

The GEModelTools Package for Solving HANK Models in Python

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Abstract

This note provides an overview of the GEModelTools package for solving general equilibrium models easily in Python. To learn how to use the package follow these steps:

1. Read this document
2. Install the package
3. Run the example notebooks
4. Read the commented code for the example notebooks
5. Implement your own model

Structure:

Section 1 gives an overview

Section 2 go through a specific example

Code:

Package: github.com/JeppeDrue Dahl/GEModelTools

Notebooks: github.com/JeppeDrue Dahl/GEModelToolsNotebooks

Requirements: Rely on EconModel and ConSav.

Packages:

github.com/NumEconCopenhagen/EconModel

github.com/NumEconCopenhagen/ConsumptionSaving

Notebooks:

github.com/NumEconCopenhagen/EconModelNotebooks

github.com/NumEconCopenhagen/ConsumptionSavingNotebooks

1 Overview

The central tool in `GEModelTools` is the `GEModelClass`, which is an add-on to the basic `EconModelClass` whose functionality is explained [here](#). An example of the setup is shown in Listing 1. The `.settings()`, `.setup()` and `.allocate()` methods are called automatically when the model is created.

A model of the `GEModelClass` consists of the following list of namespaces:

1. **Parameters:** `.par`
2. **Solution:** `.sol`
3. **Simulation:** `.sim`
4. **Steady state:** `.ss`
5. **Transition path:** `.path`
6. **Jacobians for household problem:** `.jac_hh`
7. **Full Jacobians:** `.jac`

The user is required to specify some **variable lists** in `.settings()` for:

1. **Aggregate variables:** `.varlist`. Used as `path.VARNAME`.
2. **Household variables:** `.varlist_hh`
Used as `sol.VARNAME` and `sol.path_VARNAME`
Extra I: `i` and `w` are used for saving simulation indices and weights.
Extra II: `path_i` and `path_w` are used for saving simulation indices and weights.
3. **Household grids:** `.grid_hh`. Used as `par.VARNAME_grid`.
4. **Household policy functions:** `.pols_hh`. Should be in `.varlist_hh`.
5. **Household inputs:** `.inputs_hh`. Should be in `.varlist`.
6. **Household outputs:** `.outputs_hh`. Should be in `.varlist_hh`.
7. **Exogenous inputs:** `.inputs_exo`. Should be in `.varlist`.
8. **Endogenous inputs:** `.inputs_endo`. Should be in `.varlist`.
9. **Targets:** `.targets`. Should be in `.varlist`.

The user must choose the following **settings** in `.setup()` :

1. **Number of exogenous states:** `par.Nz`
2. **Number of grid points:** `par.Nendo1`, `par.Nendo2`, ...
where `endo1`, `endo2`, ..., is in `.grids_hh`
3. **Length of transition period:** `par.transition_T`

4. **For each exogenous input:**

Initial jump: `par.jump_VARNAME`

Persistence: `par.rho_VARNAME`

5. **Optional solver settings:**

`par.max_iter_solve`, `par.max_iter_simulate`, `par.max_iter_broyden`

`par.tol_solve`, `par.tol_simulate`, `par.tol_broyden`

In the `.allocate()` method the internal **method** `.allocate_GE(sol_shape)` can now be called to allocate:

1. **Exogenous grids and transition matrices:**

`par.z_grid_ss`, `shape=(par.Nz,)`

`par.z_trans_ss`, `shape=(par.Nz,)`

`par.z_ergodic_ss`, `shape=(par.Nz,)`

`par.z_grid_path`, `shape=(par.transition_T,par.Nz)`

`par.z_trans_path`, `shape=(par.transition_T,par.Nz,par.Nz)`

2. **Distribution:**

`sim.D`, `shape=sol_shape`

`sim.path_D`, `shape=(par.transition_T,*sol_shape)`

3. **All variables in .sol**

`sol.VARNAME`, `shape=sol_shape`

`sol.path_VARNAME`, `shape=(par.transition_T,*sol_shape)`

4. **All variables in .path**

`path.VARNAME`,

`shape=(par.transition_T,len(inputs_endo)×par.transition_T)`

`ss.VARNAME`, scalar

Note on use:

(a) Normally: `path.VARNAME[0,t]` is the value in period t .

