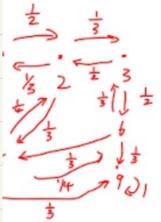
Problem 6.1. (15 points) Teddy the cat likes to hunt his pet snake. Note that his pet snake is a stuffed animal, so it stays wherever it is placed. To add some excitement to his life, today he will hunt the snake in the maze in Figure 6.1. At time 0 Teddy is placed in area 1, and his pet snake is placed in area 9. Each minute teddy moves to a new area by choosing one of the doors in his current area uniformly at random and going through it (so, for example, since he starts in area 1 at time 0, at time 1 he will be in either area 2 or area 4, and both are equally likely). Compute the expect number of minutes that it takes Teddy to reach his pet snake.



$E_1(T_9)$

FIGURE 1. The maze contains nine areas. Teddy starts in area 1 and his pet snake starts in area 9.

$$
\begin{array}{c|cccccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 0 & \frac12 & 0 & \frac12 & 0 & 0 & 0 & 0 & 0 \\
2 & \frac13 & 0 & \frac13 & 0 & \frac13 & 0 & 0 & 0 & 0 \\
3 & 0 & \frac12 & 0 & 0 & 0 & \frac12 & 0 & 0 & 0 \\
4 & \frac13 & 0 & 0 & 0 & \frac13 & 0 & \frac13 & 0 & 0 \\
5 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 \\
6 & 0 & 0 & \frac13 & 0 & \frac13 & 0 & 0 & 0 & \frac13 \\
7 & 0 & 0 & 0 & \frac12 & 0 & 0 & 0 & \frac12 & 0 \\
8 & 0 & 0 & 0 & 0 & \frac13 & 0 & \frac13 & 0 & \frac13 \\
\hline
9 & 0 & 0 & 0 & 0 & 0 & \frac12 & 0 & \frac12 & 1
\end{array}
$$

$E_1(T_9)$

$$m = e + Bm$$

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$m = (I - B)^{-1} e = \begin{pmatrix} 18 \\ 17 \\ 15 \\ 17 \\ 15 \\ 11 \\ 15 \\ 11 \end{pmatrix}$$

So is 18.

Problem 6.2. (10 points) Consider a Markov chain $\{X_n\}$ on $\{0,1,2,\dots\}$ with the following transition probability matrix P given by, for $x,y \in \{0,1,2,\dots,\}$

$$P_{x,y} = \begin{cases} p, & y = x+1 \\ 1-p, & y = 0. \end{cases}$$

For each $x \in \{1,2,\dots\}$, let $\tau_x \doteq \inf\{n \geq 0 : X_n = x\}$, and calculate $m_x \doteq \mathbb{E}[\tau_x | X_0 = 0]$.

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \cdots \\ \hline 0 & 1-p & p & 0 & 0 & 0 & \\ 1 & 1-p & 0 & p & 0 & 0 & \\ 2 & 1-p & 0 & 0 & 0 & 0 & \\ 3 & 1-p & 0 & 0 & p & 0 & \\ 4 & 1-p & 0 & 0 & 0 & p & \\ \vdots & & & & & & \end{array}$$

$$m_{x-1} = 1 + (1-p)m_0 + p\, m_x$$

$$= 1 + (1-p)m_0 + 0 = (1-p)\left(\frac{1}{1-p} + m_0\right)$$

$$m_{x-2} = 1 + (1-p)m_0 + p\, m_{x-1}$$

$$= 1 + (1-p)m_0 + p\left(\frac{1}{1-p} + m_0\right)(1-p) = (1-p^2)\left(\frac{1}{1-p} + m_0\right)$$

$$m_{x-i} = 1 + (1-p)m_0 + p\, m_{x-i}$$

$$= 1 + (1-p)m_0 + p\left(\frac{1}{1-p} + m_0\right)(1-p^i)$$

$$= (1-p)\left(\frac{1}{1-p} + m_0\right) + p\left(\frac{1}{1-p} + m_0\right)(1-p^i)$$

$$= (1-p^i)\left(\frac{1}{1-p} + m_0\right)$$

$$m_0 = m_{x-x} = (1-p^x)\left(\frac{1}{1-p} + m_0\right)$$

Problem 6.3. (15 points) You repeatedly flip a fair coin until you observe the sequence of flips $HTHT$ for the first time.

