- GEModelTools -

Two-Asset HANK Model with Capital

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Model

Households

A continuum of households of unit mass populates the economy. Each household is subject to idiosyncratic risk through stochastic changes in its productivity level. The productivity levels e with states z_t behave according to a Markov process with a fixed transition matrix Π. Utility depends on consumption and hours worked through separable preferences $u(c) - v(n)$. Households receive after-tax labour income $Z_t \equiv (1-\tau_t)w_t N_t$ that is then weighted by productivity $z_t \equiv Z_t e(z_t)$ and government lump-sum transfers T_t. They can invest in liquid assets ℓ_t and illiquid assets a_t through a financial intermediary subject to a zero borrowing constraint. The function $d\left(a_{t-1}\right)$ governs the distribution from the illiquid account to the liquid account,

$$d_t\left(a_{\_k}, k\right) = \frac{r_{ss}^a}{1+r_{ss}^a}(1+r_t^a)a_{t-1} + \chi\left((1-r_t^a)a_{t-1} - \left(1+r_{ss}^a\right)\bar{a}\right) \tag{1}$$

This formulation of the illiquid asset choice simplifies the household problem as it leaves consumption or liquid assets as the only choice variable. At the same time, it ensures to a low MPC out of illiquid assets. The first term in (1) leads to a distribution to the liquid account when the return for illiquid assets increases. Households only distribute a fraction of the increased value to the liquid account, ensuring a low MPC out of the illiquid assets. The second term, with a value of χ close to zero, leads to a slow transition back to the target value of illiquid assets $\bar{a}$.

This results in the following dynamic programming problem:

$$
\begin{aligned}
V_t(z_t, \ell_{t-1}, a_{t-1}) &= \max_{c_t, \ell_t} u(c_t) - v(N_t) + \beta \mathbb{E}_t \left[V_{t+1}(z_{t+1}, l_t, a_t) \right] \\
c_t + \ell_t &= \left(1 + r_t^{\ell}\right) l_{t-1} + Z_t e(z_t) + d\left(a_{t-1}\right) + T_t \\
a_t &= \left(1 + r_t^{a}\right) a_{t-1} - d\left(a_{t-1}\right) \\
l_t &\geq 0.
\end{aligned} \tag{2}
$$

Financial Intermediary

The financial intermediary collects liquid and illiquid savings and performs maturity transformation to invest the funds into long-term government bonds B_t and shares in firms.

Consistent with financial markets being sensitive to news, the financial intermediary is perfectly attentive to news and updates his expectations each period. When depositing liquid funds, intermediation costs occur, with a share of ξ that is fully passed on the deposit interest rate for liquid assets r_t^{ℓ}.

There exists a continuum of shares v_{jt} in firms with price p_{jt} that each pays a dividend D_{jt}. Shares bought must then sum to one. Government bonds pay a coupon of one unit of money each period and sell at a discount afterwards. Thus, they sell at a coupon rate of $(1 + \delta q_{t+1})$ where δ represents the discount for each year of maturity. The flow-of-funds constraint at the beginning of the period states that the value of liabilities must be equal to the liquidation value of the intermediaries portfolio:

$$
(1 + r_t^a)A_{t-1} + (1 + r_t^l)L_{t-1} = (1 + \delta q_t)B_{t-1} + \int (p_{jt} + D_{jt}) v_{jt-1} dj - \xi L_{t-1}.
$$

At the end of the period, the new investment in bonds and shares must equal the intermediaries liabilities that equal aggregate savings (end-of-period flow of funds constraint):

$$
\int p_{jt} v_{jt} dj + q_t B_t = A_t + L_t.
$$

The financial intermediary maximizes the return on the illiquid assets $\mathbb{E}\left[1 + r_{t+1}^a\right]$ for the households by choosing v_{jt}, B_t and L_t through adjusting the portfolio composition. The asset pricing equation states that the expected return on the different assets has to equal out as all arbitrage opportunities have to be exhausted:[1]

$$\mathbb{E}_t[1+r^a_{t+1}] = \frac{\mathbb{E}_t[1+\delta q_{t+1}]}{q_t} = \frac{\mathbb{E}_t[p_{jt+1}+D_{jt+1}]}{p_{jt}} = 1 + r^l_{t+1} + \xi \equiv 1 + r_t. \quad (3)$$

where r_t is the ex-ante real interest rate, that is the rate before shocks have realized. The model also contains nominal reserves in zero net supply with nominal policy rate i_t. Equivalently to the asset pricing equation, the real return on the nominal reserved has to equal out the ex-ante real return, which implies the Fisher equation

$$1 + r_t = (1 + i_t)\mathbb{E}_t\left[\frac{P_{t+1}}{P_t}\right].$$

Firms

Output is produced by a final good producer that combines intermediate goods. The monopolistic intermediate good firms utilize capital provided by a capital firm and labour pooled by unions.

