Chap 14.1 Two stays nested design

Crossed design : e.g. Ch. 5

Nested design ch. 14

In a treatment structure. nesting occurs when the levels of one factor occur within only one of a second factor. In that case, the levels of the factor are said to be nested within the level of the second factor.

Model:

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \varepsilon_{(ij)k} \quad \begin{matrix} i=1,2,\dots a \\ j=1,2,\dots b \\ k=1,2,\dots n. \end{matrix}$$

$$\varepsilon_{(ij)k} \overset{iid}{\sim} N(0,\sigma^2)$$

Factor A	Factor B (nested in A)	
1	1	$y_{111}, y_{112}, \cdots y_{11n}$
	2	$y_{121}, y_{122}, \cdots y_{12n}$
	$\vdots$	
	b	
2	1	$y_{211} \quad y_{212} \cdots y_{21n}$
	2	$y_{221} \quad \cdots \quad y_{22n}$
	$\vdots$	
	b	

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{\cdots})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(\underbrace{y_{ijk}-\bar{y}_{ij\cdot}}+\underbrace{\bar{y}_{ij\cdot}-\bar{y}_{\cdots}})^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij\cdot})^2 + n\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij\cdot}-\bar{y}_{\cdots})^2$$

$$= \underbrace{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-y_{ij\cdot})^2}_{\substack{SSE \\ ab(n-1)}} + \underbrace{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij\cdot}-\bar{y}_{i\cdot\cdot})^2}_{\substack{SSB(A) \\ a(b-1)}}$$

$$+ nb\sum_{i=1}^{a}\underbrace{(\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdots})^2}_{SSA \quad a-1}$$

ANOVA

Source of variation	SS	df	MS = SS/df
A	SS_A	$a-1$	MS_A
B within A	$SS_{B(A)}$	$a(b-1)$	$MS_{B(A)}$
Err	SSE	$ab(n-1)$	MSE
TTL	SS_T	$abn-1$	

A fixed B fixed

$\sum_{i=1}^{a} \tau_i = 0$ $\sum_{j=1}^{b} \beta_{j(i)} = 0$ for $i = 1, 2, \cdots a$

$$E(MSA) = \sigma^2 + \frac{bn\sum_{i=1}^{a} \tau_i^2}{a-1}$$

$$E(MS_{B(A)}) = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b} \beta_{j(i)}}{a(b-1)}$$

$$E(MSE) = \sigma^2$$

$H_0: \tau_i = 0$ for $i = 1, 2, \dots a$

$H_1: \tau_i \neq 0$ for some i

Reject H_0: if $F > F_{a-1, ab(n-1), \alpha}$

$$F = \frac{MS_A}{MSE}$$

$H_0: \beta_{j(i)} = 0$ for all i, j.

Reject H_0 if $F > F_{a(b-1), ab(n-1), \alpha}$

$$F = \frac{MS_{B(A)}}{MSE}$$

$\hat{\mu} = \bar{y}_{\dots} \longrightarrow \mu$

$\hat{\tau}_i = \bar{y}_{i\dots} - \bar{y}_{\dots} \longrightarrow \tau_i$ for all i

$\beta_{j(i)} = \bar{y}_{ij.} - \bar{y}_{i..} \longrightarrow \beta_{j,i,}$ for all i, j

A fixed B random.

$\sum_{i=1}^{a} \tau_i = 0$ $\qquad \beta_{j(i)} \overset{iid}{\sim} N(0, \sigma_\beta^2)$

$$E(MS_A) = \sigma^2 + n\sigma_\beta^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1}$$

$$E(MS_{B(A)}) = \sigma^2 + n\sigma_\beta^2$$

$$E(MSE) = \sigma^2$$

$H_0: \tau_i = 0$ for all i

$H_1: \tau_i \neq 0$ for some i

Reject H_0 if $F > F_{a-1, a(b-1), \alpha}$

$$F = \frac{MS_A}{MS_{B(A)}}$$

$H_0: \sigma_\beta^2 = 0$

$H_1: \sigma_\beta^2 \neq 0$

$F > F_{a(b-1), ab(n-1), \alpha}$

$$F = \frac{MS_{B(A)}}{MSE}$$

$\hat{\mu} = \bar{y}_{...} \longrightarrow \mu$

$\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}$ $i = 1, 2, \ldots, a$

$\hat{\sigma}^2 = MSE \longrightarrow \sigma^2$

$\hat{\sigma}_\beta^2 = \frac{1}{n}(MS_{B(A)} - MSE) \longrightarrow \sigma_\beta^2$

A random B random

$\tau_i \overset{iid}{\sim} N(0, \sigma_\tau^2)$ $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$

$E(MSE) = \sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$

$E(MS_{B(A)}) = \sigma^2 + n\sigma_\beta^2$

$E(MSE) = \sigma^2$

$H_0: \sigma_\tau^2 = 0$

$H_1: \sigma_\tau^2 \neq 0$

Reject H_0, if $F > F_{a-1, a(b-1), \alpha}$.

