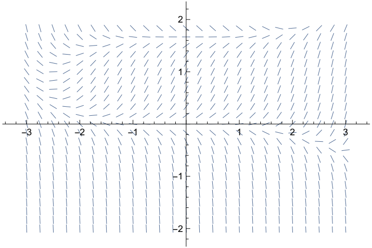
Lecture 06: Autonomous ODEs

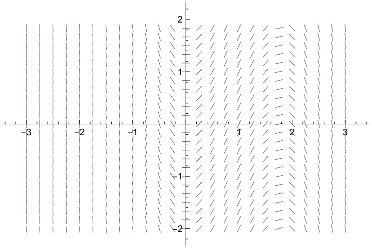
- Modeling Populations,
- Logistic Growth,
- Carrying Capacities, Thresholds, & More!

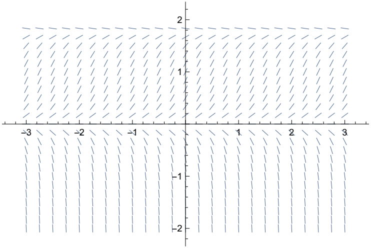
Autonomous ODEs

Question: Which of these direction fields corresponds to

$$y' = y^3/8 - 3y^2 + 5y?$$







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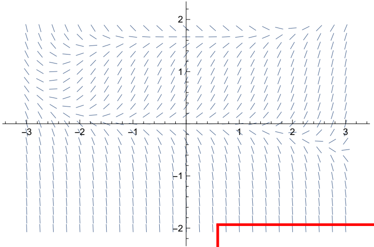

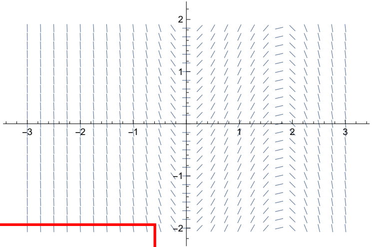

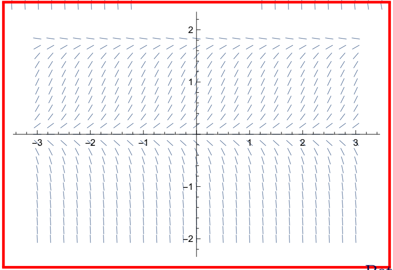

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- Example: $y' = ry$ which has solutions $y = Ce^{rt}$
- Solutions behave the same regardless of starting time.
- Ideal for population modeling

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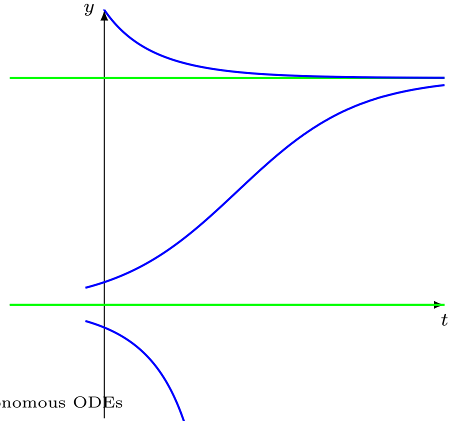

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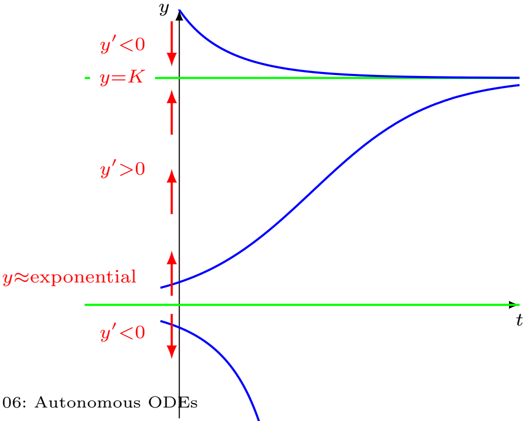

