# Algebra and Linear Algebra

Algebra: numbers and operations on numbers

$2 + 3 = 5$

$3 \cdot 7 = 21$

Linear algebra: tuples, triples, ... of numbers and operations on them

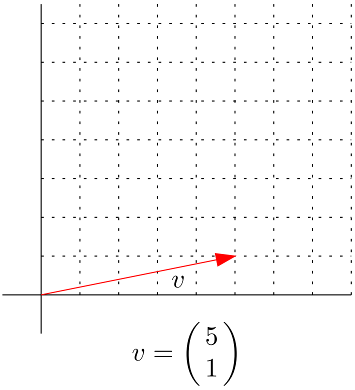
Assists in geometric computations

# Vectors: definition

A vector in $\mathbb{R}^d$ is an ordered $d$-tuple $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$.

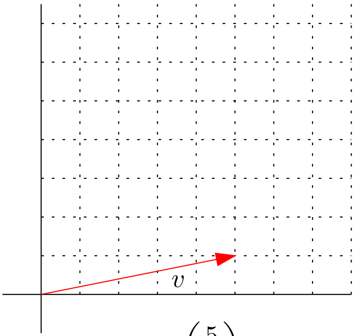
In $\mathbb{R}^3$, for example: $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

(or $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, or $\begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$, or $(v_1, v_2, v_3)$, or ... )



# Vectors: algebraic interpretation

A 2D vector $\begin{pmatrix} x_v \\ y_v \end{pmatrix}$ can be seen as the point $(x_v, y_v)$ in the Cartesian plane.




$$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

# Vectors: notation

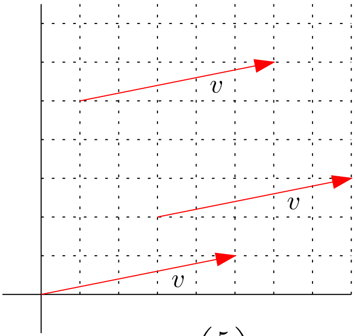
A 2D vector should be denoted as $\begin{pmatrix} x_v \\ y_v \end{pmatrix}$ or as $(x_v, y_v)^T$.
The $T$ in the exponent stands for 'transposed'.

A 2D point should be denoted as $(x_v, y_v)$.

Be aware of misused notation (mostly: point notation used for a vector).

# Vectors: geometric interpretation

A 2D vector $\begin{pmatrix} x_v \\ y_v \end{pmatrix}$ can be seen as an offset from the origin. Such an offset (arrow) can be translated.




$$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

# Vectors: length and scalar multiple

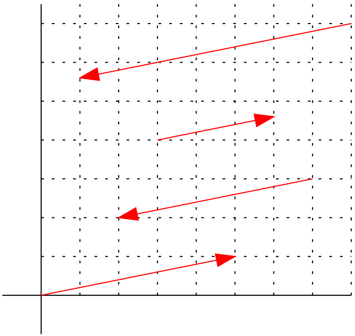
The Euclidean length of a $d$-dimensional vector $v$ is

$$||v|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_d^2}$$

A scalar multiple of a $d$-dimensional vector $v$ is

$$\lambda v = (\lambda v_1, \lambda v_2, \ldots, \lambda v_d)^T$$

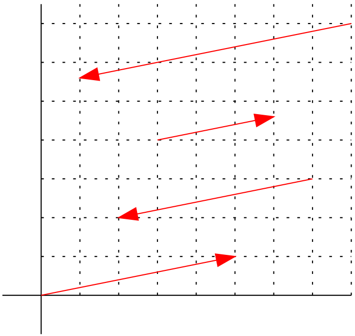
Note that $v$ and $\lambda v$ have the same direction or opposite directions



# Parallel vectors

Two vectors $v_1$ and $v_2$ are *parallel* if one is a scalar multiple of the other, i.e., there is a $\lambda \neq 0$ such that $v_2 = \lambda v_1$.

Note that if one of the vectors is the null vector, then the vectors are considered neither parallel nor not parallel.

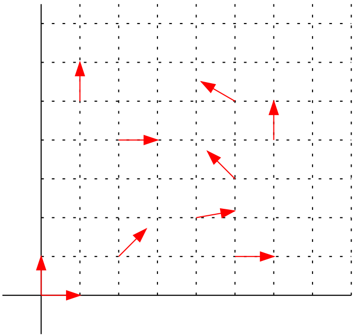


# Unit vectors

A vector $v$ is a unit vector if $||v|| = 1$.

Normalization

Q: given an arbitrary vector $v$, how do we find a unit vector parallel to $v$? Can every vector be normalized?



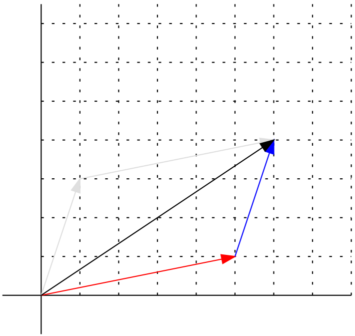
# Addition of vectors

Given two vectors in $\mathbb{R}^d$,
$v = (v_1, v_2, \ldots, v_d)^T$ and
$w = (w_1, w_2, \ldots, w_d)^T$

their sum is defined as

$$v + w = (v_1+w_1, v_2+w_2, \ldots, v_d+w_d)^T$$

Q: How would subtraction be defined?

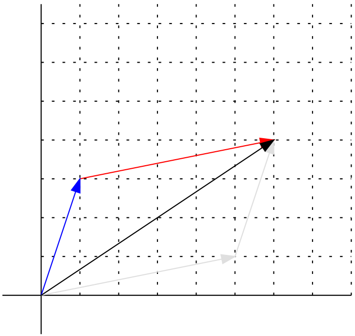


# Addition of vectors

Addition of vectors is commutative, as can be seen easily from the geometric interpretation.

Q: show algebraically that vector addition is commutative.

Q: what is the relation between $||v||$, $||w||$, and $||v + w||$?

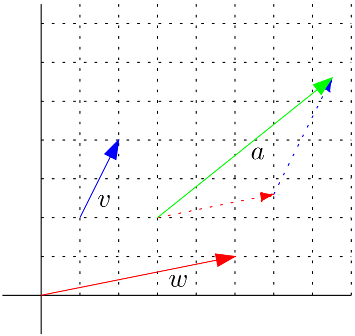


# Bases in 2D

A 2D vector can be expressed as a combination of any pair of non-parallel vectors. For instance, in the image, $a = 1.5v + 0.6w$.