`path.VARNAME[i,t]` for $i > 0$ is *undefined behavior*.

(b) In `.compute_jac()`: `path.VARNAME[i×par.transition_T+t,t]` is the value in period t when shocking i in period t

Value in period t : `path.VARNAME[0,t]`

1. **All variables in .jac_hh**

`jac_hh.OUTPUTNAME.upper()_INPUTNAME`,

`shape=(par.transition_T,par.transition_T)`

Finally, the user must also provide the following **functions**:

1. `grids.py` must contain `create_grids(model)`, which at a minimum creates the grids for the endogenous variables and the grids and transition matrices for the exogenous variable. This is called each time we solve for the steady state of the household problem.
2. `household_problem.py` must contain the jitted¹ functions:
`solve_hh_ss(par, sol, ss)`, find `sol.VARNAME` for all variables in `.outputs_hh`.
`solve_hh_path(par, sol, path)`, find `sol.path_VARNAME` for all variables in `.outputs_hh`.
3. `find_strady_state.py` must contain the function `find_ss(model, do_print)`, which fills `ss`, and solve and simulate the household problem in steady state.
4. `transition_path.py` must contain the jitted function
`evaluate_path(par, sol, sim, ss, path, jac_hh, threads, use_jac_hh)`, where
 - (a) `use_jac_hh` is a boolean for whether or not to use the household Jacobians when evaluating household behavior (used when calculating the full Jacobian).
 - (b) `threads` is 1 or `len(inputs_endo) × par.transntion_T`, where the latter is used to evaluate the path in parallel for all one step changes.
 - (c) If `threads > 1` then `use_jac_hh` should be `True`.

The following internal methods are now available:

1. `.solve_ss()`: Solve household problem at steady state,
find `sol.VARNAME`.
2. `.simulate_ss()`: Simulate household problem at steady state,
find `sim.D`.
3. `.solve_path()`: Solve household problem along transition path,
find `sol.path_VARNAME`
4. `.simulate_path()`: Simulate household problem along transition path,
find `sim.path_D`.
5. `.compute_jac_hh()`: Compute the Jacobians of household problem,
find `jac_hh`.
6. `.compute_jac()`: Compute the full Jacobian,
find `jac`.

¹ The function should be decorated with `@numba.njit`.

7. `.find_transition_path()`: Find transition for path for exogenous inputs, everything in `path`.

To solve and simulate household behavior along the transition path in a jitted function use:

```
household_problem.solve_hh_path(par, sol, path)
GEModelTools.prepare_simulation_1d_1d(par, sol, sol.
    path_endo1, par.endo1_grid)
GEModelTools.simulate_hh_path(par, sol, sim)
```

```

1
2 from EconModel import EconModelClass
3 from GEModelTools import GEModelClass
4
5 class MyModelClass(EconModelClass, GEModelClass):
6
7     def settings(self):
8         """ fundamental settings """
9
10        self.grids_hh = [] # grids
11        self.pols_hh = [] # policy functions
12        self.inputs_hh = [] # inputs to household problem
13        self.outputs_hh = [] # output of household problem
14        self.varlist_hh = [] # variables in household problem
15        self.inputs_exo = [] # exogenous inputs
16        self.inputs_endo = [] # endogenous inputs
17        self.targets = [] # targets
18        self.varlist = [] # all variables
19
20    def setup(self):
21        """ set baseline parameters """
22
23        par = self.par
24        par.NVARNAME = 100 # number of grid points
25        par.jump_VARNAME = -0.01 # initial jump in %
26        par.rho_VARNAME = 0.8 # AR(1) coefficient
27        par.transition_T = 500 # length of path
28
29    def allocate(self):
30        """ allocate model """
31
32        par = self.par
33        sol_shape = (par.Nfix, par.Nz, par.Nendo1)
34        self.allocate_GE(sol_shape)
35

```

Listing 1: Example: Setup

2 Solution method

In this section, we explain the non-linear sequence space solution method implemented in the package for a simple model.

