ASSIGNMENT 0

PSTAT 160A - FALL 2021

This assignment contains review problems from PSTAT 120A. **It will not be turned in for a grade, but it is important that you feel comfortable solving problems like these.**

Review Exercises

Exercise 0.1. Each throw of a 6-sided die lands on each of the even numbers 2, 4, 6 with probability c and on each of the odd numbers 1, 3, 5 with probability $2c$.

(a) Find c.

Suppose the die is tossed. We introduce two random variables X and Y. Let X equal 1 if the die lands on a number greater than 3, and 0 otherwise. Let Y equal 1 if the die lands on an odd number, and 0 otherwise.

(b) Find the joint probability mass function of X and Y.
(b) Are the random variables X and Y independent? Justify your answer.
(b) Compute $\mathbb{E}[X^2 \cdot \sqrt{Y}]$.

Exercise 0.2. Let X and Y be continuous random variables with the joint probability density function given by

$$f_{X,Y}(x, y) \doteq \begin{cases} 3e^{-x-3y}, & x, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal densities of X and Y.
(b) Are X and Y independent? Justify your answer.
(c) Compute the probability that $X > Y$.

Exercise 0.3. Your friend has three pet cats. You don't know the color of each of the cat's fur. Suppose that each cat is equally likely to be grey or orange, and that their fur color is independent of each other.

(a) Describe this situation with a proper probability space $(\Omega, \mathbb{P})$.

Consider the following events:

- C: all of the cats are the same color.
- D: there is at most one orange cat.
- E: there is at least one orange cat and at least one grey cat.

(b) Compute the probability of the events C, D, and E.
(b) Show that C and D are independent, and that D and E are independent, but that C and E are not independent.

Exercise 0.4. You are given a 4-sided die whose sides are numbered from 1 to 4. When rolled, we assume that each value is equally likely.

Suppose that you roll the die twice in a row.

(a) Specify the underlying probability space $(\Omega, \mathbb{P})$.

Let X denote the maximum value from the two rolls.

(b) Specify X explicitly as a mapping defined on the sample space Ω onto a properly determined state space S_X.
(b) Compute the probability mass function of X.
(b) Compute the cumulative distribution function of X.

Exercise 0.5. You have a bag which contains 100 coins. Of these coins, 99 are normal and fair, and one is fake and has two heads.

You pick a coin at random from the bag. You throw this coin $n \in \mathbb{N}$ times in a row without checking whether the coin is normal or fake. The coin lands on heads n times.

Given this information, what is the probability that you picked the fair coin?

Exercise 0.6. Three friends Afsaneh, Xiuying, and Francisca are playing the following game with a fair 6-sided die: Afsaneh throws the die. If it lands on 6, then the game ends. If it does not land on 6, she passes the die forward to Xiuying. Now, Xiuying throws the die. If it lands on 6, then the game ends. If it does not land on 6, she passes the die forward to Francisca. Now, Francisca throws the die. If it lands on 6, then the game ends. If it does not land on 6, she passes the die forward to Afsaneh, and the game restarts.

They continue playing until the first 6 is thrown, at which point the game stops. The person who rolled the first 6 is considered the winner.

What is the probability that Francisca wins the game?[1]

Exercise 0.7. You play the following game: you choose a number between 1 and 6, then throw 3 fair six-sided die. After tossing, for each die showing the number you chose, you win \$1. You must pay \$1 each round to play the game.

Let X denote your **net** winnings after one round (i.e., the amount that you win minus the amount that you must pay to play).

(a) Determine the state space S_X of X.
(b) Compute the probability mass function of X.
(c) Compute the expected value of X.

[1] *Hint:* it may be helpful to recall that if $|p| < 1$, then $\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$.

(d) In general, a game is called *fair* if the net winnings of the gambler is 0 on average. Is this game fair? If not, what should be the stake of the gambler to make this a fair game? Explain your answer.

Exercise 0.8. Compute the second moment of the Exp(1) distribution.

Exercise 0.9. Tom and Veer are waiting in two different lines at the bank. The time that each of them must wait in line follows an Exp(1) distribution, and their waiting times are independent of each other. Let T and V denote Tom's and Veer's waiting times, respectively.

(a) Let $M \doteq \min\{T, V\}$. Find the cumulative distribution function of M.
(b) Find the probability density function of M. What is the probability distribution of M?
(c) What is the probability that they both wait at least two minutes?

Exercise 0.10. You roll a fair six-sided die 1200 times. Let X denote the number of sixes that appear.

(a) What is the state space of X?
(b) Find the probability that you observe the number 6 at most 200 times.
(c) Using a limit theorem, show that the probability in (b) can be approximated by the value 1/2.[2]

Exercise 0.11. Let X be a continuous random variable with probability density function

$$f_X(t) \doteq \begin{cases} c(4t - 2t^2), & 0 < t < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of c?
(b) Find the cumulative distribution function of X.
(c) Find $\mathbb{P}(X = 1)$ and $\mathbb{P}(X > 1)$.
(d) Compute $\mathrm{Var}(X)$.
(e) Compute the moment generating function of X.

Exercise 0.12. For $\alpha, \beta > 0$, we say that a continuous random variable X has the Gamma(α, β) distribution if its probability density function is given by

$$f_X(x) \doteq \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

[2] *Hint:* which limit theorem allows us to estimate probabilities?

Here $\Gamma : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+$ is the gamma function[3], which is defined as

$$\Gamma(\alpha) \doteq \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Let $X \sim \text{Gamma}(\alpha, \beta)$.

(a) Show that for each $p > 0$, $\mathbb{E}[X^p] = \beta^{-p}\frac{\Gamma(p+\alpha)}{\Gamma(\alpha)}$.

(b) Let $\lambda > 0$, and suppose that $Y_1, Y_2 \overset{iid}{\sim} \text{Exponential}(\lambda)$, so that their probability density functions are given by

$$f_{Y_i}(y) \doteq \lambda e^{-\lambda y}.$$

Show that $Y_1 + Y_2 \sim \text{Gamma}(2, \lambda)$.

[3]The gamma function extends the notion of the factorial of an integer. In particular, if $n \in \mathbb{N}$, then $\Gamma(n) = (n-1)!$.