Lecture 16: 09/07/22

Stock price at time T: $\quad S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T} \qquad B_T \sim N(0,T)$

Proof of lemma 14.13:

Compute $\mathbb{E}[S_T]$

One quick possibility: Via moment generating function

Recall: $X \sim N(\mu, \sigma^2)$

mgf of X: $\quad m_X(u) = \mathbb{E}[e^{uX}] = e^{u\mu + \frac{1}{2}\sigma^2 u^2}$

$\Rightarrow \quad \mathbb{E}[e^X] = e^{\mu + \frac{1}{2}\sigma^2}$

Therefore, we obtain:

$$\mathbb{E}[S_T] = \mathbb{E}\left[S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T}\right] = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T} \underbrace{\mathbb{E}[e^{\sigma B_T}]}_{= e^{\frac{1}{2}\sigma^2 T}}$$

b/c $\sigma B_T \sim N(0, \sigma^2 T)$

$$= S_0 e^{\mu T}$$

Proof of Theorem 15.2:

We have
$$S_T = S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_T}$$
(here: $B_T = \sqrt{T}\, Z$ w/ $Z \sim N(0,1)$)

Arbitrage-free price of a call option (at time $t=0$):

$$\mathbb{E}^*\left[e^{-rT}(S_T - K)^+\right]$$
(risk-neutral valuation)

First:

$$\mathbb{E}^*\left[e^{-rT}\underbrace{(S_T - K)^+}_{=\max\{S_T - K,\,0\}}\right] = \mathbb{E}^*\left[e^{-rT}(S_T - K)\cdot \mathbb{1}_{\{S_T \geq K\}}\right]$$

$$= \begin{cases} S_T - K, & S_T \geq K \\ 0, & S_T < K \end{cases}$$

$$\mathbb{1}_{\{S_T \geq K\}} = \begin{cases} 1 & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

$$= e^{-rT}\underbrace{\mathbb{E}^*\left[S_T \cdot \mathbb{1}_{\{S_T \geq K\}}\right]}_{(2)} - K e^{-rT}\underbrace{\mathbb{E}^*\left[\mathbb{1}_{\{S_T \geq K\}}\right]}_{(1)}$$

For (1):

$$\mathbb{E}^*\left[\mathbb{1}_{\{S_T \geq K\}}\right] = 1\cdot \mathbb{P}^*[S_T \geq K] + 0\cdot \mathbb{P}^*[S_T < K] = \mathbb{P}^*[S_T \geq K]$$

$$= \mathbb{P}^*\left[S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z} \geq K\right]$$

$$= \mathbb{P}^*\left[Z \geq \underbrace{\frac{\log(\frac{K}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{=\beta}\right]$$

$\Phi(d) = \mathbb{P}[Z \leq d]$

$Z \overset{d}{=} -Z$ (symmetry)

$$= \mathbb{P}^*\left[-Z \geq \frac{\log(\frac{K}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]$$

$$= \mathbb{P}^*\left[Z \le \underbrace{\frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}_{= d_-(S_0, T)} \right] = \Phi(d_-(S_0,T))$$

For ②:

$$\mathbb{E}^*\left[S_T \cdot \mathbb{1}_{\{S_T \ge K\}} \right] = \mathbb{E}^*\left[S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z} \cdot \mathbb{1}_{\{Z \ge \beta\}} \right]$$

$$= S_0 e^{(r-\frac{1}{2}\sigma^2)T} \int_{\beta}^{+\infty} e^{\sigma\sqrt{T}z} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}}_{\text{density of } Z \sim \mathcal{N}(0,1)} dx$$

$$= S_0 e^{(r-\frac{1}{2}\sigma^2)T + \frac{1}{2}\sigma^2 T} \int_{\beta}^{+\infty} \frac{1}{\sqrt{2\pi}} \underbrace{e^{\sigma\sqrt{T}z - \frac{1}{2}z^2 - \frac{1}{2}\sigma^2 T}}_{= e^{-\frac{1}{2}(z - \sigma\sqrt{T})^2}} dx$$

"completing squares"

$$= S_0 e^{(r-\frac{1}{2}\sigma^2)T + \frac{1}{2}\sigma^2 T} \underbrace{\int_{\beta - \sigma\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy}$$

substitute
$z \rightsquigarrow y + \sigma\sqrt{T}$

$$= \mathbb{P}\left[Z \ge \beta - \sigma\sqrt{T} \right]$$

$$= \mathbb{P}\left[Z \le \underbrace{-\beta + \sigma\sqrt{T}}_{= d_+(S,T)} \right]$$

$$= S_0 e^{rT} \Phi(d_+(S,T))$$