2.1 Model

Overview. There is a continuum of measure one households who

1. Own stocks, a_{t-1} (measured end-of-period)
2. Supply labor with productivity z_t (exogenous and stochastic)

$$z_t = \rho z_{t-1} + \varepsilon_t^z. \quad (1)$$

$$\mathbb{E}[z_t] = 1.$$

$$\text{Var}[\varepsilon_t^z] = \sigma_z^2.$$

3. Consume, c_t

Firms rent capital, K_{t-1} , and hire labor, L_t to produce goods,

$$Y_t = Z_t K_{t-1}^\alpha L_t^{1-\alpha}, \quad (2)$$

where Z_t is technology. Capital depreciates with the rate δ .

Both households and firms are **price takers** and

1. r_t^k is the (real) rental rate for capital
2. $r_t = r_t^k - \delta$ is the implied (real) interest rate
3. w_t is the (real) wage rate

Firms. Firms maximize profits implying the standard pricing equations

$$r_t^k = \alpha Z_t (K_{t-1}/L_t)^{\alpha-1} \equiv r^k(Z_t, K_{t-1}, L_t) \quad (3)$$

$$\begin{aligned} w_t &= (1 - \alpha) Z_t (K_{t-1}/L_t)^\alpha \\ &= (1 - \alpha) Z_t \left(\frac{r_t^k}{\alpha Z_t} \right)^{\frac{\alpha}{\alpha-1}} \equiv w(r_t^k, Z_t) \end{aligned} \quad (4)$$

Households. Households have *perfect foresight* wrt. to the interest rate and the wage rate, $\{r_t, w_t\}_{t=0}^\infty$, and solve the problem

$$\begin{aligned} V_t(z_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t[V_{t+1}(z_{t+1}, a_t)] \\ \text{s.t.} \\ a_t + c_t &= (1 + r_t)a_{t-1} + w_t z_t \\ z_{t+1} &\sim \Gamma_z(z_t) \\ a_t &\geq 0, \end{aligned}$$

where

$$V_t(z_t, a_{t-1}) = V(z_t, a_{t-1}; \{r_\tau, w_\tau\}_{\tau=t}^\infty).$$

The FOC is $c_t^{-\sigma} = \beta \mathbb{E}_t[v_{a,t+1}]$ and the envelope condition is $v_{a,t} = (1 + r_t)c_t^{-\sigma}$. The optimal saving and consumption functions $a_t^*(a_{t-1}, z_t)$ and $c_t^*(a_{t-1}, z_t)$ can be found using e.g. the EGM. These solutions live on the discretized grids, $z_t \in \{z^0, \dots, z^{\#z-1}\}$ and $a_t \in \{a^0, \dots, a^{\#a-1}\}$. In `grids.py` the user must supply the function `create_grids(model)` to setup these grids.

The user should provide two jitted functions in `household_problem.py`:

1. `solve_hh_ss(par, sol, ss)`: Solve for $a_{ss}^*(\bullet)$ and $c_{ss}^*(\bullet)$ in `sol.a` and `sol.c`.
2. `solve_hh_path(par, sol, path)`: Solve for $\{a_t^*(\bullet)\}_{t=0}^{T-1}$ and $\{c_t^*(\bullet)\}_{t=0}^{T-1}$ for arbitrary sequences $\{r_t, w_t\}_{t=0}^T$, where $a_T^*(\bullet) = a_{ss}^*(\bullet)$ and $c_T^*(\bullet) = c_{ss}^*(\bullet)$, in `sol.path_a` and `sol.path_c`.

Distribution. Let D_t be the distribution of households over z_t and a_{t-1} . The supply of capital then is

$$\mathcal{K}_t = \int a_t^*(a_{t-1}, z_t) dD_t = \int a_t dD_{t+1} \quad (5)$$

The household problem implies a time-varying but non-stochastic law of motion for D_t denoted $\Gamma_{t,D}$.

In practice, this simulation problem is generic. It is beneficial to use a simulation method, where households are always on the grid. The idea here is to re-distribute

mass to grid points based on optimal decision. More precisely we calculate

$$D_{t+1}(e^k, a^l) = \sum_{i=0}^{\#e-1} \Pr[e^k | e^i] \sum_{j=0}^{\#a-1} D_t(e^i, a^j) \omega(a_t^*(e^i, a^j), a^{\max\{l-1, 0\}}, a^l, a^{\min\{l+1, \#a-1\}}), \quad (6)$$

where ω is a weight calculated using linear interpolation

$$\omega(a, \underline{a}, \tilde{a}, \bar{a}) = 1\{a \in [\underline{a}, \bar{a}]\} \begin{cases} \frac{\bar{a}-a}{\bar{a}-\tilde{a}} & \text{if } a \geq \tilde{a} \\ \frac{a-\underline{a}}{\tilde{a}-\underline{a}} & \text{if } a < \tilde{a} \end{cases}.$$

Extension to higher dimensions are straightforward. This is provided in the package as the methods `.simulate_ss()` and `.simulate_path()`.

