Math 4B: Differential Equations

Lecture 10: Second Order ODEs

- General Second Order ODEs,
- Linear Differential Equations,
- A Special Case & More!

Second Order ODEs

Today we move to study ***second order differential equations***; that is, equations of the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \qquad \text{or} \qquad y'' = f(t, y, y').$$

These are generally very complicated so we will focus on ***linear*** second order ODES, usually written

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

We'll begin our focus on ***homogenous*** ODEs; that is, ones where $g(t)$ or $G(t)$ is zero:

$$y'' + p(t)y' + q(t)y = 0 \qquad \text{or} \qquad P(t)y'' + Q(t)y' + R(t)y = 0.$$

This Chapter

In Chapter 3 we'll study homogeneous equations with constant coefficients. That is, we'll study IVPs like this one:

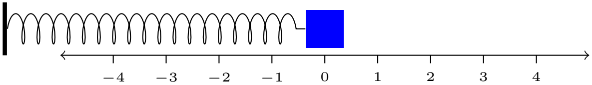
Homogeneous Second Order with Constant Coefficients

$$\begin{cases} ay'' + by' + cy = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

Notice that the initial conditions are at a single t value and specify both y and y' at this value.

Remember the Mass on a Spring

A mass on a spring:



$F = ma$ or rather $ma = \sum($ forces $)$.
Forces:

- Hooke's law says that the spring force is $F = -kx$ (where $k > 0$)
- Friction is proportional to velocity, then $F_{\text{friction}} = -\gamma v$ $(\gamma > 0)$
- External (non-constant?) forcing?

Thus

$$ma = F = F_{\text{spring}} + F_{\text{friction}} + F_{\text{external}} \quad \Longrightarrow \quad mx'' = -kx - \gamma x' + g(t)$$

or

$$mx'' + \gamma x' + kx = g(t).$$

A solution

So let's take a basic example:

$$x'' + 4x = 0 \qquad \text{or} \qquad x'' = -4x$$

Model:

- mass on a spring
- no friction
- no outside forces

Question: What kind of function would give a solution to this?

Model's natural solution: oscillating function like sine or cosine

Check: Both $x = \sin(2t)$ and $x = \cos(2t)$ work. More on this later!

Damped Oscillations

Today: Added Friction

We'll consider the differential equation

$$mx'' + \gamma x' + kx = 0$$

or

$$ay'' + by' + cy = 0.$$

Guess: $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$, so

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \qquad \text{or} \qquad \left(ar^2 + br + c\right)e^{rt} = 0.$$

How to solve SOLHDEWCC

The function $y = e^{rt}$ is a solution to the ODE $ay'' + by' + cy = 0$ if and only if $ar^2 + br + c = 0$.

The Approach

To find solutions to

$$ay'' + by' + cy = 0,$$

we find the zeroes of the ***characteristic polynomial*** $ar^2 + br + c$.

Find two roots, r_1 and r_2: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Three cases:

- $r_1 \neq r_2$, both real
- $r_1 = r_2$ (both must be real)
- r_1, r_2 complex (non-real)

Today: Two real, different roots ($r_1 \neq r_2$ real)

General Solutions

General Case

Suppose $ar^2 + br + c$ has two distinct real roots r_1 and r_2. Then the ***general solution*** to

$$ay'' + by' + cy = 0$$

is $\boxed{y = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$.

Typical IVP looks like

$$\begin{cases} ay'' + by' + cy = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y'_0 \end{cases}$$

Every solution can be written as $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for some choice of the constants c_1 and c_2.

Example #1

Example 1

Solve the IVP

$$\begin{cases} x'' + 5x' + 6x = 0 \\ x(0) = 3 \\ x'(0) = 1 \end{cases}$$

The roots of $r^2 + 5r + 6 = (r+3)(r+2)$ are $r_1 = -3$ and $r_2 = -2$.

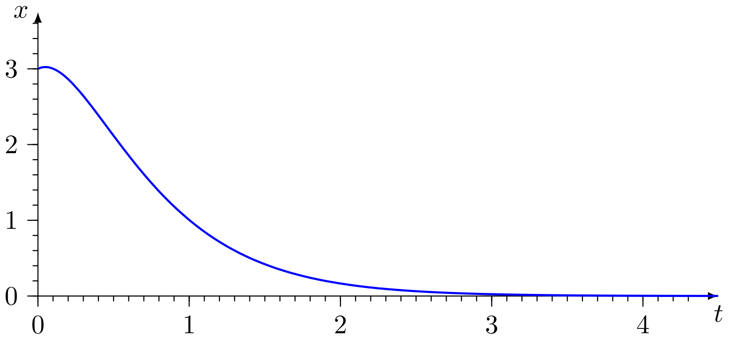
So the general solution is $x(t) = c_1 e^{-3t} + c_2 e^{-2t}$.

The initial conditions are (since $x'(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t}$)

$$\begin{aligned} x(0) &= c_1 + c_2 = 3 \\ x'(0) &= -3c_1 - 2c_2 = 1 \end{aligned} \qquad \Longrightarrow \qquad c_1 = -7,\ c_2 = 10$$

Thus $\boxed{x(t) = -7e^{-3t} + 10e^{-2t}}$.

