Financial Engineering and Risk Management

Introduction to no-arbitrage

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Contracts, prices and no-arbitrage

Consider the following contract

- Pay price p at time t=0
- Receive c_k at time t = k, $k = 1, \ldots, T$

Note that the cash flow c_k could be negative!

The no-arbitrage condition bounds the price p for this contract.

- Weak No-Arbitrage: $c_k \geq 0$ for all $k \geq 1 \Rightarrow p \geq 0$
- ullet Strong No-Arbitrage: $c_k \geq 0$ for all $k \geq 1$ and $c_\ell > 0$ for some $\ell \Rightarrow p > 0$

Essentially eliminate the possibility of a free-lunch!

Rationale for the weak no-arbitrage condition: Suppose p < 0

- Since $c_k \ge 0$ for all $k \ge 1$, the buyer receives -p > 0 at time 0, and then does not lose money thereafter. Free lunch!
- ullet Seller can increase price as long as $p \leq 0$, and still have buyers available.
- Buyers will be willing to pay a higher price in order to compete.

Assumptions underlying no-arbitrage

Rationale for the strong no-arbitrage condition

- Suppose $p \leq 0$.
- Recall that $c_{\ell} > 0$ for some $\ell \geq 1$. Therefore, a free lunch as long as $p \leq 0$.
- We can only guarantee that p > 0 but not the precise value!

Implicit assumptions underlying the no-arbitrage condition

- Markets are liquid: sufficient number of buyers and sellers
- Price information is available to all buyers and sellers
- Competition in supply and demand will correct any deviation from no-arbitrage prices

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Pricing a simple bond

What is the price p of a contract that pays A dollars in 1 year? Suppose one is able to borrow and lend unlimited amounts at an interest rate of r per year.

Construct the following portfolio

- Buy the contract at price p
- Borrow A/(1+r) at interest rate r

Cash flows associated with this portfolio

Price of portfolio	Cashflow in 1 year
$z = p - \frac{A}{1+r}$	A - A = 0

Weak No-arbitrage: $c_1 \geq 0$ implies price $z \geq 0$, i.e. $p \geq \frac{A}{1+r}$.

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Pricing a simple bond (contd.)

Next, construct the following portfolio

- **Sell** the contract at price p
- Lend A/(1+r) at interest rate r

Cash flows associated with this portfolio

Price of portfolio	Cashflow in 1 year
$z = \frac{A}{1+r} - p$	-A + A = 0

Weak No-arbitrage: $c_1 \geq 0$ implies price $z \geq 0$, i.e. $p \leq \frac{A}{1+r}$.

Two results together imply: $p = \frac{A}{1+r}$. Surprise?

The result relied on the ability to borrow and lend at rate r.

- What if borrowing and lending rates are different?
- What if the borrowing and lending markets are elastic?