ASSIGNMENT 2 - SOLUTIONS

PSTAT 160B - SPRING 2022

Instructions for the homework: Solve all of the homework problems, and submit them on GradeScope. Your reasoning has to be comprehensible and complete.

Homework Problems

Problem 2.1. Customers arrive at a coffee shop according to a Poisson process with a rate of $\lambda = 10$ customers per hour. Each customer orders exactly one beverage; fifty percent of the customers order a coffee, 30 percent order green tea, and 20 percent order herbal tea.

(a) Calculate the probability that between time $t = 5$ hours and time $t = 8$ hours that exactly 8 customers order green tea.

(b) Calculate the probability that between time $t = 5$ hours and time $t = 8$ hours that exactly 8 customers order green tea and that exactly 5 customers order herbal tea.

(c) At time $t = 4$ hours, 35 customers have visited the cafe. Given this, what is the probability that exactly 20 of the customers ordered coffee?

Solution 2.1.

(a) If we let N_t denote the number of customers that have arrived by time $t \geq 0$, then we can write

$$N_t = N_t^C + N_t^G + N_t^T,$$

where N_t^C denotes the number of customers who have ordered coffee by time t, N_t^G denotes the number of customers who have ordered coffee by time t, and N_t^T denotes the number of customers who have ordered coffee by time t. From our theorem regarding splitting of Poisson processes, we know that $\{N_t^G\}$ is a Poisson process with rate of $\lambda_G = 0.3 \cdot 10 = 3$ per hour. Then, using the stationary increments property, we have

$$\mathbb{P}(N_8^G - N_5^G = 8) = \mathbb{P}(N_3^G = 8) = e^{-\lambda_G \cdot 3} \frac{(\lambda_G \cdot 3)^8}{8!} \approx 0.132.$$

(b) From our theorem regarding split Poisson processes and the stationary increments property, we have, with $\lambda_G = 3$ and $\lambda_H = 0.2 \cdot 10 = 2$,

$$\begin{aligned}\mathbb{P}(N_8^G - N_5^G = 8, N_8^H - N_5^H = 5) &= \mathbb{P}(N_8^G - N_5^G = 8)\mathbb{P}(N_8^H - N_5^H = 5) \\ &= \mathbb{P}(N_3^G = 8)\mathbb{P}(N_3^H = 5) = e^{-\lambda_G \cdot 3} \frac{(\lambda_G \cdot 3)^8}{8!} e^{-\lambda_H \cdot 3} \frac{(\lambda_H \cdot 3)^5}{5!}.\end{aligned}$$

(c) From the proof of our theorem regarding splitting Poisson processes, we know that, for each $t \geq 0$,

$$N_t^C | N_t = 35 \sim \text{Binomial}(35, 0.5),$$

so it follows that

$$\mathbb{P}(N_4^C = 20 | N_4 = 35) = \binom{35}{20}(0.5)^{20}(0.5)^{15} \approx 0.095.$$

Problem 2.2. Let $\{N_t^1\}$ be a Poisson process with rate $\lambda_1 = 3$, and let $\{N_t^2\}$ be a Poisson process with rate $\lambda_2 = 5$. Assume that $\{N_t^1\}$ and $\{N_t^2\}$ are independent.

(a) Compute $\mathbb{P}(N_{0.5}^1 = 1, N_{0.5}^2 = 2)$.
(b) Compute $\mathbb{P}(N_2^1 + N_2^2 = 15)$.

Solution 2.2.

(a) Using the independence of the two processes, we have

$$\mathbb{P}(N_{0.5}^1 = 1, N_{0.5}^2 = 2) = \mathbb{P}(N_{0.5}^1 = 1)\mathbb{P}(N_{0.5}^2 = 2) = e^{-\lambda_1 \cdot 0.5}\frac{(\lambda_1 \cdot 0.5)^1}{1!}e^{-\lambda_2 \cdot 0.5}\frac{(\lambda_2 \cdot 0.5)^2}{2!} \approx 0.086.$$

(b) Since the process $\{N_t\}$ defined by $N_t \doteq N_t^1 + N_t^2$ is a superposition of the independent Poisson processes $\{N_t^1\}$ and $\{N_t^2\}$, we know that $\{N_t\}$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2 = 8$. Therefore

$$\mathbb{P}(N_2^1 + N_2^2 = 15) = e^{-8 \cdot 2}\frac{(8 \cdot 2)^{15}}{15!} \approx 0.099.$$