Such a pair is called linearly independent, and forms a 2D basis.

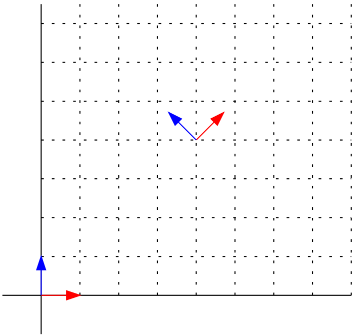
The extension to higher dimensions is straightforward.



# Orthonormal basis in 2D

Two vectors form an orthonormal basis in 2D if (1) they are orthogonal to each other, and (2) they are unit vectors.

The advantage of an orthonormal basis is that lengths of vectors, expressed in the basis, are easy to calculate.



# The null vector

The null vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is special.

It acts as the zero for addition of vectors.

It is the only vector that has length zero.

It is the only vector that does not have a direction.

It can not be used as a base vector.

# Dot product (inner product, scalar product)

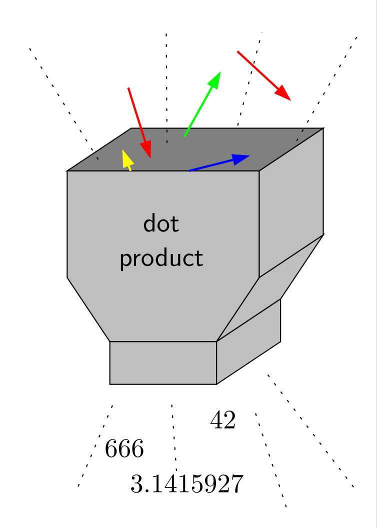
For two vectors $v, w \in \mathbb{R}^d$, the dot product is defined as

$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_d w_d,$$

or

$$v \cdot w = \sum_{i=1}^{d} v_i w_i$$

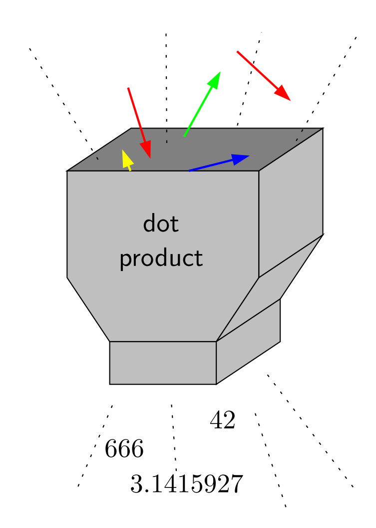
We have that $\cos\theta = \frac{v \cdot w}{||v||\,||w||}$, where $\theta$ is the angle between the two vectors.



# Dot product (inner product, scalar product)

Q: what is the inner product of an arbitrary unit vector with itself?

Q: what do we know if for two vectors $v$ and $w$ we have that $v \cdot w = 0$?



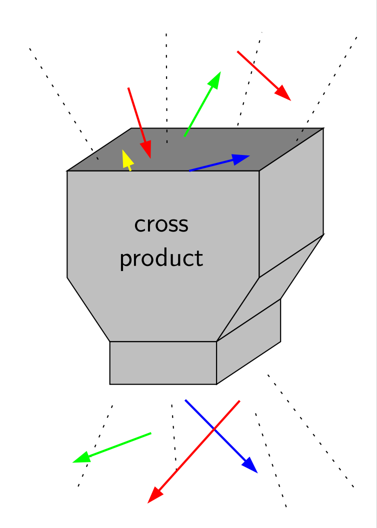
# Cross product

For two vectors $v, w \in \mathbb{R}^3$, the cross product is defined as

$$v \times w = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

Q: Show that $v \times w$ is orthogonal to both $v$ and $w$.

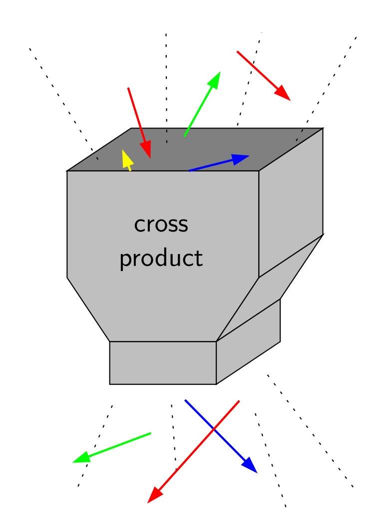
We have that $||v \times w|| = ||v||\,||w|| \sin\theta$, where $\theta$ is the angle between $v$ and $w$.



# Cross product

Q: Is it possible or necessary that $v$ and $w$ are orthogonal to form $v \times w$?

Q: What is $v \times w$ if $v$ and $w$ are parallel?



# Dot product, cross product, and the null vector

Q: What is the dot product of a vector and the null vector?

Q: What is the cross product of a vector and the null vector?

# Bases in 3D

You need three vectors to form a basis in 3D. If $u$, $v$, and $w$ form a basis, then any vector $a$ in 3D can be expressed as

$$a = \mu u + \lambda v + \rho w$$

where $\mu$, $\lambda$, and $\rho$ are scalars (ordinary numbers).

Q: Let $u$, $v$, and $w$ be three vectors (no one the null vector). Suppose that $u$ and $v$ are not parallel, $u$ and $w$ are not parallel, and $v$ and $w$ are not parallel. Do $u$, $v$, and $w$ always form a basis?

# Linear dependence in 3D

If for three vectors $u$, $v$, and $w$ in 3D (no null vectors), we have

$w = \mu u + \lambda v$

where $\mu$ and $\lambda$ are scalars, then $u$, $v$, and $w$ are linearly dependent.

If such $\mu$ and $\lambda$ do not exist, then they are linearly independent.

Any three linearly independent vectors in 3D form a 3D basis.

## Orthonormal 3D bases

Three vectors form an orthonormal basis in 3D if (1) each pair of them is orthogonal, and (2) they are unit vectors.

Q: What would you do to test if three 3D vectors form an orthonormal basis?

# Orthonormal 3D bases

Q: Suppose that two vectors $u$ and $v$ in 3D are orthogonal, and they are unit vectors.

