

- GEMODELTOOLS -

HETEROGENEOUS AGENT NEW KEYNESIAN MODEL (HANK)

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Model

We consider a *closed* economy with heterogeneous agent and *sticky prices* and *flexible wages*.

Time is discrete and indexed by t . There is a continuum of households indexed i .

Households. Households are *ex ante* homogeneous, and *ex post* heterogeneous with respect to their productivity, z_{it} , and assets, a_{it-1} . Each period household choose labor supply ℓ_{it} , and consumption c_{it} subject to a no-borrowing constraint, $a_{it} \geq 0$. Taxes, τ_t , and dividends, d_t , are for simplicity both proportional to productivity, z_{it} . The real interest rate for period $t - 1$ to t is r_t and w_t is the real wage. Households have *perfect foresight* wrt. to the aggregate variables, $\{r_t, w_t, \tau_t, d_t\}_{t=0}^{\infty}$, and solve the problem

$$\begin{aligned} v_t(z_{it}, a_{it-1}) &= \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_{it}^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t [v_{t+1}(z_{it+1}, a_{it})] \\ \text{s.t. } a_{it} &= (1 + r_t)a_{it-1} + (w_t \ell_{it} - \tau_t + d_t)z_{it} - c_{it} \geq 0 \\ \log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \mathbb{E}[z_{it}] = 1, \end{aligned} \tag{1}$$

where implicitly $v_t(z_{it}, a_{it-1}) = v(z_{it}, a_{it-1}, \{r_t, w_t, \tau_t, d_t\}_{\tau=t}^{\infty})$. We also define labor supply in efficiency units by $n_{it} = \ell_{it} z_{it}$.

We denote optimal policy functions by $a_t^*(z_{it}, a_{it-1})$, $\ell_t^*(z_{it}, a_{it-1})$, and $c_t^*(z_{it}, a_{it-1})$. The distribution of households *before* the realization of idiosyncratic shocks, i.e. over z_{it-1} and a_{it-1} , is denoted \underline{D}_t . The distribution of households *after* the realization idiosyncratic shocks, i.e. over z_{it} and a_{it-1} , is denoted \mathbf{D}_t .

Central aggregate variables are

$$A_t^{hh} = \int a_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t \quad (2)$$

$$= \mathbf{a}_t^{*'} \mathbf{D}_t$$

$$L_t^{hh} = \int \ell_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t \quad (3)$$

$$= \boldsymbol{\ell}_t^{*'} \mathbf{D}_t$$

$$C_t^{hh} = \int c_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t \quad (4)$$

$$= \mathbf{c}_t^{*'} \mathbf{D}_t$$

$$N_t^{hh} = \int \ell_t^*(z_{it}, a_{it-1}) z_{it} d\mathbf{D}_t, \quad (5)$$

where N_t^{hh} is labor supply in efficiency units.

To solve the model, we define the beginning-of-period value function as

$$\underline{v}_t(z_{it-1}, a_{it-1}) = \mathbb{E}_t[v_t(z_{it}, a_{it-1})] \quad (6)$$

The optimality conditions are

$$\text{FOC wrt. } c_{it} : 0 = c_{it}^{-\sigma} - \beta \underline{v}_{a,t+1}(z_{it}, a_{it})$$

$$\text{FOC wrt. } \ell_{it} : 0 = w_t z_{it} \beta \underline{v}_{a,t+1}(z_{it}, a_{it}) - \varphi \ell_{it}^v$$

$$\text{Envelope condition: } \underline{v}_{a,t}(z_{it}, a_{it-1}) = \mathbb{E}_t[(1 + r_t) c_{it}^{-\sigma}]$$

An endogenous grid point method can then be constructed as

$$c_{it} = (\beta \underline{v}_{a,t+1}(z_{it}, a_{it}))^{-\frac{1}{\sigma}}$$

$$\ell_{it} = \left(\frac{w_t z_{it}}{\varphi} c_{it}^{-\sigma} \right)^{\frac{1}{v}}$$

$$m_{it} = c_{it} + a_{it} - (w_t \ell_{it} - \tau_t + d_t) z_{it}.$$

If this results in $a^*(z_{it}, a_{it-1}) < 0$ then ℓ_{it}^* , n_{it}^* and c_{it}^* can be found with a *Newton solver* assuming $a_{it}^* = 0$:

1. Stop if $f(\ell_{it}^*) = \ell_{it}^* - \left(\frac{w_t z_{it}}{\varphi} \right)^{\frac{1}{v}} (c_{it}^*)^{-\frac{\sigma}{v}} < \text{tol.}$ where

$$c_{it}^* = (1 + r_t) a_{it-1} + (w_t \ell_{it}^* - \tau_t + d_t) z_{it}$$

$$n_{it}^* = \ell_{it}^* z_{it}$$

2. Set

$$\ell_{it}^* = \frac{f(\ell_{it}^*)}{f'(\ell_{it}^*)} = \frac{f(\ell_{it}^*)}{1 - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c_{it}^*)^{-\frac{\sigma}{\nu}} w_t z_{it}}$$

3. Return to step 1

Firms. A continuum of intermediary goods firms indexed by j produce differentiated goods with labor, set prices under monopolistic competition, and pay dividends to households.

