# Simplicial Homology

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**Abstract**

This paper gives an overview of the mathematical background of basic topology and group theory, including concepts such as topological spaces, homeomorphisms, quotient topology, and the first isomorphism theorem. It also introduces simplicial homology and singular homology, illustrating their applications through examples of computing the homology groups of various spaces.

# Contents

# 1 Motivation

Topology is the branch of mathematics concerned with the properties of space that are preserved under continuous transformations, such as stretching, bending, or twisting. Two spaces are considered topologically equivalent if they can be transformed into each other by such continuous transformations. Singular homology provides a way to study the topological properties of a space in a systematic and quantitative manner.

One of the key insights behind singular homology is that the topology of a space can be understood by looking at the behavior of continuous maps from the space to other spaces. Specifically, the homology groups of a space are constructed using the maps from the space to simplices, which are simple geometric objects that capture the essential features of higher-dimensional spaces. By examining the ways in which these maps can be combined and deformed, we can extract information about the holes, voids, and higher-dimensional structures that are present in the original space.
Another motivation for studying singular homology is that it provides a powerful tool for distinguishing between different topological spaces. Even seemingly simple spaces can exhibit a wide range of topological properties, and the homology groups provide a way to quantify these differences and classify spaces according to their topological features. This can have important applications in a variety of fields, from pure mathematics to engineering and physics.

# 2 Mathematical Background

## 2.1 Topological space

Topology is defined as the intent to generalize the notion of open sets. As we saw in real analysis, open sets in $\mathbb{R}$ are open intervals $(a, b)$, for $a, b \in \mathbb{R}$ and $a < b$; while the open sets in $\mathbb{R}^n$ are open balls defined as $B_\varepsilon(a) = \{x \in \mathbb{R}^n \mid \|x\| < \varepsilon\}$, where $\varepsilon > 0$, and $a \in \mathbb{R}^n$. In the sense of topology, they are called the standard topology in $\mathbb{R}^n$, but what is topology? We only studied topological objects like open sets, closed sets, and boundary points, but never introduce the definition of topology. So, let's begin with the definition of topology.

**Definition 2.1** (Munkres[1])**.** A ***Topology*** on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ having the following properties:

1. $\emptyset$ and $X$ are in $\mathcal{T}$.

2. Union of arbitrary subcollections of $\mathcal{T}$ is in $\mathcal{T}$.

3. Interseciton of finite subcollections of $\mathcal{T}$ is in $\mathcal{T}$.

A set $X$ with a topology $\mathcal{T}$ specified, $(X, \mathcal{T})$, is called a ***topological space***.

If $(X, \mathcal{T})$ is a topological space, we say $U$ is an ***open set*** if $U \in \mathcal{T}$.

**Example 2.2.** $(\mathbb{R}^n, \mathcal{T}_{\text{standard}})$ is a topological space, where $\mathcal{T}_{\text{standard}} = \{B_\varepsilon(a) \mid a \in \mathbb{R}^n, \varepsilon \geq 0\}$. The axioms of topology can be verified as we take the union of any subcollections of open balls, and the intersection of finite subcollections of open balls, both of which are elements in $\mathcal{T}_{\text{standard}}$.

With topology introduced, we then can go ahead and talk about the continuity of functions. Informally speaking, continuous functions are structure-preserving maps between topological space, it preserves the notion of open sets. If we require an additional condition that this continuous function is bijective, then we say this function is a homeomorphism between two topological spaces, and we say two topological spaces are homeomorphic. In topology, homeomorphic topological spaces can be considered equivalent.

**Definition 2.3** (Munkres[1])**.** Let $(X, \mathcal{T})$, $(X', \mathcal{T}')$ be two topological spaces. A function $f : X \to X'$ is ***continuous*** if for any $U \in \mathcal{T}'$, $f^{-1}(U) \in \mathcal{T}$. Moreover, if there exist a function $g : X' \to X$ such that $f, g$ are inverse of each and $g$ is continuous, then $f, g$ are called ***homeomorphisms***.

