Math 4B: Differential Equations

Lecture 25: Complex Eigenvalues

- Complex Eigenvalues,
- Real Solutions
- & More!

An Example

1. Find the general solution to

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{x}(t).$$

Solution: Here's the plan we used last time:

- We find the eigenvalues λ_1 and λ_2
- ...and the corresponding eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.
- Then the general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \boldsymbol{\xi}_1 + c_2 e^{\lambda_2 t} \boldsymbol{\xi}_2$.

Today's Problem: What if the λs and $\boldsymbol{\xi}$s are complex (not real)?

Details

Eigenvalues: The eigenvalues of $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ are the roots of

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = \left(\lambda^2 - 2\lambda + 1\right) + 4 = \lambda^2 - 2\lambda + 5$$

Thus $\lambda_1 = 1 + 2i$ and $\lambda_2 = \overline{\lambda_1} = 1 - 2i$.

Eigenvectors:

$$\begin{aligned} \operatorname{Null}\left(A - (1+2i)I\right) &= \operatorname{Null}\begin{pmatrix} 1-(1+2i) & -2 \\ 2 & 1-(1+2i) \end{pmatrix} \\ &= \operatorname{Null}\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} = \operatorname{Null}\begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

If $\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is in this null space, then $i\xi_1 + 1\xi_2 = 0$ or $\xi_2 = -i\xi_1$.

So $\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is our first eigenvector.

Example (continued)

Since $A\boldsymbol{\xi}_1 = \lambda_1 \boldsymbol{\xi}_1$, taking the conjugates give

$$\overline{A\boldsymbol{\xi}_1} = \overline{\lambda_1}\,\overline{\boldsymbol{\xi}_1} \qquad \Longrightarrow \qquad A\overline{\boldsymbol{\xi}_1} = \lambda_2\,\overline{\boldsymbol{\xi}_1} \quad \text{since} \quad \overline{\lambda_1} = \lambda_2.$$

Thus $\boldsymbol{\xi}_2 = \overline{\boldsymbol{\xi}_1} = \begin{pmatrix} 1 \\ +i \end{pmatrix}$.

Solution: We then get

$$\boxed{\mathbf{x}(t) = c_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}}.$$

This isn't really what we're looking for, as these seem to be ***complex*** (non-real) solutions.

Idea: Now take the real and imaginary parts of our complex solutions to find a ***real*** fundamental set of solutions.

General Complex Approach

So suppose we have two eigenvalues $\lambda = \alpha \pm i\beta$ (where $\beta \neq 0$) with corresponding eigenvectors $\boldsymbol{\xi} = \mathbf{u} \pm i\mathbf{v}$ (where α, β, $\mathbf{u}$ and $\mathbf{v}$ are real). For one eigenpair, we get

$$\begin{aligned} e^{(\alpha+i\beta)t}\left(\mathbf{u}+i\mathbf{v}\right) &= e^{\alpha t}\left(\cos(\beta t)+i\sin(\beta t)\right)\left(\mathbf{u}+i\mathbf{v}\right) \\ &= e^{\alpha t}\left(\cos(\beta t)\mathbf{u}-\sin(\beta t)\mathbf{v}\right)+i\,e^{\alpha t}\left(\sin(\beta t)\mathbf{u}+\cos(\beta t)\mathbf{v}\right). \end{aligned}$$

So instead of the fundamental set of solutions

$$\left\{e^{(\alpha+i\beta)t}\left(\mathbf{u}+i\mathbf{v}\right), e^{(\alpha-i\beta)t}\left(\mathbf{u}-i\mathbf{v}\right)\right\},$$

we'll take the ***real*** fundamental set of solutions

$$\left\{e^{\alpha t}\left(\cos(\beta t)\mathbf{u}-\sin(\beta t)\mathbf{v}\right), e^{\alpha t}\left(\sin(\beta t)\mathbf{u}+\cos(\beta t)\mathbf{v}\right)\right\}.$$

This Approach for Our Example

In our example we have $\lambda = 1 \pm 2i$ and $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ \mp i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mp i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then

$$e^{(1+2i)t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = e^t\left(\cos(2t) + i\sin(2t)\right)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= e^t\left(\cos(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + i\, e^t\left(\sin(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \cos(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

