Math 4B: Differential Equations

Lecture 03: Integrating Factors

- Solving First-order Linear ODEs,
- Some Background ODEs,
- General Cleverness & More!

Today's Goal

Remember that a first-order linear differential equation is one that looks like

$$a_1(t)y' + a_0(t)y = g(t).$$

We can write this in a ***standard form***

$$\boxed{y' + p(t)y = q(t)}\,.$$

Our **goal today** is to learn how to solve this kind of ODE.

Some Background

Solution to a Familiar ODE

Every solution to $y' = ky$ is of the form $y = Ce^{kt}$.

Note:

- We should be able to check that $y = Ce^{kt}$ solves $y' = ky$.
- Next time we'll show that every solution to $y' = ky$ is of this form.

Solution to a Less Familiar ODE

Every solution to $y' = p(t)y$ is of the form

$$y = C\exp\left(\int p(t)\,dt\right) = Ce^{\int p(t)\,dt}.$$

Finding a Solution

1. Suppose

$$t^4 y' + 4t^3 y = \sin(t).$$

Can find a solution y?

Clever students: the left-hand side is the derivative of $t^4 y$:

$$\frac{d}{dt}\big(t^4 y\big) = t^4 y' + 4t^3 y.$$

This means

$$\frac{d}{dt}\big(t^4 y\big) = \sin(t) \quad \Longrightarrow \quad t^4 y = \int \sin(t)\,dt = -\cos(t) + C.$$

Thus $\boxed{y(t) = -\frac{\cos(t)}{t^4} + \frac{C}{t^4}}$.

More Solutions

2. Suppose

$$y' + \frac{4}{t}y = \frac{\sin(t)}{t^4}.$$

Can find a solution y?

Clever students: If we multiply through by t^4, we get the same equation as **1.** :

$$t^4y' + 4t^3y = \sin(t).$$

Even more clever students: Can we do this for a more general equation? Perhaps the general first-order linear ODE

$$y' + p(t)y = q(t)?$$

Integrating Factors: Case #1

Suppose rather than the general

$$y' + p(t)y = q(t)$$

we have

$$y' + ay = 0$$

where "a" is a constant. Aside: This is $y' = ky \implies y = Ce^{kt}$.

Goal: Multiply this equation by an "integrating factor" $\mu(t)$ so that the left-hand side is $\dfrac{d}{dt}\big(\mu(t) \cdot y\big)$. We have two formulations of our left-hand side:

1. $\mu(t)$ times our original ODE: $\mu(t)y' + a\mu(t)y = 0$

2. The derivative of $\mu(t) \cdot y$: $\dfrac{d}{dt}\big(\mu(t) \cdot y\big) = \mu(t)y' + \mu'(t)y$.

Comparing these, we see we want $\mu'(t) = a\mu(t)$.

Case #1 (continued)

Remember we're trying to solve

$$y' + ay = 0$$

where "a" is a constant. We're multiplying this by $\mu(t)$, which we want to satisfy

$$\mu'(t) = a\mu(t).$$

The general solution is $\mu(t) = Ce^{at}$ – we'll take $\mu(t) = e^{at}$:

$$e^{at}\big(y' + ay = 0\big) \qquad \Longrightarrow \qquad e^{at}y' + ae^{at}y = 0.$$

The **key fact** is that the left-hand side is a derivative:

$$e^{at}y' + ae^{at}y = \frac{d}{dt}\big(e^{at}y\big) = \frac{d}{dt}\big(\mu(t)y\big).$$

So $$\frac{d}{dt}\big(e^{at}y\big) = 0 \qquad \Longrightarrow \qquad e^{at}y = C.$$

Thus the solution is $\boxed{y = Ce^{-at}}$.

Integrating Factors: Case #2

Suppose rather than the general

$$y' + p(t)y = q(t)$$

we have

$$y' + ay = t$$

where "a" is a constant.

Note: Same left-hand side as the previous example, so $\mu(t) = e^{at}$ again.

