Math 4B: Differential Equations

Lecture 16: Variation of Parameters

- Solving Nonhomogeneous ODEs: Another Approach
- Using Linear Algebra & Cramer's Rule,
- Some examples & More!

Variation of Parameters

Today: A new technique to find a particular solution to the second order linear nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t),$$

called ***variation of parameters*** .

Advantages:

- Similar to technique used in reduction of order
- An explicit formula for a particular solution

Disadvantages:

- Requires solution to the corresponding homogeneous equation
- The formula requires integrals involving $g(t)$ that may be difficult

How Would This Work?

Let's try the technique with

$$y'' + p(t)y' + q(t)y = g(t).$$

Note: We'll write a fundamental set of solutions for the corresponding homogeneous ODE as y_1 and y_2.
We'll assume that a solution to our ODE can be written as

$$Y(t) = u_1 y_1 + u_2 y_2$$

for some functions u_1 and u_2. Then

$$\begin{aligned} Y &= u_1 y_1 + u_2 y_2 \\ Y' &= u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' \\ Y'' &= u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2'' \end{aligned}$$

and so

$$\begin{aligned} &u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2'' \\ &+ p(t)\big(u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'\big) + q(t)\big(u_1 y_1 + u_2 y_2\big) = g(t). \end{aligned}$$

Yikes!

Let's Figure This Out

In order to solve for u_1 and u_2, we're going to need two equations, so we assume

$$\boxed{0 = u_1' y_1 + u_2' y_2}\,.$$

So we get

$$\begin{aligned} Y &= u_1 y_1 + u_2 y_2 \\ Y' &= u_1 y_1' + u_2 y_2' \\ Y'' &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''. \end{aligned}$$

Thus

$$\begin{aligned} &\big(u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''\big) \\ &+ p(t)\big(u_1 y_1' + u_2 y_2'\big) + q(t)\big(u_1 y_1 + u_2 y_2\big) = g(t). \end{aligned}$$

We simplify this to

$$u_1' y_1' + u_2' y_2' + u_1\big(y_1'' + p(t) y_1' + q(t) y_1\big) + u_2\big(y_2'' + p(t) y_2' + q(t) y_2\big) = g(t)$$

or

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t). \end{aligned}$$

Linear Algebra Review

How do we solve the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad \text{or} \quad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}?$$

We could use many techniques to solve this, but the most explicit rule is Cramer's Rule:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

This means the solution of

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g(t) \end{aligned} \quad \text{is} \quad u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Conclusion

So we've seen that a particular solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

is $y_p = u_1 y_1 + u_2 y_2$, given a fundamental set $\{y_1, y_2\}$ of solutions to the corresponding homogeneous equation, where

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-g(t)y_2(t)}{W[y_1, y_2](t)}$$

and

$$u_2'(t) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{g(t)y_1(t)}{W[y_1, y_2](t)}.$$

OK, Now the Conclusion

Variation of Parameters

Consider the two equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$
$$y'' + p(t)y' + q(t)y = 0, \qquad (**)$$

where p, q, and g are continuous functions on some interval I. If $\{y_1, y_2\}$ is a fundamental set of solutions of the homogeneous equation $(**)$, then a particular solution of the nonhomogeneous equation $(*)$ is

$$y_p = -y_1 \int \frac{g(t)y_2(t)}{W[y_1, y_2](t)}\, dt + y_2 \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}\, dt$$

and the general solution of $(*)$ is

$$y = c_1y_1 + c_2y_2 + y_p.$$

Example 1

1. Find the general solution to

$$y'' + 4y = 8\sec^2(2t).$$

Solution: A fundamental set of solutions of the homogeneous equation $y'' + 4y = 0$ is $y_1 = \sin(2t)$ and $y_2 = \cos(2t)$.

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{vmatrix} = -2.$$

Then

$$\begin{aligned} u_1(t) &= -\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\,dt = -\int \frac{\cos(2t)\cdot 8\sec^2(2t)}{-2}\,dt \\ &= 4\int \sec(2t)\,dt = 2\ln|\sec(2t) + \tan(2t)| + C_1 \end{aligned}$$

and...

Example 1 (Cont'd)

...and

$$u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}\,dt = \int \frac{\sin(2t)\cdot 8\sec^2(2t)}{-2}\,dt$$
$$= -4\int \tan(2t)\sec(2t)\,dt = -2\sec(2t) + C_2.$$

Thus

$$y_p = u_1y_1 + u_2y_2 = 2\sin(2t)\ln|\sec(2t)+\tan(2t)| - 2\cos(2t)\sec(2t)$$
$$= 2\sin(2t)\ln|\sec(2t)+\tan(2t)| - 2.$$

and so the general solution is

$$\boxed{y = c_1\sin(2t) + c_2\cos(2t) + 2\sin(2t)\ln|\sec(2t)+\tan(2t)| - 2}.$$

Another Example

2. Find a particular solution to

$$2t^2y'' + 3ty' - y = t^3 \qquad (t > 0),$$

given that a fundamental set of solutions to the corresponding homogeneous equation is $y_1 = \sqrt{t}$ and $y_2 = t^{-1}$.

Solution:

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ t^{-1/2}/2 & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2}$$

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\,dt = -\int \frac{t^{-1}\cdot\frac{t^3}{2t^2}}{-3t^{-3/2}/2}\,dt = \frac{1}{3}\int t^{3/2}\,dt = \frac{2}{15}\,t^{5/2} + C_1$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}\,dt = \int \frac{\sqrt{t}\cdot\frac{t^3}{2t^2}}{-3t^{-3/2}/2}\,dt = -\frac{1}{3}\int t^3\,dt = -\frac{1}{12}\,t^4 + C_2.$$

So

$$y_p = u_1y_1 + u_2y_2 = \frac{2}{15}\,t^{5/2}\cdot\sqrt{t} - \frac{1}{12}\,t^4\cdot t^{-1} = \left(\frac{2}{15} - \frac{1}{12}\right)t^3 = \boxed{\frac{1}{20}\,t^3}$$