PRACTICE FINAL - KEY - PSTAT 160A - FALL 2021

Name:______________________________

UCSB Perm Number:______________________________

Instructions: You must show all of your work and justify your answers to receive any credit. You may justify your answer by either writing brief sentences explaining your reasoning or annotating your math work with brief explanations (e.g., "by independence...").

If your work is illegible, or if it is unclear what your answer is or which work/answers you intend to be graded, you may not receive credit. For questions where it is appropriate, I suggest circling, boxing, or writing a sentence indicating your answer.

If you aren't sure how complete a problem or finish a calculation, explain clearly where and why you are stuck, and what you believe the correct approach to the problem is.

It may be helpful or necessary to use previous parts of problems to solve later parts. For example, part (a) might be useful for solving part (b). In such a situation, if you are unable to solve (a), and you believe that the answer to (a) is needed to solve (b), you may use (a) to solve (b).

Academic Integrity: *Please don't cheat!* If you are caught cheating you will receive 0 points for the midterm, and, according to university rules, the incident will be reported to the College of Letters and Sciences.

You may not discuss any of the questions on this exam with anyone besides the instructor until your graded exams have been returned. Doing so is a violation of the university's academic integrity policy.

You are allowed to have two pages of notes (front and back) during this exam. As stated previously, your note sheets may not contain any worked out examples, homework problems, or practice problems. If your note sheets have worked out problems please bring it to the instructor now.

Notation: recall that $\mathbb{N} \doteq \{1, 2, \dots\}$ denotes the set of all *positive* integers, and $\mathbb{R} \doteq (-\infty, \infty)$ denotes the set of all real numbers.

Please sign your name in the space provided below to verify that you have read and agree to the following: "I pledge that my solutions are solely from my own individual work, and that I have followed all UCSB academic integrity policies when taking this test. Additionally, I have read the **Instructions** and **Academic Integrity** sections above."

Signature: ______________________________

Problem 1. Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(0,1)$, and let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be constants. Showing all steps, determine the probability distribution of the random variable Y defined as

$$Y \doteq \sum_{i=1}^{n} \alpha_i X_i.$$

Solution 1. The mgf of the $\mathcal{N}(\mu, \sigma^2)$ distribution is given by

$$m(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).$$

For details on the calculation, see the lecture notes. This means that the mgf of $\alpha_i X_i$ is

$$m_i(t) = \mathbb{E}[e^{t\alpha_i X_i}] = m(\alpha_i t) = \exp\left(\frac{\alpha_i^2 t^2}{2}\right).$$

Since the $\alpha_i X_i$'s are independent, the mgf of Y is

$$m_Y(t) = \prod_{i=1}^{n} m_i(t) = \prod_{i=1}^{n} \exp\left(\frac{\alpha_i^2 t^2}{2}\right) = \exp\left(\frac{t^2 \cdot \sum_{i=1}^{n} \alpha_i^2}{2}\right),$$

which is the mgf of a $\mathcal{N}(0, \alpha^2)$ distribution, where

$$\alpha^2 \doteq \sum_{i=1}^{n} \alpha_i^2.$$

Since mgs uniquely determine probability distributions, it follows that $Y \sim \mathcal{N}(0, \alpha)$.

Problem 2. A machine functions until one of its two components, Component A and Component B, fails, and the joint probability density function of the lifetime of the two components (in hours) is given by

$$f(x,y) = \begin{cases} c(x+y), & x, y \in (0, 24) \\ 0, & \text{otherwise.} \end{cases}$$

Note that the components last at most one day.

(a) Find the value of the constant c.
(b) How long, on average, does it take for the first component to fail?
(c) How long, on average, does it take for both components to fail?
(d) Compute the expected lifetime of component B given the lifetime of the Component A.

Solution 2.

(a) Note that

$$\int_0^{24} \int_0^{24} (x+y)dxdy = 13824$$

so $c = \frac{1}{13824}$.

