EXAM II - PRACTICE - PSTAT 160B - SUMMER 2022

Name: ______________________________

UCSB Perm Number: ______________________________

Instructions: You must show all of your work and justify your answers to receive any credit. You may justify your answer by either writing brief sentences explaining your reasoning or annotating your math work with brief explanations (e.g., "by independence..."). If your work and reasoning for a problem is not clear, you may receive 0 points.

If your work is illegible, or if it is unclear what your answer is or which work/answers you intend to be graded, you may not receive credit. For questions where it is appropriate, I suggest circling, boxing, or writing a sentence indicating your answer. **You may write your solutions either directly on the exam, or in a blue book.**

If you aren't sure how complete a problem or finish a calculation, explain clearly where and why you are stuck, and what you believe the correct approach to the problem is.

It may be helpful or necessary to use previous parts of problems to solve later parts. For example, part (a) might be useful for solving part (b). In such a situation, if you are unable to solve (a), and you believe that the answer to (a) is needed to solve (b), you may use (a) to solve (b).

The midterm has # problems (some have multiple parts). **You only need to complete # of them.** Please select # of the problems to solve (you may select any # of the problems), and clearly indicate which # problems you would like to have graded.

Academic Integrity: *Don't cheat!* If you are caught cheating you will receive a grade of 'F' in the course, and, according to university rules, the incident will be reported to the College of Letters and Sciences.

You may not discuss any of the questions on this exam with anyone besides the instructor until your graded exams have been returned. Doing so is a violation of the university's academic integrity policy.

Notation: You do not need to convert numerical answers to decimals. For example, if the answer to a question was $7e^{-5}$ or $e^{-4}\frac{4^2}{2!}$, it would be ok to leave your answer in that form.

Please sign your name in the space provided below to verify that you have read and agree to the following: "I pledge that my solutions are solely from my own individual work, and that I have followed all UCSB academic integrity policies when taking this test. Additionally, I have read the **Instructions** and **Academic Integrity** sections above."

Signature: ______________________________

Important Formulas

You may use the following formulas without proof.

(1) If $X \sim \text{Exp}(\lambda)$, then the pdf of X is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Recall that if $X \sim \text{Exp}(\lambda)$, then $\mathbb{E}(X) = \lambda^{-1}$ and $\text{Var}(X) = \lambda^{-2}$.

(2) If $X \sim \text{Poisson}(\lambda)$, then the pmf of X is given by

$$p(x) = \begin{cases} e^{-\lambda} \frac{\lambda^x}{x!}, & x \in \{0, 1, 2, \dots\} \\ 0, & \text{otherwise.} \end{cases}$$

Recall that if $X \sim \text{Poisson}(\lambda)$, then $\mathbb{E}(X) = \lambda$ and $\text{Var}(X) = \lambda$.

(3) If $X \sim \mathcal{N}(\mu, \sigma^2)$, then the pdf of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right), \ x \in \mathbb{R}.$$

Recall that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2$.

(4) For random variables X and Y, the law of total expectation says that $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}[X|Y])$.

(5) If $X_1 \sim \text{Exp}(\lambda_1), X_2 \sim \text{Exp}(\lambda_2), \dots, X_n \sim \text{Exp}(\lambda_n)$, are independent, then, with $M = \min\{X_1, \dots, X_n\}$,

$$\mathbb{P}(M \geq x, M = X_i) = e^{-\lambda x} \frac{\lambda_i}{\lambda},$$

where $\lambda = \lambda_1 + \dots + \lambda_n$.

(6) Let $\{W_t\}$ be an SBM, and suppose that, for some functions $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$, $\{X_t\}$ is a stochastic process satisfying

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_0 \end{aligned}$$

Then, if $f : \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function with continuous second derivative, then $\{f(X_t)\}$ solves the SDE

$$\begin{aligned} df(X_t) &= f'(X_t)b(X_t)dt + f'(X_t)\sigma(X_t)dW_t + \frac{1}{2} f''(X_t)\sigma^2(X_t)dt \\ f(X_0) &= f(x_0) \end{aligned}$$

Problem 1. Let $\{W_t\}$ be an SBM and fix $T > 0$. Define the process $\{B_t\}$ on $[0, T]$ by

$$B_t \doteq W_t - tT^{-1}W_T, \quad t \in [0, T].$$

(a) Showing *all* steps, determine the probability distribution of the bivariate random variable (B_t, W_t), for $t \in [0, T]$.