Market clearing. Market clearing requires

$$\begin{aligned} \text{Capital: } K_t &= \mathcal{K}_t = \int a_t dD_{t+1} = \int a_t^*(z_t, a_{t-1}) dD_t \\ \text{Labour: } L_t &= \int e_t dD_t = 1 \\ \text{Goods: } Y_t &= \int c_t^*(z_t, a_{t-1}) dD_t + K_t - K_{t-1} + \delta K_{t-1} \end{aligned}$$

2.2 Stationary equilibrium

A **stationary equilibrium** for a given Z_{ss} is one where

1. Quantities K_{ss} and L_{ss} ,
2. prices r_{ss} and w_{ss} ,
3. a distribution D_{ss} over a_{t-1} and z_t
4. and policy functions $a_{ss}^*(z_t, a_{t-1})$ and $c_{ss}^*(z_t, a_{t-1})$

are such that

1. $a_{ss}^*(\bullet)$ and $c_{ss}^*(\bullet)$ solves the household problem with $\{r_{ss}, w_{ss}\}_{t=0}^{\infty}$
2. D_{ss} is the invariant distribution implied by the household problem
3. Firms maximize profits, $r_{ss} = r(Z_{ss}, K_{ss}, L_{ss})$ and $w_{ss} = w(r_{ss}, Z_{ss})$
4. The labor market clears, i.e. $L_{ss} = \int e_t dD_{ss} = 1$

5. The capital market clears, i.e. $K_{ss} = \int a_{ss}^*(z_t, a_{t-1}) dD_{ss}$
6. The goods market clears, i.e. $Y_{ss} = \int c_{ss}^*(z_t, a_{t-1}) dD_{ss} + \delta K_{ss}$

We can **find the stationary equilibrium** by solving a root-finding problem

1. Guess on r_{ss}
2. Calculate $w_{ss} = w(r_{ss}, Z_{ss})$
3. Solve the infinite horizon household problem
4. Simulate until convergence of D_{ss}
5. Calculate supply $\mathcal{K}_{ss} = \int a_{ss}^*(z_t, a_{t-1}) dD_{ss}$
6. Calculate demand $K_{ss} = \left(\frac{r_{ss} + \delta}{\alpha Z_{ss}} \right)^{\frac{1}{\alpha-1}} L_{ss}$
7. If for some tolerance ϵ

$$|\mathcal{K}_{ss} - K_{ss}| < \epsilon$$

then stop, otherwise update r_{ss} appropriately and return to step 2

In `find_strady_state.py` the user must supply the function `find_ss(model, do_print)` to solve the problem. In practice we guess on r_{ss} and w_{ss} and derive Z_{ss} and δ_{ss} from the implied household problem.

2.3 Transition path

A **transition path** for $t \in \{0, 1, 2, \dots\}$, given an initial distribution D_0 and a path of Z_t , is paths of quantities K_t and L_t , prices r_t and w_t , policy functions $a_t^*(\bullet)$ and $c_t^*(\bullet)$, distributions D_t , such that for all t

1. $a_t^*(\bullet)$ and $c_t^*(\bullet)$ solve the household problem given price paths
2. D_t are implied by the household problem given price paths and D_0
3. Firms maximizes profit, $r_t = r(Z_t, K_{t-1}, L_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int z_t dD_t = 1$
5. The capital market clears, i.e. $K_{t-1} = \int a_{t-1} dD_t$
6. The goods market clears, i.e. $Y_t = \int c_t^*(\bullet) dD_t + K_t - K_{t-1} + \delta K_{t-1}$

This is also called an MIT-shock \equiv »shock in a world without shocks«.

In practice we consider a *truncated* transition path of length T , and everything is back to steady state afterwards.