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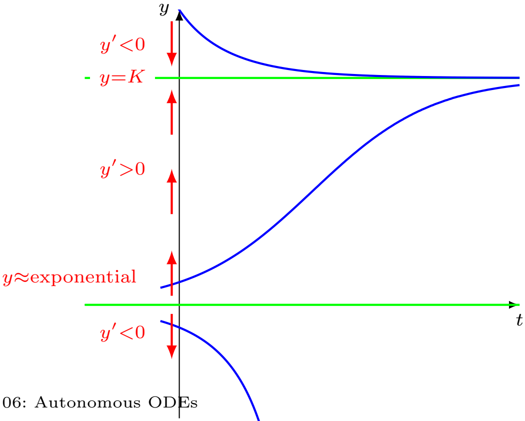

Goal: Write

$$y' = h(y) \cdot ry.$$

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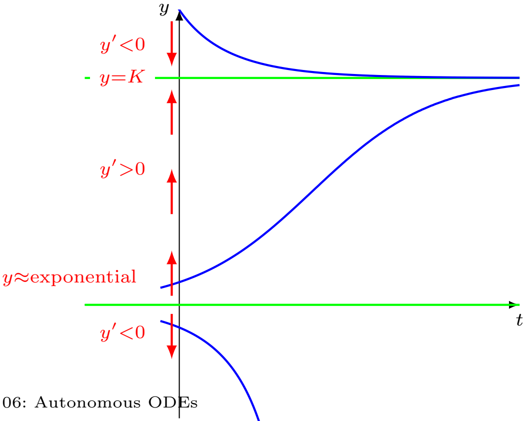

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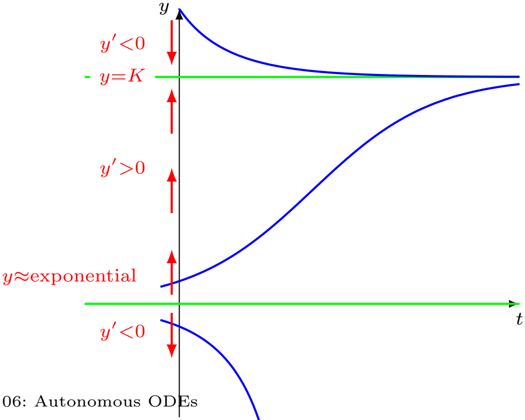

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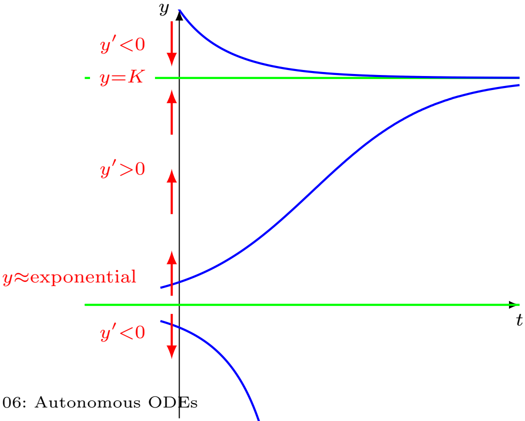

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$$y\approx K \implies h(y)\approx 0$$

Easiest:

$$h(y) = 1 - \frac{y}{K}$$

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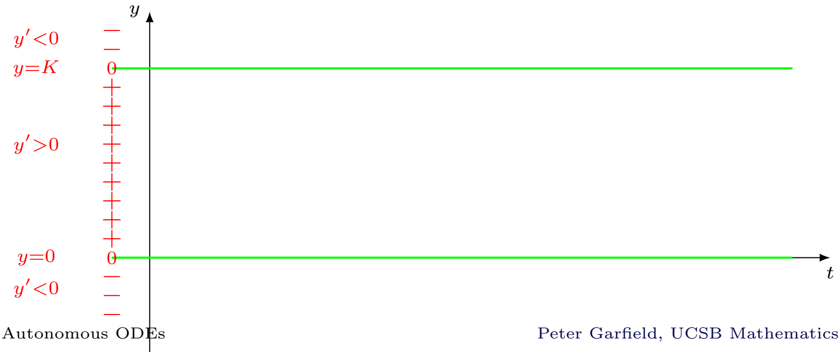