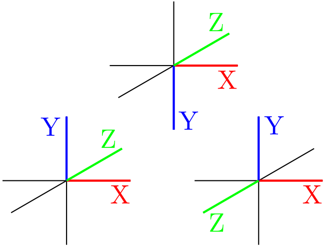
Let $w$ be the cross-product of $u$ and $v$. What can you say about $u$, $v$, and $w$ (do they form an orthonormal basis)?

# Left- and right-handed systems

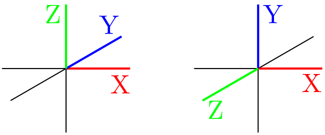
Coordinate systems in 3D come in two flavors: left-handed and right-handed.

There are arguments for both left- and right-handed systems for

- The global system
- The camera system
- Object systems




Camera coordinate system

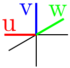


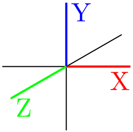

World coordinate system

# Coordinate transformations

A frequent operation in graphics is the change from one coordinate system (e.g., the $(u, v, w)$ camera system) to another (e.g., the $(x, y, z)$ global system).

Having orthonormal bases for both systems makes the transformations simpler.



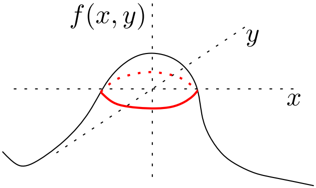



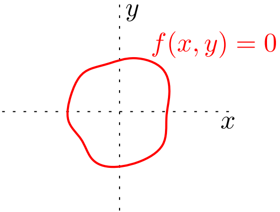
# 2D Implicit curves

An implicit curve in 2D has the form

$$f(x, y) = 0$$

$f$ maps two-dimensional points to a real value; the points for which this value is 0 are on the curve, while other points are not on the curve.



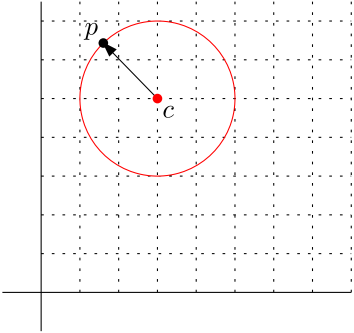
# Implicit representation of circles

The implicit representation of a 2D circle with center $c$ and radius $r$ is

$$(x - x_c)^2 + (y - y_c)^2 - r^2 = 0$$

So for any point $p$ that lies on the circle, we have that

$(p - c) \cdot (p - c) - r^2 = 0$, so
$|p - c|^2 - r^2 = 0$, which gives
$|p - c| = r$.



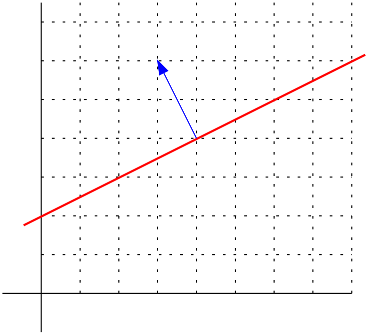
# Implicit representation of lines

A well-known representation of lines is the slope-intercept form $y = ax + b$

This can easily be converted to $-ax + y - b = 0$.

If $b = 0$, the line intersects the origin, and we have $n \cdot p = 0$, with $p = \begin{pmatrix} x \\ y \end{pmatrix}$ and $n = \begin{pmatrix} -a \\ 1 \end{pmatrix}$

Q: What if the line does not intersect the origin?

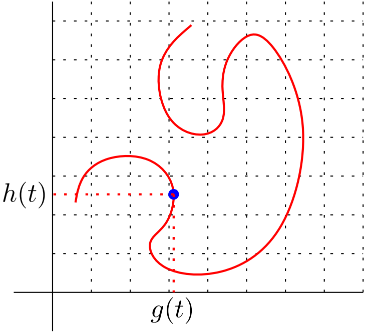


# 2D parametric curves

A parametric curve is controlled by a single parameter, and has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}$$

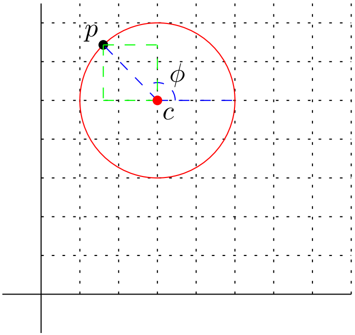
Parametric representations have some advantages over functions, even if a function would suffice to represent the curve.



# Parametric equation of a circle

The parametric equation of a 2D circle with center $c$ and radius $r$ is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c + r\cos\phi \\ y_c + r\sin\phi \end{pmatrix}$$

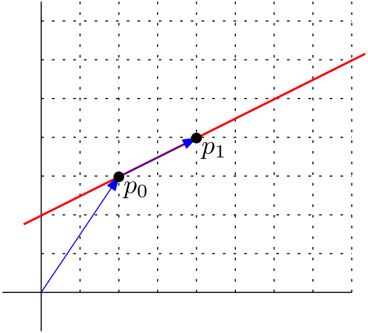


# Parametric equation of a line

The parametric equation of a line through the points $p_0$ and $p_1$ is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{p_0} + t(x_{p_1} - x_{p_0}) \\ y_{p_0} + t(y_{p_1} - y_{p_0}) \end{pmatrix}$$

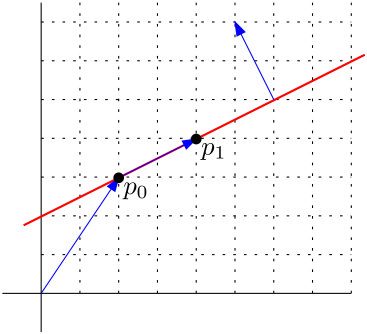
As we have seen before, this can alternatively be written as $p(t) = p_0 + t(p_1 - p_0)$.



# Conversion between representations

It is convenient to be able to convert a parametric equation of a line into an implicit equation, and vice versa.

Q: how do we do that?



