Chapter 14

Section 14.1

- Two-staged nested design
- Crossed design(e.g. Chapter 5)
- Nested Design(Chapter 14)

> [*Definition*]
> In a treatment structure, nesting occurs when the levels of one factor occurs within **only one** level of a second factor.
> In that case, level of a factor are said to be nested within the level of the second factor.

The surface finish of metal parts made on four machines is being studied. An experiment is conducted in which each machine is run by three different operators and two specimens from each operator are collected and tested. Because of the location of the machines, different operators are used on each machine, and the operators are chosen at random. The data are shown in the following table. Analyze the data and draw conclusions.

	Machine 1			Machine 2			Machine 3			Machine 4		
Operator	1	2	3	1	2	3	1	2	3	1	2	3
	79	94	46	92	85	76	88	53	46	36	40	62
	62	74	57	99	79	68	75	56	57	53	56	47

We have 4 different machines here, and different machines may be far away from each other. There will be 3 operators working on the same machine.(second factor)

Statistical Modeling

We have the following model:

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \epsilon_{(ij)k}$$

where

$$\begin{aligned} i &= 1, 2, \ldots, a \\ j &= 1, 2, \ldots, b \\ k &= 1, 2, \ldots, n \end{aligned}$$

and

$$\epsilon_{(ij)k} \sim N(0, \sigma^2)$$

Decomposition of SS_{Total}

Since $\bar{y}_{ij.} = \frac{1}{n}\sum_{k=1}^{n} y_{ijk}$

$$\begin{aligned}
SS_{Total} &= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{...})^2 \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.}+\bar{y}_{ij.}-\bar{y}_{...})^2 \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2 + n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{...})^2 \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2 + n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{i..}+\bar{y}_{i..}-\bar{y}_{...})^2 \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2 + n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{i..})^2 + nb\sum_{i=1}^{a}(\bar{y}_{i..}-\bar{y}_{...})^2 \\
&= SS_{Error} + SS_{(B)A} + SS_A
\end{aligned}$$

Sum of Square	SS_{Error}	$SS_{(B)A}$	SS_A
Degree of Freedom	$ab(n-1)$	$a(b-1)$	$a-1$

Anova Table

Source of Variation	**SS**	**DF**	**MS = SS/DF**
A	SS_A	$a-1$	MS_A
B within A	$SS_{(B)A}$	$a(b-1)$	$MS_{(B)A}$
Error	SS_{Error}	$ab(n-1)$	MS_{Error}
Total	SS_{Total}	$abn-1$	MS_{Total}

A is fixed B is fixed

Suppose A is fixed B is fixed

$$\sum_{i=1}^{a}\tau_i = 0$$

$$\sum_{j=1}^{b}\beta_{(j)i} = 0 \text{ for } i = 1, 2, \ldots, a$$

Expected Value for Mean Square

$$E(MS_A) = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1}$$
$$E(MS_{(B)A}) = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}\beta_{j(i)}^2}{a(b-1)}$$
$$E(MS_{Error}) = \sigma^2$$

Hypothesis Testing

(1) Suppose

> $$H_0 : \tau_i = 0 \text{ for } i = 1, 2, \ldots, a$$
> $$H_1 : \tau_i \neq 0 \text{ for some } i$$

If $\tau_i \to 0$, then $\frac{MS_A}{MS_{Error}}$ will be close to one. If τ_i grows too big, we reject the null hypothesis.

> Reject H_0 if $F > F_{a-1,ab(n-1),\alpha}$, where
>
> .
>
> $$F = \frac{MS_A}{MS_{Error}}$$
>
> .

(2) Suppose

> .
>
> $$H_0 : \beta_{j(i)} = 0 \text{ for all i, j}$$
> $$H_1 : \beta_{j(i)} \neq 0$$
>
> Reject H_0 if $F > F_{a(b-1),ab(n-1),\alpha}$, where
>
> $$F = \frac{MS_{(B)A}}{MS_{Error}}$$
>
> .

Estimation of Parameters

We have three parameter to estimate:

> .
>
> $$\hat{\mu} = \bar{y}_{...} \to \mu$$
> $$\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...} \to \tau_i \text{ for all i}$$
> $$\hat{\beta}_{j(i)} = \bar{y}_{ij.} - \bar{y}_{i..} \to \beta_{j(i)} \text{ for all i, j}$$

> .

A is fixed, B is random

Suppose A is fixed, B is random, we have

> .
>
> $$\sum_{i=1}^{a} \tau_i = 0$$
> $$\beta_{j(i)} \sim N(0, \sigma_\beta^2)$$
>
> .

Likewise

.

$$E(MS_A) = \sigma^2 + n\sigma_\beta^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1}$$
$$E(MS_{(B)A}) = \sigma^2 + n\sigma_\beta^2$$
$$E(MS_{Error}) = \sigma^2$$

.