In other words, $f$ is continuous if and only if the preimage of the open set is open. Notice that for a topological space, we do not have the notion of distance, consequently, we can not measure the closeness. However, if we can add an extra structure to this topological space, namely, a notion of distance, then we could show that $\varepsilon$-$\delta$ definition of continuity is equivalent to this topological definition. We skip the proof here since the $\varepsilon$-$\delta$ definition is not that important in topology.

Another comment on homeomorphisms is that if two topological spaces are homeomorphic, then we can continuously deform one space to another. This is a crucial idea in topology, take a square and a circle for example: imagine that we form a square with rope, and push the corners of the square inward to get a circle. This series of moves is continuous and we actually can recover the square from the circle just by pulling four corners outward. This is exactly what the homeomorphism describes.

Let's move to the next vital idea in topology (we will also use a lot in this paper), the quotient topology. One of the motivations for quotient topology comes from geometry which uses "cut-and-paste" techniques to construct a geometric object. Take *Torus* (surface of donut) for example, one way to construct such a surface is by identifying the opposite edges of a rectangle, where identifying means, informally, pasting, as shown in Figure 2.1.

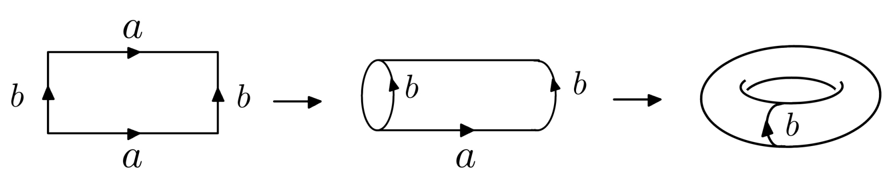


**Figure 2.1.** Identify (glue) opposite edges to get a torus.

**Definition 2.4** (Munkres[1])**.** Let $X$ and $Y$ be topological spaces, let $p : X \to Y$ be a surjective map. The map $p$ is said to be a ***quotient map*** provided a subset $U$ of $Y$ is open in $Y$ is and only if $p^{-1}(U)$ is open in $X$.

We won't talk about the quotient topology in detail, but it is important to throw out the definition of quotient topology so that we are able to talk about the gluing and a crucial consequence of quotient topology, namely, *polygonal representation*.

A polygonal representation of a surface is a polygon with its sides labeled and directed. We usually use a homeomorphism to glue the sides with the same label and along their directions. Moreover, the edges should be counted twice although they just represent one; while the vertices could be counted many times, this depends on the identification of the sides. Figure 2.1 is a good example, the edges labeled with $a$ are identified, and the edges labeled with $b$ are also identified, while the four vertices are identified to be the same one eventually. We desire to give some examples of polygonal representation which is quite important for the later calculation of homology groups of these topological objects.

**Example 2.5.** We begin with a simple one, $S^2$, also known as ***2-sphere***, defined as

$$S^2 := \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1\}.$$

Its polygonal representation is shown below in Figure 2.2. Notice that the edge $c$ is the result of identifying two sides. We can imagine that we cut the square along the diagonal and paste them back, it seems like we did nothing, but adding this extra edge $c$ is called *triangulation*, which would be important for the later discussion on homology. We also observe that the edge $a$ and the edge $b$ can be considered as one "long" edge, and the square representation becomes a bigon representation.

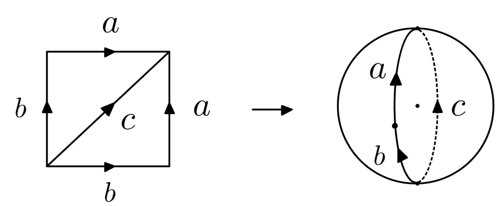


**Figure 2.2.** Polygonal representation of *2-sphere*.

These two examples we give above are called *orientable* surface, meaning that if we trace along one loop on them, the notion of orientation would not change. For example, consider $S^2$, imagine a vector starting at the north pole that points in one specific direction, moves along one of the longitude lines all the way to the south pole, and keeps going without changing the direction until it reaches the original starting point. We found that the direction it is pointing does not change after this travel, as shown in Figure 2.3 below. There are also some surfaces not behave like this, they are called *non-orientable* surfaces, as shown in the next example.