(b) Let X and Y denote the lifetimes of components A and B respectively. Then the time for the first component to fail is given by $U = \min\{X, Y\}$. Then,

$$\begin{aligned}
\mathbb{E}[U] &= c \int_0^{24} \int_0^{24} \min\{x,y\}(x+y)dxdy \\
&= c \left[\int_0^{24} \int_0^{y} \min\{x,y\}(x+y)dxdy + \int_0^{24} \int_y^{24} \min\{x,y\}(x+y)dxdy\right] \\
&= c \left[\int_0^{24} \int_0^{y} x(x+y)dxdy + \int_0^{24} \int_y^{24} y(x+y)dxdy\right] \\
&= c \cdot 138240 \\
&= 10.
\end{aligned}$$

(c) If we let $V \doteq \max\{X, Y\}$, then

$$\begin{aligned}
\mathbb{E}[U] &= c\int_0^{24}\int_0^{24} \max\{x, y\}(x+y)dxdy \\
&= c\left[\int_0^{24}\int_0^{y} \max\{x, y\}(x+y)dxdy + \int_0^{24}\int_y^{24} \max\{x, y\}(x+y)dxdy\right] \\
&= c\left[\int_0^{24}\int_0^{y} y(x+y)dxdy + \int_0^{24}\int_y^{24} x(x+y)dxdy\right] \\
&= c \cdot 248832 \\
&= 18.
\end{aligned}$$

(d) We want to calculate $\mathbb{E}[Y|X]$. We begin by noting that the marginal distribution of X is given by

$$f_X(x) = \int_0^{24} c(x+y)dy = c24(x+12).$$

Note that, for $x \in (0, 24)$, the conditional density of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{c(x+y)}{c24(x+12)} = \frac{x+y}{24(x+12)},$$

so the conditional expectation of Y given $X = x$ is

$$\begin{aligned}
\mathbb{E}[Y|X = x] &= \int_0^{24} y f_{Y|X}(y|x)dy \\
&= \int_0^{24} y\frac{x+y}{24(x+12)} \\
&= \frac{12(x+16)}{x+12},
\end{aligned}$$

so $\mathbb{E}[Y|X] = \frac{12(X+16)}{X+12}$.

Problem 3. You have a collection of k coins, and the probability of heads of the i-th coin is denoted by $p_i \in (0, 1)$. The game works as follows: you start by flipping coin 1, and you flip it repeatedly until it lands on tails. After it lands on tails, you start flipping coin 2, and you flip it repeatedly until it lands on tails. You continue this process until you reach coin k; when coin k lands on tails, you switch back to coin 1 and repeat the process. Let X_n denote the coin used on the n-th flip. Is $\{X_n\}$ a Markov chain? If so, specify its transition matrix. Otherwise, explain why the Markov property is not satisfied.

Solution 3. Yes, $\{X_n\}$ is a Markov chain. For $i \in \{1, 2, \ldots, k-1\}$, we have

$$\mathbb{P}[X_{n+1} = j | X_n = i] = \begin{cases} p_i, & j = i \\ 1 - p_i & j = i+1 \\ 0, & \text{otherwise,} \end{cases}$$

and we have

$$\mathbb{P}[X_{n+1} = j | X_n = k] = \begin{cases} p_k, & j = k \\ 1 - p_k & j = 1 \\ 0, & \text{otherwise,} \end{cases}$$

Problem 4. Let $Y_i \overset{iid}{\sim} \text{Poisson}(0.5)$. Let $X_0 \doteq 0$, and for each $n \geq 1$, let X_n denote the running maximum of $Y_1, \ldots, Y_n$, namely,

$$X_n \doteq \max\{Y_1, \ldots, Y_n\}.$$

(1) Is $\{X_n\}$ a Markov chain? If so, specify its transition matrix. Otherwise, explain why the Markov property is not satisfied.
(2) Compute $\mathbb{E}[X_{n+1}|X_n = 7]$. You may leave your answer in terms of a series.

Solution 4.

(a) Yes, $\{X_n\}$ is a Markov chain. Let p and F denote the pmf and cdf, respectively, of the Poisson(0.5) distribution. Then the transition matrix is given by

$$\mathbb{P}[X_{n+1} = y | X_n = x] = \begin{cases} F(y), & y = x \\ p(y), & y > x \\ 0, & \text{otherwise.} \end{cases}$$

(b) We have

$$\begin{aligned} \mathbb{E}[X_{n+1} | X_n = 7] &= \sum_{y=0}^{\infty} y \cdot \mathbb{P}[X_{n+1} = y | X_n = 7] \\ &= 7 \cdot F(7) + \sum_{y=8}^{\infty} y \cdot p(y). \end{aligned}$$

Problem 5. Each generation of a population contains 5 individuals. Each individual has a gene with allele type I, II, or III. Given that the current generation has α_1 individuals with allele type I, each of the 5 individuals in the next generation has a $\frac{\alpha_1}{5}$ chance of having allele type I. Alleles type II and III are passed on to the next generation similarly.

