Math 4B: Differential Equations

Lecture 28: Phase Plane Trajectories

- Phase Plane Possibilities,
- Traces & Determinants,
- The Trace-Determinant Plane & More!

Today's Plan

Set-up: We'll be talking about the ***form*** of solutions to the linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where A is an invertible matrix (so $\det(A) \neq 0$). We'll focus today on 2×2 matrices A.

We'll summarize results we've already seen using eigenvalues and eigenvectors.

We'll relate these results to the ***determinant*** and ***trace*** of A.

The Trace

Definition: The Trace of a Matrix

Suppose A is an $n \times n$ matrix, which we can write as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

The ***trace of*** A, written $\mathrm{tr}(A)$, is the sum of the elements on the main diagonal:

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Example: $\mathrm{tr} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 3 & -1 \\ -1 & 4 & 0 \end{pmatrix} = 1 + 3 + 0 = 4$

Trace & Determinant

Proposition: Trace, Determinant, and Eigenvalues

Suppose A is an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly repeated and possibly complex). Then

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

and

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

Example: $\begin{pmatrix} 1 & 2 & -2 \\ 0 & 3 & -1 \\ -1 & 4 & 0 \end{pmatrix}$ has $\operatorname{tr}(A) = 4$, $\det(A) = 0$, and $\lambda = 0, 2 \pm i$

Why?

We'll talk about why this is true for 2×2 matrices. Larger matrices are similar, but we're focused on 2×2 today.

If A is a 2×2 matrix, the characteristic polynomial has roots λ_1 and λ_2, so it is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \big(\lambda_1 + \lambda_2\big)\lambda + \lambda_1 \cdot \lambda_2.$$

On the other hand, the characteristic polynomial of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - \big(a + d\big)\lambda + \big(ad - bc\big)$$

Thus

$$\lambda_1 + \lambda_2 = a + d = \mathrm{tr}(A) \qquad \text{and} \qquad \lambda_1 \cdot \lambda_2 = ad - bc = \det(A).$$

Trace, Determinant, & Eigenvalues

If we're given a 2×2 matrix A with trace $\operatorname{tr}(A) = T$ and determinant $\det(A) = D$, then the characteristic polynomial is

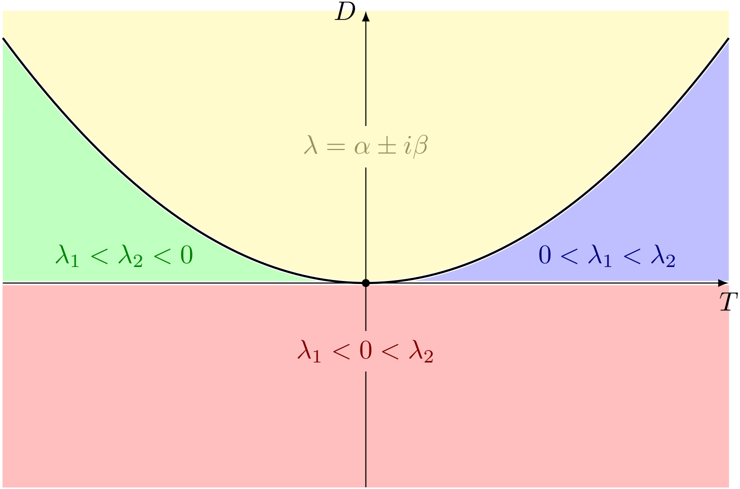
$$\lambda^2 - T\lambda + D.$$

Thus the eigenvalues are $\lambda = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$.

Today we'll talk about the behavior of solutions in different cases.

- Two distinct (real) eigenvalues $\boxed{T^2 - 4D > 0}$
  - Two positive: $T > 0$ and $D > 0$
  - Two negative: $T < 0$ and $D > 0$
  - One positive, one negative: $D < 0$
  - Skipped case: One zero $D = 0$ (other eigenvalue $= T$)
- A repeated (real) eigenvalue $\boxed{T^2 - 4D = 0}$
- Conjugate complex eigenvalues $\lambda = \alpha \pm i\beta$ $\boxed{T^2 - 4D < 0}$

The Trace-Determinant Plane

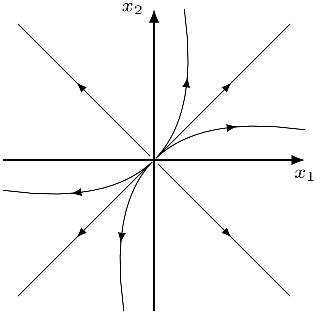


Eigenvalues with Same Sign

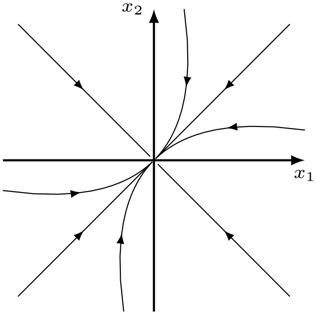
Solutions look like

$$\mathbf{x}(t) = c_1 e^{+\lambda_1 t}\boldsymbol{\xi}_1 + c_2 e^{+\lambda_2 t}\boldsymbol{\xi}_2 \qquad \text{or} \qquad \mathbf{x}(t) = c_1 e^{-\lambda_1 t}\boldsymbol{\xi}_1 + c_2 e^{-\lambda_2 t}\boldsymbol{\xi}_2$$

and are called ***nodes***.



