Math 174E
Lecture 9

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References

Hull

Chapters 5.10, 5.11, 5.12

Forward and Futures Contracts on Currencies 1/3

- forward and futures contracts on a foreign currency from the perspective of a U.S. investor (domestic currency is the U.S. dollar)
- underlying asset: one unit of the foreign currency quoted in U.S. dollars
- exchange rate = price of one unit of the foreign currency (**base currency**) quoted in U.S. dollars (**quote currency**)
  - *Examples:* EUR/USD = 1.0168, GBP/USD = 1.2103, CNY/USD = 0.1474, JPY/USD = 0.0075, CHF/USD = 1.0534 (BTC/USD = 24,045.46, ETH/USD = 1,895.12)
- a holder of a foreign currency can earn interest at the risk free interest rate prevailing in the foreign country (e.g., by investing the currency in a foreign-denominated bond)
- hence, a foreign currency can be regarded as an investment asset paying a known yield

Forward and Futures Contracts on Currencies 2/3

- spot price S_0 and forward price $F_0(T)$ is the price of one unit of the foreign currency expressed in U.S. dollars
- r_f foreign risk-free interest rate (p.a.)
- r domestic risk-free interest rate (p.a.)

Lemma 5.10

The arbitrage-free forward exchange rate (at time 0) is given by

$$F_0(T) = S_0 e^{(r-r_f)\cdot T}.$$

This relationship is also called **interest rate parity**.

Proof: Lecture Notes.

Compare Lemma 5.10 with Lemma 5.6: a foreign currency can be regarded as an **investment asset paying a known yield r_f**.

Forward and Futures Contracts on Currencies 3/3

Comment:

- note that the forward price (futures price) might be increasing or decreasing with time to maturity, depending on whether $r > r_f$ or $r < r_f$

Forward and Futures Contracts on Commodities 1/5

- commodities typically have **storage costs**
- recall that a commodity is either considered an **investment asset** (like gold and silver) or a **consumption asset** (like crude oil)

- some commodities that are investment assets (like gold and silver) can also provide **income** to the holder

  Example:

  - gold owners such as central banks charge a *gold lease rate* when they lend gold
  - gold mining companies might seek to hedge their exposure to price changes in the gold price by locking in a price to sell its gold via a short position in a forward contract with an investment bank
  - the investment bank (long position) might then want to hedge its risk by short selling (= borrowing) gold from the central bank

Forward and Futures Contracts on Commodities 2/5

Forward contract on a **commodity** which is an **investment asset** with **income** and **storage costs** (e.g., gold and silver)

- *absolute storage costs* can be treated as a *negative income*
- *proportional storage costs* can be treated as a *negative yield*

Lemma 5.11

The arbitrage-free forward price (at time 0) of a **commodity** which is an **investment asset** with spot price S_0 is given by

$$F_0(T) = (S_0 + U_0)e^{rT},$$

where U_0 represents the present value at time 0 of all storage costs, *net of income*, during the life of the forward contract.

Forward and Futures Contracts on Commodities 3/5

Lemma 5.11 (continued)

If the storage costs (net of income) incurred at any time are proportional to the price of the commodity, the arbitrage-free forward price (at time 0) is given by

$$F_0(T) = S_0 e^{(r+u)T},$$

where u denotes the storage costs per annum as a proportion of the spot price *net of any yield earned* on the commodity.

Proof: Follows from Lemma 5.4 and 5.6.

Forward and Futures Contracts on Commodities 4/5

Commodities as **consumption assets** (e.g., crude oil):

- primary for consumption
- usually no income, but significant storage costs
- individuals and companies who own a consumption commodity are usually *reluctant to sell the commodity in the spot market and buy it back in the forward market*

As a consequence:

Lemma 5.12

The arbitrage-free forward price (at time 0) of a **commodity** which is a **consumption asset** satisfies the upper bound

$$F_0(T) \leq (S_0 + U_0)e^{rT} \qquad \text{or} \qquad F_0(T) \leq S_0 e^{(r+u)T},$$

respectively, where U_0 and u are as in Lemma 5.11.

Forward and Futures Contracts on Commodities 5/5

Summary of the arbitrage opportunities:

	$F_0(T) > (S_0 + U_0)e^{rT}$	$F_0(T) \leq (S_0 + U_0)e^{rT}$
$t = 0$	borrow amount $S_0 + U_0$ at r use it to buy commodity and pay storage costs short position in forward	sell commodity save storage costs invest proceeds at r long position in forward
$t = T$	sell commodity for $F_0(T)$ repay loan with $(S_0 + U_0)e^{rT}$	buy commodity for $F_0(T)$ receive $(S_0 + U_0)e^{rT}$
net profit	$F_0(T) - (S_0 + U_0)e^{rT} > 0$	$(S_0 + U_0)e^{rT} - F_0(T) \geq 0$

- the argumentation in the case $F_0(T) \leq (S_0 + U_0)e^{rT}$ **cannot** be used for a commodity that is a **consumption asset** (rather than an investment asset)
- individuals/companies owning the commodity for consumption might simply **not be willing to sell it** (*limited arbitrage*)

Convenience Yield 1/3

- we do not necessarily have **equality** in the forward price equations in Lemma 5.12 for **consumption commodities**
- physical ownership of a consumption commodity provides benefits that are not obtained by a forward contract
  Example: oil refiner using crude oil
- the benefits from holding the physical asset are sometimes referred to as the **convenience yield** provided by the commodity

Definition 5.13

The **convenience yield** $\boldsymbol{y}$ of a commodity (or asset) is defined such that

$$F_0(T)e^{yT} = (S_0 + U_0)e^{rT} \qquad \text{or} \qquad F_0(T)e^{yT} = S_0 e^{(r+u)T},$$

holds, where U_0 and u are as in Lemma 5.12.