Problem 2.3. Subatomic particles of type α enter a chamber according to a Poisson process with a rate of $\lambda_\alpha = 10$ per second, and subatomic particles of type β enter the chamber according to a Poisson process with a rate of $\lambda_\beta = 15$ per second. Additionally, the two types of particles arrive independently of one another. Let α_t denote the number of type α particles by time t, and let β_t denote the number of type β particles by time t

(a) The 10^{th} particle of type α arrives at time $t = 1.5$. Find the probability that the 20^{th} particle of type α arrives within 0.5 seconds after the 10^{th} particle.
(b) Find the probability that, in total, exactly 20 particles enter within the first second.
(c) Denote the total number of particles that have entered the chamber at time t by N_t. Compute $\mathbb{E}[N_3 | \alpha_2 = 25]$.
(d) Each time a particle of type β enters the chamber, it has a 0.001% chance of being detected. How long, on average, will it take for the first particle of type β to be detected?

Solution 2.3.

(a) Let α_t denote the number of particles of type α that have entered the chamber by time $t \geq 0$. Then, we have, using the stationary and independent increments properties,

$$\mathbb{P}(\alpha_2 \geq 20 | \alpha_{1.5} = 10) = \mathbb{P}(\alpha_{0.5} \geq 10) = 1 - \mathbb{P}(\alpha_{0.5} \leq 9) = 1 - \sum_{k=0}^{9} e^{-\lambda_\alpha \cdot 0.5}\frac{(\lambda_\alpha \cdot 0.5)^k}{k!} \approx 0.0318.$$

(b) If we denote the total number of particles that have entered by time $t \geq 0$ as N_t, then $N_t = \alpha_t + \beta_t$, so $\{N_t\}$ is a superposition of independent Poisson processes and therefore is a Poisson process with rate $\lambda = \lambda_\alpha + \lambda_\beta = 25$. Thus

$$\mathbb{P}(N_1 = 20) = e^{-25 \cdot 1}\frac{(25 \cdot 1)^{20}}{20!} \approx 0.052.$$

(c) We have, using the fact the $\{N_t^\alpha\}$ and $\{N_t^\beta\}$ are independent, that

$$\mathbb{E}[N_3|\alpha_2 = 25] = \mathbb{E}[\alpha_3 + \beta_3|\alpha_2 = 25] = \mathbb{E}[\alpha_3|\alpha_2 = 25] + \mathbb{E}[\beta_3].$$

Using the stationary and independent increments property, we know that

$$\begin{aligned}\mathbb{E}[\alpha_3|\alpha_2 = 25] = \mathbb{E}[\alpha_3 - \alpha_2 + \alpha_2|\alpha_2 = 25] &= \mathbb{E}[\alpha_3 - \alpha_2|\alpha_2 = 25] + \mathbb{E}[\alpha_2|\alpha_2 = 25] \\ &= \mathbb{E}[\alpha_3 - \alpha_2] + 25 = \mathbb{E}[\alpha_1] + 25 = 35,\end{aligned}$$

so it follows that

$$\mathbb{E}[N_3|\alpha_2 = 25] = 35 + \mathbb{E}[\beta_3] = 35 + 45 = 80.$$

(d) We can write $N_t^\beta \doteq N_t^{\beta,d} + N_t^{\beta,u}$, where $N_t^{\beta,d}$ denotes the number of type β particles that enter the chamber and are detected by time $t \geq 0$, and $N_t^{\beta,u}$ denotes the number that enter the chamber and are undetected. Then $N_t^{\beta,d}$ is a Poisson process with rate $\lambda_\beta^d \doteq 0.00001 \cdot 15$.

Since the inter-arrival times of $\{N_t^{\beta,d}\}$ follow an Exponential distribution with rate λ_β^d, so it follows that the expected time for the first particle of type β to be detected is

$$\frac{1}{\lambda_\beta^d} = 6666.6 \text{ seconds},$$

or about 1.85 hours.

Problem 2.4. Customers arrive at a shop according to a Poisson process with rate $\lambda = 10$ per hour. The amount of money that each customer spends in the store can be modeled as an exponentially distributed random variable with a mean of 5 dollars. Additionally, we can assume that the amount of money that each customer spends is independent of the spending of the other customers.

