Problem 5.1. Let $\{W_t\}$ be an SBM, and denote the first hitting time of state $a \in \mathbb{R}$ by T_a. Calculate the following:

(a) $\mathbb{P}(W_3 \geq 2)$.

(b) $\mathbb{P}(W_3 \geq 2 | W_1 = 1.5)$.

(c) $\mathbb{E}[W_{17} | W_5 = 3]$.

a). $P(W_3 \geqslant 2) = 1 - \Phi_{0,3}(2) = 0.2524925$ by R

B) $P(W_3 \geqslant 2 \mid W_1 = 1.5) = P(W_3 - W_1 \geqslant 2 - 1.5 \mid W_1 - W_0 = 1.5 - 0)$

$= \dfrac{P(W_2 \geqslant 0.5)\, P(W_1 = 1.5)}{P(W_1 = 1.5)} = P(W_2 \geqslant 0.5)$

$= 1 - \Phi_{0,2}\left(\frac{1}{2}\right) = 0.40129$

c) $E[W_{17} - W_5 + 3 \mid W_5 = 3] = E[W_{17} - W_5] + 3 = 0 + 3 = 3$ drifted SBM

and indep increment

Problem 5.2. Fix $\alpha > 0$ and let $\{W_t\}$ be an SBM. Define the process $\{\hat{W}_t\}$ by

$$\hat{W}_t \doteq \frac{1}{\sqrt{\alpha}} W_{\alpha t}.$$

Show that $\{\hat{W}_t\}$ is an SBM.

Show Scaled SBM is also a SBM.

it can be done by showing $\hat{W}_t$ iid w/ SBM

Note that $\frac{1}{\sqrt{\alpha}} W_{\alpha t} \sim N(0, \alpha t)$

Since

$$\hat{W}_t = \frac{1}{\sqrt{\alpha}} W_{\alpha t}$$

$$\left.\begin{aligned} \hat{W}_t &\sim N(\mu t, \sigma^2 t) \\ \Rightarrow &\sim N(0, \tfrac{1}{\alpha} \cdot \alpha \cdot t) \\ \Rightarrow &\sim N(0, t) \end{aligned}\right\} \text{Prop 4.27 (3)}$$

which is iid w/ SBM

Problem 5.3. Let $\{W_t^1\}, \ldots, \{W_t^d\}$ be independent SBMs. The $\mathbb{R}^d$-valued process $\{\boldsymbol{W}_t\}$ defined as

$$\boldsymbol{W}_t \doteq \left(W_t^1 \quad \ldots W_t^d.\right)$$

What is the probability distribution of $\boldsymbol{W}_t$? Note that, for each $t \geq 0$, $\boldsymbol{W}_t$ is an $\mathbb{R}^d$-valued random variable.

Gaussian process

Multivariate Normal distribution, if $a_i = \sum_{j=1}^{d} a_j$

$$(W_t^1 \cdots W_t^d) = \sum_{j=1}^{d} a_j W_t^j \sim (\underset{\sim}{0}, \sum_{j=1}^{d} a_j^2 t) \text{ (scaling)}$$

Problem 5.4. Let $X, X_1, X_2, \ldots, X_d$ be a collection of iid $\mathcal{N}(\mu, \sigma^2)$ random variables.

(a) Let $\mathbb{X} = \begin{pmatrix} X & X & \ldots & X \end{pmatrix} \in \mathbb{R}^d$. Determine the probability distribution of $\mathbb{X}$.

(b) Let $\mathbb{Y} = \begin{pmatrix} X_1 & X_2 & \ldots & X_d \end{pmatrix} \in \mathbb{R}^d$. Determine the probability distribution of $\mathbb{Y}$.

a). Same with single term $x \sim N(\mu, \sigma^2)$

b) $(X_1 X_2 \ldots X_d) \Rightarrow$ Multivariate Normal $\Rightarrow Cov(X_1 \cdots X_d) = 0$

$Y \sim N(\mu, \Sigma)$

where

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_d \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma_{x_1}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{x_d}^2 \end{pmatrix}$$

