Math 4B: Differential Equations

Lecture 08: A Theorem Revisited

- Review of the Existence-Uniqueness Theorem,
- An Outline of the Proof,
- Some Examples & More!

The Theorem

Existence & Uniqueness for Non-linear First Order IVPs

Consider the non-linear first order initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$

If the functions f and $\frac{\partial f}{\partial y}$ are continuous on an open rectangle

$$R = \{(t, y) \,:\, \alpha < t < \beta,\ \gamma < y < \delta\}$$

containing the point (t_0, y_0), then there is an interval $t_0 - h < t < t_0 + h$ and a unique function $y = \phi(t)$ defined on that interval that satisfies the IVP for each t in $t_0 - h < t < t_0 + h$.

The Theorem (tweaked)

Existence & Uniqueness for Non-linear First Order IVPs (Simplified)

Consider the non-linear first order initial value problem

$$\begin{cases} y' = f(t, y) \\ y(0) = 0. \end{cases}$$

If the functions f and $\frac{\partial f}{\partial y}$ are continuous on an open rectangle

$$R = \{(t, y) : |t| < a, \ |y| < b\}$$

then there is an interval $|t| < h \leq a$ and a unique function $y = \phi(t)$ defined on that interval that satisfies the IVP for each t with $|t| < h$.

Idea:

Suppose $y = \phi(t)$ is our solution that solves the IVP

$$\begin{cases} y' = f(t, y) \\ y(0) = 0 \end{cases}$$

so $\phi'(t) = f\big(t, \phi(t)\big)$ and $\phi(0) = 0$. Now integrate with respect to t:

$$\boxed{\phi(t) = \int_0^t f\big(s, \phi(s)\big)\, ds.}$$

Notice:

- s is a dummy variable
- $\phi(0) = 0$

We'll solve this integral equation!

Picard's Iteration Method

Method of Successive Approximation

We start with one approximate solution. The "easiest" is

$$\boxed{\phi_0(t) = 0.}$$

Notice that this satisfies the initial condition $\phi_0(0) = 0$, but not necessarily the ODE / integral equation.

Now we get a new approximation by

$$\boxed{\phi_1(t) = \int_0^t f\big(s, \phi_0(s)\big)\,ds.}$$

Again $\phi_1(t)$ satisfies the initial condition $\phi_1(0) = 0$, but not necessarily the ODE / integral equation.

Now repeat (iterate!). In general

$$\boxed{\phi_{n+1}(t) = \int_0^t f\big(s, \phi_n(s)\big)\,ds \qquad \text{for all } n \geq 0.}$$

A Sequence of "Solutions"

Given $\phi_0(t) = 0$ and

$$\boxed{\phi_{n+1}(t) = \int_0^t f\big(s, \phi_n(s)\big)\, ds \qquad \text{for all } n \geq 0,}$$

we get a sequence of functions

$$\phi_0(t),\ \phi_1(t),\ \phi_2(t),\ \phi_3(t),\ \ldots,\ \phi_n(t),\ \ldots$$

- If $\phi_{n+1}(t) = \phi_n(t)$, then the sequence terminates (meaning $\phi_k(t) = \phi_n(t)$ for all $k \geq n$) and $\phi_n(t)$ solves the IVP.
- If the sequence never settles, then $\phi(t) = \lim\limits_{n\to\infty} \phi_n(t)$ is the solution to our IVP. (This uses tools on infinite sequences and series that we don't cover in Math 3AB. Wait for 6B!)
- Uniqueness also uses tools on infinite sequences and series. Sigh.

An Example

Let's try to solve the IVP

$$\begin{cases} y' = 1 - y \\ y(0) = 0. \end{cases}$$

via this iterative procedure.

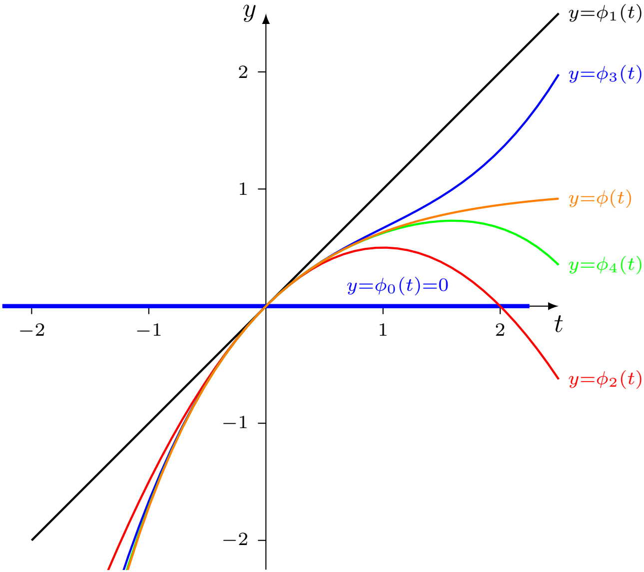
$$\phi_0(t) = 0$$

$$\phi_1(t) = \int_0^t \big(1 - \phi_0(s)\big)\,ds = \int_0^t 1\,ds = t$$

$$\phi_2(t) = \int_0^t \big(1 - \phi_1(s)\big)\,ds = \int_0^t \big(1 - s\big)\,ds = t - \frac{t^2}{2}$$

