Math 4B: Differential Equations

Lecture 19: The Laplace Transform

- Definitions, General Plan,
- Lots of Examples,
- Properties & More!

The Laplace Transform

Today we're going to introduce something called the ***Laplace transform*** which will

- turn functions of t into functions of s
- turn ODEs/IVPs in t into algebra problems in s
- allow us to solve IVPs with a new class of functions (piecewise continuous functions)

Definition of an Integral Transform

Given a ***kernel*** $K(s,t)$, an integral transform

$$F(s) = \int_{\alpha}^{\beta} K(s,t)\, f(t)\, dt \qquad (-\infty \le \alpha < \beta \le +\infty)$$

turns a function $f(t)$ into $F(s)$, the transform of f.

The Laplace Transform

Suppose $f(t)$ is defined for $t \geq 0$ and satisfies some conditions to be named later. Define the ***Laplace transform*** of f to be

$$F(s) = \mathscr{L}[\,f(t)\,] = \int_0^\infty e^{-st} f(t)\,dt.$$

turns a function $f(t)$ into $F(s)$, the transform of f.

It's now later. To make sure $\mathscr{L}[\,f(t)\,]$ makes sense, we assume:

- f be ***piecewise continuous*** (That is, f has finitely many jump discontinuities and no infinite discontinuities.)
- $\big|f(t)\big| \leq Ke^{at}$ for some s and K and for all $t \geq M$ for some M (This means f has ***exponential order*** as $t \to \infty$.)

Idea: $\mathscr{L}[\,f(t)\,] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)\,dt + \int_M^\infty e^{-st} f(t)\,dt$

$$\leq \text{constant} + \int_M^\infty K e^{(a-s)t}\,dt < \infty \text{ if } a - s < 0$$

What About ODEs and IVPs?

Our application of Laplace transforms to ODEs and IVPs is a three-step process:

1. Take the Laplace transform of an equation in t to get an algebraic equation in s.
2. Solve the algebraic equation to find the Laplace transform of the solution to the IVP.
3. Undo the Laplace transform (take the ***inverse Laplace transform***) to find the solution to the original IVP.

Our Goal: To explore this a little, find some Laplace transforms, solve a few IVPS, and move on.

Example 1

1. Suppose $f(t) = 1$ is a constant function. Find $\mathscr{L}[\,f(t)\,] = \mathscr{L}[\,1\,]$.

Solution: Remember that

$$\mathscr{L}[\,f(t)\,] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\,dt.$$

We calculate using the usual approach for improper integrals:

$$\begin{aligned}
\mathscr{L}[\,1\,] &= \lim_{b\to\infty} \int_0^b e^{-st}\,dt = \lim_{b\to\infty} \left[\frac{1}{-s}\,e^{-st}\right]_0^b = \lim_{b\to\infty} \frac{1}{-s}\left(e^{-sb} - 1\right) \\
&= \lim_{b\to\infty} \frac{1}{s}\left(1 - e^{-sb}\right) \\
&= \frac{1}{s}\left(1 - 0\right) \qquad \text{if } s > 0
\end{aligned}$$

That is, $\boxed{\mathscr{L}[\,1\,] = \frac{1}{s},\ s > 0}$.

Example 2

2. Find $\mathscr{L}[\, e^{at} \,]$.

Solution: Remember that

$$\mathscr{L}[\, e^{at} \,] = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt.$$

We calculate using the usual approach for improper integrals:

$$\begin{aligned}
\mathscr{L}[\, e^{at} \,] &= \lim_{b\to\infty} \int_0^b e^{(a-s)t}\, dt = \lim_{b\to\infty} \left[\frac{1}{a-s}\, e^{(a-s)t} \right]_0^b \\
&= \lim_{b\to\infty} \frac{1}{a-s}\left(e^{(a-s)b} - 1 \right) = \lim_{b\to\infty} \frac{1}{s-a}\left(1 - e^{(a-s)b} \right) \\
&= \frac{1}{s-a}\left(1 - 0 \right) \qquad \text{if } a - s < 0
\end{aligned}$$

That is, $\boxed{\mathscr{L}[\, e^{at} \,] = \dfrac{1}{s-a},\ s > a}$.