(a) Model the process as a Markov chain.
(b) Compute the expected number of flips it takes for you to observe the sequence $HTHT$.

(a)

$$
\begin{array}{c|ccccc}
 & 0 & H & HT & HTH & HTHT \\
\hline
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
H & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
HT & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
HTH & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
HTHT & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$M = (I - B)^{-1} e = \begin{pmatrix} 20 \\ 18 \\ 16 \\ 10 \end{pmatrix}$$

So 20 times

Problem 6.4. (15 points) Consider the Markov chain $\{X_n\}$ on $\mathcal{S} = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{array}{c} 1 \\ \\ \\ 4 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/8 & 1/4 & 1/8 \\ 1/4 & 1/2 & 1/8 & 1/8 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the chain is observed when it reaches either of the states in the set $A \doteq \{1, 4\}$.

(a) Denote the absorption time of the chain by

$$\tau_A \doteq \inf\{n \geq 0 : X_n \in A\}.$$

For each $x \in \mathcal{S}$, calculate $\mathbb{E}_x[\tau_A]$.

(b) Let E denote the event that the chain is eventually absorbed, namely,

$$E \doteq \{\tau_A < \infty\}.$$

Calculate $\mathbb{P}_x(E)$ for each $x \in \mathcal{S}$.

(c) For each $x \in \mathcal{S}$, let

$$\tau_x \doteq \inf\{n \geq 0 : X_n = x\}.$$

For each $x \in \mathcal{S}$, calculate $\mathbb{P}_x(\tau_4 < \tau_1)$.

(a) $E(\tau_A \mid X_0 = 1) = E(\tau_A \mid X_0 = 4) = 0$

$E(\tau_A \mid X_0 = 2) = \frac{1}{2}[1 + E(\tau_1)] + \frac{1}{8}[1 + E(\tau_2)] + \frac{1}{4}[1 + E(\tau_3)] + \frac{1}{8}[1 + E(\tau_4)]$

$E(\tau_A \mid X_0 = 3) = \frac{1}{2}[1 + E(\tau_1)] + \frac{1}{2}[1 + E(\tau_2)] + \frac{1}{8}[1 + E(\tau_3)] + \frac{1}{8}[1 + E(\tau_4)]$

$$\Rightarrow \begin{cases} m_1 = m_4 = 0 \\ m_2 = \frac{1}{2} + \frac{1}{8}(1 + m_2) + \frac{1}{4}(1 + m_3) + \frac{1}{8} \\ m_3 = \frac{1}{2} + \frac{1}{2}(1 + m_2) + \frac{1}{8}(1 + m_3) + \frac{1}{8} \end{cases} \Rightarrow \begin{cases} m_1 = 0 \\ m_2 = \frac{72}{41} \\ m_3 = \frac{88}{41} \\ m_4 = 0 \end{cases}$$

(b) $P_1(E) = P_4(E) = 1$

$P_2(E) = \frac{1}{2} P_1(E) + \frac{1}{8} P_2(E) + \frac{1}{4} P_3(E) + \frac{1}{8} P_4(E)$

$P_3(E) = \frac{1}{4} P_1(E) + \frac{1}{2} P_2(E) + \frac{1}{8} P_3(E) + \frac{1}{8} P_4(E)$

So $P_1(E) = P_2(E) = P_3(E) = P_4(E) = 1$

(c) $P_1 = 0$

$P_4 = 1$

$P_2 = \frac{1}{8} P_3 + \frac{1}{4} P_3 + \frac{1}{8}$

$P_3 = \frac{1}{2} P_2 + \frac{1}{8} P_3 + \frac{1}{8}$

$$\Rightarrow \begin{cases} P_1 = 0 \\ P_2 = \frac{9}{41} \\ P_3 = \frac{11}{41} \\ P_4 = 1 \end{cases}$$