Math 4B: Differential Equations

Lecture 17: Applications & Oscillations

- Electrical Circuits,
- Sinusoidal Forcing,
- Resonance, Beats & More!

Electrical Circuits

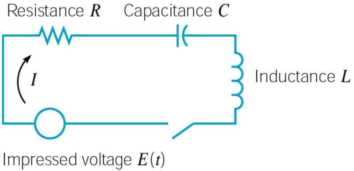


Image courtesy of the textbook

Kirchoff's Second Law: In a closed circuit, the impressed voltage $E(t)$ is equal to the sum of the voltage drops in the rest of the circuit.

- Q = charge, I = current $= \dfrac{dQ}{dt}$
- Voltage drop across the resistor is RI
- Voltage drop across the capacitor is Q/C
- Voltage drop across the inductor is $L\dfrac{dI}{dt}$
- Thus $LI' + RI + Q/C = E(t)$ or

$$L\,Q'' + R\,Q' + \frac{1}{C}\,Q = E(t) \qquad \text{or} \qquad L\,I'' + R\,I' + \frac{1}{C}\,I = E'(t).$$

Typical Sine Forcing

1. Write down the solutions of the linear nonhomogeneous second-order ODE

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \cos(3t).$$

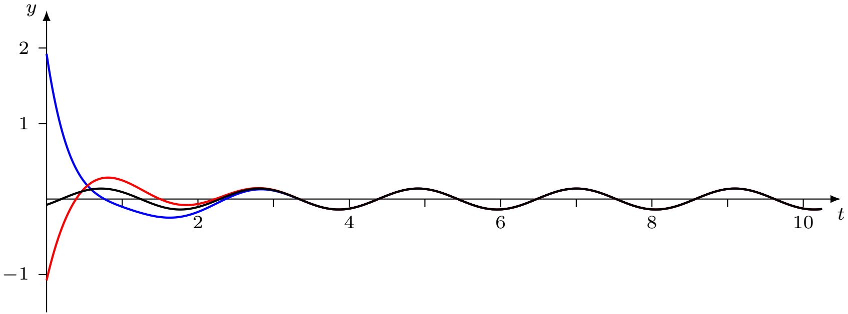
Solution: The corresponding homogeneous ODE has characteristic equation $r^2 + 2r + 5 = 0$, so $r = -2 \pm i$. So $y = e^{-2t}\left(c_1 \cos(t) + c_2 \sin(t)\right)$ is the general solution of this homogeneous ODE.

To find a particular solution, we guess $y_p = A\cos(3t) + B\sin(3t)$ and find that the full solution is

$$\boxed{y = e^{-2t}\big(c_1 \cos(t) + c_2 \sin(t)\big) + \frac{1}{26}\big(3\sin(3t) - 2\cos(3t)\big).}$$

Graph of Solutions

Here are some sketches of various solutions:



Moral: The solutions

$$y = e^{-2t}\big(c_1\cos(t) + c_2\sin(t)\big) + \frac{1}{26}\big(3\sin(3t) - 2\cos(3t)\big)$$

converges to $\frac{1}{26}\big(3\sin(3t) - 2\cos(3t)\big)$ when t is large for any c_1, c_2.

Trigonometry Algebra

Can we write

$$\frac{1}{26}\big(3\sin(3t) - 2\cos(3t)\big)$$

as $r\sin(\omega t - \theta)$? By trigonometry rules, we can write this as

$$\boxed{\frac{1}{26}\big(3\sin(3t) - 2\cos(3t)\big) = r\cos(\theta)\sin(\omega t) - r\sin(\theta)\cos(\omega t)}\,.$$

This means we want

$$r\cos(\theta) = \frac{3}{26} \qquad \text{and} \qquad r\sin(\theta) = \frac{2}{26}.$$

Hence $r = \sqrt{\left(\frac{3}{26}\right)^2 + \left(\frac{2}{26}\right)^2} = \frac{\sqrt{13}}{26}$ and $\theta = \arctan(2/3)$. That is,

$$\frac{1}{26}\big(3\sin(3t) - 2\cos(3t)\big) = \frac{\sqrt{13}}{26}\sin\big(3t - \arctan(2/3)\big).$$