# Implicit surfaces: from 2D to 3D

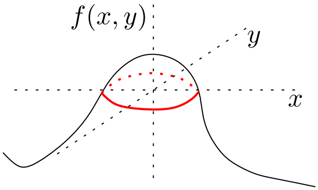
Recall that an implicit curve has the form

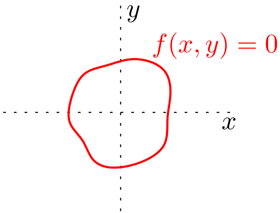
$$f(x, y) = 0$$

The 3D generalization is an implicit surface with a similar form

$$f(x, y, z) = 0$$

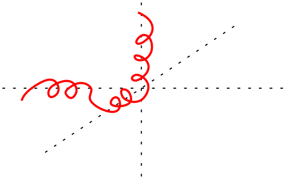
Fun project: try to draw the 4D image of the graph of such a function . . .



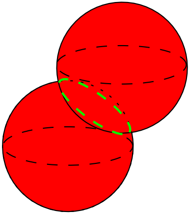
# Implicit one-dimensional curves in 3D?

Cooking up an implicit function for a one-dimensional thingy in 3D is in general not possible; such thingies are degenerate surfaces.



E.g., $x^2 + y^2 = 0$ is a cylinder with radius 0: the $Z$-axis.

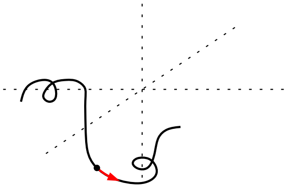
More complex curves can be described as the intersection of two or more implicit surfaces.



# Parametric curves and surfaces

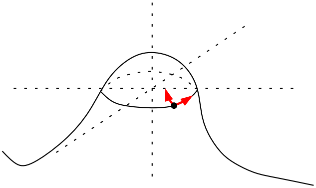
As opposed to implicit curves, it is possible to specify parametric curves in 3D:

$$x = f(t),$$
$$y = g(t),$$
$$z = h(t).$$



Parametric surfaces depend on two parameters:

$$x = f(u, v),$$
$$y = g(u, v),$$
$$z = h(u, v).$$



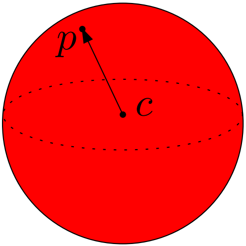
# Implicit spheres

We have already seen the sphere equation:

$$(x - c_x)^2 + (y - c_y)^2 + \\ (z - c_z)^2 - r^2 = 0$$

Just as in the circle case, this can be written in dot product form for any point $p$ on the sphere:

$$(p - c) \cdot (p - c) - r^2 = 0, \text{ or} \\ |p - c|^2 - r^2 = 0, \text{ or} \\ |p - c| = r.$$

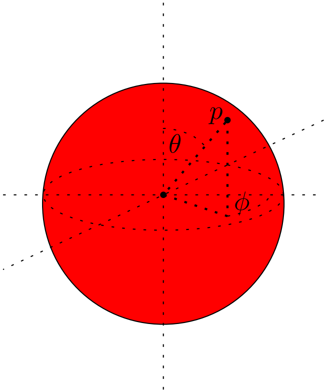


# Parametric spheres

Spheres can also be represented parametrically. For instance, a sphere with radius $r$ centered at the origin has the equation:

$$x = r \cos\phi \sin\theta,$$
$$y = r \sin\phi \sin\theta,$$
$$z = r \cos\theta$$

Q: What would the equation for a sphere with radius $r$ centered at $c = (c_x, c_y, c_z)$ be?

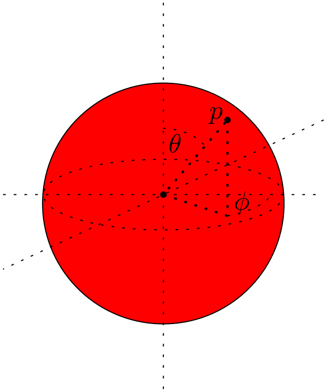


# Parametric spheres

$$x = r\cos\phi\sin\theta,$$
$$y = r\sin\phi\sin\theta,$$
$$z = r\cos\theta$$

The parametric representation of a sphere looks much more inconvenient than the implicit equation.

However, when we have to do texture mapping, the parametric representation turns out to be quite convenient.



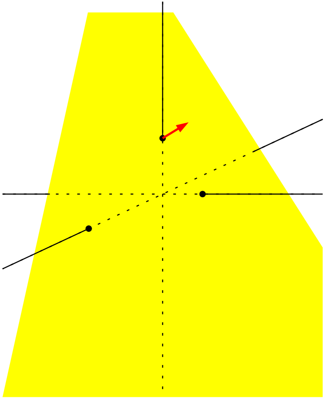
# Implicit planes

The implicit equation for a plane in 3D looks a lot like the equation for a line in 2D:

$ax + by + cz - d = 0$

Here, $(a, b, c)$ is a normal vector of the plane.

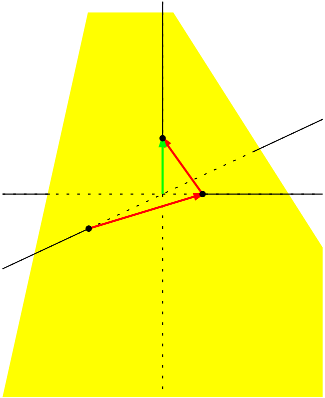
Q: what is the meaning of $d$?



# Parametric planes

Planes can also be described parametrically. Instead of one direction vector (as for lines), we need two:

$$(x, y, z) = (x_p, y_p, z_p) + s(x_v, y_v, z_v) + t(x_w, y_w, z_w).$$



# Implicit and parametric planes

Implicit:

$$ax + by + cz - d = 0$$

Parametric:

$$(x, y, z) = (x_p, y_p, z_p) + s(x_v, y_v, z_v) + t(x_w, y_w, z_w).$$

Q: Is an implicit description of a plane in 3D unique?

Q: Is a parametric description of a plane in 3D unique?

## $n \times n$ matrices

The system of $m$ linear equations in $n$ variables $x_1, x_2, \ldots, x_n$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

can be written as a matrix equation by $Ax = b$, or in full

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

# $n \times n$ matrices

The matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

has $m$ rows and $n$ columns, and is called an $m \times n$ matrix.

# Special matrices

A square matrix (for which $m = n$) is called a diagonal matrix if all elements $a_{ij}$ for which $i \neq j$ are zero. If all elements $a_{ii}$ are one, then the matrix is called an identity matrix, denoted with $I_m$ (depending on the context, the subscript $m$ may be left out).

