1. (10 points) We chose a number from the set $\{1, 2, 3, \ldots, 100\}$ uniformly at random and denote this number by X. For each of the following choices decide whether the two events in question are independent ~~or not~~.

   (a) $A = \{X \text{ is even}\}, B = \{X \text{ is divisible by } 5\}$

   (b) $C = \{X \text{ has two digits}\}, D = \{X \text{ is divisible by } 3\}$

   (c) $E = \{X \text{ is prime}\}, F = \{X \text{ has a digit } 5\}$. Note that 1 is not considered a prime number.

a) $A = \{2, 4, 6, 8, 10, 12, \ldots 100\} \Rightarrow \boxed{2}$

$B = \{5, 10, 15, 20, 25, 30, \cdots 100\} = \boxed{5}$

They are not independent of each other.

b). No they are not independent. Counter example: $C = 33$

c) They are not independent of each other. Ex: 5.

2. (10 points) Suppose there are two student assistants working as typists in the main office of the Statistics & Applied Probability Department at UCSB. The number of typos per page made by student assistant A is a Poisson random variable with parameter $\lambda_A = 1$. The number of typos per page made by student assistant B is also a Poisson random variable with an average of 10 typos per page.

   One of the professors in the department asks one of the students to type up a letter. From experience, this work will be done with 1/3 probability by student A and with 2/3 probability by student B.

   (a) What is the probability that the typewritten letter will contain **exactly one typo**?

   (b) It turns out that the typewritten letter does **not** contain **any** typos. Given this information, what is the probability that student B typewrote this letter?

a) $P(A) = \frac{1}{3}$ $\qquad$ $P(B) = \frac{2}{3}$

Poisson (1) $\qquad$ Poisson (10)

$$P(X=1) = \frac{1}{3}\left(\frac{1^1}{1!}e^{-1}\right) + \frac{2}{3}\left(\frac{10^1}{1!}e^{-10}\right)$$
$$= \frac{1}{3}e^{-1} + \frac{2}{3}(10e^{-10})$$
$$= 0.1229$$

b). $$P(X=0) = \frac{1}{3}\cdot\left(\frac{1^0}{0!}e^{-1}\right) + \frac{2}{3}\left(\frac{10^0}{0!}e^{-10}\right)$$

$= 0.1227$

$P(\text{No typos by } B \mid \text{No typo}) = \frac{2}{3}\left(\frac{10^0}{0!}e^{-10}\right)/0.1227$

$= 0.000247$

3. (10 points) Suppose you are rolling a *fair* die 600 times independently. Let X count the number of sixes that appear.

   (a) What type of random variable is X? Specify all parameters needed to characterize X as well as the state space S_X of X.

   (b) Find the probability that you observe the number 6 at most 100 times.

a). Binomial, success rate $p\ \frac{1}{6}$ $\quad S_X: \{1, 2, 3, 4, 5, 6 \ldots 600\}$.
(600 times)

b). $P_6 = \binom{600}{100}\left(\frac{1}{6}\right)^{100}\left(\frac{5}{6}\right)^{600-100} = 0.52842$

   (c) Use a famous limit theorem (which one?) to show why the probability in (b) can be approximated by the value 1/2.

   Hint: Use (without proof) the fact that

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt = \frac{1}{2}.$$

c) Central limit theorem.

$E(600) = n \cdot P = \frac{1}{6} \cdot 600 = 100,$

$Var(600) = n \cdot P(1-P) = 600\left(\frac{1}{6}\right)(5/6) = 83.3$

$SD = \sqrt{83.3} = 9.129$

$P(X \leq 100) = P\Big(\underset{\Uparrow\, Z}{\frac{X-100}{9.129}} \leq \frac{100-100}{9.129}\Big) \sim N(0,1)$

4. (10 points) Suppose that X is a continuous random variable whose probability density function is given by

$$f_X(t) = \begin{cases} c \cdot (4t - 2t^2), & 0 < t < 2, \\ 0, & \text{otherwise}, \end{cases}$$

where $c > 0$ is a constant.

(a) What is the value of c?

(b) Compute the cumulative distribution function F_X of X.

(c) Find the probabilities $\mathbb{P}(X = 1)$ and $\mathbb{P}(X > 1)$.

(d) Compute the variance of X.

a) $\int_0^2 c(4t-2t^2) = 1$

$c \int_0^2 (4t-2t^2) = 1.$

$c \quad 2t^2 - \frac{2t^3}{3} \Big|_0^2 = 1$

$c \cdot \quad 8/3 = 1$

$\boxed{c = 3/8}$

b). $\int 3/8\,(4t-2t^2) = -\frac{3}{4}\left(\frac{t^3}{3} - t^2\right)$

$\begin{cases} -\frac{3}{4}\left(\frac{t^3}{3} - t^2\right) & 0<t<2 \\ 0 & \text{o.w.} \end{cases}$

c). $P(X=1) \Rightarrow -\frac{3}{4}\left(\frac{1}{3} - 1\right) = -\frac{3}{4}\left(-\frac{2}{3}\right) = +\frac{1}{2}$

$P(X>1) \Rightarrow 1 - P(X \leq 1) = 1 - \int_0^1 \frac{3}{8}(4t-2t^2) = 1 - 0.5 = 0.5$

d) $E(X) = \int_0^2 x\, f(x)\,dx = \int_0^2 t \cdot 3/8\,(4t-2t^2)\,dt = 1$

$Var(X) = E(X^2) - (E(x))^2 = \int_0^2 x^2 f(x)\,dx - (1)^2$

$= 1.2 - 1 = 0.2$