Let X_n denote the number of individuals with allele type I in the n-th generation.

(a) If the population starts with 3 individuals who have allele type I, what is the probability that allele type I is eventually the only allele present in the population?
(b) Is $\{X_n\}$ irreducible?
(c) Classify each of the communication classes of $\{X_n\}$ as either open or closed.
(d) Compute $\mathbb{P}_2[X_n = 2]$ for $n \in \{5, 50, 500\}$.
(e) Does $\{X_n\}$ have a stationary distribution? If so, is it unique?
(f) Does $\{X_n\}$ have a limiting distribution? If so, write down the system of equations you would need to solve to calculate the limiting distribution. If not, explain why.

Solution 5.

(a) Let $\tau_5 \doteq \inf\{n \geq 0 : X_n = 5\}$, and for $x \in \{0, 1, \ldots, 5\}$, let $\alpha_x \doteq \mathbb{P}_x[\tau_5 < \infty]$. Then $\alpha_0 = 0$, $\alpha_5 = 1$, and

$$\begin{aligned} \alpha_1 &= \sum_{x=0}^{5} \mathbb{P}_1[\tau_5 < \infty | X_1 = x] \mathbb{P}_1[X_1 = x] \\ &= \sum_{x=1}^{4} \mathbb{P}_x[\tau_5 < \infty] \mathbb{P}_1[X_1 = x] + 1 \cdot \mathbb{P}_1[X_1 = 5] \\ &= \sum_{x=1}^{4} \alpha_x \binom{5}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{5-x} + \binom{5}{5} \left(\frac{1}{5}\right)^5. \end{aligned}$$

Repeating this for $\alpha_2, \alpha_3, \alpha_4$, we obtain a system of 4 linear equations with 4 unknowns, which we can solve using linear algebra.

(b) $\{X_n\}$ is not irreducible, as $\mathbb{P}[X_{n+1} = x | X_n = 0] = 0$ for all $x \in \{1, 2, 3, 4, 5\}$.
(c) The communication class $\mathcal{C}_1 = \{1, 2, 3, 4\}$ is open, while $\mathcal{C}_0 = \{0\}$ and $\mathcal{C}_5 = \{5\}$ are closed.
(d) To compute this, numerically evaluate

$$(P^n)_{2,2}$$

for $n \in \{5, 50, 500\}$, where

$$P_{x,y} = \binom{5}{y} \left(\frac{x}{5}\right)^y \left(\frac{5-x}{5}\right)^{5-y}, \quad x, y \in \{0, 1, 2, 3, 4, 5\}$$

(e) $\{X_n\}$ has infinitely many stationary distributions; they are of the form

$$\lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\lambda \in [0,1]$.

(f) In (e) we saw that $\{X_n\}$ has multiple stationary distributions, so it follows that it has no limiting distribution.

Problem 6. Let $\{X_n\}$ be a Markov chain on $\mathcal{S} = \{a, b, c\}$ with transition matrix

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 3/8 & 1/8 & 1/2 \\ 1/10 & 7/10 & 1/5 \end{pmatrix}.$$

(a) If $\{X_n\}$ has initial distribution $\mu = \begin{pmatrix} 1/4 & 3/4 & 0 \end{pmatrix}$, what is its probability distribution when $n = 73$?
(b) Write down a system of equations that you could solve to calculate the expected number of steps that it takes for the chain to reach state 3 if it starts in state 1.
(c) Write down a system of equations that you could solve to calculate the limit, as $n \to \infty$, of P^n. Explain why solving this system of equations is equivalent to calculating this limit.

Solution 6.