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

If all matrix entries are zero, then the matrix is called a zero matrix or null matrix, denoted with 0.

# Matrix addition

For two matrices $A$ and $B$, we have $A + B = C$, with $c_{ij} = a_{ij} + b_{ij}$:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 14 \\ 10 & 16 \\ 12 & 18 \end{bmatrix}$$

Q: what are the conditions for the dimensions of the matrices $A$ and $B$?

# Matrix multiplication

Multiplying a matrix with a scalar is defined as follows: $cA = B$, with $b_{ij} = ca_{ij}$. For example,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

# Matrix multiplication

Multiplying two matrices is a bit more involved.
We have $AB = C$ with $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. For example,

$$\begin{bmatrix} 6 & 5 & 1 & -3 \\ -2 & 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 2 \\ 37 & 5 & 16 \end{bmatrix}$$

Q: what are the conditions for the dimensions of the matrices $A$ and $B$? What are the dimensions of $C$?

# Properties of matrix multiplication

Matrix multiplication is associative and distributive over addition:

$$\begin{aligned} (AB)C &= A(BC) \\ A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \end{aligned}$$

However, matrix multiplication is not commutative:
in general, $AB \neq BA$.

Also: if $AB = AC$, it doesn't necessarily follow that $B = C$ (even if $A$ is not the zero matrix).

# Zero and identity matrix

The zero matrix $0$ has the property that if you add it to another matrix $A$, you get precisely $A$ again.

$$A + 0 = 0 + A = A$$

The identity matrix $I$ has the property that if you multiply it with another matrix $A$, you get precisely $A$ again.

$$AI = IA = A$$

# Matrix multiplication as a linear transformation: 2D

The matrix multiplication of a $2 \times 2$ square matrix and a $2 \times 1$ matrix gives a new $2 \times 1$ matrix, e.g.:

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

We can interpret a $2 \times 1$ matrix as a vector; the $2 \times 2$ matrix transforms any vector into another vector.

More later . . .

# Transposed matrices

The transpose $A^T$ of an $m \times n$ matrix $A$ is an $n \times m$ matrix that is obtained by interchanging the rows and columns of $A$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

# Transposed matrices

for example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

For the transpose of the product of two matrices we have

$$(AB)^T = B^T A^T$$

# The dot product revisited

If we regard (column) vectors as matrices, we see that the inproduct of two vectors can be written as $u \cdot v = u^T v$:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 32$$

(A $1 \times 1$ matrix is simply a number, and the brackets are omitted.)

# Inverse matrices

The inverse of a matrix $A$ is a matrix $A^{-1}$ such that $AA^{-1} = I$.

Only square matrices possibly have an inverse.

Note that the inverse of $A^{-1}$ is $A$, so we have $AA^{-1} = A^{-1}A = I$

# Gaussian elimination

Matrices are a convenient way of representing systems of linear equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

If such a system has a unique solution, it can be solved with Gaussian elimination.

# Gaussian elimination

Permitted operations in Gaussian elimination are

- interchanging two rows.
- multiplying a row with a (non-zero) constant.
- adding a multiple of another row to a row.

# Gaussian elimination

Matrices are not necessary for Gaussian elimination, but very convenient, especially augmented matrices. The augmented matrix corresponding to the system of equations on the previous slides is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

# Gaussian elimination: example

Suppose we want to solve the following system:

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

Q: what is the geometric interpretation of this system? And what is the interpretation of its solution?

# Gaussian elimination: example

Applying the rules in a clever order, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 2 & 1 & 1 & 15 \\ 1 & 2 & 3 & 26 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & -1 & -3 & -19 \\ 0 & 1 & 1 & 9 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 3 & 19 \\ 0 & 1 & 1 & 9 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & -2 & -10 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

# Gaussian elimination: example

The interpretation of the last augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$

is the very convenient system of linear equations $\begin{array}{rcl} x & = & 3 \\ y & = & 4 \\ z & = & 5 \end{array}$

In other words, the point $(3, 4, 5)$ satisfies all three equations.

# Gaussian elimination: geometric interpretation

We started with three equations, which are implicit representations of planes:

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

We ended with three other equations, which can also be interpreted as planes:

$$\begin{aligned} x &= 3 \\ y &= 4 \\ z &= 5 \end{aligned}$$

The steps in Gaussian elimination preserve the location of the solution.

## Gaussian elimination: possible outcomes in 3D

Since any linear equation in three variables is a plane in 3D, we can interpret the possible outcomes of systems of three equations.

1. Three planes intersect in one point: the system has one unique solution
2. Three planes do not have a common intersection: the system has no solution
3. Three planes have a line in common: the system has many solutions

The three planes can also coincide, then the equations are equivalent.

# Gaussian elimination: inverting matrices

The same procedure can also be used to invert matrices:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 1 & -1 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{6} & -\frac{1}{6} & -\frac{2}{6} \\ 0 & 1 & 0 & -\frac{5}{6} & \frac{5}{6} & -\frac{2}{6} \\ 0 & 0 & 1 & \frac{2}{6} & -\frac{2}{6} & \frac{2}{6} \end{array}\right]$$

# Gaussian elimination: inverting matrices

The last augmented matrix tells us that the inverse of

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

equals

$$\frac{1}{6}\begin{bmatrix} 7 & -1 & -2 \\ -5 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

# Gaussian elimination: inverting matrices

When does a (square) matrix have an inverse?

If and only if its columns, seen as vectors, are linearly independent.

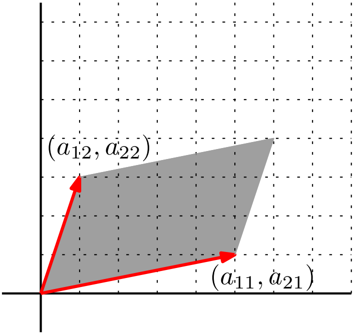
Equivalently, if and only if its rows, seen as transposed vectors, are linearly independent.

# Determinants

The determinant of a matrix is the signed volume spanned by the column vectors. The determinant $det\, A$ of a matrix $A$ is also written as $|A|$. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$det\, A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$



# Computing determinants

Determinants can be computed as follows:

The determinant of a matrix is the sum of the products of the elements of any row or column of the matrix with their cofactors

If only we knew what cofactors are...

# Cofactors

Take a deep breath. . .