Example 3

3. Find $\mathscr{L}[\, t^n \,]$ for positive integers n.

Claim: $\mathscr{L}[\, t^n \,] = \dfrac{n}{s} \mathscr{L}[\, t^{n-1} \,]$

Remember that

$$\mathscr{L}[\, t^n \,] = \int_0^\infty e^{-st} t^n \, dt.$$

Integrate by parts ($u = t^n$, $dv = e^{-st}\, dt$) to get

$$\mathscr{L}[\, t^n \,] = -\frac{1}{s} t^n e^{-st} \bigg|_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} \, dt = 0 + \frac{n}{s} \mathscr{L}[\, t^{n-1} \,].$$

Since $\mathscr{L}[\, 1 \,] = \mathscr{L}[\, t^0 \,] = \frac{1}{s}$, we get

$$\mathscr{L}[\, t \,] = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2} \qquad \mathscr{L}[\, t^2 \,] = \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3} \qquad \mathscr{L}[\, t^3 \,] = \frac{3}{s} \cdot \frac{2}{s^3} = \frac{3!}{s^4}$$

The general rule is thus $\boxed{\mathscr{L}[\, t^n \,] = \dfrac{n!}{s^{n+1}}}$ for integers $n \geq 0$.

Example 4

4. Find $\mathscr{L}[\sin bt]$.

Solution: Remember that

$$F(s) = \mathscr{L}[\sin(bt)] = \int_0^\infty e^{-st}\sin(bt)\,dt.$$

Integrate by parts ($u = e^{-st}$, $dv = \sin(bt)\,dt$) to get

$$F(s) = e^{-st}\cdot\frac{-\cos(bt)}{b}\bigg|_0^\infty - \frac{s}{b}\int_0^\infty e^{-st}\cos(bt)\,dt$$

Let's do it again ($u = e^{-st}$, $dv = \cos(bt)\,dt$) and see

$$F(s) = \left(0 - \frac{-1}{b}\right) - \frac{s}{b}\left(e^{-st}\cdot\frac{\sin(bt)}{b}\bigg|_0^\infty + \frac{s}{b}\int_0^\infty e^{-st}\sin(bt)\,dt\right)$$

or

$$F(s) = \frac{1}{b} - \frac{s}{b}\left(0 - 0 + \frac{s}{b}F(s)\right) = \frac{1}{b} - \frac{s^2}{b^2}F(s).$$

Solving, we find that $F(s) = \boxed{\mathscr{L}[\sin(bt)] = \frac{b}{s^2+b^2},\ s > 0}$.

Example 5

5. Find $\mathscr{L}[\, e^{at} \sin bt \,]$.

Solution: Remember that

$$\mathscr{L}[\, e^{at} \sin(bt) \,] = \int_0^\infty e^{-st} \cdot e^{at} \sin(bt)\, dt = \int_0^\infty e^{-(s-a)t} \sin(bt)\, dt$$

and we *just* saw that

$$\mathscr{L}[\, \sin(bt) \,] = \int_0^\infty e^{-st} \sin(bt)\, dt = \frac{b}{s^2 + b^2}.$$

Replacing "s" in this formula with "$s - a$" we get

$$\mathscr{L}[\, e^{at} \sin(bt) \,] = \frac{b}{(s-a)^2 + b^2}.$$

That is, $\boxed{\mathscr{L}[\, e^{at} \sin(bt) \,] = \frac{b}{(s-a)^2 + b^2},\ s > a}$.

General Fact

Proposition: Multiplying by e^{at} Leads to a Shift

Suppose $f(t)$ has Laplace transform $\mathscr{L}[\,f(t)\,] = F(s)$ for $s > 0$. Then the Laplace transform of $e^{at}f(t)$ is

$$\mathscr{L}[\,e^{at}f(t)\,] = F(s-a) \qquad \text{for } s > a.$$

Idea: The Laplace transform $\mathscr{L}[\,e^{at}f(t)\,]$ is an integral:

$$\mathscr{L}[\,e^{at}f(t)\,] = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt.$$

Compare this to

$$F(s) = \mathscr{L}[\,y(t)\,] = \int_0^\infty e^{-st}f(t)\,dt,$$

and we see that $\mathscr{L}[\,e^{at}f(t)\,] = F(s-a)$.

