Math 4B: Differential Equations

Lecture 05: Modeling / (Non-)Linear

- Modeling with Differential Equations,
- Linear vs Non-linear ODEs,
- Existence-Uniqueness Theorems & More!

Models

- Mathematical models characterize physical systems, often using differential equations.
- Model Construction: Translating physical situation into mathematical terms. Clearly state physical principles believed to govern process. Differential equation is a mathematical model of process, typically an approximation.
- Analysis of Model: Solving equations or obtaining qualitative understanding of solution. May simplify model, as long as physical essentials are preserved.
- Comparison with Experiment or Observation: Verifies solution or suggests refinement of model.

Example 1: Salt Solution

Problem: We have a physical situation:

- At time $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gallons of water.
- Water containing 1/4 lb of salt/gal enters tank at rate of r gal/min
- Well-mixed water leaves tank at same rate.

Model this situation with an IVP.

What do you expect the limiting amount of salt in the tank to be? Should it depend on Q_0?

The IVP

The general set-up of these kinds of mixing problems is

$$\frac{dQ}{dt} = (\text{ rate in }) - (\text{ rate out }).$$

rate in: Salt enters the tank at

$$\text{rate in} = \left(\frac{1}{4} \frac{\text{lb}}{\text{gal}} \right) \left(r \, \frac{\text{gal}}{\text{min}} \right) = \frac{r}{4} \, \frac{\text{lb}}{\text{min}}.$$

rate out: Salt leaves the tank at

$$\text{rate out} = \left(\frac{Q(t)\,\text{lb}}{100 \text{ gal}} \right) \left(r \, \frac{\text{gal}}{\text{min}} \right) = \frac{r}{100} \, Q(t) \, \frac{\text{lb}}{\text{min}}.$$

So the IVP is
$$\begin{cases} \dfrac{dQ}{dt} = \dfrac{r}{4} - \dfrac{r}{100} \, Q(t) \\ Q(0) = Q_0. \end{cases}$$

Long-term Behavior

Remember our IVP is

$$\begin{cases} \dfrac{dQ}{dt} = \dfrac{r}{4} - \dfrac{r}{100}\,Q(t) \\ Q(0) = Q_0. \end{cases}$$

What happens to solutions as $t \to \infty$?

If we assume that $Q(t)$ approaches some limiting value, then we expect that

$$\frac{dQ}{dt} = 0 \qquad \Longrightarrow \qquad 0 = \frac{r}{4} - \frac{r}{100}\,Q(t).$$

Solving, we get $Q(t) = 25$ lbs.

Is this reasonable?

Note that it is independent of Q_0, the starting amount.

Escape Velocity

Problem: Physical situation:

- Body of mass m is projected vertically away from the earth
- Initial velocity v_0
- Assume no air resistance.

How large should v_0 be to escape earth's gravity?

Physics: The gravitational force acting on the mass is

$$\text{force} = -\frac{mgR^2}{(R+x)^2}$$

where

- R is the radius of the earth
- x is the altitude of the body above the earth's surface
- $g = 9.8 \text{ m/s}^2$

Using $F = ma$ we get $$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}.$$

A Problem

Our ODE is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}.$$

But we have three variables: v, x, and t.

Chain rule to the rescue!

$$\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot v.$$

Making this substitution and canceling the mass terms, we get the IVP

$$\begin{cases} v\dfrac{dv}{dx} = -\dfrac{gR^2}{(R+x)^2} \\ v(0) = v_0. \end{cases}$$

A Solution

Our IVP

$$\begin{cases} v\dfrac{dv}{dx} = -\dfrac{gR^2}{(R+x)^2} \\ v(0) = v_0. \end{cases}$$

is separable:

$$\frac{1}{2}v^2 = \frac{gR^2}{R+x} + C$$

Initial condition implies $C = \frac{v_0^2}{2} - gR$, so

$$v^2 = \frac{2gR^2}{R+x} + v_0^2 - 2gR.$$

The maximum altitude occurs when $v = 0$, so $x_{\text{max}} = \dfrac{v_0^2 R}{2gR - v_0^2}$.

Solving for v_0, we get $v_0 = \sqrt{2gR\dfrac{x_{\text{max}}}{R + x_{\text{max}}}} \to \sqrt{2gR}$ as $x_{\text{max}} \to +\infty$

First Order ODEs Recap

Remember that we're assuming that a general first order ODE can be written as

$$y' = f(t, y)$$

while a linear first order ODE can be written

$$y' + p(t)y = q(t).$$

Differences:

- Existence and uniqueness theorems. When solutions must exist and when they are unique varies between the two cases.
- General solutions. A linear first order ODE has a general solution that encompasses all possible solutions. This generally isn't true for non-linear ODEs.
- A Formula. We have an explicit way to construct the solutions to a linear first order ODE, but not generally for a nonlinear one.

Existence & Uniqueness

Existence & Uniqueness for Linear First Order IVPs

Consider the linear first order initial value problem

$$\begin{cases} y' + p(t)y = q(t) \\ y(t_0) = y_0. \end{cases}$$

If the functions p and q are continuous on an open interval I: $a < t < b$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the IVP for each t in I.

Reason: $y(t) = y_0 + \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(x) q(x)\, dx$ where $\mu(t) = \exp\left(\int_{t_0}^{t} p(x)\, dx\right)$

Existence & Uniqueness Part II

Existence & Uniqueness for Non-linear First Order IVPs

Consider the non-linear first order initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$

If the functions f and $\frac{\partial f}{\partial y}$ are continuous on an open rectangle

$$R = \{(t, y) : \alpha < t < \beta,\ \gamma < y < \delta\}$$

containing the point (t_0, y_0), then there is an interval

$$t_0 - h < t < t_0 + h$$

and a unique function $y = \phi(t)$ defined on that interval that satisfies the IVP for each t in $t_0 - h < t < t_0 + h$.