The cofactor of an entry $a_{ij}$ in an $n \times n$ matrix $a$ is the determinant of the $(n-1) \times (n-1)$ matrix $A'$ that is obtained from $A$ by removing the $i$-th row and the $j$-th column, multiplied by $-1^{i+j}$.

Right: long live recursion!

Q: what is the bottom of this recursion?

# Cofactors

Example: for a $4 \times 4$ matrix $A$, the cofactor of the entry $a_{13}$ is

$$a_{13}^c = \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

and $|A| = a_{12}a_{12}^c + a_{22}a_{22}^c + a_{32}a_{32}^c + a_{42}a_{42}^c$

# Determinants and cofactors

Example

$$\begin{aligned}\begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} &= 0\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} \\ &= 0(32-35) - 1(24-30) + 2(21-24) \\ &= 0.\end{aligned}$$

## Systems of linear equations and determinants

Consider our system of linear equations again:

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

Such a system of $n$ equations in $n$ unknowns can be solved by using determinants. In general, if we have $Ax = b$, then $x_i = \frac{|A^i|}{|A|}$. where $A^i$ is obtained from $A$ by replacing the $i$-th column with $b$.

# Determinants and systems of linear equations

So for our system

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

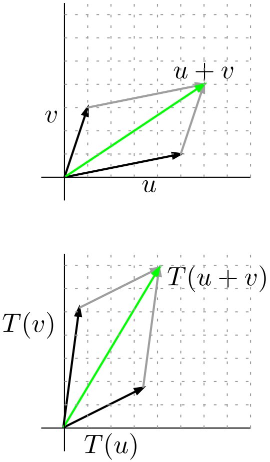
we have

$$x = \frac{\begin{vmatrix} 17 & 1 & 2 \\ 15 & 1 & 1 \\ 26 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 1 & 17 & 2 \\ 2 & 15 & 1 \\ 1 & 26 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} 1 & 1 & 17 \\ 2 & 1 & 15 \\ 1 & 2 & 26 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}}$$

# Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it satisfies

1. $T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$.
2. $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and all scalars $c$.



# Linear transformations in graphics

Many transformations that we use in graphics are linear transformations.

Linear transformations can be represented by matrices.

A sequence of linear transformations can be represented with a single matrix.

With some tricks, we can represent translations and perspective projections with matrices as well.

# Matrices and linear transformations

A $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the linear transformation that maps the vector $(x \quad y)^T$ to the vector $(ax + by \quad cx + dy)^T$.

Or (more readable): $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$

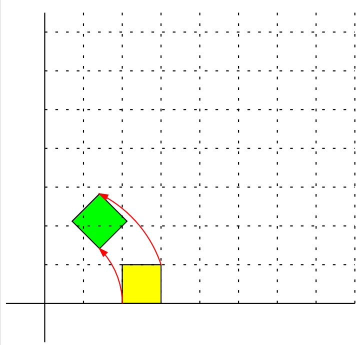
A $2 \times 3$ matrix is a linear transformation that maps a 3D vector to a 2D vector (from some 3-dim. space to some 2-dim. plane).

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \end{bmatrix}$$

# Example: rotation

To rotate $45^\circ$ about the origin, we apply the matrix

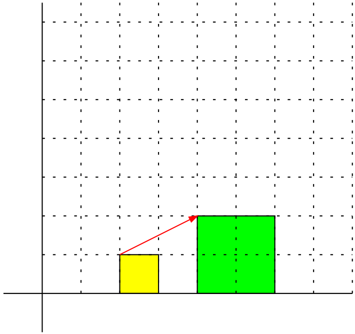
$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$



# Example: scaling

To scale with a factor two with respect to the origin, we apply the matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

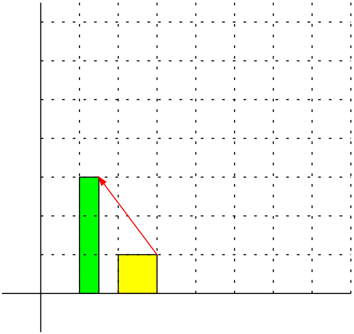


# Example: scaling

Scaling doesn't have to be uniform. Here, we scale with a factor one half in $x$-direction, and three in $y$-direction:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}$$

Q: what is the inverse of this matrix?

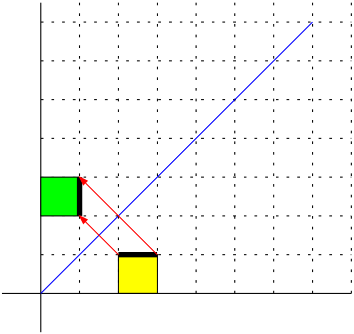


# Example: reflection

Reflection in the line $y = x$ boils down to swapping $x$- and $y$-coordinates:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Q: what is the inverse of this matrix?

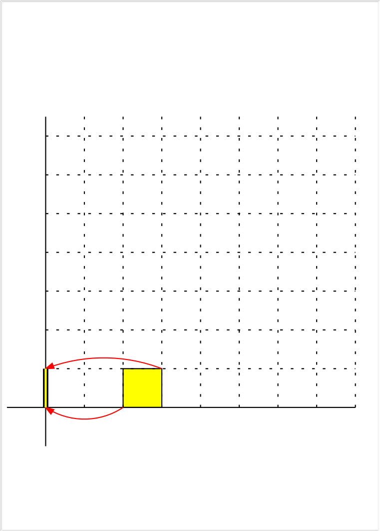


# Example: projection

We can also use matrices to do orthographic projections, for instance, onto the $Y$-axis:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Q: what is the inverse of this matrix?

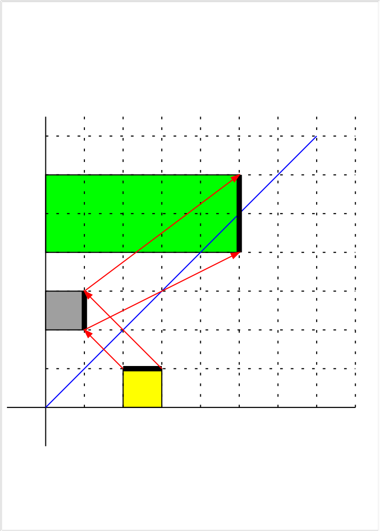


# Example: reflection and scaling

Multiple transformations can be combined into one. Here, we first do a reflection in the line $y = x$, and then we scale with a factor 5 in $x$-direction, and a factor 2 in $y$-direction:

$$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}$$

Q: Why is the transformation that is done first rightmost?