(a) Compute the probability that the first two customers combined spend more than 12 dollars.[1]
(b) The shop opens at 8am and closes at 5pm. Compute the expected amount of money that customers will spend at the store over the course of the day.
(c) Compute the variance of the amount of money that customers will spend at the store over the course of the day.
(d) Compute the probability that, over the course of the day, every customer spends at least 2 dollars.

Solution 2.4.

[1] Hint: to do this part of the question, you don't even need to know what a Poisson process is.

(a) Let X_1 and X_2 denote the amount that the first and second customer spend, respectively. Then $X_1, X_2 \overset{iid}{\sim} \text{Exp}(1/5)$, so

$$\begin{aligned}
\mathbb{P}(X_1 + X_2 > 12) &= \int_0^\infty \int_{\max\{12-x_1,0\}}^\infty \frac{1}{5}e^{-\frac{1}{5}x_1}\frac{1}{5}e^{-\frac{1}{5}x_2}dx_2dx_1 \\
&= \frac{1}{5}\int_0^{12} e^{-\frac{1}{5}x_1}\int_{12-x_1}^\infty \frac{1}{5}e^{-\frac{1}{5}x_2}dx_2dx_1 + \frac{1}{5}\int_{12}^\infty e^{-\frac{1}{5}x_1}\int_0^\infty \frac{1}{5}e^{-\frac{1}{5}x_2}dx_2dx_1 \\
&= \frac{1}{5}\int_0^{12} e^{-\frac{1}{5}x_1}e^{-\frac{1}{5}(12-x_1)}dx_1 + \frac{1}{5}\int_{12}^\infty e^{-\frac{1}{5}x_1}dx_1 \\
&= \frac{1}{5}e^{-\frac{12}{5}}\int_0^{12} dx_1 + e^{-\frac{12}{5}} \\
&= \left(\frac{12}{5}+1\right)e^{-\frac{12}{5}} \\
&\approx 0.308.
\end{aligned}$$

(b) Let X_i denote the amount that the i^{th} customer spends, and let N_t denote the number of customers that have arrived by time $t \geq 0$. Then, since the shop is open for 9 hours, we can calculate, using our theorem regarding the expected value of a compound Poisson process,

$$\mathbb{E}\left[\sum_{i=1}^{N_9} X_i\right] = 10 \cdot 9 \cdot 5 = 450,$$

so we expect customers to spend 450 dollars over the course of the day.

(c) Using the same reasoning and notation as in (b), we have

$$\text{Var}\left(\sum_{i=1}^{9} X_i\right) = 10 \cdot \frac{2}{\left(\frac{1}{5}\right)^2} \cdot 9 = 4500.$$

(d) We would like to calculate

$$\mathbb{P}(\min\{X_1, \ldots, X_{N_9}\} \geq 2).$$

Using the law of total probability, we have

$$\begin{aligned}
\mathbb{P}(\min\{X_1, \ldots, X_{N_9}\} \geq 2) &= \sum_{k=0}^\infty \mathbb{P}\left[\min\{X_1, \ldots, X_{N_9}\} \geq 2 | N_9 = k\right]\mathbb{P}(N_9 = k) \\
&= \sum_{k=0}^\infty \mathbb{P}\left[\min\{X_1, \ldots, X_k\} \geq 2 | N_9 = k\right]\mathbb{P}(N_9 = k) \qquad (1) \\
&= \sum_{k=0}^\infty \mathbb{P}\left[\min\{X_1, \ldots, X_k\} \geq 2\right]\mathbb{P}(N_9 = k).
\end{aligned}$$

Since each customer's spendings are independent of the other customers', we can use our theorem regarding the distribution of the minimum of independent exponentially-distributed random variables to see that, for each $k \geq 0$,

$$\min\{X_1, \ldots, X_k\} \sim \text{Exp}\left(\frac{k}{5}\right),$$

from which we see that

$$\mathbb{P}\left[\min\{X_1, \ldots, X_k\} \geq 2\right] = e^{-2\frac{k}{5}}. \qquad (2)$$

Combining (1) and (2), we have

$$\begin{aligned}
\mathbb{P}(\min\{X_1, \ldots, X_{N_9}\} \geq 2) &= \sum_{k=0}^{\infty} e^{-\frac{2k}{5}} e^{-10\cdot 9} \frac{(10\cdot 9)^k}{k!} \\
&= e^{-90} \sum_{k=0}^{\infty} \frac{(90 \cdot e^{-\frac{2}{5}})^k}{k!} \\
&= e^{-90} e^{90 e^{-\frac{2}{5}}} \\
&= e^{\left(e^{-\frac{2}{5}} - 1\right) 90} \\
&\approx 1.3 \times 10^{-13}.
\end{aligned}$$