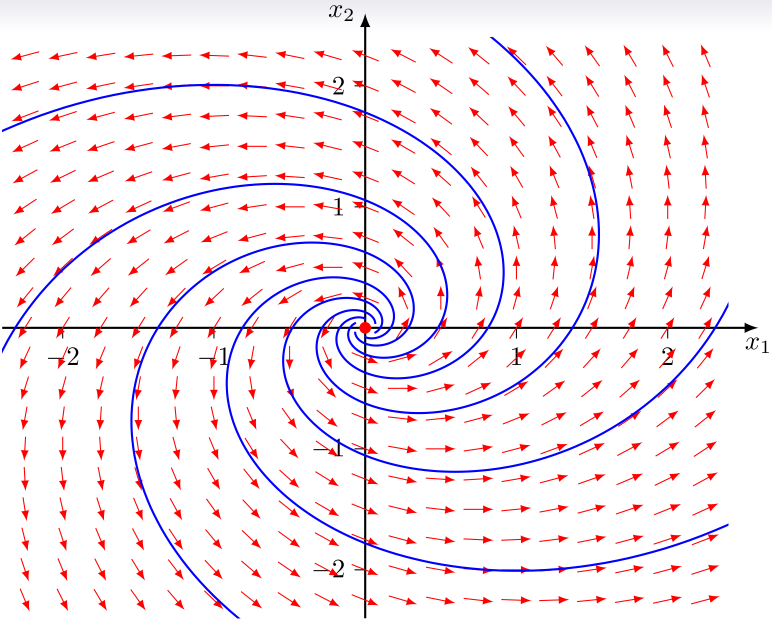
$$= \begin{pmatrix} e^t\cos(2t) \\ e^t\sin(2t) \end{pmatrix} + i\begin{pmatrix} e^t\sin(2t) \\ -e^t\cos(2t) \end{pmatrix}$$

So instead of the fundamental set of solutions

$$\left\{ e^{(1+2i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}, e^{(1-2i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} \right\},$$

we'll take the ***real*** fundamental set of solutions

$$\left\{ \begin{pmatrix} e^t\cos(2t) \\ e^t\sin(2t) \end{pmatrix}, \begin{pmatrix} e^t\sin(2t) \\ -e^t\cos(2t) \end{pmatrix} \right\}.$$


x_2
x_1
-2
-1
1
2
2
1
-1
-2

Another Example

2. Find the general solution to

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 5 \\ -2 & 0 \end{pmatrix} \mathbf{x}(t).$$

Solution: Here's the plan we used last time:

- We find the eigenvalues $\lambda = \alpha \pm i\beta$
- …and the corresponding eigenvectors $\boldsymbol{\xi} = \mathbf{u} \pm i\mathbf{v}$.
- Then a real fundamental set of solution is

$$\left\{ \mathrm{Re}\left(e^{(\alpha+i\beta)t} \left(\mathbf{u} + i\mathbf{v}\right) \right), \mathrm{Im}\left(e^{(\alpha+i\beta)t} \left(\mathbf{u} + i\mathbf{v}\right) \right) \right\}$$

or

$$\left\{ e^{\alpha t} \left(\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v} \right), e^{\alpha t} \left(\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v} \right) \right\}.$$

Details

Eigenvalues: The eigenvalues of $A = \begin{pmatrix} -2 & 5 \\ -2 & 0 \end{pmatrix}$ are the roots of

$$0 = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 5 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 10.$$

Thus $\lambda_1 = -1 + 3i$ and $\lambda_2 = \overline{\lambda_1} = -1 - 3i$.

Eigenvectors:

$$\begin{aligned} \operatorname{Null}(A - (-1+3i)I) &= \operatorname{Null}\begin{pmatrix} -2-(-1+3i) & 5 \\ -2 & -(-1+3i) \end{pmatrix} \\ &= \operatorname{Null}\begin{pmatrix} -1-3i & 5 \\ -2 & 1-3i \end{pmatrix} = \operatorname{Null}\begin{pmatrix} -1-3i & 5 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

If $\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is in this null space, then $(-1-3i)\xi_1 + 5\xi_2 = 0$ or...

