# ASSIGNMENT 6

PSTAT 160B - SPRING 2022

**Instructions for the homework:** This assignment does not need to be submitted.

## Homework Problems

**Problem 6.1.** In this problem you may use without proof the following fact: if $\{Z_n\}$ is a sequence of random variables, and $Z$ is a random variable satisfying

$$\mathbb{E}[|Z|] < \infty,$$

then, for all $n \in \mathbb{N}_0$,

$$\mathbb{E}[\mathbb{E}[Z|Z_0, \ldots, Z_n, Z_{n+1}]|Z_0, \ldots, Z_n] = \mathbb{E}[Z|Z_0, \ldots, Z_n].$$

Let $\{Y_n\}$ be a sequence of random variables, and let $Y$ be a random variable satisfying

$$\mathbb{E}(|Y|) < \infty.$$

Define a sequence of random variables $\{X_n\}$ by

$$X_n \doteq \mathbb{E}[Y|Y_0, \ldots, Y_n], \quad n \in \mathbb{N}_0.$$

(a) Is $\{X_n\}$ a martingale with respect to the filtration generated by $\{Y_n\}$? If so, show that it is. If not, give an example of random variables $Y$, $\{Y_n\}$ such that $\{X_n\}$ is not a martingale with respect to the filtration generated by $\{Y_n\}$.

(b) Is $\{X_n\}$ a Markov chain? If so, show that it is. If not, give an example of random variables $Y$, $\{Y_n\}$ such that $\{X_n\}$ is not a Markov chain.

**Solution 6.1.**

(a) Note that

$$\mathbb{E}[|X_n|] = \mathbb{E}[|\mathbb{E}[Y|Y_0, \ldots, Y_n]] \leq \mathbb{E}[\mathbb{E}[|Y||Y_0, \ldots, Y_n]] = \mathbb{E}[|Y|] < \infty.$$

Additionally,

$$\mathbb{E}[X_{n+1}|Y_0, \ldots, Y_n] = \mathbb{E}[\mathbb{E}[Y|Y_0, \ldots, Y_{n+1}]|Y_0, \ldots, Y_n] = \mathbb{E}[Y|Y_0, \ldots, Y_n] = X_n,$$

so $\{X_n\}$ is a martingale.

(b) Don't worry about this part of the question.

**Problem 6.2.** Let $\{X_n\}$ be a sequence of iid random variables such that

$$\mathbb{P}(X_n \geq 0) = 1, \quad \mathbb{E}(X_n) = 1.$$

Let $M_n \doteq \prod_{i=1}^{n} X_i$. Show that $\{M_n\}$ is a martingale with respect to $\{X_n\}$.

**Solution 6.2.** Note that, since the $\{X_n\}$ are independent and non-negative,

$$\mathbb{E}[|M_n|] = \mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} \mathbb{E}[X_i] = 1 < \infty.$$

Additionally,

$$\begin{aligned}
\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] &= \mathbb{E}\left[X_{n+1}\left(\prod_{i=1}^{n} X_i\right)|X_0, \ldots, X_n\right] \\
&= \left(\prod_{i=1}^{n} X_i\right)\mathbb{E}[X_{n+1}|X_0, \ldots, X_n] \\
&= M_n\mathbb{E}[X_{n+1}] \\
&= M_n,
\end{aligned}$$

so $\{M_n\}$ is a martingale with respect to $\{X_n\}$.

**Problem 6.3.** Let $\{W_t\}$ be an SBM.

(a) Using Itô's lemma, show that

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}.$$

(b) Using Itô's lemma, show that

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds.$$

(c) What is the probability distribution of $\int_0^t W_s ds$?

**Solution 6.3.**

(a) See the lecture notes; we use Itô's formula to evaluate

$$f(W_t) = \frac{W_t^2}{2}.$$

(b) Let $f(x) \doteq \frac{x^3}{3}$. Then, Itô's formula tells us that

$$\begin{aligned}
d\left(\frac{W_t^3}{3}\right) &= df(W_t) \\
&= f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt \\
&= W_t^2 dW_t + W_t dt.
\end{aligned}$$

Thus,

$$\frac{W_t^3}{3} = \int_0^t W_s^2 dW_s + \int_0^t W_s ds,$$

so rearranging yields

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds.$$

(c) For notational convenience, we consider the case when $t = 1$. Then, using the definition of the Riemann integral and the fact that $\{W_t\}$ is continuous and therefore integrable, we have that,

$$\lim_{n\to\infty} I_n = \int_0^1 W_s ds,$$

where

$$I_n \doteq \sum_{i=0}^{n-1} W_{\frac{i}{n}} \left(\frac{i+1}{n} - \frac{i}{n}\right) = \frac{1}{n}\sum_{i=0}^{n-1} W_{\frac{i}{n}}.$$

Note that

$$I_n = \frac{1}{n}\sum_{i=0}^{n-1} W_{\frac{i}{n}} = \frac{1}{n}\sum_{i=0}^{n-1}(n-i)\left(W_{\frac{i+1}{n}} - W_{\frac{i}{n}}\right) = \frac{1}{n}\sum_{i=0}^{n-1} Z_i.$$

where

$$Z_i^n \doteq (n-i)\left(W_{\frac{i+1}{n}} - W_{\frac{i}{n}}\right) \sim \mathcal{N}\left(0, \frac{(n-i)^2}{n}\right).$$