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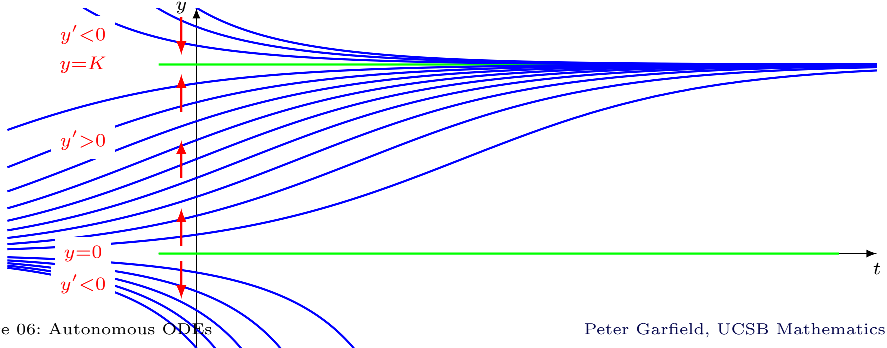

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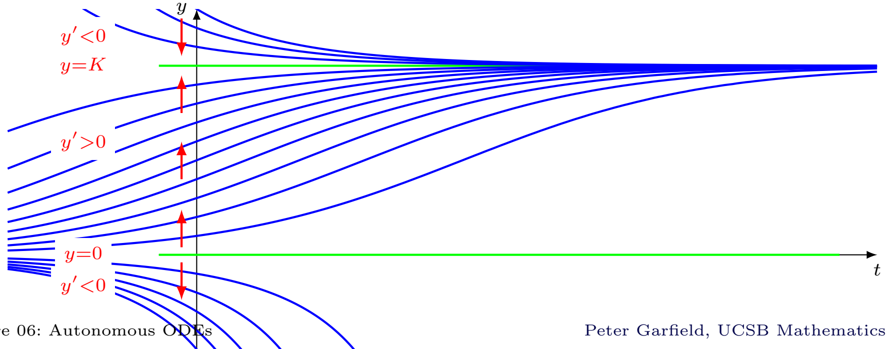
$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$$

- $y = K$ is a ***stable*** equilibrium; $y = 0$ is ***unstable***



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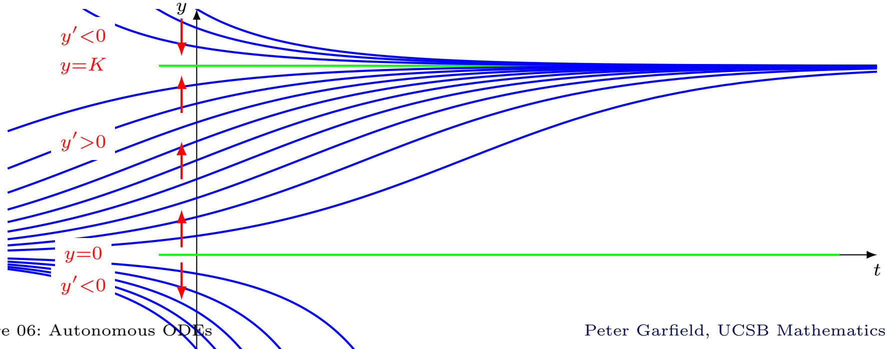
- $y = K$ is the ***carrying capacity*** of the system



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Also $\boxed{y = 0}$

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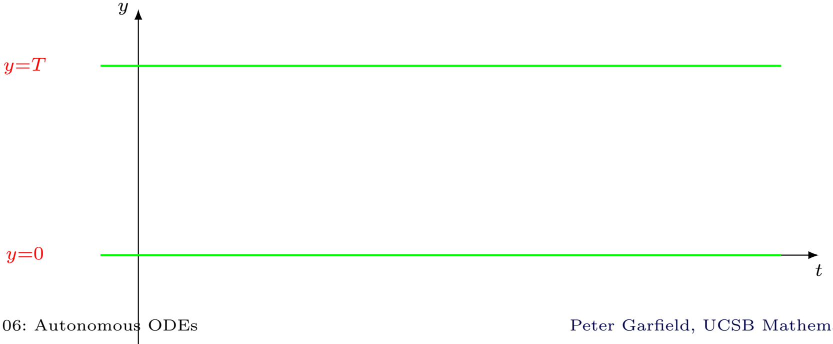
- Equilibrium solutions where $y' = 0$. Here that's $y = 0$ and $y = K$