2.4 Sequence space method

We can think of the model in terms of inputs are targets:

1. **1 exogenous input:** $\{Z_t\}_{t=0}^{T-1}$
2. **1 endogenous input:** $\{K_t\}_{t=0}^{T-1}$
3. **1 target:** Asset market clearing

The model is then captured by the equation system

$$\begin{aligned}
 \mathbf{H}(\{K_t, Z_t\}_{t=0}^T) &= \mathbf{0} \Leftrightarrow \\
 \left[\begin{array}{c} \text{Asset market clearing} \end{array} \right] &= \mathbf{0} \\
 \left[\begin{array}{c} K_t - \mathcal{K}_t \end{array} \right] &= \left[\begin{array}{c} 0 \end{array} \right] \\
 \forall t \in \{0, 1, \dots, T-1\}
 \end{aligned}$$

where we have

$$\begin{aligned}
 L_t &= 1 \\
 r_t &= \alpha Z_t (K_{t-1}/L_t)^{\alpha-1} \\
 w_t &= (1-\alpha) Z_t \left(\frac{r_t + \delta}{\alpha Z_t} \right)^{\frac{\alpha}{\alpha-1}} \\
 D_t &= \Gamma_{t-1,D}(D_{t-1}), \forall t > 0 \\
 \mathcal{K}_t &= \int a_t^*(z_t, a_{t-1}) dD_t \\
 K_{-1} &= K_{ss} \\
 D_0 &= D_{ss}
 \end{aligned}$$

In `evaluate_transition_path.py` the user must supply the jitted function `evaluate_transition_path_distribution(...)`, which given the inputs updates the value of all targets.

Jacobian. Defining $\mathbf{K} = (K_0, K_1, \dots)$ and $\mathbf{Z} = (Z_0, Z_1, \dots)$ we can write the equation system in time-stacked form

$$\mathbf{H}(\mathbf{K}, \mathbf{Z}) = \mathbf{0}$$

Total differentiation implies

$$\mathbf{H}_K d\mathbf{K} + \mathbf{H}_Z d\mathbf{Z} = 0 \Leftrightarrow d\mathbf{K} = -\mathbf{H}_K^{-1} \mathbf{H}_Z d\mathbf{Z}$$

where

$$\mathbf{H}_K = \begin{bmatrix} \frac{\partial H_0}{\partial K_0} & \frac{\partial H_0}{\partial K_1} & \dots \\ \frac{\partial H_1}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \mathbf{H}_Z = \begin{bmatrix} \frac{\partial H_0}{\partial Z_0} & \frac{\partial H_0}{\partial Z_1} & \dots \\ \frac{\partial H_1}{\partial Z_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{H}_K &= \mathcal{J}^{\mathcal{K},r} \mathcal{J}^{r,K} + \mathcal{J}^{\mathcal{K},w} \mathcal{J}^{w,K} - \mathbf{I} \\ \mathbf{H}_Z &= \mathcal{J}^{\mathcal{K},r} \mathcal{J}^{r,Z} + \mathcal{J}^{\mathcal{K},w} \mathcal{J}^{w,Z} \end{aligned}$$

where generically

$$\mathcal{J}^{x,y} = \begin{bmatrix} \frac{\partial x_0}{\partial y_0} & \frac{\partial x_0}{\partial y_1} & \dots \\ \frac{\partial x_1}{\partial y_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

Once the Jacobian \mathbf{H}_K , also referred to as “the full Jacobian”, is calculated, the equation system can be solved with a quasi-Newton equation solver such as the Broyden-solver. This is provided in the package with the method `.find_transition_path()`.

In the package there are two methods which both needs to be run to calculate the Jacobian:

1. `.compute_jac_hh()`: Compute the Jacobians of household problem $(\mathcal{J}^{\mathcal{K},r}, \mathcal{J}^{\mathcal{K},w})$ using the fast fast news algorithm (see below).
2. `.compute_jac()`: Compute the full Jacobian using simple numerical differentiation relying on the Jacobians of the household problem.

2.5 Fake new algorithm

Consider the following notation:

1. **Productivity:** z_t , indexed by i , lives on $\mathcal{G}_z = \{z^0, z^1, \dots, z^{\#z-1}\}$ with transition matrix Π^e with elements

$$\pi_{[i,i+]}^z = \Pr[z_{t+1} = z^{i+1} | z_t = z^i].$$

2. **Assets:** a_t , indexed by j , lives on $\mathcal{G}_a = \{a^0, a^1, \dots, a^{\#_a-1}\}$.