Problem 5.5. Let $\{W_t\}$ be an SBM. For $s < t$, what is the probability distribution of the $\mathbb{R}^2$-valued random variable (W_s, W_t)?

$$\{W_t\} = (W_1 \cdots W_s \cdots W_t)^T$$
$$\underset{\sim}{a} = (a_1 \cdots a_s \cdots a_t)^T$$

By def 4.3

$$\boxed{\underset{\sim}{a}^T W_t = \Sigma a_i W_{t_i} \sim N(\mu, \sigma^2)}$$

For $(W_s, W_t) \Rightarrow a_s W_s + a_t W_t$

$\Rightarrow a_s W_s + a_t (W_t - W_s + W_s)$

$\Rightarrow a_s W_s + a_t W_t - a_t W_s + a_t W_s$

$\Rightarrow (a_s + a_t) W_s + a_t (W_t - W_s)$

Above is of form BM w/ drift & Scaling

B/C there is no t or s exist in above formula

$\Rightarrow \mu = 0$

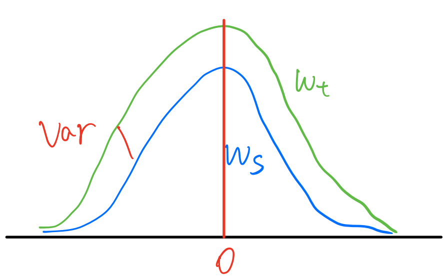
Both $(a_s + a_t)$, a_t are Scaling terms

$\Rightarrow \quad W_s \sim N(0, (a_s+a_t)^2 t)$

$\quad\quad W_t \sim N(0, a_t^2 (t-s))$

Thus

$(W_s, W_t) \sim N(0, (a_s+a_t)^2 t + a_t^2 (t-s))$



Problem 5.6. Let $\{W_t\}$ be an SBM. Define the process $\{B_t\}$ on the time interval $[0, 1]$ by

$$B_t \doteq W_t - tW_1.$$

(a) What is the probability distribution of B_t?

(b) Briefly explain why $\mathbb{P}(B_1 = 0) = 1$.

(c) At what time is the variance of the process maximized?

a) Multivorlate normal. $\sim N(\mu, \sigma^2)$

b) B/C $B_0 = W_0 - 0$ and $B_1 = W_1 - W_1 = 0$

c) $Var = t(1-t) \Rightarrow$ $t = \frac{1}{2}$ gives max value of $\frac{1}{4}$.

Problem 5.7. Let $\{X_n\}$ be a sequence of iid random variables such that

$$\mathbb{P}(X_n \geq 0) = 1, \quad \mathbb{E}(X_n) = 1.$$

Let $M_n \doteq \prod_{i=1}^{n} X_i$. Show that $\{M_n\}$ is a martingale with respect to $\{X_n\}$.

B/C $E(X_n) = 1 \Rightarrow E(X_2) = P(X_1 \geqslant 0) = 1$

and $X_n \perp\!\!\!\perp X_{n-1}$, $n \in \mathbb{Z}$

Let $M_n = \prod_{i=1}^{n} X_i$, $M_{n-1} = \prod_{i=1}^{n-1} X_i$

$E[M_n] = E[\prod_{i=1}^{n} X_i] = E(X_1)\,E(X_2) \cdots E(X_n) = 1 \cdot 1 \cdot 1 \cdot 1 \cdots 1 = 1$

$P[M_{n-1}] = P(\prod_{i=1}^{n-1} X_i) = P(X_1) \cdots P(X_{n-1}) = 1 \cdot 1 \cdot 1 \cdots 1 = 1$

Since $E[M_n] = P(M_{n-1}) \Rightarrow M_n$ is Martingale w/ respect to X_n.

Problem 5.8. Let $\{W_t\}$ be an SBM.