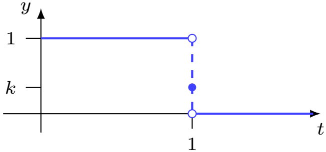
Moral: $\mathscr{L}[\,\cos(bt)\,] = \dfrac{s}{s^2+b^2} \quad \Longrightarrow \quad \mathscr{L}[\,e^{at}\cos(bt)\,] = \dfrac{s-a}{(s-a)^2+b^2}.$

Piecewise Continuous Functions

Remember that we've defined ***piecewise continuous functions*** to be functions that are continuous except possibly at a finite number of jump discontinuities. In particular, these functions have no infinite discontinuities (vertical asymptotes).

6. Find $\mathscr{L}[\,f(t)\,]$, the Laplace transform of the function

$$f(t) = \begin{cases} 1 & \text{if } t < 1 \\ k & \text{if } t = 1 \\ 0 & \text{if } t > 1 \end{cases}$$



Solution: We do the usual thing:

$$\mathscr{L}[\,f(t)\,] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \int_0^1 e^{-st}\,dt + \int_1^1 e^{-st} k\,dt + \int_1^\infty 0\,dt$$

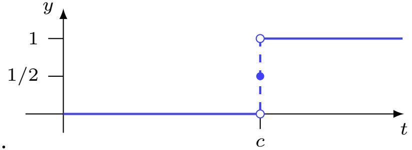
$$= \frac{1}{-s} e^{-st}\Big|_0^1 = \frac{1}{-s}\left(e^{-s} - 1\right) = \frac{1 - e^{-s}}{s}.$$

That is, $\boxed{\mathscr{L}[\,f(t)\,] = \frac{1 - e^{-s}}{s},\ s > 0}$.

The Heaviside Function

7. Find $\mathscr{L}[\,u_c(t)\,]$, the Laplace transform of the ***Heaviside function*** or ***unit step function***

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ \frac{1}{2} & \text{if } t = c \\ 1 & \text{if } t > c \end{cases}$$



Solution: We do the usual thing:

$$\mathscr{L}[\,u_c(t)\,] = \int_0^\infty e^{-st} \cdot u_c(t)\,dt = \int_0^c 0\,dt + \int_c^c e^{-st}\,\frac{1}{2}\,dt + \int_c^\infty e^{-st}\,dt$$
$$= \frac{1}{-s}e^{-st}\Big|_c^\infty = \frac{e^{-cs}}{s} \text{ if } s > 0.$$

That is, $\boxed{\mathscr{L}[\,u_c(t)\,] = \dfrac{e^{-cs}}{s},\ s > 0}$.

Properties of the Laplace Transform

Remember that

$$\mathscr{L}[\, f(t)\,] = \int_0^\infty e^{-st} f(t)\, dt,$$

so

- $\mathscr{L}[\, c \cdot f(t)\,] = c \cdot \mathscr{L}[\, f(t)\,]$, and
- $\mathscr{L}[\, f(t) + g(t)\,] = \mathscr{L}[\, f(t)\,] + \mathscr{L}[\, g(t)\,]$

Thus we have

The Laplace Transform is a Linear Operator

The Laplace transform is a linear operator in the sense that

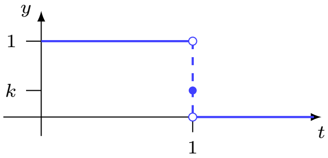
$$\mathscr{L}[\, c_1 f_1(t) + c_2 f_2(t)\,] = c_1 \mathscr{L}[\, f_1(t)\,] + c_2 \mathscr{L}[\, f_2(t)\,]$$

for all constants c_1 and c_2 and appropriate functions $f_1(t)$ and $f_2(t)$.

