Financial Engineering and Risk Management

Floating rate bonds and term structure of interest rates

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Linear pricing

Theorem. (Linear Pricing) Suppose there is no arbitrage. Suppose also

- Price of cash flow $\mathbf{c}_A$ is p_A
- Price of cash flow $\mathbf{c}_B$ is p_B

Then the price of cash flow that pays $\mathbf{c} = \mathbf{c}_A + \mathbf{c}_B$ must be $p_A + p_B$.

Let p denote the price of the total cash flow $\mathbf{c}$. Suppose $p < p_A + p_B$, i.e. $\mathbf{c}$ is cheap! Will create an arbitrage portfolio, i.e. a free-lunch portfolio.

- Purchase $\mathbf{c}$ at price p
- Sell cash flow $\mathbf{c}_A$ and $\mathbf{c}_B$ separately

Price of the portfolio $= p - p_A - p_B < 0$, i.e. net income at time $t = 0$.

The cash flows cancel out at all times. Future cash flows $=$ **zero**. Free lunch!

No arbitrage $\equiv$ no free lunch. Therefore, $p \geq p_A + p_B$

We can reverse the argument if $p > p_A + p_B$

- Note that we need a liquid market for buying/selling all the cash flows.

Simple example of linear pricing

Cash flow $\mathbf{c} = (c_1, \ldots, c_T)$ is a portfolio of T separate cash flows

- $\mathbf{c}^{(t)}$ pays c_t at time t and zero otherwise.

Suppose the cash flows are annual and the annual interest rate is r.

Price of cash flow $\mathbf{c}^{(t)} = \frac{c_t}{(1+r)^t}$.

Price of cash flow $\mathbf{c} = \sum_{t=1}^{T}$ Price of cash flow $\mathbf{c}^{(t)} = \sum_{t=1}^{T} \frac{c_t}{(1+r)^t}$

Floating interest rates

Interest rates are **random** quantities ... they fluctuate with time.

Let r_k denote the per period interest rate over period $[k, k+1)$

- The exact value of r_k becomes known only at time k
- 1-period loans issued in period k to be repaid in period $k+1$ are charged r_k

Cash flow of floating rate bond

- coupon payment at time k: $r_{k-1}F$
- face value at time n: F

Goal: Compute the arbitrage-free price P_f of the floating rate bond

Split up the cash flows of floating rate bond into simpler cash flows

- $p_k =$ Price of contract paying $r_{k-1}F$ at time k
- $P =$ Price of Principal F at time $n = \frac{F}{(1+r)^n}$

Price of floating rate bond $P_f = P + \sum_{k=1}^{n} p_k$

Price of contract that pays $r_{k-1}F$ at time k

Goal: Construct a portfolio that has a **deterministic** cash flow

- The price of a deterministic cash flow at time $t = 0$ is given by the NPV

	$t=0$	$t=k-1$	$t=k$
Buy contract	$-p_k$		$r_{k-1}F$
Borrow α over $[0, k-1]$	α	$-\alpha(1+r_0)^{k-1}$	
Borrow $\alpha(1+r_0)^{k-1}$ over $[k-1, k]$		$\alpha(1+r_0)^{k-1}$	$-\alpha(1+r_0)^{k-1}(1+r_{k-1})$
Lend α from $[0, k]$	$-\alpha$		$\alpha(1+r_0)^k$

Cash flow at time k

$$\begin{aligned} c_k &= r_{k-1}F - \alpha(1+r_0)^{k-1}(1+r_{k-1}) + \alpha(1+r_0)^k \\ &= \underbrace{\left(F - \alpha(1+r_0)^{k-1}\right) r_{k-1}}_{\text{random}} + \underbrace{\alpha r_0 (1+r_0)^{k-1}}_{\text{deterministic}} \end{aligned}$$

Set $\alpha = \frac{F}{(1+r_0)^{(k-1)}}$. Then the random term is 0.

Net cash flow is now deterministic ... $c_k = \alpha r_0 (1+r_0)^{k-1} = F r_0$

Price of floating rate bond (contd)

Price of the portfolio $= p_k - \alpha + \alpha = p_k = \frac{c_k}{(1+r)^k} = \frac{Fr_0}{(1+r)^k}$

Recall that

$$\begin{aligned}
P_f &= \frac{F}{(1+r_0)^n} + \sum_{k=1}^{n} p_k \\
&= \frac{F}{(1+r_0)^n} + \sum_{k=1}^{n} \frac{Fr_0}{(1+r_0)^k} \\
&= \frac{F}{(1+r_0)^n} + \frac{Fr_0}{(1+r_0)} \sum_{k=1}^{n} \frac{1}{(1+r_0)^{k-1}} \\
&= \frac{F}{(1+r_0)^n} + \frac{Fr_0}{(1+r_0)} \cdot \frac{1 - \frac{1}{(1+r_0)^n}}{1 - \frac{1}{1+r_0}} \\
&= F
\end{aligned}$$

The price P_f of a floating rate bond is equal to its face value F

Term structure of interest rates

Interest rates depend on the term or duration of the loan. Why?

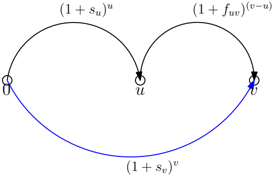
- Investors prefer their funds to be liquid rather than tied up.
- Investors have to be offered a higher rate to lock in funds for a longer period.
- Other explanations: expectation of future rates, market segmentation.

Spot rates: s_t = interest rate for a loan maturing in t years

$$A \text{ in year } t \quad \Rightarrow \quad PV = \frac{A}{(1+s_t)^t}$$

Discount rate $d(0,t) = \frac{1}{(1+s_t)^t}$. Can infer the spot rates from bond prices.

Forward rate f_{uv}: interest rate quoted today for lending from year u to v.



$$(1+s_v)^v = (1+s_u)^u (1+f_{uv})^{(v-u)} \Rightarrow f_{uv} = \left(\frac{(1+s_v)^v}{(1+s_u)^u}\right)^{\frac{1}{v-u}} - 1$$

Relation between spot and forward rates

$$(1+s_t)^t = \prod_{k=0}^{t-1} (1+f_{k,k+1})$$