- GEMODELTOOLS -

HANK WITH STICKY PRICES

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Model

We consider a *closed* economy with heterogeneous agent and *sticky prices* and *flexible wages*.

Time is discrete and indexed by t. There is a continuum of households indexed i.

Households. Households are *ex ante* homogeneous, and *ex post* heterogeneous with respect to their productivity, z_{it}, and assets, a_{it-1}. Each period household choose labor supply ℓ_{it}, and consumption c_{it} subject to a no-borrowing constraint, $a_{it} \geq 0$. Taxes, τ_t, and dividends, d_t, are for simplicity both proportional to productivity, z_{it}. The real interest rate for period $t-1$ to t is r_t and w_t is the real wage. Households have *perfect foresight* wrt. to the aggregate variables, $\{r_t, w_t, \tau_t, d_t\}_{t=0}^{\infty}$, and solve the problem

$$
\begin{aligned}
v_t(z_{it}, a_{it-1}) &= \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_{it}^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[v_{t+1}(z_{it+1}, a_{it}) \right] \qquad (1) \\
\text{s.t. } a_{it} &= (1+r_t)a_{it-1} + (w_t \ell_{it} - \tau_t + d_t) z_{it} - c_{it} \geq 0 \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1} \ , \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \ \mathbb{E}[z_{it}] = 1,
\end{aligned}
$$

where implicitly $v_t(z_{it}, a_{it-1}) = v(z_{it}, a_{it-1}, \{r_t, w_t, \tau_t, d_t\}_{\tau=t}^{\infty})$. We also define labor supply in efficiency units by $n_{it} = \ell_{it} z_{it}$.

We denote optimal policy functions by $a_t^*(z_{it}, a_{it-1})$, $\ell_t^*(z_{it}, a_{it-1})$, and $c_t^*(z_{it}, a_{it-1})$. The distribution of households *before* the realization of idiosyncratic shocks, i.e. over z_{it-1} and a_{it-1}, is denoted $\underline{\boldsymbol{D}}_t$. The distribution of households *after* the realization idiosyncratic shocks, i.e. over z_{it} and a_{it-1}, is denoted $\boldsymbol{D}_t$.

Central aggregate variables are

$$
\begin{aligned}
A_t^{hh} &= \int a_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t && (2) \\
&= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\
L_t^{hh} &= \int \ell_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t && (3) \\
&= \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\
C_t^{hh} &= \int c_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t && (4) \\
&= \boldsymbol{c}_t^{*\prime} \boldsymbol{D}_t \\
N_t^{hh} &= \int \ell_t^*(z_{it}, a_{it-1}) z_{it} d\boldsymbol{D}_t, && (5)
\end{aligned}
$$

where N_t^{hh} is labor supply in efficiency units.
To solve the model, we define the beginning-of-period value function as

$$
\underline{v}_t \left(z_{it-1}, a_{it-1}\right) = \mathbb{E}_t \left[v_t(z_{it}, a_{it-1}\right] \tag{6}
$$

The optimality conditions are

$$
\begin{aligned}
\text{FOC wrt. } c_{it} &: 0 = c_{it}^{-\sigma} - \beta \underline{v}_{a,t+1}(z_{it}, a_{it}) \\
\text{FOC wrt. } \ell_{it} &: 0 = w_t z_{it} \beta \underline{v}_{a,t+1}(z_{it}, a_{it}) - \varphi \ell_{it}^{\nu} \\
\text{Envelope condition:} & \, \underline{v}_{a,t}(z_{it}, a_{it-1}) = \mathbb{E}_t \left[(1 + r_t) c_{it}^{-\sigma}\right]
\end{aligned}
$$

An endogenous grid point method can then be constructed as

$$
\begin{aligned}
c_{it} &= \left(\beta \underline{v}_{a,t+1}(z_{it}, a_{it})\right)^{-\frac{1}{\sigma}} \\
\ell_{it} &= \left(\frac{w_t z_{it}}{\varphi} c_{it}^{-\sigma}\right)^{\frac{1}{\nu}} \\
m_{it} &= c_{it} + a_{it} - (w_t \ell_{it} - \tau_t + d_t) z_{it}.
\end{aligned}
$$

If this results in $a^*(z_{it}, a_{it-1}) < 0$ then ℓ_{it}^*, n_{it}^* 9and c_{it}^* can be found with a *Newton solver* assuming $a_{it}^* = 0$:

1. Stop if $f(\ell_{it}^*) = \ell_{it}^* - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} \left(c_{it}^*\right)^{-\frac{\sigma}{\nu}} <$ tol. where

$$
\begin{aligned}
c_{it}^* &= (1 + r_t) a_{it-1} + (w_t \ell_{it}^* - \tau_t + d_t) z_{it} \\
n_{it}^* &= \ell_{it} z_{it}
\end{aligned}
$$

2. Set

$$\ell_{it}^* = \frac{f(\ell_{it}^*)}{f'(\ell_{it}^*)} = \frac{f(\ell_{it}^*)}{1 - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) \left(c_{it}^*\right)^{-\frac{\sigma}{\nu}} w_t z_{it}}$$

3. Return to step 1

Firms. A continuum of intermediary goods firms indexed by j produce differentiated goods with labor, set prices under monopolistic competition, and pay dividends to households.

