Math 174E
Lecture 11

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References

Hull
Chapter 11.4, 11.5, 11.6, 11.7

Put-Call Parity: European Case 1/2

For arbitrage-free **European call** and **put** option prices on the same **non-dividend** paying stock:

Lemma 11.5

We have

$$
\begin{aligned}
& C_t(K,T) - P_t(K,T) = S_t - Ke^{-r(T-t)} \\
\Leftrightarrow \quad & \underbrace{C_t(K,T) + Ke^{-r(T-t)}}_{\text{portfolio A}} = \underbrace{S_t + P_t(K,T)}_{\text{portfolio B}}
\end{aligned}
$$

for all $t \in [0,T]$.

Proof: See next slide and lecture notes for a sketch.

Note: Put-call parity allows to deduce the price of a European call option with a certain strike K and maturity T from the price of a European put with same strike K and maturity T, and vice versa!

Put-Call Parity: European Case 2/2

Value of portfolio A and B at time T:

		$S_T > K$	$S_T < K$
portfolio A	long call	$S_T - K$	0
	long zero-coupon bond w/ principal K	K	K
	total	S_T	K
portfolio B	long put	0	$K - S_T$
	long share	S_T	S_T
	total	S_T	K
	value of portfolio A and B	$\max\{S_T, K\}$	

- since portfolio A and B have **identical values at time T** (and the European options can not be exercised prior to time T), they must have **identical values at any time $t < T$** (**"Law of one price"**)
  - otherwise, an arbitrageur could *buy the less expensive* portfolio and *(short) sell the more expensive* portfolio
  - because the portfolio values are guaranteed to cancel each other out at time T, this would lock in an arbitrage profit equal to the difference in the value of the two portfolios

Numerical Example

Example 11.6 (Put-Call parity)

Suppose the current stock price (of a non-dividend paying stock) is \$31, the risk-free interest rate is 10% p.a. and the price of a 3-month European call option with exercise price \$30 is \$3.

Then, the price of a European put option written on the same stock with same strike K and maturity T must be

$$P_0(K,T) = C_0(K,T) - S_0 + Ke^{-rT} = 3 - 31 + 30e^{-0.1 \cdot 0.25} = 1.26.$$

American vs. European Call option

Lemma 11.7

For a **non-dividend** paying stock it is never optimal to exercise an **American call option** before the expiration date. In particular,

$$C_t^{\mathrm{am}}(T, K) = C_t(T, K)$$

for all $t \in [0, T]$.

Proof: See Assignment 5.

Remark:

- for an American put option on a non-dividend paying stock it can be optimal to exercise before maturity
- in case of dividend-paying stocks it can be optimal to exercise an American put and call option before maturity

Arbitrage-free Bounds: American Call

For arbitrage-free **American call option** prices on a **non-dividend** paying stock:

Lemma 11.8

We have
$$(S_t - Ke^{-r(T-t)})^+ < \boldsymbol{C_t^{\mathrm{am}}(K, T)} < S_t$$
for all $t \in [0, T)$.

Proof: Follows from Lemma 11.7 and Lemma 11.2.

Arbitrage-free Bounds: American Put

For arbitrage-free **American put option** prices on a **non-dividend** paying stock:

Lemma 11.9

We have

$$(K - S_t)^+ \leq \boldsymbol{P_t^{\mathrm{am}}(K, T)} < K$$

for all $t \in [0, T)$.

Remark:

- when early exercise at time $t < T$ is optimal, the value of the American put option is actually $K - S_t$ ($=$ intrinsic value)

Put-Call Parity: American Case

For arbitrage-free **American call** and **put** option prices on the same **non-dividend** paying stock:

Lemma 11.10

We have

$$S_t - K \leq C_t^{\text{am}}(K, T) - P_t^{\text{am}}(K, T) \leq S_t - Ke^{-r(T-t)}.$$

for all $t \in [0, T]$.

Proof: Problem 11.18 in Hull.

Effect of Dividends

Dividend-paying stocks: Suppose the dividends that will be paid during the life of the option are known and denote with D_t their present value at time $t \in [0, T]$.

- European call option:

$$(S_t - D_t - Ke^{-r(T-t)})^+ < C_t(K, T) < S_t$$

- European put option:

$$(D_t + Ke^{-r(T-t)} - S_t)^+ < P_t(K, T) < Ke^{-r(T-t)}$$

- European put-call parity:

$$C_t(T, K) - P_t(T, K) = S_t - D_t - Ke^{-r(T-t)}$$

- American put-call parity

$$S_t - D_t - K \leq C_t^{\mathrm{am}}(K, T) - P_t^{\mathrm{am}}(K, T) \leq S_t - Ke^{-r(T-t)}.$$

Factors Affecting Option Prices

Summary of the effect on the price of a stock option of **increasing** one variable while keeping all others fixed:

variable (factor)	call	put
current stock price S_0	$\uparrow$	$\downarrow$
strike price K	$\downarrow$	$\uparrow$
time to maturity T (European/American)	?/$\uparrow$	?/$\uparrow$
volatility	$\uparrow$	$\uparrow$
risk free interest rate r	$\uparrow$	$\downarrow$
expected future dividends	$\downarrow$	$\uparrow$

Source of table: Hull, Chapter 11.1, Table 11.1, page 232.