Final good firms produce a final good with intermediary goods taking the price as given under perfect competition. The profit maximization problem is

$$\max_{y_{jt} \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^\mu, \quad (7)$$

and implies the demand curve

$$\forall j : y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t, \quad (8)$$

where the price index can be derived from profits being zero due to perfect competition.

Intermediary goods firms solve the problem

$$\begin{aligned} J_t(p_{jt-1}) &= \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\} \\ \text{s.t. } y_{jt} &= Z_t n_{jt}, \quad y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t \\ \Omega(p_{jt}, p_{jt-1}) &= \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2, \end{aligned} \quad (9)$$

where $\frac{\mu}{\mu-1} \frac{1}{2\kappa}$ is the price adjustment cost.

Assuming symmetry ($p_{jt} = P_t$, $y_{jt} = Y_t$, $n_{jt} = N_t$), the New Keynesian Wage Phillips Curve (NKPC) becomes

$$\log(1 + \pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \quad (10)$$

where inflation is $\pi_t \equiv P_t/P_{t-1} - 1$ and dividends are

$$d_t = Y_t - w_t N_t - \frac{\mu}{\mu - 1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t. \quad (11)$$

Central bank. Set the nominal interest from period t to $t + 1$ following a Taylor rule

$$i_t = i_t^* + \phi \pi_t + \phi^Y (Y_t - Y_{ss}), \quad (12)$$

where i_t^* is exogenous.

The Fisher relationship implies

$$r_t = (1 + i_{t-1}) / (1 + \pi_t) - 1.$$

Government The government chooses τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t.$$

Market clearing.

1. Assets: $B_{ss} = \int a_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t$
2. Labor: $N_t = \int \ell_t^*(z_{it}, a_{it-1}) z_{it} d\mathbf{D}_t$ (in effective units)
3. Goods: $Y_t = \int c_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t + G_t + \frac{\mu}{\mu - 1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t$

Representative agent For a representative agent version of the model, we can replace the market clearing conditions with the first order conditions

$$C_t^{-\sigma} = \beta(1 + r_{t+1}) C_{t+1}^{-\sigma} \quad (13)$$

$$\varphi N_t^\nu = w_t C_t^{-\sigma}. \quad (14)$$

Consumption can be derived from the resource constraint

$$C_t = Y_t - G_t - \frac{\mu}{\mu - 1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t. \quad (15)$$

To ensure the same steady, we can choose the parameters by

$$\beta^{RA} = \frac{1}{1 + r_{ss}}$$

$$\varphi^{RA} = \frac{w_{ss} (C_{ss}^{hh})^{-\sigma}}{(N_{ss})^v}.$$

Defining aggregate household income as $Y_t^{RA} = w_t N_t + d_t - \tau_t$, the intertemporal budget constraint becomes

$$C_0 + \frac{C_1}{1 + r_1} + \dots = (1 + r_0) A_{-1} + Y_0^{RA} + \frac{Y_1^{RA}}{1 + r_1} \dots \quad (16)$$

Equation system

1. Shocks: $\mathbf{Z} = \{\Gamma, \mathbf{i}^*, G\}$
2. Unknowns: $\mathbf{U} = \{Y, w, \pi\}$
3. Targets: NKPC, asset market clearing and labor market clearing
4. Aggregate variables: $\mathbf{X} = \{\dots\}$
5. Household inputs: $\mathbf{X}_t^{hh} = \{r, w, \tau, d\}$
6. Household outputs: $\mathbf{Y}_t^{hh} = \{A^{hh}, C^{hh}, L^{hh}, N^{hh}\}$

This implies the equation system

$$\mathbf{H}(\pi, w, Y, \mathbf{i}^*, \Gamma, \underline{\mathbf{D}}_0) = \mathbf{0} \Leftrightarrow \begin{bmatrix} \log(1 + \pi_t) - \left[\kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}} \right] \\ N_t - \int \ell_t^*(z_{it}, a_{it-1}) z_{it} d\mathbf{D}_t \\ B_{ss} - \int a_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t \end{bmatrix} = \mathbf{0}.$$