Using the independent and stationary increments property of Brownian motion, we know that, $\{Z_i^n\}$ are independent, so

$$\sum_{i=0}^{n-1} Z_i^n \sim \mathcal{N}\left(0, \sum_{i=0}^{n-1}\frac{(n-i)^2}{n}\right).$$

Observe that

$$\sum_{i=0}^{n-1}\frac{(n-i)^2}{n} = \frac{2n^2+3n+1}{6},$$

so it follows that

$$I_n = \frac{1}{n}\sum_{i=0}^{n-1} Z_i^n \sim \mathcal{N}\left(0, \frac{2n^2+3n+1}{6n^2}\right).$$

Thus, as $n \to \infty$,

$$I_n \xrightarrow{d} Z,$$

where

$$Z \sim \mathcal{N}(0, 1/3).$$

It follows that

$$\int_0^1 W_s ds \sim \mathcal{N}(0, 1/3).$$

A similar argument shows that

$$\int_0^t W_s ds \sim \mathcal{N}(0, t/3).$$

**Problem 6.4.** Recall that if $\{W_t\}$ is an SBM, and $\{Y_t\}$ is a process for which the Itô integral

$$I_t \doteq \int_0^t Y_s dW_s,$$

is defined, then $\{I_t\}$ is a martingale. Using this and Itô's formula, show that the process $\{X_t\}$ defined by

$$X_t \doteq \exp\left(\frac{t}{2}\right)\cos(W_t), \quad t \geq 0,$$

is a martingale.

**Hint:** apply Itô's formula to the function $f(t,x) \doteq \exp\left(\frac{t}{2}\right)\cos(x)$.

**Solution 6.4.** Note that

$$\dot{f}(t,x) = \frac{1}{2}\exp\left(\frac{t}{2}\right)\cos(x), \quad f'(t,x) = -\exp\left(\frac{t}{2}\right)\sin(x), \quad f''(t,x) = -\exp\left(\frac{t}{2}\right)\cos(x),$$

so Itô's formula tells us that

$$\begin{aligned}
df(t,W_t) &= d\left(\exp\left(\frac{t}{2}\right)\cos(W_t)\right) \\
&= \dot{f}(t,W_t)dt + f'(t,W_t)dW_t + \frac{1}{2}f''(t,W_t)dt \\
&= \frac{1}{2}\exp\left(\frac{t}{2}\right)\cos(W_t)dt - \frac{1}{2}\exp\left(\frac{t}{2}\right)\sin(W_t)dW_t - \frac{1}{2}\exp\left(\frac{t}{2}\right)\cos(W_t)dt \\
&= -\frac{1}{2}\exp\left(\frac{t}{2}\right)\sin(W_t)dW_t.
\end{aligned}$$

Thus, using the fact that $f(0,W_0)=1$, we have

$$X_t = \exp\left(\frac{t}{2}\right)\cos(W_t) = 1 - \int_0^t \frac{1}{2}\exp\left(\frac{s}{2}\right)\sin(W_s)dW_s.$$

Since Itô integrals are martingales with respect to the filtration generated by $\{W_t\}$, it follows that $\{X_t\}$ is as well.

**Problem 6.5.** Let $\{W_t\}$ be an SBM. Show that the process $\{X_t\}$ defined by

$$X_t = \mu + (x_0 - \mu)\exp(-rt) + \sigma\int_0^t \exp(-r(t-s))dW_s,$$

satisfies the SDE

$$\begin{aligned}
dX_t &= -r(X_t - \mu)dt + \sigma W_t \\
X_0 &= x_0.
\end{aligned}$$

**Hint:** apply Itô's formula to the function $f(t,x) \doteq \exp(rt)x$.

**Solution 6.5.** Note that

$$\dot{f}(t,x) = r\exp(rt)x, \quad f'(t,x) = \exp(rt), \quad f''(t,x) = 0,$$

and

$$d\langle X\rangle_t = \sigma^2 dt.$$

Applying Itô's formula, we obtain

$$\begin{aligned}
df(t,X_t) &= d\left(\exp(rt)X_t\right) \\
&= \dot{f}(t,X_t)dt + f'(t,X_t)dX_t + \frac{1}{2}f''(t,X_t)d\langle X\rangle_t \\
&= r\exp(rt)X_t dt + \exp(rt)(-r(X_t-\mu))dt + \exp(rt)\sigma dW_t + 0 \\
&= r\mu\exp(rt)dt + \exp(rt)\sigma dW_t,
\end{aligned}$$

so, using the fact that $f(0, X_0) = \exp(r \cdot 0)X_0 = x_0$, we have

$$\begin{aligned}\exp(rt)X_t &= x_0 + \int_0^t r\mu \exp(rs)ds + \int_0^t \exp(rs)\sigma dW_s \\ &= x_0 + \mu(\exp(rt) - 1) + \sigma \int_0^t \exp(rs)dW_s.\end{aligned}$$

Thus,

$$\begin{aligned}X_t &= x_0 \exp(-rt) + \mu - \mu \exp(-rt) + \sigma \exp(-rt) \int_0^t \exp(rs)dW_s \\ &= \mu + (x_0 - \mu) \exp(-rt) + \sigma \int_0^t \exp(-r(t-s))dW_s.\end{aligned}$$