“The Formal Definition of Reference Priors”

by Berger, Bernardo and Sun (2009)

Hans Niederhausen

Dept. of Physics and Astronomy
Stony Brook University

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Motivation

Objectivity is a fundamental scientific desideratum

"Bayesians address the question everyone is interested in by using assumptions no-one believes" - Louis Lyons (2006)

(particle physicists mostly use classical inference: "profile likelihood"

cf. presentation by Hechuan Wang)

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- Reference priors offer objective Bayesian inference (Bernardo, 1979)
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- Bernardo, Berger and Sun (2009) give rigorous definition for single parameter models (this talk!)
- Why do I care about Bayes?
  $\rightarrow$ include scientific information/knowledge
  (e.g. about nuisance parameters - e.g. modeling of detector response)

Definitions

DEFINITION 8 (Reference Prior):
A function $\pi\left(\theta\right) = \pi\left(\theta|M, P\right)$ *is a reference prior for model* M *given a class of prior functions* P *if it is permissible and has the MMI property (Maximizing Missing Information)*

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some properties

- independent of sample size
- compatible with sufficient statistics
- consistent under re-parametrization (invertible, one-to-one)
- reduce to Jeffrey's priors (if regularity conditions are met) $\pi(\theta) = \sqrt{I(\theta)}$ with $I(\theta) = -E_x\left\{\frac{\partial^2}{\partial\theta^2}\log\left[f(x|\theta)\right]\right\}$

Permissibility

$\rightarrow$ view improper prior (defined on Θ) as approximation to the analysis based on the “true but difficult to specify” large bounded space

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DEFINITION 5 (Permissible Prior):

A strictly positive continuous function $\pi\left(\theta\right)$ is a permissible prior if 1.) $\int_\Theta f(x|\theta)\pi\left(\theta\right)d\theta < \infty$ for all $x \in X$ and 2.) for some approximating compact sequence $\{\Theta\}_{i=1}^{\infty}$ converging to Θ, the corresponding posterior sequence is expected logarithmically convergent to $\pi\left(\theta|x\right) \propto f\left(x|\theta\right)\pi\left(\theta\right)$

Permissibility

DEFINITION 4 (Expected Logarithmic Convergence):

[...] The corresponding sequence of posteriors $\{\pi_i(\theta|x)\}_{i=1}^{\infty}$ *is expected logarithmically convergent to* $\pi(\theta|x)$ *if*

$$\lim_{i\to\infty} \int_X \kappa\{\pi(\theta|x) \,|\, \pi_i(\theta|x)\}\, p_i(x)\, dx = 0 \qquad (1)$$

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The expectation in eq. (1) is needed to ensure consistency of the corresponding interval estimates

(K-L divergence was also covered in Chen Mingshen's presentation)

MMI property

"MMI property" uses concept of missing information

DEFINITION 6 (Expected Information):

The information to be expected from one observation from Model $M \equiv \{f(x|\theta), \theta \in \Theta, x \in X\}$ *with prior* $q(\theta)$ *is*

$$I(q|M) = \int\int_{X\times\Theta} f(x|\theta)\, q(\theta) \log \frac{p(\theta|x)}{q(\theta)} dx d\theta \qquad (2)$$

$$= \int_X \kappa\{q(\theta) \,|\, p(\theta|x)\}\, p(x)\, dx \qquad (3)$$

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$\rightarrow$ interpret as expected K-L divergence between prior and posterior distributions from one observation

$\rightarrow$ measure of information gain from one observation

MMI property

Infinitely many observations $\{x_1, ..., x_k\}$ ($k \to \infty$) provide any missing information about the value of θ

$\to I\left(q|M^k\right)$ is measure of missing information (as $k \to \infty$).

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DEFINITION 7 (Maximizing Missing Information):

The function $\pi(\theta)$ is said to have the MMI property for model M given P if, for any compact set $\Theta_0 \in \Theta$ and any $p \in P$,

$$\lim_{k\to\infty}\left\{I\left(\pi_0|M^k\right) - I\left(p_0|M^k\right)\right\} \geq 0 \tag{4}$$

where π_0 and p_0 are the normalized restrictions of π and p to Θ_0.