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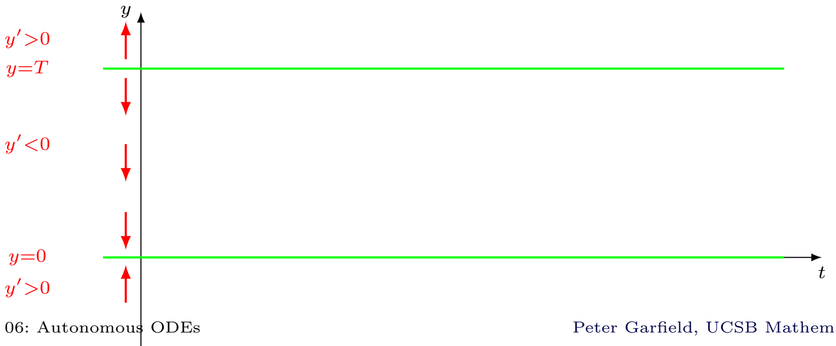

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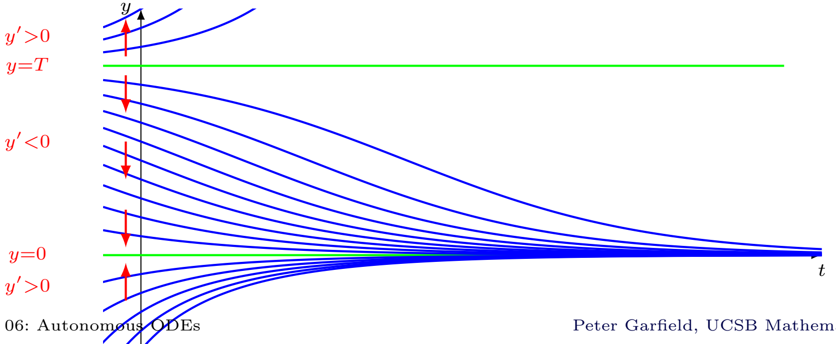

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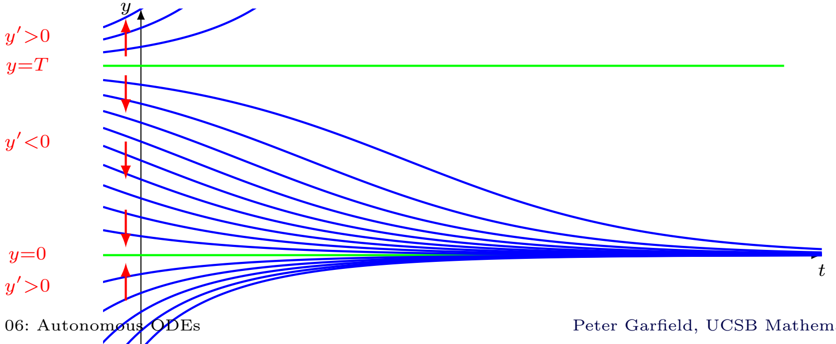
- $y = T$ is an ***unstable*** equilibrium; $y = 0$ is ***stable***



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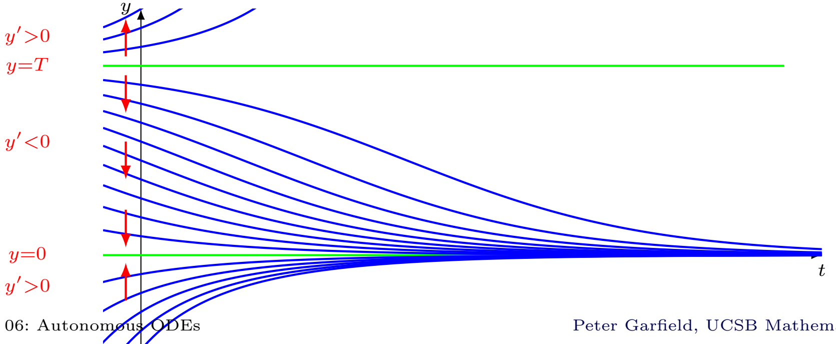