We begin by showing that $\boldsymbol{X} \doteq (B_t, W_t)$ is MVN. For $a, b \in \mathbb{R}$, we have

$$aB_t + bW_t \sim \mathcal{N}\left(0, \left(\frac{at}{T}\right)^2 (T-t) + \left(a + b - \frac{at}{T}\right)^2 t\right),$$

where the distributional equivalence is due to the independent increments property of Brownian motion. Observe that this shows that $\boldsymbol{X}$ is MVN. Note that

$$\mathbb{E}(\boldsymbol{X}) = \mathbb{E}((B_t, W_t)) = (0, 0),$$

and, using the fact that $\mathrm{Cov}(W_s, W_t) = \min\{s, t\}$, we have

$$\mathrm{Cov}(B_t, W_t) = \mathbb{E}(B_tW_t) - \mathbb{E}(B_t)\mathbb{E}(W_t) = t\left(1 - \frac{t}{T}\right),$$

while

$$\mathrm{Var}(W_t) = t,$$

and

$$\mathrm{Var}(B_t) = t - \frac{t^2}{T},$$

so it follows that

$$\boldsymbol{X} \sim \mathcal{N}_2\left((0,0), \begin{pmatrix} \mathrm{Var}(B_t) & \mathrm{Cov}(B_t, W_t) \\ \mathrm{Cov}(B_t, W_t) & \mathrm{Var}(W_t) \end{pmatrix}\right),$$

where those quantities are defined as above.

(b) Does $\{B_t\}$ have independent increments? If so, prove that it does. If not, show that it does not.

Note that for each $x \in \mathbb{R}$,

$$\mathbb{P}(B_T - B_{T/2} = -x | B_{T/2} - B_0 = x) = 1,$$

which says that $\{B_t\}$ does not have independent increments.

Problem 2. Let $\{W_t\}$ be an SBM. Fix $\sigma > 0$ and consider the SDE

$$dS_t = \sigma S_t dW_t$$
$$S_0 = 1.$$

(a) Using Itô's formula, solve this SDE.

(b) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Showing *all* steps, derive the moment generating function of X.

(c) Using (a) and (b), compute $\mathbb{E}(S_t)$, where $\{S_t\}$ is a solution to the SDE above.

If $Z \sim \mathcal{N}(0, 1)$, note that

$$e^{-\frac{z^2}{2}}e^{tz} = e^{-\frac{(z-t)^2}{2}}e^{\frac{t^2}{2}},$$

so

$$\mathbb{E}(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz} f_Z(z)dz = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(z-t)^2}{2}}e^{\frac{t^2}{2}}dz$$
$$= e^{\frac{t^2}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-t)^2}{2}}dz = e^{\frac{t^2}{2}}\int_{-\infty}^{\infty} f_{Z_t}(z)dz = e^{\frac{t^2}{2}},$$

where we have used the fact that $f_{Z_t}(z) \doteq \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}}$ is the pdf of the $\mathcal{N}(t,1)$ distribution, and therefore that

$$\int_{-\infty}^{\infty} f_{Z_t}(z)dz = 1.$$

Now, recall that if $Z \sim \mathcal{N}(0,1)$ and $X \sim \mathcal{N}(\mu, \sigma^2)$, then $X \stackrel{d}{=} \mu + \sigma Z$. Using this, you can easily show that $\mathbb{E}[e^{tX}] = e^{\mu t + \frac{\sigma^2 t^2}{2}}$.

Problem 3. Let $\{W_t\}$ be an SBM.

(a) Show that

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds.$$

This follows immediately on applying Itô's formula to the function $f(W_t) = \frac{W_t^3}{3}$

(b) Using (a), compute $\mathbb{E}\left(\int_0^t W_s ds\right)$.

Recall that, for an process $\{X_t\}$ for which Itô integration is well defined, we have

$$\mathbb{E}\left(\int_0^t X_s dW_s\right) = 0.$$

Thus,

$$\mathbb{E}\left(\int_0^t W_s ds.\right) = \mathbb{E}\left(\frac{W_t^3}{3}\right) - \mathbb{E}\left(\int_0^t W_s^2 dW_s\right) = 0 - 0 = 0.$$

Hint: you may use without proof that $\mathbb{E}(W_t^3) = 0$. You should also use the fact that Itô integrals with respect to an SBM have mean 0.