The **key fact** is again that the left-hand side is a derivative:

$$e^{at}y' + ae^{at}y = \frac{d}{dt}\big(e^{at}y\big) = \frac{d}{dt}\big(\mu(t)y\big).$$

So $$\frac{d}{dt}\big(e^{at}y\big) = te^{at} \qquad \Longrightarrow \qquad e^{at}y = \int te^{at}\,dt = \frac{1}{a}te^{at} - \frac{1}{a^2}e^{at} + C.$$

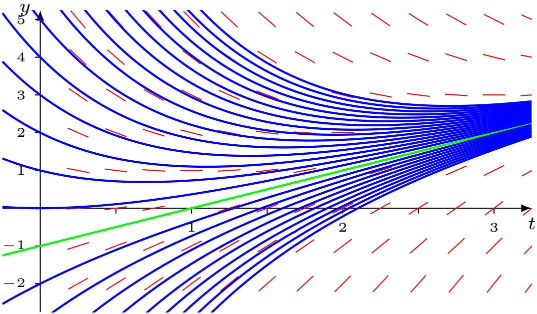
Thus the solution is $\boxed{y = \frac{t}{a} - \frac{1}{a^2} + Ce^{-at}}$.

Case #2 Concluded

We saw the solutions of

$$\frac{dy}{dt} + ay = t \qquad \text{were} \qquad y(t) = \frac{t}{a} - \frac{1}{a^2} + Ce^{-at}.$$

Here's a picture of direction fields and solutions when $a = 1$:



You Try It!

3. Find the general solution to

$$\frac{dy}{dt} - 7y = 3e^{6t}.$$

Hint: What is $\mu(t)$?

Solution: $a = -7$, so $\mu(t) = e^{-7t}$. Multiplying, the equation becomes

$$e^{-7t}\left(\frac{dy}{dt} - 7y = 3e^{6t}\right) \implies e^{-7t}\frac{dy}{dt} - 7e^{-7t}y = 3e^{-t}$$
$$\implies \frac{d}{dt}\left(e^{-7t}y\right) = 3e^{-t}.$$

Thus

$$e^{-7t}y = -3e^{-t} + C \implies \boxed{y = -3e^{6t} + Ce^{7t}}.$$

Integrating Factors: General Case

Our plan: Let's return to finding solutions to

$$y' + p(t)y = q(t)$$

using an "integrating factor" $\mu(t)$.

Goal: Multiply this equation by an "integrating factor" $\mu(t)$ so that the left-hand side is $\frac{d}{dt}\big(\mu(t) \cdot y\big)$. We have two formulations of our left-hand side:

1. $\mu(t)$ times our original ODE: $\mu(t)y' + \mu(t)p(t)y = \mu(t)q(t)$

2. The derivative of $\mu(t) \cdot y$: $\frac{d}{dt}\big(\mu(t) \cdot y\big) = \mu(t)y' + \mu'(t)y$.

Comparing these, we see we want $\mu'(t) = \mu(t)p(t)$.

General Case (continued)

Remember we're trying to solve

$$y' + p(t)y = q(t).$$

We're multiplying this by $\mu(t)$, which we want to satisfy

$$\mu'(t) = \mu(t)p(t).$$

The general solution is $\mu(t) = Ce^{\int p(t)\,dt}$ – we'll take $\mu(t) = e^{\int p(t)\,dt}$:

$$e^{\int p(t)\,dt}\big(y' + p(t)y = q(t)\big) \quad\Longrightarrow\quad e^{\int p(t)\,dt}y' + p(t)e^{\int p(t)\,dt}y = q(t)e^{\int p(t)\,dt}.$$

The **key fact** is that the left-hand side is a derivative:

$$e^{\int p(t)\,dt}y' + p(t)e^{\int p(t)\,dt}y = \frac{d}{dt}\big(e^{\int p(t)\,dt}y\big) = \frac{d}{dt}\big(\mu(t)y\big).$$

So $\quad \frac{d}{dt}\big(e^{\int p(t)\,dt}y\big) = q(t)e^{\int p(t)\,dt} \quad\Longrightarrow\quad e^{\int p(t)\,dt}y = \int q(t)e^{\int p(t)\,dt}\,dt.$

Thus the solution is $\boxed{y = e^{-\int p(t)\,dt}\int q(t)e^{\int p(t)\,dt}\,dt}$.