(a) Using Itô's lemma, show that

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}.$$

(b) Using Itô's lemma, show that

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds.$$

a) From Example 6.7

Let $X_t = \int_0^t W_s dW_s$, $t \geq 0$

then $dX_t = W_t dW_t$

From Corollary 6.6. $\boxed{df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt}$

$\Rightarrow f'(W_t)dW_t = df(W_t) - \frac{1}{2}f''(W_t)dt$

if $\boxed{f(W) = \frac{W^2}{2}}$ $f'(W) = W$, $f''(W) = 1$

then, $W_t dW_t = d\left(\frac{W_t^2}{2}\right) - \frac{1}{2}dt$

$X_t = \int_0^t d\left(\frac{W_t^2}{2}\right) - \int_0^t \frac{1}{2}dt$

$$\boxed{X_t = \frac{W_t^2}{2} - \frac{1}{2}t}$$

(b) Using Itô's lemma, show that

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds.$$

Let $X_t = \int_0^t W_s dW_s$, $t \geq 0$

then $dX_t = W_t dW_t$

From Corollary 6.6. $\boxed{df(W_t^2) = f'(W_t^2)dW_t + \frac{1}{2}f''(W_t^2)dt}$

$\Rightarrow f'(W_t)dW_t = df(W_t) - \frac{1}{2}f''(W_t)dt$

if $\boxed{f(W^2) = \frac{W^3}{3}}$ $f'(W) = W^2$; $f''(W) = 2W$

then, $W_t^2 dW_t = d\left(\frac{W_t^3}{3}\right) - \frac{1}{2} 2W dt$

$X_t = \int_0^t d\left(\frac{W_t^3}{3}\right) - \int_0^t W dt$

$$\boxed{X_t = \frac{W_t^3}{3} - \int_0^t W_s ds}$$

Problem 5.9. Let $\{W_t\}$ be an SBM. Consider a process $\{X_t\}$ satisfying the SDE

$$dX_t = \alpha dW_t + \beta dt$$
$$X_0 = x_0,$$

GBM

where $\alpha, \beta, x_0 > 0$. Let $Y_t \doteq \exp(\gamma X_t)$, where $\gamma > 0$. By applying Itô's formula, find the SDE solved by $\{Y_t\}$. That is, "calculate" dY_t.

$f'(x) = \gamma e^{\gamma x}$ $\qquad$ $f''(x) = \gamma \cdot \gamma e^{\gamma x} = \gamma^2 e^{\gamma x}$

$$dY_t = f'(X_t)dX_t + \frac{1}{2} f''(X_t)\, dX_t$$
$$= \gamma e^{\gamma x} + \frac{1}{2}\gamma^2 e^{\gamma x}$$

?

Problem 5.10. Let $\{W_t\}$ be an SBM. Solve the SDE

$$dX_t = 3X_t^{\frac{2}{3}} dW_t + 3X_t^{\frac{1}{3}} dt$$
$$X_0 = 0.$$

diffusion GBM

Ito's formula

$$df(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, \sigma^2(X_t)\, dt$$
$$= f'(X_t)\, b(X_t)\, dt + \frac{1}{2} f''(X_t)\, \sigma^2(X_t)\, dt$$

where $dX_t = b(X_t)dt + \sigma(X_t)dW_t$

GBM formula

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad , \quad X_0 = x_0$$

$$\sigma = 3 \qquad \mu = 3$$

$$\Rightarrow df(X_t) = f'(X_t)\, 3X_t^{\frac{2}{3}} dW_t + \frac{1}{2} f''(X_t)\, 9\,(X_t^{\frac{1}{3}})\, dt$$

Let $f'(X_t)\, 3X_t^{\frac{2}{3}} = 1$

$$f'(X_t) = \frac{1}{3X_t^{\frac{2}{3}}}$$

$$f''(X_t) = \frac{-2}{9x^{5/3}}$$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, b(X_s)\, ds + \int_0^t f'(X_s)\, \sigma(X_s)\, dW_s + \frac{1}{2}\int_0^t f''(X_s)\, \sigma^2(X_s)\, ds$$

$$= 0 + \int_0^t \frac{1}{3X_t^{2/3}}\, 3X_t^{\frac{2}{3}}\, ds + \int_0^t \frac{1}{3X_t^{2/3}}\, 3x^{\frac{1}{3}}\, dW_s + \frac{1}{2}\int_0^t \frac{-2}{9x^{5/3}}\, 9x^{\frac{1}{3}}\, ds$$