Final good firms produce a final good with intermediary goods taking the price as given under perfect competition. The profit maximization problem is

$$\max_{y_{jt}\,\forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj\right)^{\mu}, \tag{7}$$

and implies the demand curve

$$\forall j : y_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} Y_t, \tag{8}$$

where the price index can be derived from profits being zero due to perfect competition.

Intermediary goods firms solve the problem

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\} \tag{9}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log\left(\frac{p_{jt}}{p_{jt-1}}\right)\right]^2,$$

where $\frac{\mu}{\mu-1}\frac{1}{2\kappa}$ is the price adjustment cost.

Assuming symmetry ($p_{jt} = P_t, y_{jt} = Y_{jt}, n_{jt} = N_t$), the New Keynesian Wage Phillips Curve (NKPC) becomes

$$\log(1 + \pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \tag{10}$$

where inflation is $\pi_t \equiv P_t / P_{t-1} - 1$ and dividends are

$$d_t = Y_t - w_t N_t - \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log\left(1 + \pi_t\right)\right]^2 Y_t. \tag{11}$$

Central bank. Set the nominal interest from period t to $t+1$ following a Taylor rule

$$i_t = i_t^* + \phi \pi_t + \phi^Y \left(Y_t - Y_{ss}\right), \tag{12}$$

where i_t^* is exogenous.

The Fisher relationship implies

$$r_t = (1 + i_{t-1}) / (1 + \pi_t) - 1.$$

Government The government chooses τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t.$$

Market clearing.

1. Assets: $B_{ss} = \int a_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t$
2. Labor: $N_t = \int \ell_t^*(z_{it}, a_{it-1}) z_{it} d\boldsymbol{D}_t$ (in effective units)
3. Goods: $Y_t = \int c_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t + G_t + \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log\left(1 + \pi_t\right)\right]^2 Y_t$

Representative agent For a representative agent version of the model, we can replace the market clearing conditions with the first order conditions

$$C_t^{-\sigma} = \beta(1 + r_{t+1}) C_{t+1}^{-\sigma} \tag{13}$$

$$\varphi N_t^{\nu} = w_t C_t^{-\sigma}. \tag{14}$$

Consumption can be derived from the resource constraint

$$C_t = Y_t - G_t - \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log\left(1 + \pi_t\right)\right]^2 Y_t. \tag{15}$$

To ensure the same steady, we can choose the parameters by

$$\beta^{RA} = \frac{1}{1+r_{ss}}$$
$$\varphi^{RA} = \frac{w_{ss}\left(C_{ss}^{hh}\right)^{-\sigma}}{\left(N_{ss}\right)^{\nu}}.$$

Defining aggregate household income as $Y_t^{RA} = w_t N_t + d_t - \tau_t$, the intertemporal budget constraint becomes

$$C_0 + \frac{C_1}{1+r_1} + \ldots = \left(1+r_0\right) A_{-1} + Y_0^{RA} + \frac{Y_1^{RA}}{1+r_1} \ldots \tag{16}$$

Equation system

1. Shocks: $\boldsymbol{Z} = \{\boldsymbol{\Gamma}, \boldsymbol{i}^*, \boldsymbol{G}\}$
2. Unknowns: $\boldsymbol{U} = \{\boldsymbol{Y}, \boldsymbol{w}, \boldsymbol{\pi}\}$
3. Targets: NKPC, asset market clearing and labor market clearing
4. Aggregate variables: $\boldsymbol{X} = \{...\}$
5. Household inputs: $\boldsymbol{X}_t^{hh} = \{\boldsymbol{r}, \boldsymbol{w}, \boldsymbol{\tau}, \boldsymbol{d}\}$
6. Household outputs: $\boldsymbol{Y}_t^{hh} = \{\boldsymbol{A}^{hh}, \boldsymbol{C}^{hh}, \boldsymbol{L}^{hh}, \boldsymbol{N}^{hh}\}$

This implies the equation system

$$\boldsymbol{H}\left(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^*, \Gamma, \underline{\boldsymbol{D}}_0\right) = \mathbf{0} \Leftrightarrow$$

$$\begin{bmatrix} \log(1+\pi_t) - \left[\kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1}}{1+r_{t+1}})\right] \\ N_t - \int \ell_t^*(z_{it}, a_{it-1}) z_{it} d\boldsymbol{D}_t \\ B_{ss} - \int a_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t \end{bmatrix} = \mathbf{0}.$$