Example #1 Graph



$$x(t) = -7e^{-3t} + 10e^{-2t}$$

This is an ***overdamped*** system.

Example #1 Tweaked

Example 1

Solve the IVP

$$\begin{cases} x'' + 5x' + 6x = 0 \\ x(0) = 5 \\ x'(0) = -25 \end{cases}$$

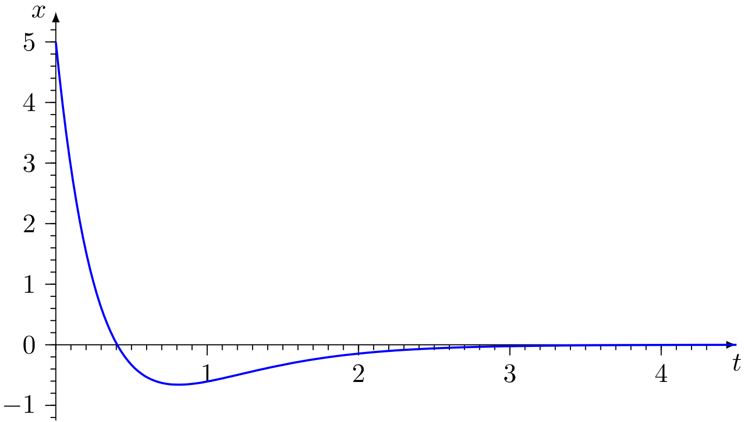
The general solution is still $x(t) = c_1 e^{-3t} + c_2 e^{-2t}$.

The initial conditions are (since $x'(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t}$)

$$\begin{aligned} x(0) &= c_1 + c_2 = 5 \\ x'(0) &= -3c_1 - 2c_2 = -25 \end{aligned} \qquad \Longrightarrow \qquad c_1 = 15,\ c_2 = -10$$

Thus $\boxed{x(t) = 15e^{-3t} - 10e^{-2t}}$.

Example #1 Tweaked Graph



$$x(t) = 15e^{-3t} - 10e^{-2t}$$

This is still an ***overdamped*** system.

Example #2

Example 2

Solve the IVP

$$\begin{cases} x'' + 5x' + 4x = 0 \\ x(0) = 8 \\ x'(0) = -5 \end{cases}$$

The roots of $r^2 + 5r + 4 = (r+4)(r+1)$ are $r_1 = -4$ and $r_2 = -1$.

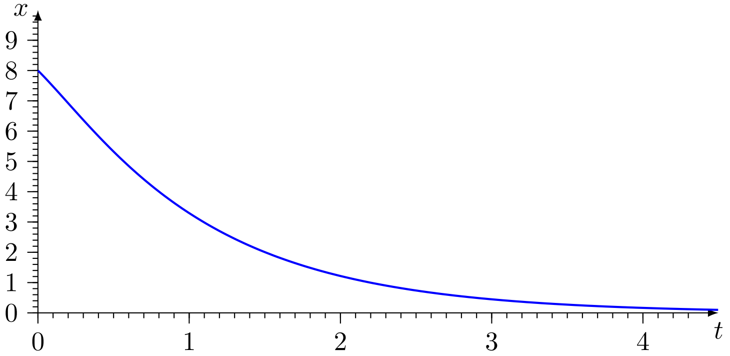
So the general solution is $x(t) = c_1 e^{-4t} + c_2 e^{-t}$.

The initial conditions are (since $x'(t) = -4c_1 e^{-3t} - c_2 e^{-t}$)

$$\begin{aligned} x(0) &= c_1 + c_2 = 8 \\ x'(0) &= -4c_1 - c_2 = -5 \end{aligned} \qquad \Longrightarrow \qquad c_1 = -1, \; c_2 = 9$$

Thus $\boxed{x(t) = -e^{-4t} + 9e^{-t}}$.

Example #2 Graph



$$x(t) = -e^{-4t} + 9e^{-t}$$

This is another ***overdamped*** system.

Analogy with Math 4A

In **Linear Algebra**, to solve the ***homogeneous linear system*** $A\mathbf{x} = \mathbf{0}$, we did the following:

1. Row reduce A to $\mathrm{rref}(A)$

2. Each column of $\mathrm{rref}(A)$ without a pivot (a "free variable" column) gives a vector $\mathbf{x}_i$ that solves $A\mathbf{x} = \mathbf{0}$.

3. The ***general solution*** of $A\mathbf{x} = \mathbf{0}$ is $\mathrm{Span}\left(\mathbf{x}_1, \ldots, \mathbf{x}_k\right)$.

In **Differential Equations**, we do the same thing to solve the ***homogeneous second order ODE with constant coefficients:***

1. From $ax'' + bx' + cx = 0$, form the quadratic equation $ar^2 + br + c = 0$.

2. Each root r_i of the quadratic equation $ar^2 + br + c = 0$ gives a solution $e^{r_i t}$.

3. The ***general solution*** of $ax'' + bx' + cx = 0$ is $\mathrm{Span}\left(e^{r_1 t}, e^{r_2 t}\right)$. That is, any solution to our ODE is a linear combination of those solutions from Step **2.**