So $\boldsymbol{\xi}_1 = \begin{pmatrix} 5 \\ 1+3i \end{pmatrix}$ is our first eigenvector and $\boldsymbol{\xi}_2 = \overline{\boldsymbol{\xi}_1} \begin{pmatrix} 5 \\ 1-3i \end{pmatrix}$.

Example (continued)

We found $\lambda = -1 \pm 3i$ and $\boldsymbol{\xi} = \begin{pmatrix} 5 \\ 1 \pm 3i \end{pmatrix}$. Thus $\alpha = -1$, $\beta = 3$, $\mathbf{u} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

Thus a fundamental set of solutions is

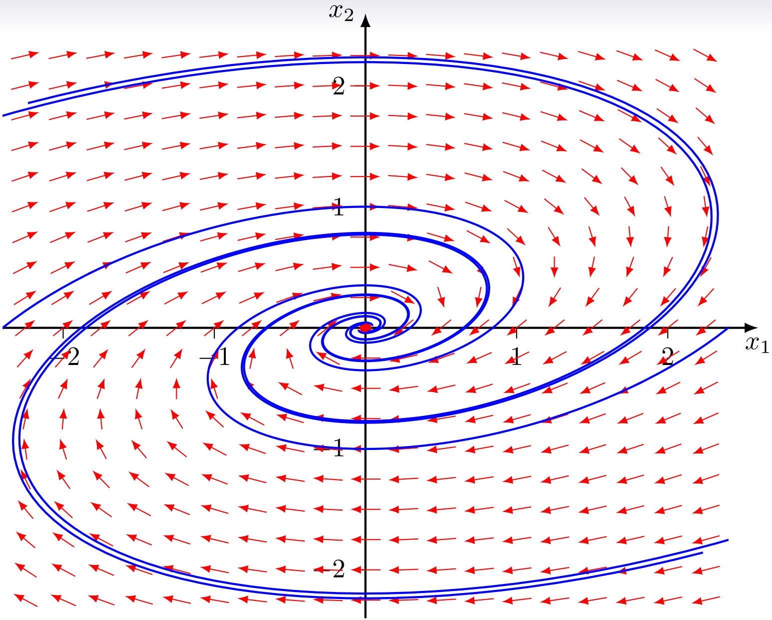
$$\left\{ e^{\alpha t} \left(\cos(\beta t) \mathbf{u} - \sin(\beta t) \mathbf{v} \right), e^{\alpha t} \left(\sin(\beta t) \mathbf{u} + \cos(\beta t) \mathbf{v} \right) \right\}$$

or

$$\left\{ e^{-t} \left(\cos(3t) \begin{pmatrix} 5 \\ 1 \end{pmatrix} - \sin(3t) \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right), e^{-t} \left(\sin(3t) \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \cos(3t) \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right) \right\}$$

or

$$\left\{ e^{-t} \begin{pmatrix} 5\cos(3t) \\ \cos(3t) - 3\sin(3t) \end{pmatrix}, e^{-t} \begin{pmatrix} 5\sin(3t) \\ \sin(3t) + 3\cos(3t) \end{pmatrix} \right\}.$$


x_2
2
1
−1
−2
x_1
−2
−1
1
2

A Third Example

3. Find the general solution to

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} \mathbf{x}(t).$$

Solution: Here's the plan again:

- We find the eigenvalues $\lambda = \alpha \pm i\beta$
- ...and the corresponding eigenvectors $\boldsymbol{\xi} = \mathbf{u} \pm i\mathbf{v}$.
- Then a real fundamental set of solution is

$$\left\{ \operatorname{Re}\left(e^{(\alpha+i\beta)t} \left(\mathbf{u} + i\mathbf{v}\right) \right), \operatorname{Im}\left(e^{(\alpha+i\beta)t} \left(\mathbf{u} + i\mathbf{v}\right) \right) \right\}$$

or

$$\left\{ e^{\alpha t} \left(\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v} \right), e^{\alpha t} \left(\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v} \right) \right\}.$$

Details

Eigenvalues: The eigenvalues of $A = \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}$ are the roots of

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 5 \\ -1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)^2 + 5 = \lambda^2 + 4$$

Thus $\lambda_1 = 2i$ and $\lambda_2 = \overline{\lambda_1} = -2i$.