Hypothesis Testing

(1) Suppose

> $$H_0 : \tau_i = 0 \text{ for } i = 1, 2, \ldots, a$$
> $$H_1 : \tau_i \neq 0 \text{ for some } i$$
>
> Reject H_0 if $F > F_{a-1,a(b-1),\alpha}$, where
>
> $$F = \frac{MS_A}{MS_{(B)A}}$$

On the other hand, for the randomly selected operators, we are testing

> .
>
> $$H_0 : \sigma_\beta^2 = 0$$
> $$H_1 : \sigma_\beta^2 \neq 0$$
>
> Reject H_0 if $F > F_{a(b-1),ab(n-1),\alpha}$, where

> $$F = \frac{MS_{(B)A}}{MS_{Error}}$$
>
> .

Estimation of Parameters

We have four parameter to estimate:

> .
>
> $$\hat{\mu} = \bar{y}_{...} \to \mu$$
> $$\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...} \to \tau_i \text{ for all i}$$
> $$\hat{\sigma}^2 = MSE \to \sigma^2$$
> $$\hat{\sigma}_\beta^2 = \frac{1}{n}(MS_{(B)A} - MSE) \to \sigma_\beta^2$$
>
> .

A random, B random

Suppose A is random, B is random, we have

> .
>
> $$\tau_i \sim N(0, \sigma_\tau^2)$$
> $$\beta_{j(i)} \sim N(0, \sigma_\beta^2)$$
>
> .

Expected Value for Mean Square

> .
>
> $$E(MS_A) = \sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$$
> $$E(MS_{(B)A}) = \sigma^2 + n\sigma_\beta^2$$
> $$E(MS_{Error}) = \sigma^2$$
>
> .

Hypothesis Testing

(1) Suppose

> $$H_0 : \sigma_\tau^2 = 0$$
> $$H_1 : \sigma_\tau^2 \neq 0$$
>
> Reject H_0 if $F > F_{a-1,a(b-1),\alpha}$, where

$$F = \frac{MS_A}{MS_{(B)A}}$$

Likewise:

> .
>
> $$H_0 : \sigma_\beta^2 = 0$$
> $$H_1 : \sigma_\beta^2 \neq 0$$
>
> Reject H_0 if $F > F_{a(b-1),ab(n-1),\alpha}$, where
>
> $$F = \frac{MS_{(B)A}}{MS_{Error}}$$
>
> .

Estimation of Parameters

We have four parameter to estimate:

> .
>
> $$\hat{\mu} = \bar{y}_{...} \rightarrow \mu$$
> $$\hat{\sigma}^2 = MSE \rightarrow \sigma^2$$
> $$\hat{\sigma}_\beta^2 = \frac{1}{n}(MS_{(B)A} - MSE) \rightarrow \sigma_\beta^2$$
> $$\hat{\sigma}_\tau^2 = \frac{1}{bn}(MS_A - MS_{(B)A}) \rightarrow \sigma_\tau^2$$
>
> .

A random, B fixed

Suppose A is random, B is fixed, we have

> .
>
> $$\tau_i \sim N(0, \sigma_\tau^2)$$
> $$\sum_{j=1}^{b} \beta_{(j)i} = 0 \text{ for } i = 1, 2, \ldots, a$$
>
> .

Expected Value for Mean Square

> .

> $$E(MS_A) = \sigma^2 + bn\sigma_\tau^2$$
> $$E(MS_{(B)A}) = \sigma^2 + \frac{n\sum_{i=1}^a \sum_{j=1}^b \beta_{j(i)}^2}{a(b-1)}$$
> $$E(MS_{Error}) = \sigma^2$$
>
> .

Hypothesis Testing

(1) Suppose

> $$H_0 : \sigma_\tau^2 = 0$$
> $$H_1 : \sigma_\tau^2 \neq 0$$
>
> Reject H_0 if $F > F_{a-1,ab(n-1),\alpha}$, where
>
> $$F = \frac{MS_A}{MS_E}$$

Likewise

> .
>
> $$H_0 : \beta_{j(i)} = 0 \text{ for all i, j}$$
> $$H_1 : \beta_{j(i)} \neq 0$$
>
> Reject H_0 if $F > F_{a(b-1),ab(n-1),\alpha}$, where
>
> $$F = \frac{MS_{(B)A}}{MS_{Error}}$$
>
> .

Estimation of Parameters

We have four parameter to estimate:

> .
>
> $$\hat{\mu} = \bar{y}_{...} \to \mu$$
> $$\hat{\sigma}^2 = MSE \to \sigma^2$$
> $$\hat{\sigma}_\tau^2 = \frac{1}{bn}(MS_A - MS_E) \to \sigma_\tau^2$$
> $$\hat{\beta}_{j(i)} = \bar{y}_{ij.} - \bar{y}_{i..} \to \beta_{j(i)} \text{ for all i, j}$$
>
> .