Modified Logistic with Threshold

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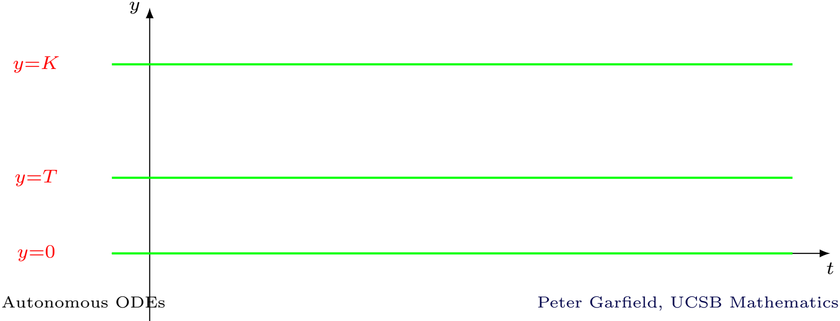

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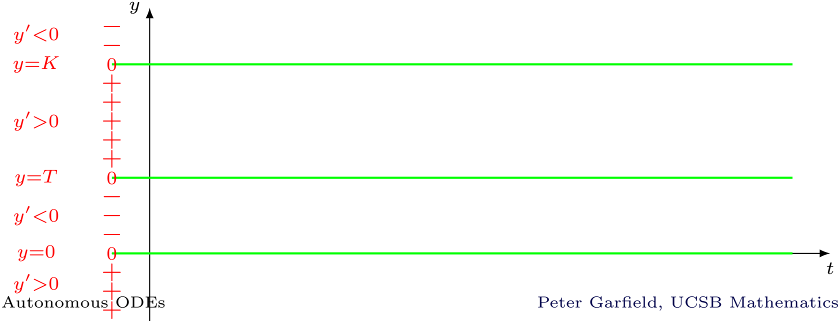

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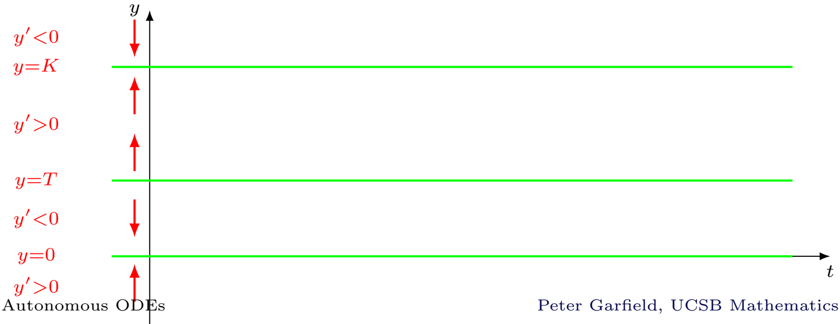

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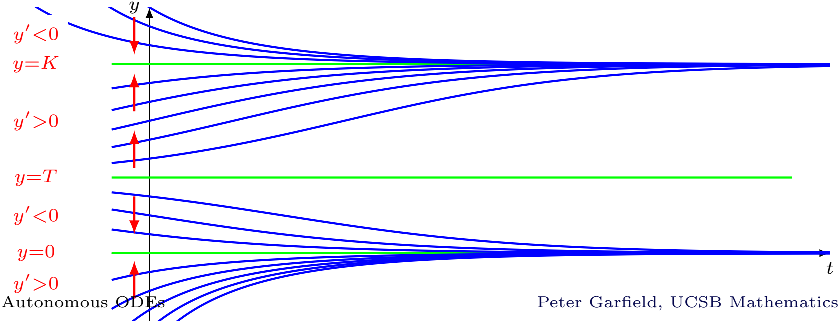

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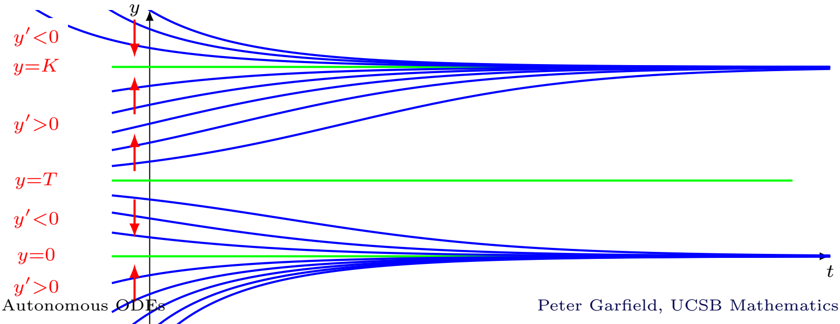

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- Both a ***threshold*** T and ***carrying capacity*** K