or



Nodal Source

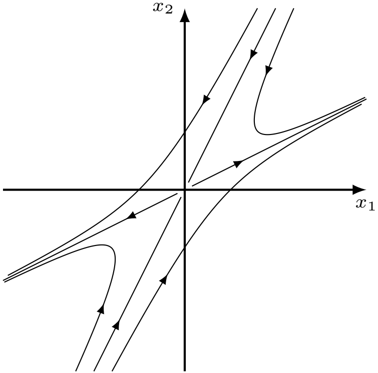
Nodal Sink

Eigenvalues with Opposite Signs

Solutions look like

$$\mathbf{x}(t) = c_1 e^{+\lambda_1 t}\boldsymbol{\xi}_1 + c_2 e^{-\lambda_2 t}\boldsymbol{\xi}_2$$

and are called ***saddle points***. The origin is not a stable equilibrium point.

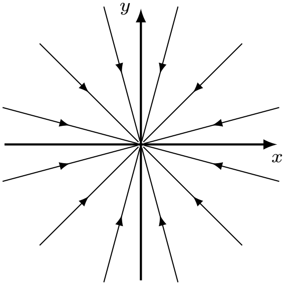


Saddle Point

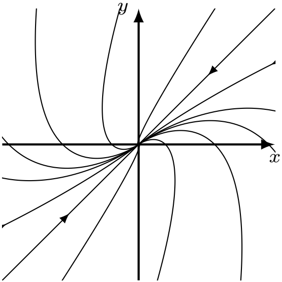
Repeated Real Eigenvalues

Now solutions look like either

$$\mathbf{x}(t) = \mathbf{x}_0 e^{\lambda t} \qquad \text{or} \qquad \mathbf{x}(t) = c_1 e^{\lambda t}\boldsymbol{\xi} + c_2\left(te^{\lambda t}\boldsymbol{\xi} + e^{\lambda t}\boldsymbol{\eta}\right).$$



Proper Node / Star Point

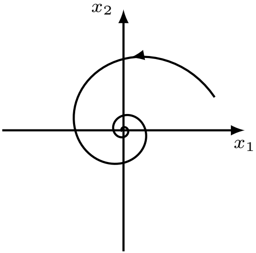


Improper (Degenerate) Node

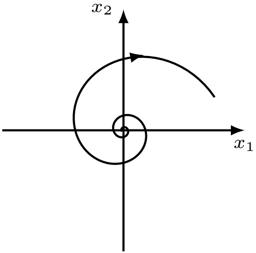
Complex Eigenvalues

If $\lambda = \alpha \pm i\beta$ with $\boldsymbol{\xi} = \mathbf{u} + i\mathbf{v}$, then

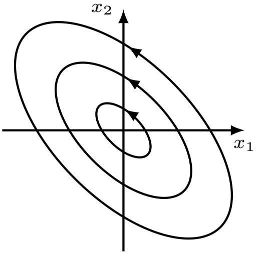
$$\mathbf{x}(t) = c_1 e^{\alpha t}\left(\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}\right) + c_2 e^{\alpha t}\left(\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}\right).$$



Spiral ($\alpha < 0$)



Spiral ($\alpha > 0$)



Center ($\alpha = 0$)

Summary

Eigenvalues	Type of Critical Point	Stability	Trace/Det
Distinct Eigenvalues: $(T^2 - 4D > 0)$			
$0 < \lambda_1 < \lambda_2$	Node	Unstable	$T > 0,\ D > 0$
$\lambda_1 < \lambda_2 < 0$	Node	Asymp. Stable	$T < 0,\ D > 0$
$\lambda_1 < 0 < \lambda_2$	Saddle	Unstable	$D < 0$
Repeated Eigenvalues: $(T^2 - 4D = 0)$			
$\lambda_1 = \lambda_2 > 0$	(Im)Proper Node	Unstable	$T > 0$
$\lambda_1 = \lambda_2 < 0$	(Im)Proper Node	Asymp. Stable	$T < 0$
Complex Eigenvalues: $\lambda = \alpha \pm i\beta$ $(T^2 - 4D < 0)$			
$\alpha > 0$	Spiral Source	Unstable	$T > 0$
$\alpha < 0$	Spiral Sink	Asymp. Stable	$T < 0$
$\alpha = 0$	Center	Stable	$T = 0$