Example: Linear

Problem: Describe the behavior of solutions to the ODE

$$ty' + 2y = t^2$$

In standard form this is $y' + \frac{2}{t}\,y = t$, so $p(t) = 2/t$ is ***not continuous*** at $t = 0$. This means we expect solutions to exist on $(-\infty, 0)$ or $(0, +\infty)$. Solving, we find

$$\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = \exp\big(2\ln(t)\big) = \exp(\ln t^2) = t^2.$$

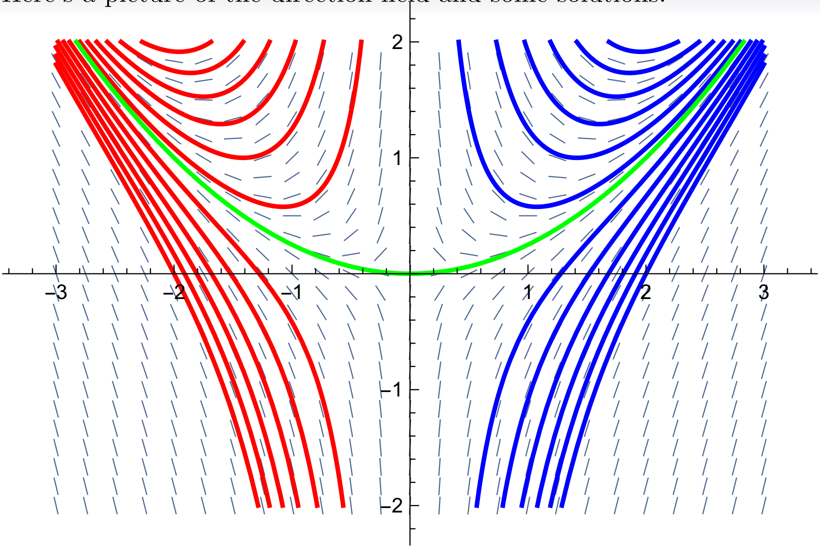
Multiplying through, this becomes

$$t^2\left(y' + \frac{2}{t}\,y = t\right) \qquad \Longrightarrow \qquad \big(\,t^2 y\,\big)' = t^3$$

Hence

$$t^2 y = \int t^3\,dt = \frac{1}{4}\,t^4 + C \qquad \text{and so} \qquad y = \frac{t^2}{4} + \frac{C}{t^2}.$$

Here's a picture of the direction field and some solutions:



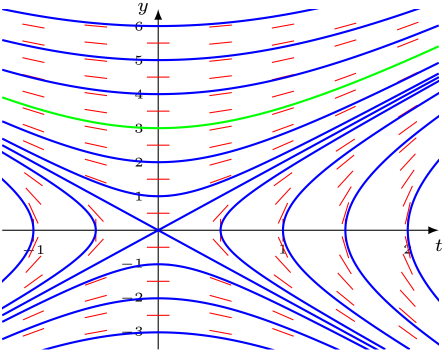
Note: Solutions with $y(t_0) = y_0$ for $t_0 > 0$ are defined for all $t > 0$;
Solutions with $y(t_0) = y_0$ for $t_0 < 0$ are defined for all $t < 0$.

A Non-linear Example

In Lecture 04, we solved

$$\frac{dy}{dt} = \frac{4t}{y}.$$

Here $f(t, y) = 4t/y$ and $\frac{\partial f}{\partial y} = -4t/y^2$ are continuous in any rectangle not crossing the t-axis ($y = 0$). Here are the solutions we found:



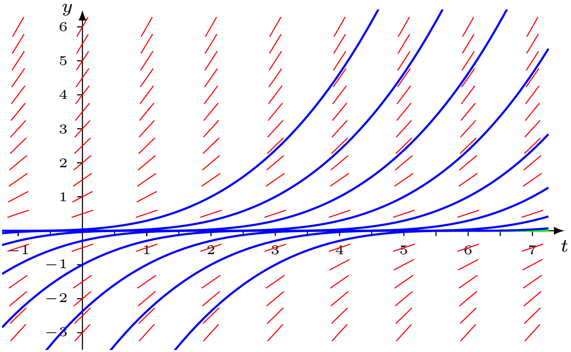
Another Non-linear Example

Solve the IVP

$$\begin{cases} y' = y^{2/3} \\ y(t_0) = 0. \end{cases}$$

You can easily check that both these solutions would work:

$$y_1(t) = 0 \qquad \text{and} \qquad y_2(t) = (t - t_0)^3/27.$$



Another Non-linear Example (cont'd)

Solve the IVP

$$\begin{cases} y' = y^{2/3} \\ y(t_0) = 0. \end{cases}$$

So: We ***don't*** have a unique solution to this IVP. What went wrong?

Here $f(t, y) = y^{2/3}$, so $\dfrac{\partial f}{\partial y} = \dfrac{2}{3}\, y^{-1/3} = \dfrac{2}{3\sqrt[3]{y}}$ is ***discontinuous*** at $y = 0$.

That is, the hypotheses of the theorem are not met, so we shouldn't expect a unique solution for any initial point $(t, y) = (t_0, 0)$.

One More Non-linear Example

Solve the IVP

$$\begin{cases} y' = y^2 \\ y(0) = 1. \end{cases}$$

- Note that $f(t, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are as nice as you could ask for.
- Separate: $y^{-2}\,dy = dt$ and so $-y^{-1} = t + C$ or $y = \frac{-1}{t+C}$.
- $y(0) = 1$ means $C = -1$, so $y = \frac{1}{1-t}$.
- So this solution is defined on $-\infty < t < 1$. Why $t = 1$?