2. Write down the solutions of the linear nonhomogeneous second-order ODE

$$\frac{d^2y}{dt^2} + 3y = \sin(\sqrt{3}\,t).$$

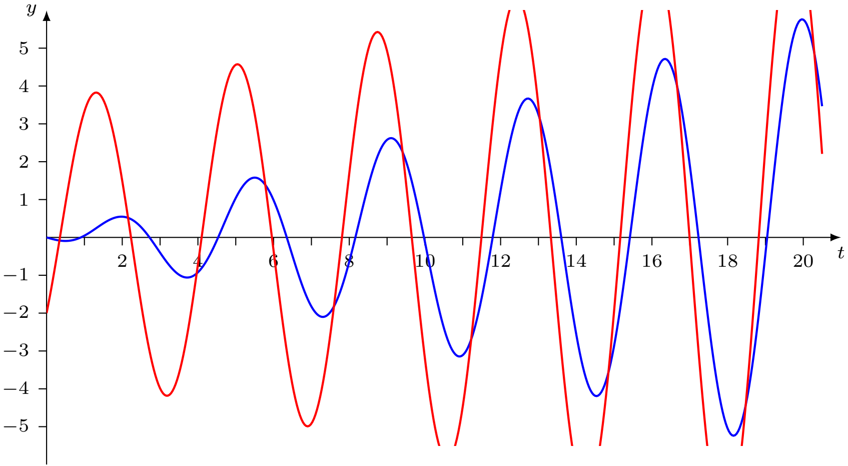
Solution: The corresponding homogeneous ODE has characteristic equation $r^2 + 3 = 0$, so $r = \pm\sqrt{3}\,i$. So $y = c_1\cos(\sqrt{3}\,t) + c_2\sin(\sqrt{3}\,t)$ is the general solution of this homogeneous ODE.

To find a particular solution, we guess $y_p = At\cos(\sqrt{3}\,t) + Bt\sin(\sqrt{3}\,t)$ and find that the full solution is

$$\boxed{y = c_1\cos(\sqrt{3}\,t) + c_2\sin(\sqrt{3}\,t) - \frac{1}{2\sqrt{3}}t\cos(\sqrt{3}t).}$$

Graph of Solutions

Here are some sketches of various solutions:



Beats

3. Write down the solutions of the linear nonhomogeneous second-order ODE

$$\frac{d^2y}{dt^2} + 3y = \sin(\omega\, t)$$

where ω is about $\sqrt{3}$ (but $\omega \neq \sqrt{3}$).

Solution: The corresponding homogeneous ODE again has $y = c_1 \cos(\sqrt{3}\, t) + c_2 \sin(\sqrt{3}\, t)$ as the general solution.

To find a particular solution, we guess $y_p = A\cos(\omega\, t) + B\sin(\omega\, t)$ and find that the full solution is

$$\boxed{y = c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}\, t) + \frac{1}{3-\omega^2}\sin(\omega\, t).}$$

Why is this slide called "beats"?

An Example

Let's look at an example when $\omega = 1.75$. (We've picked this because $\sqrt{3} \approx 1.73$.) Then

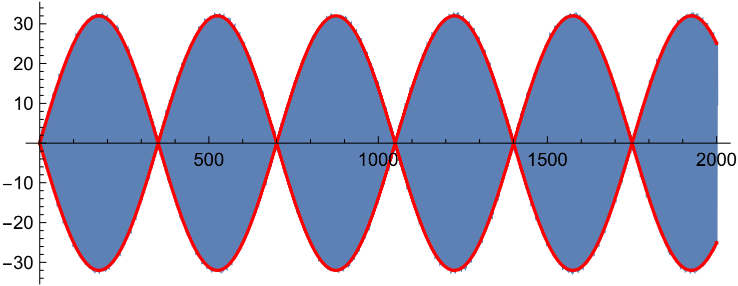
$$\frac{1}{3-\omega^2} = \frac{1}{3-1.75^2} = -16,$$

so we get a solution $y = c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}\,t) - 16\sin(1.75\,t)$. Let's take $c_1 = 0$ and $c_2 = 16$ as an **example**. Then

$$\begin{aligned}
y &= 16\left(\sin(\sqrt{3}\,t) - \sin(1.75t)\right) \\
&= 16\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right) \quad \text{where } \alpha = \tfrac{\sqrt{3}+1.75}{2}\,t \text{ and } \beta = \tfrac{\sqrt{3}-1.75}{2}\,t \\
&= 16\left[\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) - (\sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha))\right] \\
&= 32\sin(\beta)\cos(\alpha) \\
&= 32\sin\left(\frac{\sqrt{3}-1.75}{2}\,t\right)\cos\left(\frac{\sqrt{3}+1.75}{2}\,t\right).
\end{aligned}$$

Beats Graph

Here are some sketches of this solution:



$$\text{Graph of } y = 32\sin\left(\frac{\sqrt{3}-1.75}{2}\,t\right)\cos\left(\frac{\sqrt{3}+1.75}{2}\,t\right)$$

$$\text{with } y = 32\sin\left(\frac{\sqrt{3}-1.75}{2}\,t\right)$$

Beats Graph

One more sketch of this solution: