Math 4B: Differential Equations

Lecture 07: Euler's Method

- Approximating Solutions to IVPs,
- Euler's Method,
- Different Time Steps, & More!

The Big Picture:

What can we say about the solutions of first order IVPs?

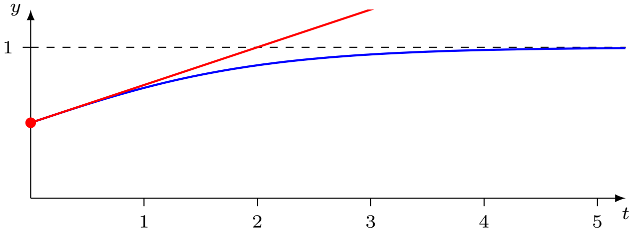
$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

- If f and $\partial f / \partial y$ are continuous, then solutions exist and are unique (locally).
- We can draw direction fields and sketch *approximate* solution curves.
- Many solutions can't be found, even with "easy" ODEs:
  - $y' = e^{-t^2}$
  - $y' = \dfrac{1}{1+t^3}$
  - $y' = \cos(t^4)$
- So we need another approach

How 'bout a Tangent Line?

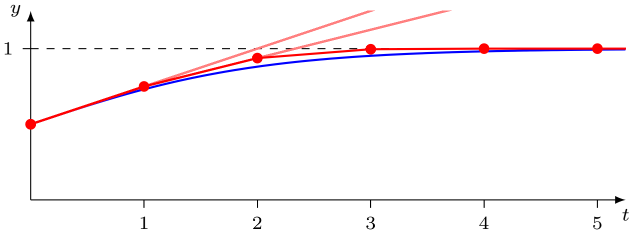
Here's an IVP we can solve:

$$\begin{cases} y' = y(1-y) \\ y(0) = 0.5 \end{cases}$$



Suppose I couldn't solve this. We could approximate this with a tangent line at $t = 0$. We know $y(0) = 0.5$, $y'(0) = 0.5(1-0.5) = 0.25$. Formula: $y = y(0) + y'(0)(t-0)$ or $\boxed{y = 0.5 + 0.25(t-0)}$.

Better Than a Tangent Line



The tangent line is a really good approximation near $t = 0$, but really bad far away from $t = 0$.

So what to do?
Plan: Change to the tangent line at $t = 1$ to continue on! Then keep going!!

Formalization

To approximate the solution to

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

we use the iteration

$$\begin{aligned}
y_1 &= y_0 + f_0(t_1 - t_0) && \text{where } f_0 = f(t_0, y_0),\ t_1 = t_0 + h \\
y_2 &= y_1 + f_1 \cdot h && \text{where } f_1 = f(t_1, y_1) \\
y_3 &= y_2 + f_2 \cdot h && \text{where } f_2 = f(t_2, y_2) \\
&\ \vdots \\
y_{n+1} &= y_n + f_n \cdot h && \text{where } f_n = f(t_n, y_n)
\end{aligned}$$

Idea: If $y = \phi(t)$ is a solution to the IVP above, then $\phi(t_n) \approx y_n$.

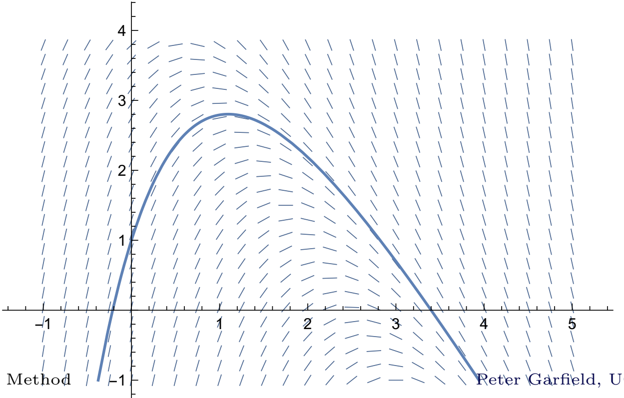
Questions: How good are these approximations?
Can we find error bounds? (See Chapter 8.)

Example 1

Solve the IVP

$$\begin{cases} y' = 5 - 2t - y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 1$



Example 1 (continued)

Solve the IVP

$$\begin{cases} y' = 5 - 2t - y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 1$

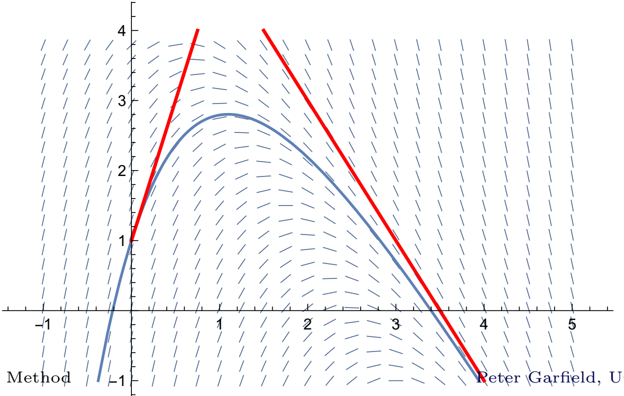
n	t_n	y_n	$f_n = f(t_n, y_n)$	$y_{n+1} = y_n + f_n \cdot h$
0	0	1	4	5
1	1	5	-2	3
2	2	3	-2	1
3	3	1	-2	-1
4	4	-1	-2	-3
5	5	-3	-2	-5