Problem 4. Let $\{W_t\}$ be an SBM. Define $\{X_t\}$ by

$$X_t \doteq W_t^{2022}.$$

Show that $\{X_t\}$ satisfies the SDE

$$\begin{aligned} dX_t &= 2022 X_t^{\frac{2021}{2022}} dW_t + 2043231 X_t^{\frac{2020}{2022}} \\ X_0 &= 0 \end{aligned}$$

The result follows immediately on applying Itô's formula to $f(W_t) = W_t^{2022}$, then using that $X_t = W_t^{2022}$.

Problem 5. A store has 4 cashiers, each of which can serve one customer at a time. Each cashier independently serves customers at a rate of 12 customers per hours. Additionally, there is a line in which customers can wait before they are served. If there are 3 or fewer people in line, then, upon arrival, a new customer will join the end of the line. If there are 4 or more people in line, then any new customers that arrive will immediately depart. Customers independently arrive at the store at a rate of 11 customers per hour.

(a) Model this system as a CTMC by specifying its state space and generator matrix.

See the homework solutions.

(b) Write down a system of equations that you would solve to calculate the stationary distribution of this CTMC.

Letting Q denote the generator of the chain, we would need to solve $\pi Q = \mathbf{0}$.

(c) Let π denote the solution to the system of equations you wrote down in (b). In terms of π, in the long term, what is the probability that all cashiers are busy and that there are at least 2 people in line?

If we denote the state space by $\mathcal{S} = \{0, 1, \dots, 7\}$, so that state i corresponds to there being i people in the system, this probability is given by $\pi_6 + \pi_7$.

Problem 6. Let $\{W_t\}$ be an SBM. Define the process $\{X_t\}$ by

$$X_t \doteq W_t^2 - t, \quad t \geq 0.$$

Is $\{X_t\}$ a martingale with respect to the filtration generated by $\{X_t\}$? Justify your answer.

Note that

$$\mathbb{E}[X_t | X_u, u \in [0, s]] = \mathbb{E}[W_t^2 - t | W_u^2 - u, u \in [0, s]] = \mathbb{E}[W_t^2 - t | W_u^2, u \in [0, s]] = \mathbb{E}[W_t^2 | W_u^2, u \in [0, s]] - t.$$

Also,

$$\begin{aligned}\mathbb{E}[W_t^2 | W_u^2, u \in [0, s]] &= \mathbb{E}[(W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 | W_u^2, u \in [0, s]] \\ &= \mathbb{E}[(W_{t-s})^2] + 0 + W_s^2 = t - s + W_s^2.\end{aligned}$$

The fact that $\{X_t\}$ is a martingale with respect to $\{X_t\}$ follows on combining the last two displays and noting that

$$\mathbb{E}|X_t| \leq \mathbb{E}(W_t^2) + t = 2t < \infty.$$

Problem 7. There is a light switch on the wall. People walk by the light switch according to a Poisson process with a rate of 3 people per hour. If the light switch is 'ON' when someone walks by, there is a 75% chance they leave it 'ON,' and a 25% chance they turn it 'OFF'. On the other hand, if the light switch is 'OFF' when someone walks by, there is a 50% chance they turn it 'ON,' and a 50% chance they leave it 'OFF'.

(a) Write down a system of equations that you would solve to calculate the probability that, if the light switch is currently 'ON', that it will be 'ON' in 3 hours.

Let $\{N_t\}$ denote the sequence of arrivals (i.e., people walking by the switch). Then, if we let Y_n denote the position of the switch after the nth person walks by, if we let

$$X_t \doteq Y_{N_t}, \quad t \geq 0,$$

then X_t describes the position of the switch at time t. Observe that $\{N_t\}$ is a Poisson process with rate λ and Y_n is a DTMC with transition matrix

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}.$$

Thus, $\{X_t\}$ is a CTMC with generator $Q = \lambda(P - I)$, where I is the identity matrix. Thus, to calculate the transition function of $\{X_t\}$, we need to solve the system of ODEs given by

$$P'(t) = QP(t)$$
$$P(0) = I.$$

Then, the probability is given by $P(3)_{1,1}$.

(b) Write down a system of equations that you would solve to calculate, in the long term, the proportion of time in which the light switch is 'ON'.

The CTMC is irreducible and therefore ergodic. Thus, it suffices to calculate the stationary distribution by solving

$$\pi Q = \mathbf{0}.$$

Then, in the long term, the proportion of time in which the switch is on is π_1.