Eigenvectors:

$$\begin{aligned} \operatorname{Null}(A - 2i\, I) &= \operatorname{Null}\begin{pmatrix} 1-2i & 5 \\ -1 & -1-2i \end{pmatrix} \\ &= \operatorname{Null}\begin{pmatrix} 1-2i & 5 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

So $\boldsymbol{\xi}_1 = \begin{pmatrix} 5 \\ -1+2i \end{pmatrix}$ is our first eigenvector and $\boldsymbol{\xi}_2 = \overline{\boldsymbol{\xi}_1} = \begin{pmatrix} 5 \\ -1-2i \end{pmatrix}$.

Example (continued)

We found $\lambda = \pm 2i$ and $\boldsymbol{\xi} = \begin{pmatrix} 5 \\ -1 \pm 2i \end{pmatrix}$. Thus $\alpha = 0$, $\beta = 2$,
$\mathbf{u} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.
Thus a fundamental set of solutions is

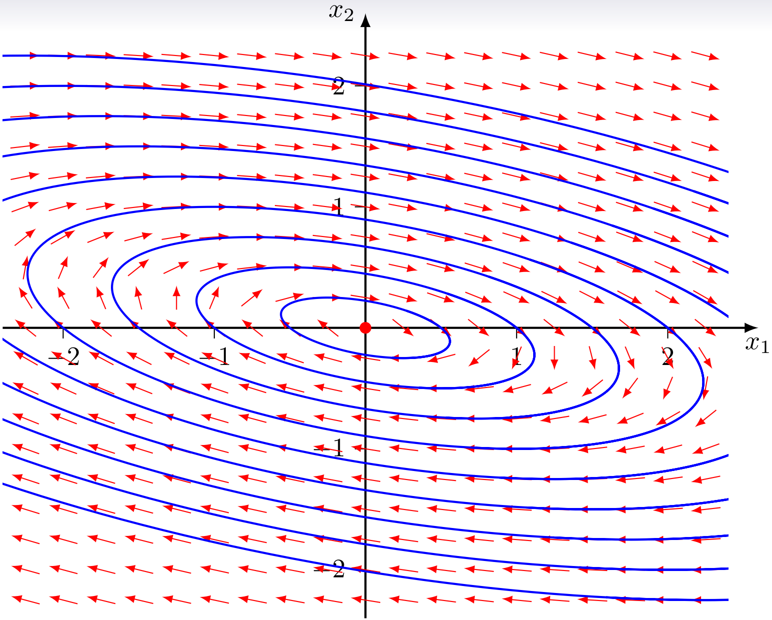
$$\left\{ e^{\alpha t} \left(\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v} \right), e^{\alpha t} \left(\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v} \right) \right\}$$

or

$$\left\{ \cos(2t) \begin{pmatrix} 5 \\ -1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \sin(2t) \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \cos(2t) \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$$

or

$$\left\{ \begin{pmatrix} 5\cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{pmatrix}, \begin{pmatrix} 5\sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{pmatrix} \right\}.$$



x_2
x_1
2
1
−1
−2
−2
−1
1
2

Equilibrium Points

In all these examples, $(0,0)$ is an ***equilibrium point*** because $\mathbf{x}' = 0$ at $\mathbf{x} = \mathbf{0}$. (In our examples, each A in $\mathbf{x}' = A\mathbf{x}$ is invertible, so $\mathbf{x} = \mathbf{0}$ is the ***only*** equilibrium point in these examples.)

- In Example 1, solutions spiraled out from the origin. We call this a ***spiral point***. Since solutions spiral out, this equilibrium point is ***not*** asymptotically stable.

  This happens when $\mathrm{Re}(\lambda) > 0$ (that is, $\alpha > 0$).

- In Example 2, solutions spiraled in toward the origin. This is again a ***spiral point***. Since solutions spiral in, this equilibrium point ***is*** asymptotically stable.

  This happens when $\mathrm{Re}(\lambda) < 0$ (that is, $\alpha < 0$).

- In Example 3, solutions are periodic and orbit the origin. In this case the origin is called a ***center***. This equilibrium point is ***stable*** but ***not*** asymptotically stable.

  This happens when $\mathrm{Re}(\lambda) = 0$ (that is, $\alpha = 0$).