FINAL EXAM

Math 174E – Summer Session C 2022
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Instructions:

- Please submit your exam together with this cover sheet and signature below via **Gradescope** and match your pages with the problems. Alternatively, you can also copy the honor statement below on a separate sheet of paper, sign and date, and upload it on Gradescope together with your exam.
- Submission deadline: **Saturday, September 10, 2022, at 8 a.m. (PT).**
- This is an open notes and open book exam and you are welcome to refer to our lecture slides, lecture notes, discussion notes and assignment sheets.
- This is **not a collaborative exam**. You are expected to take this exam in isolation with no communication with other people. **Communicating with others about the exam problems is considered cheating**.
- You must show all of your work and justify your answers to receive any credit. You may justify your answers by either writing brief sentences explaining your reasoning or annotating your math work with brief explanations.
- You must write legibly and clearly mark the answers you want graded. If the work is illegible or it is unclear which answers you intend to be graded, credit may not be given. I strongly suggest circling, boxing, or writing a sentence indicating your answer.
- Use a calculator to compute numerical values as final answers (if applicable). Round your numerical values to 4 decimal places.
- Academic dishonesty: Any student caught cheating will get 0 points for the midterm. According to University rules, cheating must be reported to the College of Physical Sciences. Further action may be taken.

Please sign the honor statement:

> *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

Signature & Date

Good Luck!

1. (10 points) Consider a 2-year 8% coupon bond with a face value of \$100. Suppose that the yield on the bond is 10% p.a. with continuous compounding. Coupon payments of \$4 are made every 6 months.
   (a) Compute the price of the bond.
   (b) Compute the duration of the bond.
   (c) Suppose the yield on the bond decreases by 5 basis points (= 0.05%). Use the duration relationship to estimate how the bond price will approximately change due to the change in the yield.

2. (10 points) The spot price of gold today is \$1,700 per ounce. The current storage costs for gold are \$0.30 per ounce per year, payable quarterly in advance. Suppose the risk-free interest rate is 4% per annum (for all maturities).

   Consider a forward contract on gold for delivery in 9 months.
   (a) Compute today's arbitrage-free forward price of the 9-month forward contract.
   (b) Suppose you are entering today into a short position in the 9-month forward contract on gold with the forward price computed in (a). What will be the value of the forward contract in 6 months if the gold spot price will be at \$1,650 per ounce at that time?

3. (10 points) Companies AAACorp and BBBCorp have been offered the following rates per annum on a \$50 million 3-year loan:

	fixed rate	floating rate
AAACorp	10.0%	6-month LIBOR
BBBCorp	10.5%	6-month LIBOR + 1%

   Suppose that AAACorp requires a fixed-rate loan, whereas BBBCorp requires a floating-rate loan.

   Design a swap contract directly between AAACorp and BBBCorp that is equally attractive to both companies. Determine the net interest rates (p.a.) both companies end up paying on their \$50 million 3-year loan thanks to the swap contract.

4. (10 points) Suppose that the price of a stock today is at \$30. For a strike price of \$25 a 6-month European call option on that stock is quoted with a price of \$12, and a 6-month European put option on the same stock is quoted at \$4. Assume that the risk-free interest rate is 1% per annum (with continuous compounding).
   (a) Does the put-call parity hold?
   (b) Is there an arbitrage opportunity? If yes, carefully explain how the arbitrage strategy would look like. Provide all necessary details of the strategy and the arbitrage gain.

5. (10 points) A stock price is currently at \$50. Assume that at the end of one month the price will be either \$60 or \$40 (and *nothing else!*). The risk-free interest rate is 5% per annum with continuous compounding.
   (a) Compute today's arbitrage-free price of a 1-month European put option written on the stock with a strike price of \$50.

(b) Carefully explain the arbitrage opportunity which arises if the price of the put option considered in (a) was \$6. Provide all necessary details of the strategy and the arbitrage gain.

6. (10 points) A *binary option* (also called *digital option*) is an exotic option for which the payoff at maturity to the holder of the option is either some US dollar amount or nothing at all.

   One example of a binary option is a *cash-or-nothing option* written on a stock. It pays the holder of the option the fixed US dollar amount H at maturity $T > 0$ ("*cash*"), but only if the stock price at time T is above a certain threshold K. Otherwise, the payoff is zero ("*nothing*").

   Suppose a stock price is currently at \$100. Assume that over each of the next two one-month periods the stock price will either go up by 10% or go down by 10%. The risk-free interest rate is 1% p.a. with continuous compounding.

   Use a two-step binomial tree model to compute today's arbitrage-free price of a cash-or-nothing option written on that stock which pays the holder \$10 in two months, but only if the stock price at that time is above today's price of \$100, and nothing otherwise.

7. (10 points) A stock price is currently at \$50 and has a volatility of 30% p.a. The risk-free interest rate is 1% p.a. with continuous compounding.

   (a) Use a three-step binomial tree model with step size 3 months to compute the arbitrage-free price of an *American put option* written on that stock with strike price of \$50 and maturity in 9 months.

   (b) At which nodes in the tree would it be optimal for the holder to early exercise the American put option before maturity T?

8. (10 points) Consider the Black-Scholes-Merton model where the stock price of a (non-dividend-paying) stock $(S_t)_{t\geqslant 0}$ with initial spot price S_0 (today at time $t = 0$) is modeled by

$$S_t = S_0 \cdot e^{(\mu - \frac{\sigma^2}{2})\cdot t + \sigma \cdot B_t} \qquad (t \geqslant 0)$$

   with expected return $\mu \in \mathbb{R}$, volatility $\sigma > 0$ and (standard) Brownian motion $(B_t)_{t\geqslant 0}$.

   Compute the dynamics (stochastic differential equation) of the *inverse* of stock price process

$$I_t = \frac{1}{S_t} \qquad (t \geqslant 0).$$

   Hint: Use the dynamics of the stock price process $(S_t)_{t\geqslant 0}$ and Itô's formula with a suitable function $g(x)$.

9. (10 points) A *power option* written on a stock is a financial derivative contract that pays off the US dollar amount S_T^n for some $n \in \mathbb{N}$ to the holder of the option at time $T > 0$, where S_T is the price of the underlying stock at that time.

   Let $r > 0$ denote the risk-free interest rate (p.a. and continuously compounded).

   Compute today's ($t = 0$) arbitrage-free price of this power option in the Black-Scholes-Merton model by using the risk-neutral valuation approach.

   Hint: You can use (without proof) the fact that $\mathbb{E}[e^Z] = e^{\frac{1}{2}\sigma^2}$ for a normally distributed random variable $Z \sim \mathcal{N}(0, \sigma^2)$ with mean zero and variance σ^2.