Final Good Firm

The competitive and representative final good firm produces a homogeneous good Y_t with intermediate goods Y_{jt} using a CES aggregator

$$Y_t = \left(\int_0^1 Y_{jt}^{\frac{ep-1}{ep}}\, dj\right)^{\frac{ep}{ep-1}},$$

where ϵ_p is the substitution elasticity between intermediary goods.

Intermediate Good Firm

There is a continuum of monopolistic intermediate good firms that produce heterogeneous goods with constant productivity Θ and the production function

$$Y_{jt} = \Theta K_{jt}^{\alpha} N_{jt}^{1-\alpha}.$$

Because firms are identical and rent capital and labour from a common market, they all have the same capital-to-labour ratio $\frac{K_t}{N_t} = \frac{K_{jt}}{N_{jt}}$ and marginal costs s_t. Factor prices

[1] For an extended derivation of the first-order conditions of the financial intermediary problem, see Appendix C.2 in Auclert, Rognlie, Straub (2020).

are taken as given and satisfy

$$\begin{aligned} w_t &= s_t(1-\alpha)\Theta K_t^{\alpha} N_t^{-\alpha} = s_t(1-\alpha)\frac{Y_t}{N_t} \\ r_t^K &= s_t \alpha \Theta K_t^{\alpha-1} N_t^{1-\alpha} = s_t \alpha \frac{Y_t}{K_t}. \end{aligned} \tag{4}$$

Each firm sells its good for price P_{jt} to the final good firm and is subject to price stickiness a la Calvo. Thus, a fraction of $1-\xi_p$ firms reset their prices each period. When choosing their optimal reset price, the intermediate firms maximize their stock prices p_{jt} and dividends

$$D_{jt} = \left(\frac{P_{jt}}{P_t} - s_t\right) Y_{jt}.$$

Inflation, $\pi_t = log\left(\frac{P_t}{P_{t-1}}\right)$, evolves according to the following New Keynesian Philipps Curve (NKPC)

$$\pi_t = \underbrace{\frac{(1-\xi_p)\left(1-\frac{\xi_p}{1+r}\right)}{\xi_p} \frac{\epsilon_p}{\epsilon_p - 1}}_{\kappa} \left(s_t - \frac{\epsilon_p - 1}{\epsilon_p}\right) + \frac{1}{1+r}\mathbb{E}_t\left[\pi_{t+1}\right]. \tag{5}$$

Inflation increases when firms expect markups $\frac{1}{s_t}$ below their steady state value $\frac{\epsilon_p}{\epsilon_p - 1}$. When resetting, they increase their price to bring markups back to their desired value.

Capital Firms

Capital for the production of goods is provided by a capital firm that owns the capital stock K_t and rents it out at rate r_t^k. The capital stock evolves according to

$$K_{t+1} = (1-\delta) K_t + I_t,$$

where δ is depreciation and I_t investment.
Investment takes one period to build up and is subject to quadratic and convex adjustment costs

$$S\left(\frac{I_{t+1}}{I_t}\right) = \frac{\phi}{2}\left(\frac{I_{t+1}}{I_t} - 1\right)^2,$$

such that $S(1) = S'(1) = 0$ and $S''(1) = \phi$. This form implies that firms adjust investment continuously and smoothly, where the degree of smoothness depends on the parameter ϕ.

Firms maximize dividends

$$D_t^K = r_t^K K_t - I_t \left(1 + S\left(\frac{I_t}{I_{t-1}}\right)\right),$$

and stock prices

$$p_t^K = \frac{D_{t+1}^K + p_{t+1}^K}{1+r},$$

by choosing next periods investment

$$\max_{I_{t+1}} \left\{ \mathbb{E}_t \left[D_{t+1}^K(K_{t+1}, I_t, I_{t+1}) + \max_{I_{t+2}} \; p_{t+1}^K \left(K_{t+2}, I_{t+1}, I_{t+2}\right)\right]\right\}$$

Appendix C.4 in **?** shows that investment dynamics are described by the following set of first order conditions

$$1 + S\left(\frac{I_{t+1}}{I_t}\right) + \frac{I_{t+1}}{I_t} S'\left(\frac{I_{t+1}}{I_t}\right) = Q_t + \mathbb{E}\left[\frac{1}{1+r_{t+1}}\left(\frac{I_{t+2}}{I_{t+1}}\right)^2 S'\left(\frac{I_{t+2}}{I_{t+1}}\right)\right] \tag{6}$$

$$Q_t = \mathbb{E}\left[\frac{1}{1+r_{t+1}}(r_{t+2}^K + (1-\delta)Q_{t+1})\right]$$

We can interpret $Q_t \equiv \mathbb{E}_t \left[\frac{\partial p_{t+1}^K}{\partial K_{t+2}}\right]$ in the form of Tobin's Q. Thus, when $Q > 1$ firms can increase their future stock price by building up more capital.

