Math 174E
Lecture 8

Moritz Voss

August 17, 2022

References

Hull
Chapters 5.3, 5.4, 5.5, 5.6, 5.7, 5.9

Standing Assumptions

We assume that the following holds true for some (key) market participants (e.g., large derivatives dealers):

1. The market participants are subject to no transaction costs when they trade.
2. The market participants are subject to the same tax rate on all net trading profits.
3. The market participants can borrow money at the same risk-free rate of interest as they can lend money.
4. The market participants take advantage of arbitrage opportunities as they occur.

Notation

- T: maturity date for a forward or futures contract (in years)
- S_t: Spot price of the underlying asset at time $t \in [0, T]$
- $F_t(T)$: Forward or futures price at time $t \in [0, T]$ (with maturity T)
- r: Zero-coupon risk-free rate of interest for an investment maturing in T years (p.a., continuous compounding)

The **risk-free rate *r*** is the rate at which money can be **borrowed** or **lent** when there is no credit risk.

Investment Asset without Income 1/4

Simplest case: Forward contract on an **investment asset** without income and no storage costs (e.g., non-dividend paying stocks, zero-coupon bonds)

Lemma 5.2

The arbitrage-free forward price (at time 0) on an investment asset with spot price S_0 that provides no income is given by

$$F_0(T) = S_0 e^{rT}.$$

Proof: See Lecture Notes.

Investment Asset without Income 2/4

Example 5.3

Consider a 4-month forward contract to buy a zero-coupon bond that will mature 1 year from today. The current price of the bond is \$930. We assume that the 4-month risk-free rate of interest (continuously compounded) is 6% per annum.

The arbitrage-free forward price is given by

$$F_0(T) = S_0 e^{rT} = 930 \cdot e^{0.06 \cdot 4/12} = 948.79.$$

Investment Asset without Income 3/4

Summary for the arbitrage opportunities:

	$F_0(T) > S_0 e^{rT}$	$F_0(T) < S_0 e^{rT}$
$t = 0$	borrow S_0 at r buy 1 unit of the asset short position in forward	short sell 1 unit of asset invest proceeds S_0 at r long position in forward
$t = T$	sell asset for $F_0(T)$ repay loan with $S_0 e^{rT}$	buy asset for $F_0(T)$ close out short position receive $S_0 e^{rT}$
net profit	$F_0(T) - S_0 e^{rT} > 0$	$S_0 e^{rT} - F_0(T) > 0$

- if one of these two arbitrage opportunities arises in the market, arbitrageurs will exploit it (investment asset), the spot price S_0 will adjust accordingly and the opportunity will disappear

Investment Asset without Income 4/4

Note:

$$F_0(T) = S_0 e^{rT} > S_0$$

- forward price $F_0(T)$ is higher than spot price S_0 because of the **cost of financing** the spot purchase of the asset during the life of the forward contract
- forward price $F_0(T) = S_0 e^{rT}$ is **"model-free"**:
  - determined from the spot price S_0 and the risk-free rate r (both observable market variables)
  - derived from a simple *static arbitrage argument* (static arbitrage strategy)
  - no mathematical model for the evolution of the asset's spot price $(S_t)_{0 \leq t \leq T}$ or the value S_T at maturity T required

What if short sales are not possible?

- ultimately, this does not matter to derive the formula in Lemma 5.2
- all that is required is that there are market participants who hold the asset purely for investment
- by definition this is always true for an **investment asset**
- if the forward price $F_0(T)$ is too low (i.e., smaller than $S_0 e^{rT}$), they will find it attractive to sell the asset in the spot market and buy it back in the forward market (take a long position in a forward contract)
- as we will see, this will *not necessarily be the case for consumption assets* (like crude oil)

Investment Asset with known Income 1/2

Forward contract on an **investment asset** with a *perfectly predictable cash income* and no storage costs (e.g., stocks paying known dividends, coupon-bearing bonds).

Lemma 5.4

The arbitrage-free forward price (at time 0) on an investment asset with spot price S_0 that provides a known cash income *during the life of the forward contract* with a present value of I_0 at time 0 is given by

$$F_0(T) = (S_0 - I_0)e^{rT}.$$

Proof: See related exercise on Assignment 4. Similar idea as in the proof of Lemma 5.2, but match the asset's cash income with borrowing/investing ("asset-liability matching").