(a) To solve this question you would numerically evaluate

$$\mathbb{P}_\mu(X_{73} \in \cdot) = \mu^T P^{73}.$$

(b) We would have to solve

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/4 & 1/4 \\ 3/8 & 1/8 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}.$$

(c) The chain is ergodic so its unique stationary distribution *pi* is its limiting distribution. Therefore we know

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \pi^T \\ \pi^T \\ \pi^T \end{pmatrix},$$

where π is the unique solution to

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} 1/4 & 1/4 & 1/2 & 1 \\ 3/8 & 1/8 & 1/2 & 1 \\ 1/10 & 7/10 & 1/5 & 1 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & 1 \end{pmatrix}.$$

Problem 7. Let $\{X_n\}$ be a Markov chain on $\mathcal{S}$ such that for some $x \in \mathcal{S}$, for each $n \in \mathbb{N}$,

$$\mathbb{P}_x[X_n = x] = \frac{1}{(n+1)^2}.$$

Is state x transient or recurrent?

Solution 7. We have

$$\sum_{n=1}^{\infty} (P^n)_{x,x} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} < \infty,$$

so x is transient.

Problem 8. The *total variation distance* between two probability measures μ and ν on a set $\mathcal{S} = \{s_0, s_1, s_2, \dots\}$ is defined as

$$d_{TV}(\mu, \nu) = \max_{S \subseteq \mathcal{S}} |\mu(S) - \nu(S)|.$$

Note that the maximum above is taken over all subsets S of $\mathcal{S}$. Show that d_{TV} is a distance function, namely that:

(a) If μ and ν are probability measures on $\mathcal{S}$, then $d_{TV}(\mu, \nu) \geq 0$.
(b) If μ and ν are probability measures on $\mathcal{S}$, then $d_{TV}(\mu, \nu) = 0$ if and only if $\mu = \nu$.
(c) If μ and ν are probability measures on $\mathcal{S}$, then $d_{TV}(\mu, \nu) = d_{TV}(\nu, \mu)$.
(d) If μ, ν, and η are probability measures on $\mathcal{S}$, then $d_{TV}(\mu, \nu) \leq d_{TV}(\mu, \eta) + d_{TV}(\eta, \nu)$.

Hint: this problem only requires that you carefully apply the definition of a probability measure.

Solution 8.

(a) For each $S \subseteq \mathcal{S}$, $|\mu(S) - \nu(S)| \geq 0$, so it follows that $d_{TV}(\mu, \nu) \geq 0$.

(b) For each $S \subseteq \mathcal{S}, |\mu(S) - \mu(S)| = 0$, so $d_{TV}(\mu, \mu) = 0$. On the other hand, suppose that $\mu \neq \nu$. Then there is some $x \in \mathcal{S}$ such that $\mu(\{x\}) \neq \nu(\{x\})$, which means $|\mu(\{x\}) - \nu(\{x\})| > 0$. Thus

$$d_{TV}(\mu, \nu) = \sup_{S \subseteq \mathcal{S}} |\mu(S) - \nu(S)| \geq |\mu(\{x\}) - \nu(\{x\})| > 0.$$

(c) For each $S \subseteq \mathcal{S}$, $|\mu(S) - \nu(S)| = |\nu(S) - \mu(S)|$, so $d_{TV}(\mu, \nu) = d_{TV}(\nu, \mu)$.

(d) This follows from the fact that for any numbers a, b, c, $|a - b| \leq |a - c| + |b - c|$.

Problem 9. You are playing a game of chance. Each round, the probability that you win is $\frac{18}{37}$ and the probability that you lose is $\frac{19}{37}$. If you win, you earn \$1. Similarly, if you lose, you lose \$1. Note that you have to stop playing once you have \$0. Let $\{X_n\}$ denote your wealth after round n.

(a) Calculate $\mathbb{P}_{100}(X_{12} = 98)$.

(b) You start with \$100. Calculate the probability that between rounds 60 and 70 (so, in rounds 60, 61,..., 70) you lose \$1 in total.

(c) You start with \$100. Calculate your expected wealth after 30 rounds.

(d) Is $\{X_n\}$ a Markov chain? If so, is it irreducible?

(e) Write down a system of equations that you would need to solve to calculate the probability that you eventually stop playing.

(f) Write down a system of equations that you would need to solve to calculate the expected number of rounds that it will take you to stop playing.

Solution 9.