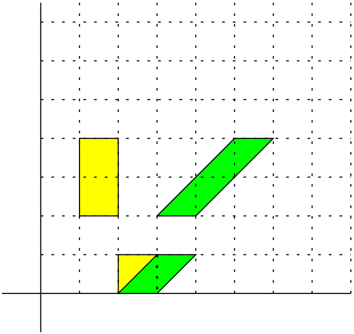


# Example: shearing

Shearing in $x$-direction pushes things sideways:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Q: What happens with the $x$-coordinate of points that are transformed with this matrix? And what with the $y$-coordinates? What is the inverse of this matrix?

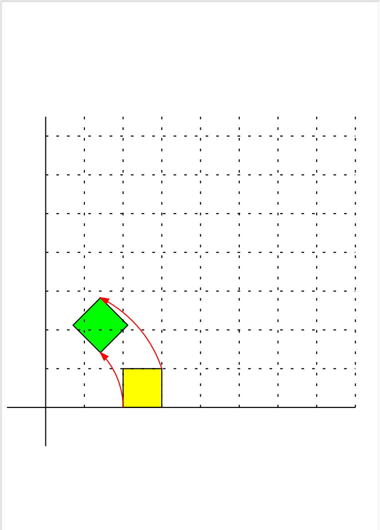


# Finding matrices

Applying matrices is pretty straightforward, but how do we find the matrix for a given linear transformation?

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Q: what is the significance of the column vectors of $A$?



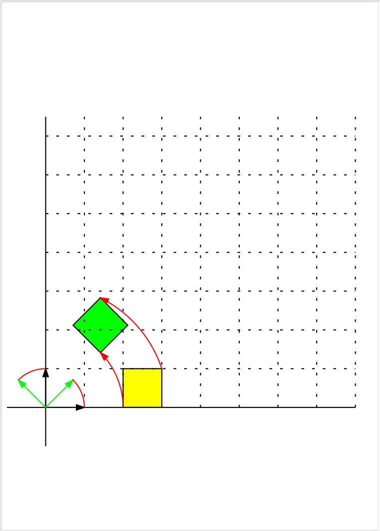
# Finding matrices

Aha! The column vectors of a transformation matrix are the images of the base vectors!

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

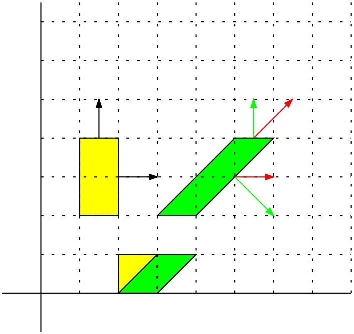
That gives us an easy method of finding the matrix for a given linear transformation.



# Transposing normal vectors

Unfortunately, normal vectors are not always transformed properly. To transform a normal vector $n$ under a given linear transformation $A$, we have to apply the matrix $(A^{-1})^T$.

Q: obviously, for shearing, normal vectors "behave funny". But what about rotations? And scalings (uniform and non-uniform)?



# Area and determinant

For any linear transformation, the determinant represents the size change (actually: the absolute value of the determinant).

For example, if a $2 \times 2$ matrix has determinant $3$ or $-3$, then the linear transformation transforms a unit square to a shape with area $3$.

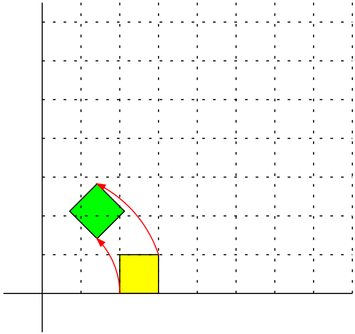
Q: What is going on when the determinant is zero?

# Example: rotation

To rotate $45°$ about the origin, we apply the matrix

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Q: What is the determinant?

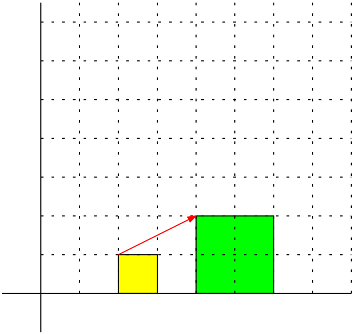


# Example: scaling

To scale with a factor two with respect to the origin, we apply the matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Q: What is the determinant?

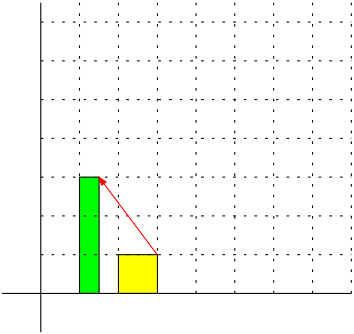


# Example: scaling

Scaling doesn't have to be uniform. Here, we scale with a factor one half in $x$-direction, and three in $y$-direction:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}$$

Q: What is the determinant?

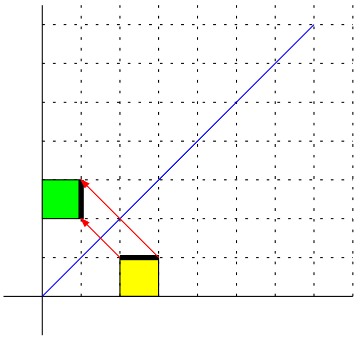


# Example: reflection

Reflection in the line $y = x$ boils down to swapping $x$- and $y$-coordinates:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Q: What is the determinant?

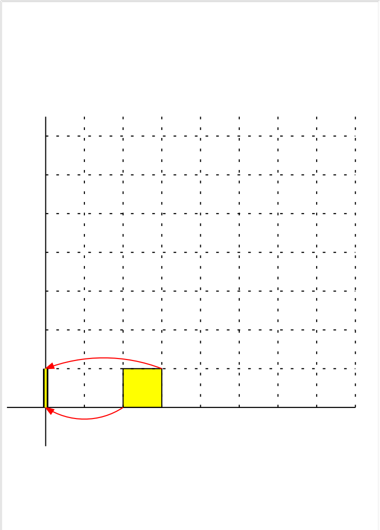


# Example: projection

We can also use matrices to do orthographic projections, for instance, onto the $Y$-axis:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Q: What is the determinant?