Chapter 13: Binomial Trees

Hull
Chapter 13.1

Introduction & Motivation 1/2

Last goal: Pricing and hedging of European and American stock options.

Question:

1. How can we compute *arbitrage-free prices* of stock options?
2. What can the *seller* of a stock option do in order to *hedge* herself against the financial liability (risk) of paying the payoff to the option holder?

As we will see questions 1.) and 2.) are closely related.

Introduction & Motivation 2/2

Problem: Arbitrage-free lower and upper bounds for stock option prices derived in Chapter 11 are too far apart.

Remedy: Mathematical modeling

Approach: We impose a **probabilistic model** for the evolution of the underlying stock price $(S_t)_{t\in[0,T]}$

- stock price process is modeled as a **stochastic process** or random process (= evolves randomly over time)
- Discrete time model: **Binomial tree model** or **Cox-Ross-Rubinstein model (1979)** $(S_n)_{n=0,1,\dots,T}$
  - stock price process is driven by a *random walk*
- Continuous time model: **Black-Scholes-Merton model (1973)** $(S_t)_{t\in[0,T]}$
  - stock price process is driven by a *Brownian motion*

One-Step Binomial Model 1/11

Example 13.1

Suppose we want to value a **European call option** (i.e., compute its price $C_0(K,T)$) on a stock with current stock price $S_0 = \$20$, strike price $K = \$21$ and maturity $T = 3/12$ (3 months). We assume that the risk-free rate is 4% p.a. (continuously compounded).

Payoff at maturity T:

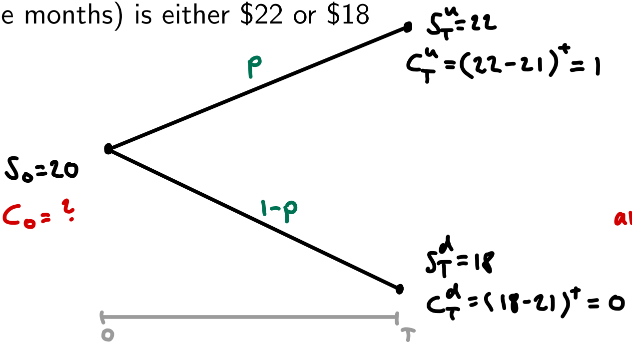
$$C_T(K,T) = (S_T - K)^+ = \max\{S_T - K, 0\} = \begin{cases} S_T - K & S_T \geq K \\ 0 & S_T < K \end{cases}$$

What is $\boldsymbol{C_0(K,T)}$? Arbitrage-free?

One-Step Binomial Model 2/11

Approach: Assume a simple **one-step** probabilistic model for the stock price

- suppose that the stock price S_T at time $T = 3/12$ (at the end of three months) is either \$22 or \$18



arbitrage-free ?

S_T = simple random variable , $C_T = f(S_T)$ = simple random variable

Naive idea: $C_0 = \mathbb{E}\left[e^{-rT}(S_T-K)^+\right] = e^{-rT}\Big(1 \cdot \underbrace{\mathbb{P}[S_T=22]}_{=p\ ?} + 0 \cdot \underbrace{\mathbb{P}[S_T=18]}_{=1-p\ ?}\Big) = p\,e^{-rT}$

One-Step Binomial Model 3/11

Key idea: "replication argument"

- construct a **portfolio** which consists of a *trading strategy in the underlying stock* and a *cash account* (bank account with interest rate r) such that the value of this portfolio at time T **perfectly replicates** (= matches) the **payoff** of the call option C_T
- due to **no-arbitrage** ("Law of one price") the value of this portfolio at time 0 and the price of the call option $C_0(K, T)$ must be the same
- note that this approach implies valuing the call option *relative* to the price/value of the underlying stock and the risk-free bank account

But: Can we actually construct such a strategy?
Yes! In the *binomial model* we can build such a strategy!

One-Step Binomial Model 4/11

Key idea (cont.):

- Introduce:

  $V_0 =$ **initial capital** ($=$ value) of the portfolio/strategy at time 0

  $\Delta_0 =$ **number of shares** to hold at time 0

  (long position if $\Delta_0 > 0$, short position if $\Delta_0 < 0$)

- **Value of the portfolio at time *T***

$$V_T = \underbrace{(V_0 - \Delta_0 S_0) \cdot e^{rT}}_{\substack{\text{cash account} \\ \text{balance at } T}} \quad + \quad \underbrace{\Delta_0 S_T}_{\substack{\text{value of stock} \\ \text{position at time } T}}$$

- Find V_0, Δ_0 such that

$$V_T = (V_0 - \Delta_0 S_0) \cdot e^{rT} + \Delta_0 S_T = (S_T - K)^+ = C_T(K, T)$$

- **No-arbitrage** implies

$$\boldsymbol{V_0 = C_0(K, T)}$$