(a) Note that $\{X_n\}$ is a simple random walk. Therefore

$$\mathbb{P}_{100}(X_{12} = 98) = \mathbb{P}_0(X_{12} = -2) = \binom{12}{5} \left(\frac{18}{37}\right)^5 \left(\frac{19}{37}\right)^7.$$

(b) This probability is given by

$$\binom{11}{5} \left(\frac{18}{37}\right)^5 \left(\frac{19}{37}\right)^6.$$

(c) We have

$$\mathbb{E}[X_{30}] = 100 + 30 \cdot \left(2 \cdot \frac{18}{37} - 1\right).$$

(d) Yes, $\{X_n\}$ is a Markov chain, with transition matrix

$$\mathbb{P}[X_{n+1} = y | X_n = x] = \begin{cases} \frac{18}{37}, & y = x + 1 \\ \frac{19}{37}, & y = x - 1. \end{cases}$$

(e) Let $E = \{\tau_0 < \infty\}$, where $\tau_0 \doteq \inf\{n \geq 0 : X_n = 0\}$. For each $x \in \mathbb{N}_0$, let $\alpha_x \doteq \mathbb{P}_x(E)$. Then we would need to solve

$$\begin{cases} \alpha_0 &= 1 \\ \alpha_i &= \alpha_{i-1} \frac{19}{37} + \alpha_{i+1} \frac{18}{37}, \quad i \in \mathbb{N}. \end{cases}$$

(f) Let $\tau_0 \doteq \inf\{n \geq 0 : X_n = 0\}$, and for each $x \in \mathbb{N}_0$, let $m_x \doteq \mathbb{E}_x[\tau_0]$. Then we would need to solve the system of equations given by

$$\begin{cases} m_0 &= 0 \\ m_i &= 1 + \frac{18}{37} m_{i+1} - \frac{19}{37} m_{i-1}, \quad i \in \mathbb{N}. \end{cases}$$

Problem 10. Each day eduroam is either up, down, or being repaired. If eduroam is up at the start of the day, then it will be up at the start of the next day with probability 0.7, and it will be down at the start of the next day with probability 0.3. When eduroam breaks, it takes 2 days for the IT department to fix it.

(a) Write down a system of equations that you would need to solve to determine, in the long-term, the proportion of time eduroam spends in each state.

(b) In terms of the solution to the system of equations you wrote down above, what is proportion of the time is eduroam down (including the time when it is being repaired)?

Solution 10.

(a) We can model the status of eduroam as a Markov chain on $\mathcal{S} = \{U, D, R_1, R_2\}$, with transition matrix

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that the Markov chain is ergodic, as it is irreducible and $P_{U,U} > 0$. Thus the limiting distribution is given by the unique stationary distribution, which is the unique solution to

$$\begin{pmatrix} \pi_U & \pi_D & \pi_{R_1} & \pi_{R_2} \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_U & \pi_D & \pi_{R_1} & \pi_{R_2} & 1 \end{pmatrix}.$$

Since π is the limiting distribution, it describes the long-term proportion of time spent in each state.

(b) The proportion of the time is eduroam down, including the time when it is being repaired, is given by $\pi_D + \pi_{R_1} + \pi_{R_2}$.

Problem 11. Consider a Markov chain $\{X_n\}$ on $\mathcal{S} = \{a, b, c\}$ with transition matrix

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.8 & 0.1 & 0.1 \end{pmatrix}.$$

(a) Draw the transition diagram of the Markov chain.

(b) Identify the communication class(es) of the Markov chain and specify whether they are open or closed. Briefly justify your answer.

(c) Is there a system of equations that you could solve to determine, on average, in the long-term, what proportion of time the Markov chain spends in each state of $\mathcal{S}$? If so, write down the system of equations. If not, briefly explain why.

Solution 11.

(a)

(b) The Markov chain is irreducible and therefore has a single communication class. To see that it is irreducible, note that $P_{a,b} > 0, P_{b,c} > 0$, and $P_{c,a} > 0$.

(c) The chain is irreducible and $P_{a,a} > 0$, so it is aperiodic, and therefore ergodic. The ergodic theorem tells us that the unique stationary distribution is the limiting distribution, and that it is given by the unique solution to

$$\begin{pmatrix} \pi_a & \pi_b & \pi_c \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 & 0 & 1 \\ 0.3 & 0.3 & 0.4 & 1 \\ 0.4 & 0.2 & 0.4 & 1 \\ 0.8 & 0.1 & 0.1 & 1 \end{pmatrix} = \begin{pmatrix} \pi_a & \pi_b & \pi_c & 1 \end{pmatrix}.$$

Then π_a tells us, on average, the long-term proportion of time spent in a, and the other states are similar.