Aggregate Firm Value

Aggregate dividends are gives as

$$D_t = \int D_{jt} dj + D_t^K = Y_t - w_t L_t - I_t \left(1 + S\left(\frac{I_t}{I_{t-1}}\right)\right).$$

Shares of intermediate and capital goods firms have unit mass such that the aggregate stock market has value $p_t = \int p_{jt} dj + p_t^K$. The arbitrage condition (3) ensures that

$$p_{jt} = \frac{1}{1+r_t} \mathbb{E}_t[D_{j,t+1} + p_{j,t+1}]$$

$$p_t^K = \frac{1}{1+r_t} \mathbb{E}_t[D_{t+1}^K + p_{t+1}^K],$$

and for the value of the aggregate stock market

$$p_t = \frac{1}{1+r_t} \mathbb{E}_t[D_{t+1} + p_{t+1}].$$

Unions

Labour for the intermediate goods firm is provided by unions that package households' labour services and set wages monopolistically subject to wage stickiness a la Calvo.

There is a continuum of unions $j \in [0,1]$ with mass one that bundle labour from households into a union-specific task N_{jt}. Households supply hours n_{ijt} to the respective union j union such that $n_{it} = \int n_{ijt} di$ and $N_{jt} = \int e\left(s_{it}\right) n_{ijt} di$. Total labour is then aggregated by a competitive labour packer such that each union faces the demand schedule

$$N_{jt} = N_t \left(\frac{W_{jt}}{W_t}\right)^{-\epsilon_w},$$

where W_t is the aggregate wage defined as

$$W_t = \left(\int W_{jt}^{1-\epsilon_w} dj\right)^{\frac{1}{1-\epsilon_w}}.$$

The union specific wage W_{jt} set by unions and paid by firms, where ϵ_w is the elasticity of substitution between labour types. Unions can reset their wages with probability $1-\xi_w$ each period leading to the New Keynesian Wage Phillips Curve

$$\pi_{w,t} = \kappa_w \left(s_{w,t} - \frac{\epsilon_w - 1}{\epsilon_w}\right) + \beta^w \mathbb{E}_t\left[\pi_{w,t+1}\right], \tag{7}$$

where $\pi_{w,t} = log\left(\frac{W_t}{W_{t-1}}\right)$. $s_{w.t}$ is the inverse wage markup and β^w the discount factor specific to the wage NKPC.

Government

The government conducts consumption spending of size G_t and pays lump-sum transfers T_t to households. It finances these expenses by issuing long-term government bonds B_t and collecting labour income taxes τ_t in the total amount of $\tau_t \frac{W_t}{P_t} N_t \mathbb{E}[\bar{e}_g e(s)] = \tau_t \frac{W_t}{P_t} N_t$ such that the government budget constraint is fulfilled:

$$q_t B_t + \tau_t \frac{W_t}{P_t} N_t = G_t + T_t + (1 + \delta q_t) B_{t-1}.$$

When the government issues additional bonds at price q_t by issuing nominal bonds, the tax rate will adjust gradually to bring the debt level back to its steady state following the rule

$$\tau_t = \phi^{\tau} q^{ss} \frac{(B_{t-1} - B^{ss})}{Y^{ss}} + \tau^{ss},$$

where $\phi^{\tau} \in (0, 1]$ governs the rate of adjustment. A higher ϕ^{τ} leads to a stronger tax rate adjustment and a faster transition back to the steady state level of debt.

Monetary policy is conducted by an independent central bank that sets the nominal interest rate i_t according to the Taylor rule:

$$1 + i_t = \rho_m i_{t-1} + (1 - \rho_m) \left(r^{ss} + \phi^{\pi} \pi_t \right),$$

ρ_m describes the persistence coefficient that smoothed the monetary reaction. The reaction to inflation depends on the Taylor coefficient ϕ^{π}.

Market clearing

The asset markets for liquid and illiquid assets both clear,

$$\int \ell_t d\boldsymbol{D}_t = L_t$$
$$\int a_t d\boldsymbol{D}_t = A_t.$$