Convenience Yield 2/3

- the convenience yield simply measures the extent to which the left-hand side is less than the right-hand side in the inequalities of Lemma 5.12
- for **investment assets** the convenience yield is always zero, otherwise there would be an arbitrage opportunity

- the convenience yield reflects the **market's expectations** concerning the future availability of the commodity
- the greater the possibility that shortages will occur, the higher the convenience yield
- if users of the commodity have high inventories, there is very little chance for shortages in the near future and the convenience yield tends to be low

Convenience Yield 3/3

Example 5.14

Suppose the current crude oil spot price is $S_0 = \$92.24$ per barrel and storage costs are \$0.10 per barrel per month (payable in advance). Moreover, assume that the current futures price for crude oil with maturity in 3 months is $F_0(T) = \$87.19$ per barrel and that the risk-free rate is 2% p.a.

Therefore, the present value of the storage costs (per barrel) today is

$$U_0 = 0.10 + 0.10 \cdot e^{-0.02 \cdot 1/12} + 0.10 \cdot e^{-0.02 \cdot 2/12} = 0.2992,$$

and the arbitrage-free upper bound for the futures price is

$$(U_0 + S_0)e^{rT} = (0.2992 + 92.24)e^{0.02 \cdot 3/12} = 93.0030.$$

Consequently, the convenience yield y for crude oil satisfies

$$87.19 \cdot e^{y \cdot 3/12} = 93.0030 \quad \Leftrightarrow \quad y = 0.2582.$$

The Cost of Carry

- relationship between spot prices and forward (futures) prices can be summarized by the **cost of carry c**
- measures the storage cost plus the interest that is paid to finance the asset less the income earned on the asset

  Examples:
  - non-dividend paying stock: $c = r$
  - stock index: $c = r - q$
  - foreign currency: $c = r - r_f$
  - commodity providing income at rate q and requiring storage costs at rate u: $c = r + u - q$

- **investment asset:** $F_0(T) = S_0 e^{cT}$
- **consumption asset:** $F_0(T) = S_0 e^{(c-y)T}$

- note that forward prices (futures prices) can be increasing or decreasing with time to maturity T, depending on the sign of c and $c - y$

Chapter 7: Swaps

Hull
Chapter 7.1

Introduction

Definition 7.1

A **swap** is an over-the-counter derivatives agreement between two parties to *exchange cash flows* in the future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated.

- exchange of cash flows on several future dates, involving the future value of an interest rate, an exchange rate, or any other market variables ...
- these cash flows are not known in advance
- used for hedging and speculation
- a forward contract can be seen as a simple example of a swap with *only one exchange of cash flows* at maturity

Examples: interest rate swaps, currency swaps, ...

Interest Rate Swaps 1/2

“plain vanilla” interest rate swap:

- most popular type of interest rate swap
- basic mechanism
  - a company agrees to pay cash flows equal to interest at a predetermined *fixed rate* on a notional principal for a predetermined period of time
  - in return company receives cash flows equal to interest at a *floating rate* on the same notional principal for the same period of time
  - interest payments are exchanged but not notional principal
- exchange unknown **floating rate** against a predetermined **fixed rate** (= **swap rate**)

Interest Rate Swaps 2/2

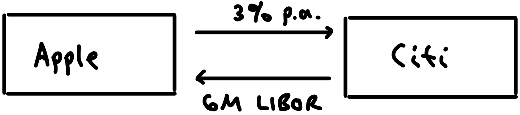
- **swap rate** (fixed rate) is determined such that there are no arbitrage opportunities and such that the value of the swap contract is zero when initiated (same as with forward contracts and forward prices)
- **floating rate** in most interest rate swaps: **LIBOR rate**
  - LIBOR = reference rate of interest for borrowing/lending in international financial markets ("market rate")
  - see Chapter 4
- each exchange of cash flows is a **forward rate agreement** (see Chapter 4)

Interest Rate Swap Example 1/2

- consider a 3-year swap between Apple and Citigroup initiated on March 8, 2017

- Apple agrees to pay to Citigroup an interest rate of 3% p.a. on a notional principal of $100 million (**fixed rate payer**)
- Citigroup agrees to pay Apple the 6-month LIBOR rate on the same notional principal (**floating rate payer**)

- swap agreement specifies that payments are to be exchanged every 6 months and that all interest rates are quoted p.a. with semiannual compounding



Interest Rate Swap Example 2/2

Swap cash flows from **Apple's perspective** ($ millions) for *one possible outcome* of the 6-month LIBOR rates:

Date	6-month LIBOR	Floating cash flow received	Fixed cash flow paid	Net cash flow
Mar. 8, 2017	2.20			
Sept. 8, 2017	2.80	+ 1.10	−1.50	−0.40
Mar. 8, 2018	3.30	+ 1.40	−1.50	−0.10
Sept. 8, 2018	3.50	+ 1.65	−1.50	+0.15
Mar. 8, 2019	3.60	+ 1.75	−1.50	+0.25
Sept. 8, 2019	3.90	+ 1.80	−1.50	+0.30
Mar. 8, 2020		+ 1.95	−1.50	+0.45

Source of table: Hull, Chapter 7.1, Table 7.1 page 157.

Note that the $100 million principal is *not* exchanged (used only for calculating the interest payments)