Lecture 15: 09/01/22

Example 14.9:

$(B_t)_{t\geq 0}$ Brownian motion, $\mu \in \mathbb{R}$, $\sigma > 0$, S_0

Show: $S_t = S_0 \, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$ solves the SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$$

Proof:

We have $dS_t = \underbrace{\mu S_t}_{=K_t} \, dt + \underbrace{\sigma S_t}_{=H_t} \, dB_t$ Itô process

Moreover:

$$g(x) = \log(x)$$

$$g'(x) = \frac{1}{x}$$

$$g''(x) = -\frac{1}{x^2}$$

Itô's formula: $g(S_t)$ is again an Itô process w/ dynamics

$$dg(S_t) = \Big(\underbrace{g'(S_t)}_{=\frac{1}{\cancel{S_t}}} \underbrace{K_t}_{=\mu \cancel{S_t}} + \frac{1}{2} \underbrace{g''(S_t)}_{=-\frac{1}{\cancel{S_t^2}}} \underbrace{H_t^2}_{=\sigma^2 \cancel{S_t^2}} \Big) dt + \Big(\underbrace{g'(S_t)}_{=\frac{1}{\cancel{S_t}}} \underbrace{H_t}_{=\sigma \cancel{S_t}} \Big) dB_t$$

$$= (\mu - \tfrac{1}{2}\sigma^2) \, dt + \sigma \, dB_t$$

$$\Rightarrow \quad \underbrace{g(S_t) - g(S_0)}_{} = (\mu - \tfrac{1}{2}\sigma^2)(t - 0) + \sigma (B_t - B_0) = (\mu - \tfrac{1}{2}\sigma^2)t + \sigma B_t$$

$$= \log(S_t) - \log(S_0)$$

$$= \log\left(\frac{S_t}{S_0}\right)$$

Solve for S_t :

$$\frac{S_t}{S_0} = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

$$\Rightarrow \quad S_t = S_0 \, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$