General Formulation

So we can remember either

1. **The process**: We multiply $y' + p(t)y = q(t)$ by $\mu(t)$ chosen so that
$$\mu(t)y' + \mu(t)p(t)y = \frac{d}{dt}\left(\mu(t)y\right),$$
which is $\mu(t) = \exp\left(\int p(t)\,dt\right)$. Then solve
$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)q(t).$$

2. **The formula**: The solution to $y' + p(t)y = q(t)$ is
$$y = e^{-\int p(t)\,dt} \int q(t)e^{\int p(t)\,dt}\,dt.$$

General Examples

4. Solve the initial value problem

$$\begin{cases} (1+t^2)y' = 2t(y+1) \\ y(0) = 6 \end{cases}$$

Hint: What is $p(t)$? What is $\mu(t)$?

Solution: We write this as $y' - \frac{2t}{1+t^2}y = \frac{2t}{1+t^2}$, so $p(t) = -2t/(1+t^2)$. Thus

$$\mu(t) = \exp\left(\int \frac{-2t}{1+t^2}\,dt\right) = \exp\left(-\ln(1+t^2)\right) = \frac{1}{1+t^2}.$$

So

$$\frac{1}{1+t^2}\left(y' - \frac{2t}{1+t^2}y = \frac{2t}{1+t^2}\right)$$

becomes

$$\frac{d}{dt}\left(\frac{1}{1+t^2}\,y\right) = \frac{2t}{(1+t^2)^2}.$$

Example 4 (concluded)

Solving

$$\begin{cases} (1+t^2)y' = 2t(y+1) \\ y(0) = 6 \end{cases}$$

we found

$$\frac{d}{dt}\left(\frac{1}{1+t^2}\,y\right) = \frac{2t}{(1+t^2)^2}.$$

Integrating, we get

$$\frac{1}{1+t^2}\,y = -\frac{1}{1+t^2} + C \qquad \text{and so} \qquad y = -1 + C(1+t^2).$$

$y(0) = 6 \implies C = 7$ and so $y = -1 + 7(1+t^2)$ or $\boxed{y = 7t^2 + 6}$.

Check:

- $(1+t^2)y' = (1+t^2)\cdot 14t = 14t^3 + 14t.$
- $2t(y+1) = 2t(7t^2+6+1) = 14t^3 + 14t$
- $y(0) = 7(0)^2 + 6 = 6$

Another General Example

5. Solve the initial value problem

$$\begin{cases} ty' + 2y = 8t^2 \\ y(1) = 0 \end{cases}$$

Solution: We write this as $y' + \frac{2}{t}y = 8t$, so $p(t) = 2/t$. Thus

$$\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = \exp\left(2\ln(t)\right) = t^2$$

So

$$t^2\left(y' + \frac{2}{t}y = 8t\right) \qquad \text{or} \qquad t^2y' + 2ty = 8t^3$$

becomes

$$\frac{d}{dt}\left(t^2\,y\right) = 8t^3$$

General Example (concluded)

Solving

$$\begin{cases} ty' + 2y = 8t^2 \\ y(1) = 0 \end{cases}$$

we found

$$\frac{d}{dt}\left(t^2\, y\right) = 8t^3$$

Integrating, we get

$$t^2\, y = 2t^4 + C \qquad \text{and so} \qquad y = 2t^2 + \frac{C}{t^2}.$$

$y(1) = 0 \implies C = -2$ and so $\boxed{y = 2t^2 - \frac{2}{t^2}}$.

Check:

- $ty' + 2y = t\left(4t + \dfrac{4}{t^3}\right) + 2\left(2t^2 - \dfrac{2}{t^2}\right) = 8t^2.$
- $y(1) = 2 \cdot 1^2 - \frac{2}{1^2} = 0$