Investment Asset with known Income 2/2

Example 5.5

Consider a 10-month forward contract on a stock when the stock price is \$50. We assume that the risk-free rate of interest (continuously compounded) is 8% per annum for all maturities and also that dividends of \$0.75 per share are expected after 3 months, 6 months, and 9 months. The present value of the dividends is

$$I_0 = 0.75 \cdot e^{-0.08 \cdot 3/12} + 0.75 \cdot e^{-0.08 \cdot 6/12} + 0.75 \cdot e^{-0.08 \cdot 9/12} = 2.162.$$

The arbitrage-free forward price is given by

$$F_0(T) = (S_0 - I_0)e^{rT} = (50 - 2.162) \cdot e^{0.08 \cdot 10/12} = 51.14.$$

Investment Asset with known Yield 1/2

Forward contract on an **investment asset** that provides a *known yield* (rather than a known cash income) and no storage costs (e.g., a stock index).

Lemma 5.6

Define q as the *average yield per annum* on the asset during the life of the forward contract with continuous compounding. The arbitrage-free forward price (at time 0) is given by

$$F_0(T) = S_0 e^{(r-q)\cdot T}.$$

Investment Asset with known Yield 2/2

Example 5.7

Consider a 6-month forward contract on an asset that is expected to provide income equal to 2% of the asset price once during a 6-month period. The risk-free rate of interest (with continuous compounding) is 10% per annum. The asset price is \$25.

In this case, $S_0 = 25$, $r = 0.10$, and $T = 0.5$.

The yield is 4% per annum with semiannual compounding. The equivalent yield q (p.a.) with continuous compounding is given by

$$\left(1 + \frac{0.04}{2}\right)^2 = e^{q \cdot 1} \quad \Leftrightarrow \quad q = 0.0396$$

The arbitrage-free forward price is given by

$$F_0(T) = S_0 e^{(r-q) \cdot T} = 25 \cdot e^{(0.10-0.0396) \cdot 6/12} = 25.77.$$

Valuing Forward Contracts 1/2

- the value of a forward contract at the time it is first entered is **zero** (this is how the forward price is actually fixed)
- at a later stage, the value of the forward contract (*value of the position* in the forward contract) may prove to be **positive** or **negative**
- this is simply because the underlying spot price $(S_t)_{0 \leq t \leq T}$ changes over time (as well as the risk-free rate r), the time-to-maturity $T - t$ changes, and therefore the forward price changes too

- it is important for banks and other financial institutions to **value the contract each day** (*marking to market* the contract)
- required by regulators (e.g., in order to compute the value at risk, discussed in Math 179)

- recall that for futures contracts the valuation of the positions is automatically done daily due to the daily settlement

Valuing Forward Contracts 2/2

Lemma 5.8

Consider a forward contract with a forward price $F_0(T)$ initiated at time $t = 0$ on some underlying asset.

The value of this forward contract at time $t \in [0, T]$ for the long and short position is given by

$$f_t^{\text{long}} = (F_t(T) - F_0(T)) \cdot e^{-r(T-t)}$$

and

$$f_t^{\text{short}} = (F_0(T) - F_t(T)) \cdot e^{-r(T-t)},$$

where $F_t(T)$ denotes the arbitrage-free forward price at time t.

Proof: See Lecture Notes (Lecture 9).

Compare the formulas with the P&L of futures contracts, which are initiated at time 0 and closed out at time t.

Futures Price of Stock Indices 1/2

- a stock index futures contract is a useful tool in managing equity portfolios (see Chapter 3.5 in Hull)

- a stock index can usually be regarded as the price of an investment asset (= portfolio of stocks underlying the index) that pays dividends (which would be received by the holder of this portfolio)
- it is usually assumed that the dividends provide a *known yield* rather than a *known cash income*
- if q is the dividend yield rate (expressed with continuous compounding), Lemma 5.6 gives the stock index futures price

$$F_0(T) = S_0 e^{(r-q)\cdot T}$$

- in practice, the chosen value of q should represent the *average annualized dividend yield* during the life of the futures contract (i.e., of those dividends for which the ex-dividend date is during the life of the futures contract)

Futures Price of Stock Indices 2/2

Example 5.9

Consider a 3-month futures contract on an index. Suppose that the stocks underlying the index provide a dividend yield of 1% per annum (continuously compounded), that the current value of the index is 1,300, and that the continuously compounded risk-free interest rate is 5% per annum.

The arbitrage-free futures price is given by

$$F_0(T) = S_0 e^{(r-q)\cdot T} = 1,300 \cdot e^{(0.05-0.01)\cdot 3/12} = 1,313.07.$$