Problem 2.5. You throw darts at a square dartboard, denoted by

$$\square = \{(x, y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\},$$

whose sides all have a length of 2 feet; you throw darts according to a Poisson process with rate $\lambda = 1$ per minute, and each dart you throw hits the square dartboard at a uniformly chosen position. Additionally, all of your throws are independent of one another. Within the square dartboard, there is a circular target with a radius of 1 foot, denoted by

$$\circ = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Denote the total number of darts that you throw at the dartboard by time $t \geq 0$ by $N_t(\square)$.

(a) Let $N_t(\circ)$ denote the number of darts that land in the circular target by time $t \geq 0$. What is the probability mass function of $N_t(\circ)$?
(b) Compute $\mathbb{P}[N_5(\circ) = 4 | N_5(\square) = 6]$ and $\mathbb{P}[N_{20}(\circ) = 4 | N_{20}(\square) = 6]$.
(c) Compute $\mathbb{E}[N_t(\circ) | N_t(\square)]$.
(d) Using (3), compute $\mathbb{E}(N_t(\circ))$.

Solution 2.5.

(a) If we let $p \doteq \frac{\pi}{4}$, then

$$N_t(\circ) | N_t(\square) = k \sim \text{Binom}(k, p),$$

so the conditional probability mass function of $N_t(\circ)$ given that $N_t(\square) = k$ is given by

$$p_{N_t(\circ)|N_t(\square)}(j|k) = \binom{k}{j} p^j (1-p)^{k-j}, \quad 0 \leq j \leq k.$$

Since $N_t(\square) \sim \text{Poisson}(t)$, the joint probability mass function of $N_t(\circ)$ and $N_t(\square)$ is given by, for $0 \leq j \leq k$,

$$\begin{aligned}
p_{N_t(\circ), N_t(\square)}(j, k) &= p_{N_t(\circ)|N_t(\square)}(j|k) \cdot p_{N_t(\square)}(k) \\
&= \binom{k}{j} p^j (1-p)^{k-j} \cdot e^{-t} \frac{t^k}{k!},
\end{aligned}$$

and equals 0 when $j > k$. It follows that the marginal probability mass function of $N_t(\circ)$ is given by

$$\begin{aligned}
p_{N_t(\circ)}(j) &= \sum_{k=0}^{\infty} p_{N_t(\circ),N_t(\Box)}(j,k) \\
&= \sum_{k=j}^{\infty} p_{N_t(\circ),N_t(\Box)}(j,k) \\
&= \frac{1}{j!}p^j e^{-t} t^j \sum_{k=j}^{\infty} \frac{1}{(k-j)!}(1-p)^{k-j} t^{k-j} \\
&= \frac{1}{j!}p^j e^{-t} t^j \sum_{k=0}^{\infty} \frac{((1-p)t)^k}{k!} \\
&= \frac{1}{j!}p^j e^{-t} t^j e^{(1-p)t} \\
&= e^{-pt}\frac{(pt)^j}{j!},
\end{aligned}$$

which says that $N_t(\circ) \sim \text{Poisson}(pt)$. Note also that $\{N_t(\circ)\}$ is a Poisson process with rate p.

(b) With p as in (a), we know that

$$N_t(\circ)|N_t(\Box) = k \sim \text{Binom}(k,p),$$

so

$$\mathbb{P}[N_5(\circ) = 4|N_5(\Box) = 6] = \binom{6}{4} p^4 (1-p)^2 \approx 0.2629.$$

Similarly, we have that

$$\mathbb{P}[N_{20}(\circ) = 4|N_{20}(\Box) = 6] = \binom{6}{4} p^4 (1-p)^2 \approx 0.2629.$$

(c) Using the fact that

$$N_t(\circ)|N_t(\Box) = k \sim \text{Binom}(k,p),$$

we have that

$$\mathbb{E}[N_t(\circ)|N_t(\Box)] = pN_t(\Box).$$

(d) Since $N_t(\Box)$ is a Poisson process with rate 1, we have that $\mathbb{E}[N_t(\Box)] = t$, so it follows from (c) and the law of total expectation that

$$\mathbb{E}(N_t(\circ)) = \mathbb{E}[\mathbb{E}[N_t(\circ)|N_t(\Box)]] = p\mathbb{E}[N_t(\Box)] = pt.$$

Problem 2.6. A company releases a new product; at first they receive complaints about the product very infrequently, but, as time progresses and more people realize how poorly the product functions. The number of complaints that the company receives can be modeled according a non-homogeneous Poisson process with intensity function $\lambda(t) = e^t$, where $t \geq 0$ denotes the time in days since the product was released.