3. **Value and policy functions:** v , \mathbf{a}^* and \mathbf{c}^* lives on $\mathcal{G}_z \times \mathcal{G}_a$ with

$$v_{[i,j]} = u(\mathbf{c}_{[i,j]}^*) + \sum_{j_+=0}^{\#_a-1} \mathbf{Q}_{[j,j_+]}^i \beta \sum_{k=0}^{\#_z-1} \pi_{[i,i_+]}^e v_{[i+j_+,k]},$$

where $\mathbf{c}_{[i,j]}^* = c^*(z_i, a_j)$ and $\mathbf{Q}_{[j,k]}^i$ are the weights implied by linear interpolation of $a^*(z_t, a_{t-1})$ at $\mathbf{a}_{[i,j]}^* = a^*(z_i, a_j)$ given by

$$\mathbf{Q}_{[j,k]}^i = \begin{cases} \frac{a_{ij}^* - a^{j+1}}{a^{j+1} - a^{j+1-1}} & \text{if } j_+ > 0, \text{ and } a_{ij}^* \in [a^{j+1-1}, a^{j+1}] \\ \frac{a_{ij}^* - a^{j+}}{a^{j+1} - a^{j+}} & \text{if } j_+ < \#_a - 1, \text{ and } a_{ij}^* \in [a^{j+}, a^{j+1}] \\ 0 & \text{else} \end{cases}.$$

Let \vec{x} be the row-stacked version of the matrix x . The Bellman equation can be written

$$\vec{v}_t = u(\vec{c}_t^*) + \beta \mathbf{Q}_t \tilde{\Pi}^e \vec{v}_{t+1} m \quad (7)$$

where $\tilde{\Pi} = \Pi \otimes \mathbf{I}_{\#_a \times \#_a}$ and \mathbf{Q}_t is the policy matrix given by

$$\mathbf{Q}_t = \begin{bmatrix} \mathbf{Q}_t^0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_t^{\#_e-1} \end{bmatrix}, \quad \mathbf{Q}_t^i = \begin{bmatrix} \ddots & \vdots & \ddots \\ \cdots & q_{[j,j_+]}^i & \cdots \\ \ddots & \vdots & \ddots \end{bmatrix}. \quad (8)$$

Simulation is now the inverse operation:

$$\vec{D}_{t+1} = \tilde{\Pi}^{e\prime} \mathbf{Q}_t' \vec{D}_t, \quad (9)$$

where \prime denoted transpose.

The fake new algorithm now is:

Step 1: Solve backwards $T - 1$ periods from a shock Δ_x to price x .

$\mathbf{a}_s^{*,x}$ is the optimal saving policy with s periods until shock arrival

\mathbf{Q}_s^x is the associated policy matrix

Step 2: Numerical derivatives,

$$\Delta_{D,x}^s = \frac{\tilde{\Pi}^{e\prime} \mathbf{Q}_s^{x\prime} \vec{D}_{ss} - \vec{D}_{ss}}{\Delta_x}, \quad \Delta_{a,x}^s = \frac{\vec{a}_s^{*,x\prime} \vec{D}_{ss} - \vec{a}_{ss}^{\prime\prime} \vec{D}_{ss}}{\Delta_x}$$

Step 3: *Expectation factors*, $\mathcal{E}_t = \begin{cases} \mathbf{a}_{ss}^* & \text{if } t = 0 \\ \mathbf{Q}_{ss} \tilde{\Pi}^e \mathcal{E}_{t-1} & \text{else} \end{cases}$

Step 4: *Fake news matrix*, $\mathcal{F}_{[t,s]}^a = \begin{cases} \Delta_{a,x}^s & \text{if } t = 0 \\ \vec{\mathcal{E}}_{t-1} \Delta_{D,x}^s & \text{else} \end{cases}$

Step 5: *Jacobian*, $\mathcal{J}_{[t,s]}^{\mathcal{K},x} = \begin{cases} \mathcal{F}_{[t,s]}^a & \text{if } t = 0 \vee s = 0 \\ \sum_{k=0}^{\min\{t,s\}} \mathcal{F}_{[t-k,s-k]}^a & \text{else} \end{cases}$