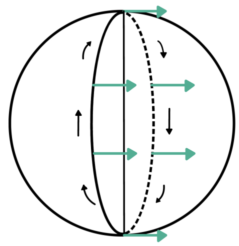

**Figure 2.3.** Vector moves along the longitude.

**Example 2.6.** Klein bottle, a non-orientable surface, is obtained by making a small change in the polygonal representation of the torus. In particular, we switch the direction of one $b$ edge but keep another one the same, then when we want to match the direction of $b$ edges, we need to twist this square to achieve this, and this twisting action will bring some weird behaviors. One thing to note here is that the Klein bottle can not be embedded into 3-dimensional space, the one shown in Figure 2.4 is the projection from 4-dimensional space.

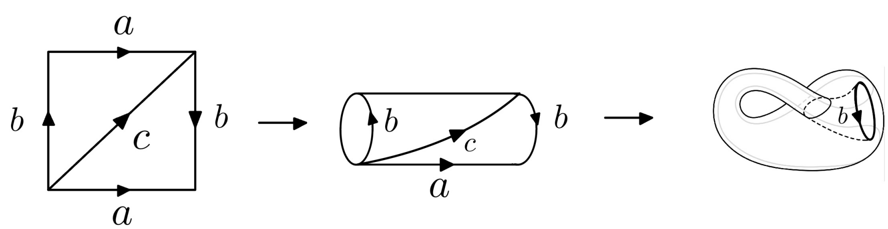


**Figure 2.4.** Polygonal representation of Klein bottle [2].

**Example 2.7.** Mobius strip is another non-orientable surface, the polygonal representation of Mobius strip is just identifying one pair of opposite edges but we need to twist it. Non-orientability is more obvious here. Consider a vector that starts somewhere on the surface, and lets it move along the red line. After it finishes one loop, it switches its direction, and the vector looks like is located in the "back" of the surface.

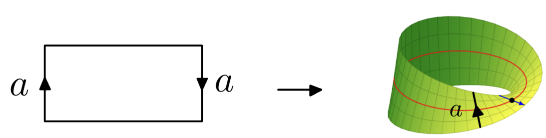


**Figure 2.5.** Polygonal representation of Mobius strip [3].

## 2.2 Group Theory

The main motivation of group theory is that mathematicians want to abstractly think about symmetries. They desire to use math language to abstract the main information instead of using words to describe it, and it turns out that this formal abstraction actually has a profound influence in many other fields. Before getting into the definition, we first look at an example. The rotation of an equilateral triangle is a good example. Each time we rotate the triangle 120 degree, and if we rotate 3 times, we will get back to the original triangle. Let's denote $e$ to be the trivial action, meaning that do nothing on this triangle and denote $r$ to be the rotation of 120 degree, then $r^2$ will be the rotation of 240 degree. One observation is that $r^3 = e$, since we can not tell any differences between rotation 360 degree and do nothing. Let me pause the example here, and state the definition, and we will see the close connection between this example and group theory later.

**Definition 2.8** (Fraleigh[4])**.** A ***group*** $(G, *)$ is a set $G$ together with a binary operation $*$, such that the following axioms are satisfied:

(i) For every $a, b, c \in G$, we have

$$(a * b) * c = a * (b * c).$$

(ii) There is an element $e$ in $G$ such that for all $g \in G$,

$$e * g = g = g * e.$$

(iii) Corresponding to each $g \in G$, there is an element $g'$ in $G$ such that

$$g * g' = g' * g = e.$$

Moreover, if $g * h = h * g$ for every $g, h \in G$, then $G$ is called an ***Abelian*** group.

