ASSIGNMENT 4

PSTAT 160B - SUMMER 2022

Instructions for the homework: Solve all of the homework problems, and submit them on GradeScope. Your reasoning has to be comprehensible and complete.

Homework Problems

Problem 5.1. Let $\{W_t\}$ be an SBM, and denote the first hitting time of state $a \in \mathbb{R}$ by T_a. Calculate the following:

(a) $\mathbb{P}(W_3 \geq 2)$.

(b) $\mathbb{P}(W_3 \geq 2 | W_1 = 1.5)$.

(c) $\mathbb{E}[W_{17} | W_5 = 3]$.

Problem 5.2. Fix $\alpha > 0$ and let $\{W_t\}$ be an SBM. Define the process $\{\hat{W}_t\}$ by

$$\hat{W}_t \doteq \frac{1}{\sqrt{\alpha}} W_{\alpha t}.$$

Show that $\{\hat{W}_t\}$ is an SBM.

Problem 5.3. Let $\{W_t^1\}, \ldots, \{W_t^d\}$ be independent SBMs. The $\mathbb{R}^d$-valued process $\{\boldsymbol{W}_t\}$ defined as

$$\boldsymbol{W}_t \doteq \begin{pmatrix} W_t^1 & \ldots W_t^d . \end{pmatrix}$$

What is the probability distribution of $\boldsymbol{W}_t$? Note that, for each $t \geq 0$, $\boldsymbol{W}_t$ is an $\mathbb{R}^d$-valued random variable.

Problem 5.4. Let $X, X_1, X_2, \ldots, X_d$ be a collection of iid $\mathcal{N}(\mu, \sigma^2)$ random variables.

(a) Let $\mathbb{X} = \begin{pmatrix} X & X & \ldots & X \end{pmatrix} \in \mathbb{R}^d$. Determine the probability distribution of $\mathbb{X}$.

(b) Let $\mathbb{Y} = \begin{pmatrix} X_1 & X_2 & \ldots & X_d \end{pmatrix} \in \mathbb{R}^d$. Determine the probability distribution of $\mathbb{Y}$.

Problem 5.5. Let $\{W_t\}$ be an SBM. For $s < t$, what is the probability distribution of the $\mathbb{R}^2$-valued random variable (W_s, W_t)?

Problem 5.6. Let $\{W_t\}$ be an SBM. Define the process $\{B_t\}$ on the time interval $[0, 1]$ by

$$B_t \doteq W_t - tW_1.$$

(a) What is the probability distribution of B_t?

(b) Briefly explain why $\mathbb{P}(B_1 = 0) = 1$.

(c) At what time is the variance of the process maximized?

Problem 5.7. Let $\{X_n\}$ be a sequence of iid random variables such that

$$\mathbb{P}(X_n \geq 0) = 1, \quad \mathbb{E}(X_n) = 1.$$

Let $M_n \doteq \prod_{i=1}^{n} X_i$. Show that $\{M_n\}$ is a martingale with respect to $\{X_n\}$.

Problem 5.8. Let $\{W_t\}$ be an SBM.

(a) Using Itô's lemma, show that

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}.$$

(b) Using Itô's lemma, show that

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds.$$

Problem 5.9. Let $\{W_t\}$ be an SBM. Consider a process $\{X_t\}$ satisfying the SDE

$$dX_t = \alpha dW_t + \beta dt$$
$$X_0 = x_0,$$

where $\alpha, \beta, x_0 > 0$. Let $Y_t \doteq \exp(\gamma X_t)$, where $\gamma > 0$. By applying Itô's formula, find the SDE solved by $\{Y_t\}$. That is, "calculate" dY_t.

Problem 5.10. Let $\{W_t\}$ be an SBM. Solve the SDE

$$dX_t = 3X_t^{\frac{2}{3}} dW_t + 3X_t^{\frac{1}{3}} dt$$
$$X_0 = 0.$$

Optional Problems (these use a version of Itô's formula we will see Tuesday)

Problem 5.11. Recall that if $\{W_t\}$ is an SBM, and $\{Y_t\}$ is a process for which the Itô integral

$$I_t \doteq \int_0^t Y_s dW_s,$$

is defined, then $\{I_t\}$ is a martingale. Using this and Itô's formula, show that the process $\{X_t\}$ defined by

$$X_t \doteq \exp\left(\frac{t}{2}\right)\cos(W_t), \quad t \geq 0,$$

is a martingale.

Hint: apply Itô's formula to the function $f(t,x) \doteq \exp\left(\frac{t}{2}\right)\cos(x)$.

Problem 5.12. Let $\{W_t\}$ be an SBM. Show that the process $\{X_t\}$ defined by

$$X_t = \mu + (x_0 - \mu)\exp(-rt) + \sigma \int_0^t \exp(-r(t-s))dW_s,$$

satisfies the SDE

$$\begin{aligned} dX_t &= -r(X_t - \mu)dt + \sigma W_t \\ X_0 &= x_0. \end{aligned}$$

Hint: apply Itô's formula to the function $f(t,x) \doteq \exp(rt)x$.