Problem 12. Show that if a Markov chain has two distinct stationary distributions, then it has no limiting distribution.

Solution 12. Denote the transition matrix of the Markov chain by P. Let μ and ν be distinct stationary distributions. Then for each $n \in \mathbb{N}$,

$$\mu^T P^n = \mu^T, \quad \nu^T P^n = \nu^T,$$

so

$$\lim_{n\to\infty} \mu^T P^n = \mu^T, \quad \lim_{n\to\infty} \nu^T P^n = \nu^T.$$

Since limiting distributions are independent of initial distributions, it follows that the Markov chain has no limiting distribution.

Problem 13. Consider a Markov chain $\{X_n\}$ on $\mathcal{S} = \{1,2,3,4\}$ with transition matrix

$$P = \begin{pmatrix} 0.1 & 0.1 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.1 & 0.3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Let $\tau_3 \doteq \inf\{n \in \mathbb{N} : X_n = 3\}$. Write down a system of equations that you would solve to calculate $\mathbb{P}_1[\tau_3 < \infty]$.

(b) Show that $\{X_n\}$ has no limiting distribution.

Solution 13.

(a) For each $x \in \mathcal{S}$, let $\alpha_x \doteq \mathbb{P}_x[\tau_3 < \infty]$. Then we would need to solve the system of equations given by

$$\begin{cases} \alpha_1 = \alpha_1 \cdot 0.1 + \alpha_2 \cdot 0.1 + \alpha_3 \cdot 0.2 + \alpha_4 \cdot 0.6 \\ \alpha_2 = \alpha_1 \cdot 0.3 + \alpha_2 \cdot 0.3 + \alpha_3 \cdot 0.1 + \alpha_4 \cdot 0.3 \\ \alpha_3 = \alpha_4 \cdot 1 \\ \alpha_4 = 1. \end{cases}$$

(b) If we let $\mu^T = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$, then

$$(\mu^T P^n)_3 = \{1, 0, 1, 0, 1, 0, \dots\},$$

which does not converge. This shows that $\{X_n\}$ has no limiting distribution.

Problem 14. Consider a Markov chain $\{X_n\}$ on $\mathcal{S} = \{1,2,3,4,5\}$ with transition matrix

$$P = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.7 & 0 \\ 0.1 & 0.1 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0.3 & 0.3 & 0.1 & 0.1 & 0.2 \end{pmatrix}$$

(a) Does $\{X_n\}$ have a stationary distribution? If so, explain why and write down the equations you would solve to calculate the stationary distribution. If not, explain why.

(b) Does $\{X_n\}$ have a limiting distribution? Justify your answer.

Solution 14.

(a) Note that we can partition $\mathcal{S}$ as $\mathcal{S} = \mathcal{C} \cup T$, where $\mathcal{C} = \{1,2,3,4\}$ is a closed communication class of recurrent states, and $T = \{5\}$ is an open communication class of transient states. The ergodic theorem for unichains tells us that if $\tilde{\pi}$ denotes the unique solution to

$$\begin{pmatrix} \tilde{\pi}_1 & \tilde{\pi}_2 & \tilde{\pi}_3 & \tilde{\pi}_4 \end{pmatrix} \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.7 & 1 \\ 0.1 & 0.1 & 0.4 & 0.4 & 1 \\ 0 & 0 & 0.8 & 0.2 & 1 \\ 0 & 0 & 0.5 & 0.5 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\pi}_1 & \tilde{\pi}_2 & \tilde{\pi}_3 & \tilde{\pi}_4 & 1 \end{pmatrix},$$

then the stationary distribution of $\{X_n\}$ is given by

$$\pi^T = \begin{pmatrix} \tilde{\pi}_1 & \tilde{\pi}_2 & \tilde{\pi}_3 & \tilde{\pi}_4 & 0 \end{pmatrix}.$$

(b) The ergodic theorem for unichains also tells us that stationary distribution π given in part (a) is the limiting distribution of $\{X_n\}$.