# Determinant $= 0$

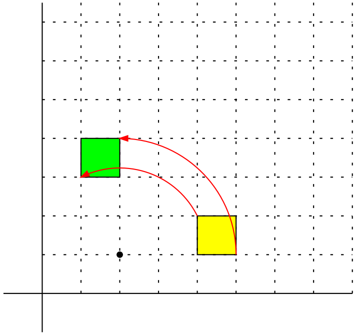
The following statements are equivalent for an $n \times n$ matrix $A$ and the linear transformation it represents:

1. The determinant of $A$ is zero.
2. The $n$ column vectors of $A$ are linearly dependent.
3. The image space of the transformation is at most $(n-1)$-dimensional (the transformation is a projection).

# More complex transformations

So now we know how to determine matrices for a given transformation. Let's try another one:

Q: What is the matrix for a rotation of $90^\circ$ about the point $(2, 1)$?

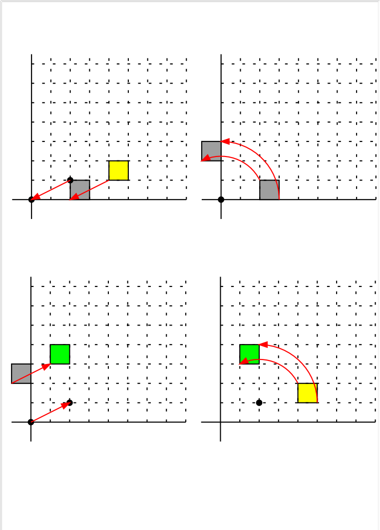


# More complex transformations

We can 'construct' our transformation by composing three simpler transformations:

- Translate everything such that the center of rotation maps to the origin.
- Rotate about the origin.
- Revert the translation from the first step.

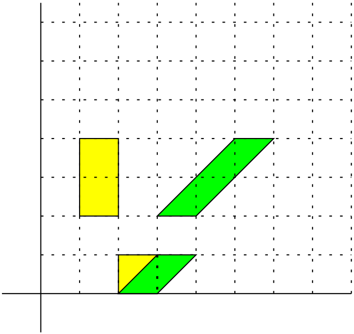
Q: But what is the matrix for a translation?



# Homogeneous coordinates

Translation is not a linear transformation. A combination of linear transformations and translations is called an affine transformation.

But...shearing in 2D smells a lot like translation in 1D



# Homogeneous coordinates

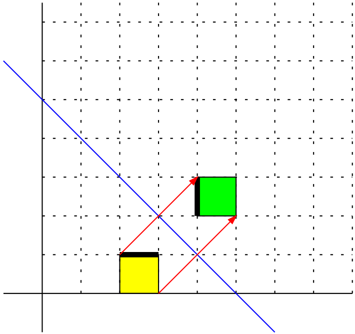
Translations in 2D can be represented by a shearing in 3D, by looking at the plane $z = 1$. The matrix for a translation over the vector $t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix}$

Q: How should we represent points? And vectors?

# Affine transformations

Q: What is the matrix for reflection in the line $y = -x + 5$?

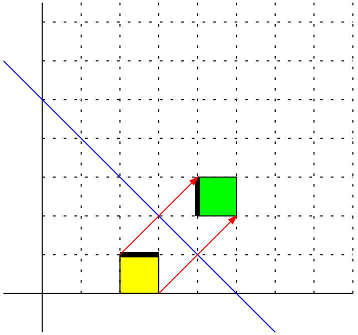
Hint: move the line to the origin, reflect, and move the line back.



# Affine transformations

The matrix for reflection in the line $y = -x + 5$ is

$$\begin{bmatrix} 0 & -1 & 5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$



# Affine transformations

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

The rightmost matrix of the three translates over $(-5 \quad 0)^T$, the leftmost matrix translates back over $(5 \quad 0)^T$.

# Affine transformations

Q: But what if we translate by $(-4 \quad -1)^T$? This also makes the line $y = -x + 5$ go through the origin...

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

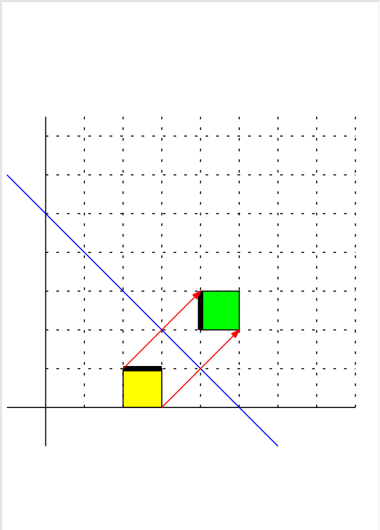
# Affine transformations

The matrix for reflection in the line $y = -x + 5$ is

$$\begin{bmatrix} 0 & -1 & 5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

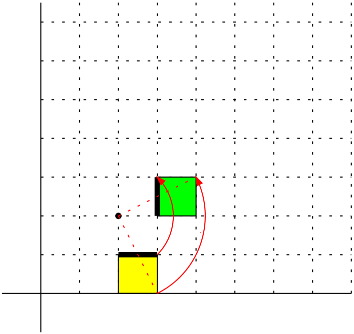
Q: What is the significance of the columns of the matrix?

Does that give us a faster way to find matrices for affine transformations?



# Affine transformations

Q: What is the matrix for rotation about the point $(2, 2)$?



# Transformations in 3D

Transformations in 3D are very similar to those in 2D:

- For scaling, we have three scaling factors on the diagonal of the matrix.
- Reflection is done with respect to planes.
- Shearing can be done in either $x$-, $y$-, or $z$-direction (or a combination thereof).
- Rotation is done about directed lines.
- For translations (and affine transformations in general), we use $4 \times 4$ matrices.

# Affine transformations in 3D

A matrix for affine transformations in 3D looks like:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the linear part and $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$ is where the origin ends up due to the affine transformation.

# Extra terminology (final slide)

Some other terms that are important in linear algebra:

- Linear subspace: Lower-dimensional linear space that includes the origin (or the whole space).
- Kernel and image of a linear transformation: What maps to the origin, and the linear subspace where all vectors are mapped to.
- Rank of a matrix: Number of linearly independent columns.
- Eigenvalue, eigenvector.

When you need to know more, look in any linear algebra book.