$$\phi_3(t) = \int_0^t \big(1 - \phi_2(s)\big)\,ds = \int_0^t \left(1 - s + \frac{s^2}{2}\right)\,ds = t - \frac{t^2}{2} + \frac{t^3}{6}$$

$$\phi_4(t) = \int_0^t \big(1 - \phi_3(s)\big)\,ds = \int_0^t \left(1 - s + \frac{s^2}{2} - \frac{s^3}{6}\right)\,ds = t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24}$$



y
t
$y\!=\!\phi_1(t)$
$y\!=\!\phi_3(t)$
$y\!=\!\phi(t)$
$y\!=\!\phi_4(t)$
$y\!=\!\phi_0(t)\!=\!0$
$y\!=\!\phi_2(t)$
2
1
−1
−2
−2
−1
1
2

Example (concluded)

So we've seen the solution to

$$\begin{cases} y' = 1 - y \\ y(0) = 0. \end{cases}$$

is

$$\begin{aligned} \phi(t) &= \lim_{n\to\infty} \phi_n(t) \\ &= t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \frac{t^5}{5!} - \cdots \\ &= 1 - \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \\ &= 1 - e^{-t}. \end{aligned}$$

(Of course we could have solved this as the ODE is separable.)

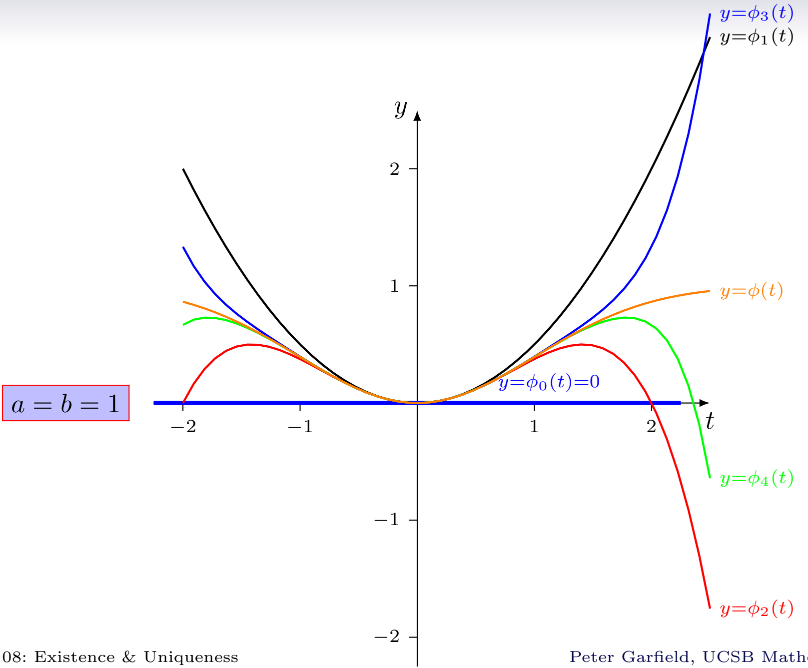
Another Example

Let's try to solve the IVP

$$\begin{cases} y' = at(b-y) \\ y(0) = 0. \end{cases}$$

where a and b are constants.

$$\begin{aligned}
\phi_0(t) &= 0 \\
\phi_1(t) &= \int_0^t as\big(b - \phi_0(s)\big)\,ds = \frac{ab}{2}t^2 \\
\phi_2(t) &= \int_0^t as\left(b - \phi_1(s)\right)\,ds = \int_0^t as\left(b - \frac{ab}{2}s^2\right)\,ds = \frac{ab}{2}t^2 - \frac{a^2b}{8}t^4 \\
\phi_3(t) &= \int_0^t as\big(b - \phi_2(s)\big)\,ds = \int_0^t as\left(b - \frac{ab}{2}s^2 + \frac{a^2b}{8}s^4\right)\,ds \\
&= \frac{ab}{2}t^2 - \frac{a^2b}{8}t^4 + \frac{a^3b}{48}t^6
\end{aligned}$$


$y=\phi_3(t)$
$y=\phi_1(t)$
y
2
1
$y=\phi(t)$
$y=\phi_0(t)=0$
$a=b=1$
-2
-1
1
2
t
$y=\phi_4(t)$
-1
$y=\phi_2(t)$
-2

Second Example (concluded)

So we've seen the solution to

$$\begin{cases} y' = at(b-y) \\ y(0) = 0. \end{cases}$$

is

$$\begin{aligned} \phi(t) &= \lim_{n\to\infty} \phi_n(t) \\ &= b\left(\frac{at^2}{2} - \frac{a^2}{8}t^4 + \frac{a^3}{48}t^6 - \cdots\right) \\ &= b\left(\frac{at^2}{2} - \frac{1}{2!}\left(\frac{at^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{at^2}{2}\right)^3 - \cdots\right) \\ &= b\Big(1 - \sum_{n=0}^{\infty} \frac{(-at^2/2)^n}{n!}\Big) \\ &= b\Big(1 - e^{-at^2/2}\Big). \end{aligned}$$