Math 4B: Differential Equations

Lecture 23: Basic Theory of Systems

- Generalizing Theorems...
- ...From First-Order Linear ODEs...
- ...To First-Order Linear Systems & More!

General Idea:

Today we're going to talk about a system of n first-order linear equations

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned} \qquad \text{or} \qquad \mathbf{x}'(t) = P(t)\mathbf{x} + \mathbf{g}(t).$$

If $\mathbf{g}(t) = \mathbf{0}$, we call this a ***homogeneous system***. It's simplest to deal with this type of system first.

We get many of the same theorems as we had earlier.

Principle of Superposition

Principle of Superposition

If the vector functions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to the homogeneous system of first-order linear equations

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t),$$

then so is the linear combination $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$.

Note 1: Our textbook uses $\mathbf{x}^{(1)}(t)$ rather than $\mathbf{x}_1(t)$.

Note 2: We can apply this repeatedly. If we have k solutions $\mathbf{x}_1, \ldots, \mathbf{x}_k$ of this system, then the linear combination

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

is also a solution.

Question: So, how many solutions are enough?

We Need n Solutions

Theorem: n is Enough

If the vector functions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are n linearly independent solutions of $\mathbf{x}' = P\mathbf{x}$ (an $n \times n$ system) in the interval $\alpha < t < \beta$, then any IVP (with $\mathbf{x}(t_0) = \mathbf{x}_0$) has a unique solution

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t).$$

That is, any solution $\mathbf{x}(t) = \phi(t)$ can be written uniquely in terms of the ***fundamental set of solutions*** $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$.

Question: So how can we tell if our set of n solutions is, in fact, a fundamental set? How do we know if this set of solutions is linearly independent?

The Wronskian Returns

How can we tell if $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly independent?

Answer: They are independent when the ***Wronskian*** is non-zero:

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix}.$$

Abel's Theorem

Suppose $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are solutions to the equation $\mathbf{x}'(t) = P(t)\mathbf{x}(t)$ on an interval $\alpha < t < \beta$. Then

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = c \exp \int \big(p_{11}(t) + p_{22}(t) + \cdots + p_{nn}(t)\big)\, dt$$

is either identically zero or never vanishes.

Are There Fundamental Sets?

Question: Can we always find a fundamental set of solutions?

Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

By the existence-uniqueness theorem, we can find $\mathbf{x}_k(t)$ that solves

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t) \qquad \text{and} \qquad \mathbf{x}(t_0) = \mathbf{e}_k$$

on some common interval I given by $\alpha < t < \beta$. Then the solution to

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t) \qquad \text{and} \qquad \mathbf{x}(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

is $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$.

Moral: $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is a fundamental set on I.

What About Complex Solutions?

Question: What if we find a complex solution to a real system? That is, suppose

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t)$$

is a real system (so all the functions in the entries of $P(t)$ are real-valued). Further suppose that $\mathbf{x}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$ is a solution, where $\mathbf{u}$ and $\mathbf{v}$ are real-valued functions and $\mathbf{v} \neq \mathbf{0}$. Can we find ***real*** solution(s) to this system?

Theorem: Real Solutions from Complex

If $\mathbf{x}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$ is a solution to the real-valued homogeneous linear system

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t),$$

then both $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions as well.