Lecture 11: 08/24/22

Sketch of proof of Lemma 11.2: $\left(S_t - Ke^{-r(T-t)}\right)^+ \overset{(ii)}{\le} C_t(K,T) \overset{(i)}{\le} S_t$

(i) $\underbrace{C_t(K,T)}_{\text{"short"}} \ge \underbrace{S_t}_{\text{"long"}}$: "covered call"

arbitrage: sell/write call buy stock invest remaining money @ r

(ii) $\underbrace{C_t(K,T)}_{\text{"long"}} \le \underbrace{S_t - Ke^{-r(T-t)}}_{\text{"short"}}$; $\left(\Leftrightarrow \left(S_t - C_t(K,T)\right)e^{r(T-t)} - K \ge 0\right) \quad (*)$

arbitrage: buy call short sell stock invest remaining money @ r

More precisely:

	t	T: $S_T \le K$	T: $S_T \ge K$
long call	$+C_t(K,T)$	0	$S_T - K$
short stock	$-S_t$	$-S_T$	$-S_T$
invest	$S_t - C_t(K,T)$	$(S_t - C_t(K,T))e^{r(T-t)}$	$(S_t - C_t(K,T))e^{r(T-t)}$
net value	0	$(S_t - C_t(K,T))e^{r(T-t)} - S_T$ $\ge (S_t - C_t(K,T))e^{r(T-t)} - K$ ≥ 0 b/c of $(*)$	$(S_t - C_t(K,T))e^{r(T-t)} - K$ ≥ 0 b/c of $(*)$

$\Rightarrow$ arbitrage opportunity

Sketch of proof of Lemma 11.3: $\left(Ke^{-r(T-t)} - S_t\right)^+ \overset{(ii)}{<} P_t(K,T) \overset{(i)}{<} Ke^{-r(T-t)}$

(i) $\underbrace{P_t(T,K)}_{\text{"short"}} \geq \underbrace{Ke^{-r(T-t)}}_{\text{"long"}}$:

arbitrage : sell/write put invest premium @ r

(ii) $\underbrace{P_t(T,K)}_{\text{"long"}} \leq \underbrace{Ke^{-r(T-t)} - S_t}_{\text{"short"}}$:

arbitrage : buy put buy share borrow money needed @ r

Lemma 11.5. Put-Call Parity $C_t(T,K) - P_t(T,K) = S_t - Ke^{-r(T-t)}$

$$\Leftrightarrow \quad \underbrace{C_t(T,K) + K e^{-r(T-t)}}_{\text{portfolio A}} = \underbrace{S_t + P_t(T,K)}_{\text{portfolio B}}$$

Sketch of proof:

(i) $\underbrace{C_t(T,K) + K e^{-r(T-t)}}_{\text{portfolio A}} < \underbrace{S_t + P_t(T,K)}_{\text{portfolio B}}$:

arbitrage : short sell portfolio B

$\Rightarrow$ short sell stock, write/sell put

buy portfolio A

$\Rightarrow$ buy call, invest $Ke^{-r(T-t)}$ @ r

invest difference between A and B @ r

(ii) $\underbrace{C_t(T,K) + K e^{-r(T-t)}}_{\text{portfolio A}} > \underbrace{S_t + P_t(T,K)}_{\text{portfolio B}}$:

arbitrage : short sell portfolio A

$\Rightarrow$ write/sell call, borrow $Ke^{-r(T-t)}$ @ r

buy portfolio B

$\Rightarrow$ buy stock, buy put

invest difference between A and B @ r