ASSIGNMENT 1

PSTAT 160A - FALL 2021
DUE DATE: TUESDAY, OCTOBER 5

The first part of this assignment sheet contains exercises for the sections, which will not be turned in. The second part consists of homework problems which have to be turned in on the due date.

Instructions for the homework: Please write your **full name**, **perm number**, **your TA's name**, and **your section's time and date** legibly on the front page of your homework assignment. Solve all of the homework problems. Your reasoning has to be comprehensible and complete. To receive full credit, sufficient explanations need to be provided. **Four** of the homework problems will be graded for correctness. The other homework problems will be graded for completion.

Section Exercises

Exercise 1.1. Suppose X and Y are two discrete random variables with joint probability mass function $p_{X,Y}$ whose values $p_{X,Y}(x,y)$ are given in the table below:

x \ y	0	1	2
1	0	1/9	0
2	1/3	2/9	1/9
3	0	1/9	1/9

(a) Find the marginal probability mass functions of X and Y. Are X and Y independent?
(b) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Cov}(X,Y)$.
(c) Find the conditional probability mass function of X given $Y=y$ for all $y \in \{0,1,2\}$. That is, find $p_{X|Y=y}(x|y)$ for each $(x,y) \in \{1,2,3\} \times \{0,1,2\}$.

Exercise 1.2. Suppose X and Y are jointly continuous random variables with joint probability density function $f_{X,Y}$ given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{12}{7}(xy+y^2), & x,y \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

(a) Verify that f is in fact a probability density function.
(b) Find the marginal density functions of X and Y. Are X and Y independent?
(c) Calculate $\mathbb{P}(X < Y)$.
(d) Calculate $\mathbb{E}[X^2 Y^{1/3}]$.
(e) Find the conditional density function of X given $Y=y$.

Exercise 1.3. Suppose X and Y are jointly continuous random variables with joint probability density function $f_{X,Y}$ given by

$$f_{X,Y}(x,y) = \begin{cases} 2xe^{x^2-y}, & 0 \le x \le 1 \text{ and } y \ge x^2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the conditional density function of Y given $X = x$.
(b) Calculate $\mathbb{P}(Y \ge 1/4 | X = x)$.
(c) Very that $\mathbb{P}(Y \ge 1/4) = \int_{-\infty}^{\infty} \mathbb{P}(Y \ge 1/4 | X = x) f_X(x) dx$.

Homework Problems

Problem 1.1. (10 points) Suppose X and Y are discrete random variables with joint probability mass function $p_{X,Y}$ whose values $p_{X,Y}(x,y)$ are given in the table below:

x \ y	0	1
1	1/4	0
2	0	1/2
3	1/6	1/12

(a) (5 points) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Cov}(X,Y)$.
(b) (5 points) Let $Z \doteq (X - Y)/2$. Compute $\mathbb{E}[Z]$, $\mathrm{Var}(Z)$, and $\mathrm{Cov}(X,Z)$.

Problem 1.2. (10 points) Suppose that X and Y are jointly continuous with joint probability density function

$$f_{X,Y}(x,y) \doteq \begin{cases} xe^{-x(1+y)}, & x, y \in \mathbb{R}_+ \\ 0, & \text{otherwise.} \end{cases}$$

(a) (3 points) Find the marginal density functions of X and Y.
(b) (3 points) Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, and $\mathbb{E}(XY)$.
(c) (4 points) Find the conditional density function of X given $Y = y$.

Problem 1.3. (10 points) Let X have the density function

$$f_X(x) \doteq \begin{cases} 2x, & x \in [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

and let $Y \sim \mathrm{Unif}(1,2)$ be independent of X.

(a) (5 points) Give the joint density function $f_{X,Y}$ of (X,Y).
(b) (5 points) Calculate $\mathbb{P}(Y - X \ge 1/2)$.

Problem 1.4. (10 points) Let X and Y be jointly continuous with the join probability density function

$$f_{X,Y}(x,y) \doteq \begin{cases} c(x+3y), & x,y \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 points) Find the value of c to make $f_{X,Y}$ a proper probability density function.
(b) (3 points) Find the marginal density functions of X and Y.
(c) (2 points) Are X and Y independent?
(d) (3 points) Find the conditional density function of Y given $X = x$.

Problem 1.5. (10 points) For an event A in our sample space Ω, the indicator function of A is the function $1_A : \Omega \to \{0,1\}$ defined as

$$1_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \in A^c. \end{cases}$$

Observe that 1_A is in fact a very simple random variable; it takes on a value of 1 if the event A occurs, and it takes on a value of 0 if the event A does not occur.

(a) (2 points) Let A be an event. Show that $1_A = 1_A^2$, namely for each $\omega \in \Omega$, show that

$$1_A(\omega) = (1_A(\omega))^2.$$

(b) (2 points) Show that for an event A, $\mathbb{E}[1_A] = \mathbb{P}(A)$.
(c) (2 points) Show that for an event A, $\mathrm{Var}(1_A) = \mathbb{P}(A)(1 - \mathbb{P}(A))$.
(d) (2 points) Let $\{A_n\}_{n=1}^{\infty}$ be a mutually disjoint collection of events, so that $A_i \cap A_j = \emptyset$ whenever $i \neq j$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Show that

$$1_A = \sum_{n=1}^{\infty} 1_{A_n}.$$

(e) (2 points) Let X be a non-negative random variable. Using indicator functions, prove the following inequality: for each $\epsilon > 0$,

$$\mathbb{P}(X \geq \epsilon) \leq \frac{\mathbb{E}[X]}{\epsilon}.$$

This inequality is known as Markov's inequality and is a fundamental example of a concentration inequality. [1]

[1]**Hint:** begin by showing that for each $\epsilon > 0$, $\epsilon 1_{\{X \geq \epsilon\}} \leq X$.