Lecture 6: 08/11/22

Proof of Prop. 3.8:

Without loss of generality: short hedge

Hedger's effective cashflow received from selling asset S at time t_2:

$$N_A \cdot S_{t_2} + N_F (F_{t_1} - F_{t_2}) = N_A S_{t_1} + N_A (\underbrace{S_{t_2} - S_{t_1}}_{=\Delta S}) + N_F (\underbrace{F_{t_1} - F_{t_2}}_{=-\Delta F})$$

$$= N_A S_{t_1} + N_A \Big(\Delta S - \underbrace{\frac{N_F}{N_A}}_{=h} \Delta F\Big) \qquad (*)$$

At time t_1: Choose h s.t. variance of $(*)$ is minimized

Specifically: Minimize

$$\mathrm{Var}(\Delta S - h \Delta F) = f(h)$$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \in [-1,1]$$

Compute

$$f(h) = \mathrm{Var}(\Delta S - h\Delta F) = \underbrace{\mathrm{Var}(\Delta S)}_{=\sigma_S^2} + \underbrace{\mathrm{Var}(-h\Delta F)}_{=h^2 \underbrace{\mathrm{Var}(\Delta F)}_{=\sigma_F^2}} + 2\underbrace{\mathrm{Cov}(\Delta S, -h\Delta F)}_{=-2h\underbrace{\mathrm{Cov}(\Delta S, \Delta F)}_{=\rho\sigma_F\sigma_S}}$$

$$= \sigma_S^2 + h^2\sigma_F^2 - 2h\rho\sigma_S\sigma_F$$

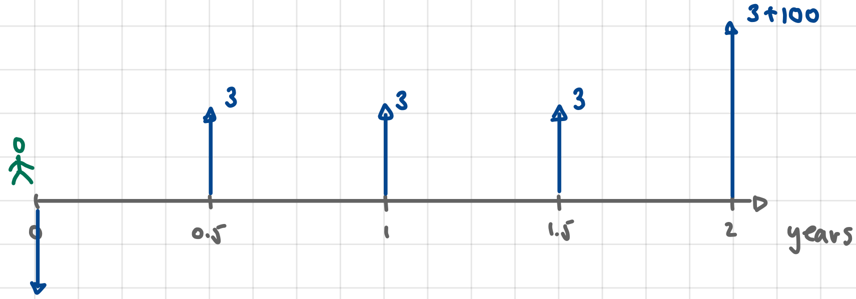
First order opt. cond.: $f'(h) = 2h\sigma_F^2 - 2\rho\sigma_S\sigma_F = 0 \iff h = \rho\frac{\sigma_S}{\sigma_F}$

Second order opt. cond.: $f''(h) = 2\sigma_F^2 > 0$

Minimum variance hedge ratio: $h^* = \rho\frac{\sigma_S}{\sigma_F}$

Example 4.5:

2-year bond: principal \$100, coupons at 6% p.a. (semiannually paid)



$$
\begin{aligned}
98.39 &= 3e^{-0.05\cdot\frac{1}{2}} && \text{5\% p.a.}\\
&+ 3e^{-0.058\cdot 1} && \text{5.8\% p.a.}\\
&+ 3e^{-0.064\cdot\frac{3}{2}} && \text{6.4\% p.a.}\\
&+ 103e^{-0.068\cdot 2} && \text{6.8\% p.a.}
\end{aligned}
$$