Reference Prior

THEOREM 7 (Explicit Form):

Let $\pi^*(\theta)$ be strictly positive, continuous, s.t. posterior distribution $\pi^*(\theta|t_k) \propto f(t_k|\theta)\,\pi^*(\theta)$ is proper and consistent. For any interior point $\theta_0 \in \Theta$ define

$$f_k(\theta) = \exp\left\{\int_{T_k} p(t_k|\theta) \log\left[\pi^*(\theta|t_k)\right] dt_k\right\} \tag{5}$$

$$f(\theta) = \lim_{k\to\infty} \frac{f_k(\theta)}{f_k(\theta_0)} \tag{6}$$

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$\pi(\theta) = f(\theta)$ *is the <u>reference prior</u> for model* M*, if*

1) $f_k(\theta)$ *is continuous*
2) $\left\{f_k^0(\theta)/f_k^0(\theta_0)\right\}$ *monotonic in* k *or bounded above (in* θ*)*
3) $f(\theta)$ *is permissible*

+ authors describe routine for numerical computation of $\pi(\theta)$ on discrete points in parameter space

Numerical Calculations

A C++ implementation (requires BOOST) of the described algorithm available from:

▸ https://github.com/HansN87/ReferencePriors

applied to example 10 - unif $\left(\theta, \theta^2\right)$:

$$f\left(x|\theta\right) = \frac{1}{\theta^2 - \theta}, \quad \theta < x < \theta^2, \quad \theta > 1 \tag{7}$$

$$\pi\left(\theta\right) \propto \frac{2\theta - 1}{\theta\left(\theta - 1\right)} \exp\left\{\psi\left(\frac{2\theta}{2\theta - 1}\right)\right\} \tag{8}$$

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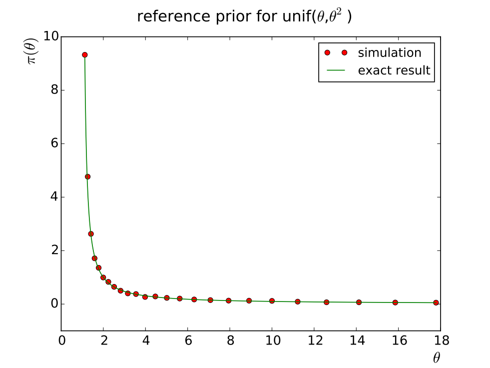

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applied to example 11 - triangular distribution (no exact solution known):

$$f\left(x|\theta\right) = \begin{cases} \frac{2x}{\theta} & \text{for } 0 < x \leq \theta \\ \frac{2(1-x)}{1-\theta} & \text{for } \theta < x < 1 \end{cases} \quad \text{and } \; 0 < \theta < 1 \tag{9}$$

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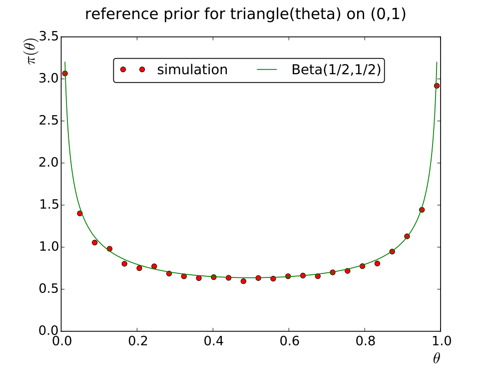

Conclusion

- Reference priors offer objective Bayesian inference
- Based on concept of maximizing missing information
- Presented analytic formula + numerical algorithm ...
- ... that provide solutions even in non-regular cases
- Multivariate generalizations exist but appear more involved
- Check out [1] for more details

Thank you!

[1] Berger, Bernardo and Sun, Ann. Statist. Volume 37, Number 2 (2009) + references therein