$$F = \frac{MS_A}{MS_{B(A)}}$$

$H_0: \sigma_\beta^2 = 0$

$H_1: \sigma_\beta^2 \neq 0$

Reject H_0 if $F > F_{a(b-1), ab(n-1), \alpha}$

$$F = \frac{MSB(A)}{MSE}$$

$$\hat{\mu} = \bar{y}_{\cdots} \longrightarrow \mu$$

$$\hat{\sigma}^2 = MSE \longrightarrow \sigma^2$$

$$\hat{\sigma}_\beta^2 = \frac{1}{n}(MSB(A)) - MSE \longrightarrow \sigma_\beta^2$$

$$\hat{\sigma}_\tau^2 = \frac{1}{bn}(MSA - MSB(A)) \longrightarrow \sigma_\tau^2$$

A random B fixed

$$\tau_i \overset{iid}{\sim} N(0, \sigma_\tau^2) \qquad \sum_{j=1}^{b} \beta_{j(i)} = 0 \quad \text{for } i = 1, 2, \dots a.$$

$$E(MSA) = \sigma^2 + bn\sigma_\tau^2$$

$$E(MSB(A)) = \sigma^2 + \frac{n \sum_{i=1}^{a} \sum_{j=1}^{b} \beta_{j(i)}^2}{a(b-1)}$$

$$E(MSE) = \sigma^2$$

$H_0 : \sigma_\tau^2 = 0$

$H_1 : \sigma_\tau^2 \neq 0$

Reject H_0 if $F > F_{a-1,\, ab(n-1),\, \alpha}$

$$F = \frac{MSA}{MSE}$$

$H_0: \beta_{j(i)} = 0$ for all i, j

Reject H_0 if $F > F_{a(b-1),\, ab(n-1),\, \alpha}$

$$F = \frac{MS_{B(A)}}{MSE}$$

$\hat{\mu} = \bar{y}_{\cdots} \longrightarrow \mu$

$\hat{\sigma}^2 = MSE \longrightarrow \sigma^2$

$\hat{\sigma}_\tau = \frac{1}{bn} (MS_A - MSE)$

$\hat{\beta}_{j(i)} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} \longrightarrow \beta_{j(i)}$ for i, j

	100	125	150	
1.	572.67	1687.33	1386	1015.33
2	553	1035	1313	967
3	573.33	1054.67	886.67	838.22
	566.33	1059	1195.22	940.19.

X = temperature.

$H_0: L = \bar{\mu}_{.3} - \bar{\mu}_{.1} = 0$

$H_1: L = \bar{\mu}_{.3} - \bar{\mu}_{.1} \neq 0$

$$SS_L = \frac{(\bar{y}_{.3} - \bar{y}_{.1})^2}{\frac{1^2}{9} + \frac{1^2}{9}} = 17703467.5$$

$H_0: Q = \bar{\mu}_{.1} - 2\bar{\mu}_{.2} + \bar{\mu}_{.3} = 0$

$H_1: Q \neq 0$

$$SS_Q = \frac{(\bar{y}_{1.} - 2\bar{y}_{.2} + \bar{y}_{.3})}{\frac{1^2}{9} + \frac{2^2}{9} + \frac{1^2}{9}}$$

$SS_L + SS_Q = SS$ Temperature.

$H_0: C = \bar{\mu}_{1.} - \bar{\mu}_{2.} = 0$

$H_1: C = \bar{\mu}_{1.} - \bar{\mu}_{2.} \neq 0$

$$SS_C = \frac{(\bar{y}_{1..} - \bar{y}_{2..})^2}{\frac{1^2}{9} + \frac{1^2}{9}}$$

$H_0: D = \bar{\mu}_{1.} + \bar{\mu}_{2.} - 2\bar{\mu}_{3.} = 0$

$D = (\bar{y}_{1..} + \bar{y}_{2..} - 2\bar{y}_{3..}) = 0$

$C \times L$

$H_0: CL =$

	L: -1	0	1
-1	-1	0	1
-1	1	0	-1
0	0	0	0

$\Rightarrow H_0 : CL = -\mu_{11} + \mu_{13} + \mu_{21} - \mu_{23} = 0$

$H_1\ CL \neq 0$

$SS_C + SS_Q = SS_{Tem}$

$\Rightarrow H_1 : CL \neq 0$

$H_0 : DL = 0$

$H_1\ DL \neq 0$

$H_0\ \ DQ = 0$

$H_1\ \ DQ \neq 0$

$$F = \frac{(SS_{C \times L} + SS_{C \times Q})/2}{MSE}$$

$$SS_{CL} = \frac{(-\bar{y}_{11.} + \bar{y}_{13.} + \bar{y}_{21.} - \bar{y}_{23.})^2}{\frac{1}{n}^2 + \frac{1}{n}^2 + \frac{1}{n}^2 + \frac{1}{n}^2}$$

$$= \frac{\left(\sum_{i=1}^{a} \sum_{j=1}^{b} c_{ij} \bar{y}_{ij.}\right)^2}{\sum_{i=1}^{a} \sum_{j=1}^{b} c_j^2 \frac{1}{n}}$$

$$\Rightarrow \frac{(SS_{CL} + SS_{DL} + SS_{CQ} + SS_{DQ})/4}{MSE}$$

$\rightarrow$ any interaction