(a) Find the probability mass function of the number of complaints the company has received within the first month of the product's release.

(b) Calculate expected number of complaints that the store will receive between the first and second month after the product's release.

Solution 2.6.

(a) Let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$\Lambda(t) \doteq \int_0^t \lambda(s)ds = \int_0^t e^s ds = e^t - 1.$$

Then, denoting the number of complaints received by day t by N_t, we have

$$\mathbb{P}(N_t = k) = e^{-\Lambda(t)} \frac{\Lambda(t)^k}{k!}.$$

(b) This is given by

$$\mathbb{E}(N_{60}) - \mathbb{E}(N_{30}) = \Lambda(60) - \Lambda(30).$$

Problem 2.7. Thunderstorms arrive according to a Poisson process with a rate of $\lambda = 1$ storm per week. The amount of rain that falls during each thunderstorm follows an Exponential distribution with a mean of 0.5 inches. Find the probability that, within four weeks, at least one rainstorm deposits more than one inch of rain.

Solution 2.7. Let O_t denote the number of rainstorms that have deposited more than one inch of rain by week t. Note that, if R_i denotes the amount of rain deposited by a rainstorm, then $R_i \sim \text{Exp}(2)$, so

$$\mathbb{P}(R_i \geq 1) = e^{-2}.$$

Note also that, due to our results regarding split Poisson processes, that $\{O_t\}$ is a Poisson process with rate $\lambda_O \doteq 1 \cdot e^{-2} = e^{-2}$. Thus, the probability that at least one rain storm deposits more than one inch of rain within four weeks is given by

$$\mathbb{P}(O_4 \geq 1) = 1 - \mathbb{P}(O_4 = 0) = 1 - e^{\lambda_O 4} \frac{(\lambda_O 4)^0}{0!} = 1 - e^{4e^{-2}}.$$

Problem 2.8. Let $\{X_t\}$ be a CTMC on $\mathcal{S} = \{1, 2\}$ with transition function

$$P(t) \doteq \frac{1}{\lambda + \mu} \begin{pmatrix} \mu + \lambda e^{-(\lambda+\mu)t} & \lambda - \lambda e^{-(\lambda+\mu)t} \\ \mu - \mu e^{-(\lambda+\mu)t} & \lambda + \mu e^{-(\lambda+\mu)t} \end{pmatrix},$$

where $\lambda, \mu > 0$.

(a) Calculate $\mathbb{P}[X_{1.7} = 2 | X_{1.2} = 1]$.

(b) Calculate $\mathbb{P}[X_{2.1} = 1, X_{1.9} = 2 | X_{1.1} = 1]$.

Solution 2.8.

(a) This is given by $P(0.5)_{1,2}$.

(b) This is given by

$$\begin{aligned}
\mathbb{P}[X_{2.1} = 1, X_{1.9} = 2 | X_{1.1} = 1] &= \mathbb{P}(X_{2.1} = 1 | X_{1.9} = 2, X_{1.1} = 1)\mathbb{P}(X_{1.9} = 2 | X_{1.1} = 1) \\
&= \mathbb{P}(X_{2.1} = 1 | X_{1.9} = 2)\mathbb{P}(X_{1.9} = 2 | X_{1.1} = 1) \\
&= (P(0.2))_{2,1} \cdot (P(0.8))_{1,2}
\end{aligned}$$

Problem 2.9. Let $\{X_t\}$ be a CTMC on $\mathcal{S} = \{1, 2\}$ with transition function $P(t)$. Suppose that

$$P(2) = \begin{pmatrix} 0.4 & 0.6 \\ 0.1 & 0.9, \end{pmatrix}$$

Do we have enough information to calculate $\mathbb{P}[X_4 = 2 | X_0 = 1]$? If so, calculate it. If not, explain what additional information would be needed.

Solution 2.9. Yes; this probability is given by $P(4)_{1,2} = [P(2)P(2)]_{1,2} = 0.78$.