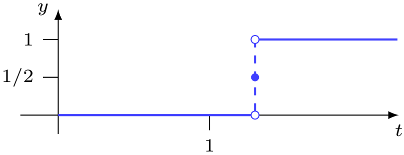
Example

8. We saw in **6.** and **7.** that

$$f(t) = \begin{cases} 1 & \text{if } t < 1 \\ k & \text{if } t = 1 \\ 0 & \text{if } t > 1 \end{cases}$$



and $u_c(t) = \begin{cases} 0 & \text{if } t < c \\ \frac{1}{2} & \text{if } t = c \\ 1 & \text{if } t > c \end{cases}$



had $\mathscr{L}[\, f(t)\,] = \dfrac{1 - e^{-s}}{s}$ and $\mathscr{L}[\, u_c(t)\,] = \dfrac{e^{-cs}}{s}$. Thus

$$\mathscr{L}[\, f(t) + u_1(t)\,] = \frac{1}{s} = \mathscr{L}[\,1\,].$$

This makes sense because, except at $t = 1$, $f(t) + u_1(t) = 1$.

Can We Compute $F(s)$?

In fact, we've found ***lots*** of Laplace Transforms, but here's a table. See Table 6.2.1 in your book.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}[f(t)]$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n > 0$ an integer	$\frac{n!}{s^{n+1}}$, $s > 0$
$t^n e^{at}$, $n > 0$ an integer	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$	$\frac{e^{-cs}}{s}$, $s > 0$

Can We Find $\mathscr{L}[\,y''\,]$ and $\mathscr{L}[\,y'\,]$?

Let's try to compute $\mathscr{L}[\,y'\,]$.

$$\begin{aligned}\mathscr{L}[\,y'\,] &= \int_0^\infty e^{-st} y'\,dt \\ &= e^{-st} y\Big|_0^\infty + s\int_0^\infty e^{-st} y\,dt \\ &= \big(0 - y(0)\big) + s\mathscr{L}[\,y\,].\end{aligned}$$

Theorem on the Laplace Transform of a Derivative

Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that f is of exponential order (so $|f(t)| \le Ke^{at}$ for $t \ge M$ for constants a, K, M). Then $\mathscr{L}[\,f'(t)\,]$ exists for $s \ge a$ and

$$\mathscr{L}[\,f'(t)\,] = s\mathscr{L}[\,f(t)\,] - f(0).$$

Generalize $\mathscr{L}[\,f'\,] = s\mathscr{L}[\,f\,] - f(0)$

So what about $\mathscr{L}[\,f''(t)\,]$? Since $f''(t) = \big(f'(t)\big)'$, the previous theorem says

$$\mathscr{L}[\,f''(t)\,] = s\mathscr{L}[\,f'(t)\,] - f'(0)$$

and so

$$\begin{aligned}\mathscr{L}[\,f''(t)\,] &= s\big(s\mathscr{L}[\,f(t)\,] - f(0)\big) - f'(0) \\ &= s^2\mathscr{L}[\,f(t)\,] - sf(0) - f'(0).\end{aligned}$$

Theorem on the Laplace Transform of Derivatives

Suppose that $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that $f, f', \ldots, f^{(n-1)}$ are of exponential order. Then $\mathscr{L}[\,f^{(n)}(t)\,]$ exists for $s \ge a$ and is given by

$$\begin{aligned}\mathscr{L}[\,f^{(n)}(t)\,] = s^n\mathscr{L}[\,f(t)\,] &- s^{n-1}f(0) - s^{n-2}f'(0) \\ &- \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).\end{aligned}$$

Remember The Plan

Remember: Our application of Laplace transforms to ODEs and IVPs is a three-step process:

1. Take the Laplace transform of an equation in t to get an algebraic equation in s.

   We can do this now!

2. Solve the algebraic equation to find the Laplace transform of the solution to the IVP.

   We can do this now, too!

3. Undo the Laplace transform (take the ***inverse Laplace transform***) to find the solution to the original IVP.

   We can figure out how to do this, as well!

Next Time:

We'll actually solve a few IVPS!