Example 1 (continued)

Solve the IVP

$$\begin{cases} y' = 5 - 2t - y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 1$

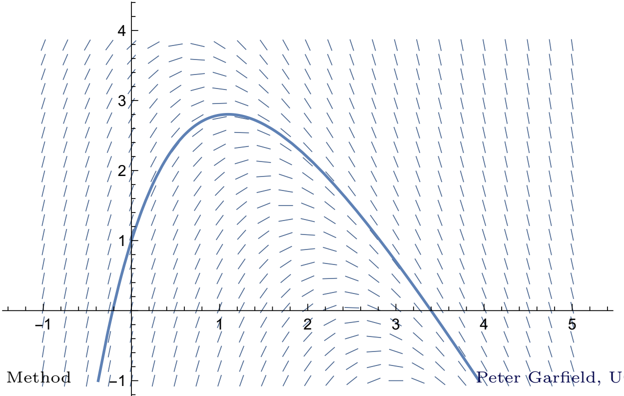


Example 1 (continued more!)

Solve the IVP

$$\begin{cases} y' = 5 - 2t - y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 0.5$



Example 1 (continued)

Solve the IVP

$$\begin{cases} y' = 5 - 2t - y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 0.5$

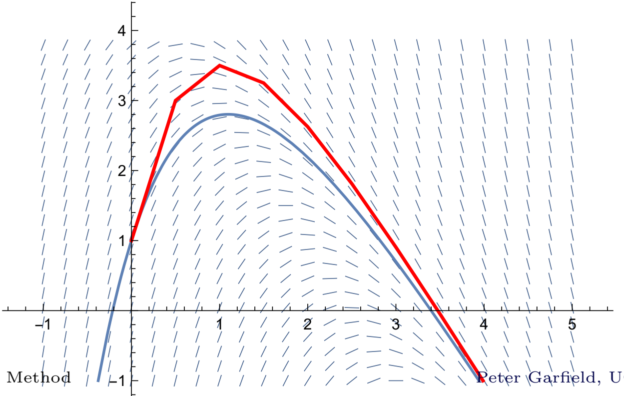
n	t_n	y_n	$f_n = f(t_n, y_n)$	$y_{n+1} = y_n + f_n \cdot h$
0	0.	1.	4.	3.
1	0.5	3.	1.	3.5
2	1.	3.5	−0.5	3.25
3	1.5	3.25	−1.25	2.625
4	2.	2.625	−1.625	1.8125
5	2.5	1.8125	−1.8125	0.90625
6	3.	0.90625	−1.90625	−0.046875
7	3.5	−0.046875	−1.95313	−1.02344
8	4.	−1.02344	−1.97656	−2.01172
9	4.5	−2.01172	−1.98828	−3.00586
10	5.	−3.00586	−1.99414	−4.00293
11	5.5	−4.00293	−1.99707	−5.00146

Example 1 (continued)

Solve the IVP

$$\begin{cases} y' = 5 - 2t - y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 0.5$ Now $h = 0.1$

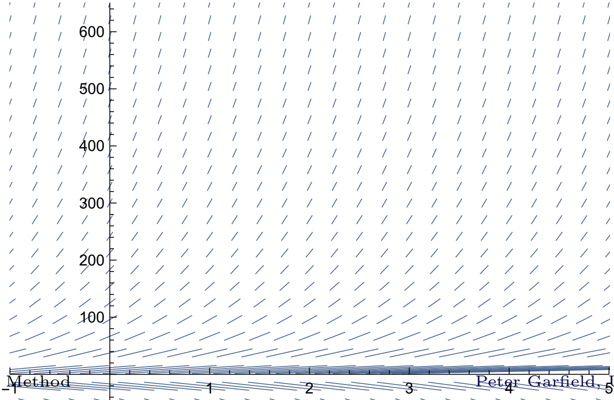


Example 2

Solve the IVP

$$\begin{cases} y' = 5 - 2t + y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 0.5$



Example 2 (continued)

Solve the IVP

$$\begin{cases} y' = 5 - 2t + y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 0.5$

n	t_n	y_n	$f_n = f(t_n, y_n)$	$y_{n+1} = y_n + f_n \cdot h$
0	0.	1.	6.	4.
1	0.5	4.	8.	8.
2	1.	8.	11.	13.5
3	1.5	13.5	15.5	21.25
4	2.	21.25	22.25	32.375
5	2.5	32.375	32.375	48.5625
6	3.	48.5625	47.5625	72.3438
7	3.5	72.3438	70.3438	107.516
8	4.	107.516	104.516	159.773
9	4.5	159.773	155.773	237.66
10	5.	237.66	232.66	353.99
11	5.5	353.99	347.99	527.985

Example 2 (continued)

Solve the IVP

$$\begin{cases} y' = 5 - 2t + y \\ y(0) = 1 \end{cases}$$

using Euler's method with a step of $h = 0.5$ Now $h = 0.1$