**Example 2.9.** Now let's see a more comprehensive example of group. $(S_3, \cdot)$ is called a ***symmetric group***, where $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$, and the group operation is defined by the composition of maps. Here, we can relate the numbers in permutation to the vertices of the triangle in the previous example, and $e, (1\ 2\ 3), (1\ 3\ 2)$ will correspond to do nothing, rotate 120 degree, rotate 240 degree, respectively. Namely, if $r = (1\ 2\ 3)$, then $r^2 = (1\ 3\ 2)$, $r^3 = e$ as you can verify. Furthermore, these permutations with length 2 correspond to the reflection of the triangle with respect to three different axes. Each of which passes through a vertex and is perpendicular to the third side.

With the definition of the group provided, we now have some thoughts about what a group looks like, but what if we have two groups? Usually, mathematicians desire to study maps between two abstract structures, more precisely, the structure-preserving maps, so that they can study totally different objects but share a similar abstract structure. It turns out that a different view of a problem sometimes provides a more concise and neater description. Therefore, we will also study this kind of map between groups, and the formal definition is given below.

**Definition 2.10** (Artin[5])**.** Let $G$ and $G'$ be groups, written with multiplicative notation. A ***homomorphism*** $\varphi : G \to G'$ is a map from $G$ to $G'$ such that for every $a$ and $b$ in $G$,

$$\varphi(ab) = \varphi(a)\varphi(b).$$

We say $\varphi$ is an ***isomorphism*** is $\varphi$ is bijective.

In group theory, when two groups are isomorphic, meaning that there exists an isomorphism between them, we can say these two groups are equivalent. In the later part of this paper, we will discuss homology groups, but to define such a special group, we need more tools.

**Definition 2.11** (Artin[5])**.** Let $\varphi : G \to G'$ be a group homomorphism. The ***image*** of a homomorphism, often denoted by im$\varphi$, is simply the image of $\varphi$ as a map of sets:

$$\operatorname{im}\varphi = \{x \in G' \mid x = \varphi(a) \text{for some } a \text{ in } G\}.$$

The ***kernel*** of a homomorphism is more subtle and also more important. The kernel of $\varphi$, often denoted by ker $\varphi$, is the set of elements of $G$ that are mapped to the identity in $G'$:

$$\ker\varphi = \{a \in G \mid \varphi(a) = 1\}.$$

Let $G$ be a group, if $H$ is a subgroup of $G$ and if $a$ is an element of $G$, the subset

$$aH = \{ah \mid h \in H\}$$

is called a *left coset*. The subgroup $H$ is a particular left coset because $H = 1H$. It turns out that the cosets of $H$ are equivalence classes in $G$[5]. In general, when we have an equivalence relation on hand, we might want to use some structures to mod out this equivalence relation. But here we are in group theory, we still want the quotient space to have a group structure.

**Definition 2.12** (Artin[5])**.** A subgroup $N$ of a group $G$ is a ***normal subgroup*** if for every $a$ in $N$ and every $g$ in $G$, the conjugate $gag^{-1}$ is in $N$.

**Proposition 2.1.** *The kernel of a homomorphism is a normal subgroup.*

*Proof.* Let $\varphi : G \to G'$ be a group homomorphism. Consider $a \in \ker(\varphi)$, for any $g \in G$, have

$$\varphi(gag^{-1}) = \varphi(g)\varphi(a)\varphi(g^{-1}) = \varphi(g)\varphi(g)^{-1} = 1,$$

implying that $gag^{-1} \in \ker \varphi$. □

**Theorem 2.2.** *Let $\varphi : G \to G'$ be a group homomorphism. If $\varphi$ is surjective, then there exists a unique isomorphism*

$$\tilde{\varphi} : G/\ker(\varphi) \to G'.$$

*Proof.* This is quite an illuminating result, but we skip the proof here. □

# 3 Simplicial Homology

In algebraic topology, homology is a way of associating algebraic objects such as abelian groups to topological spaces. Using the techniques of homology, one can calculate the homology groups of a topological space, which intuitively count the number of n-dimensional holes. A relatively simple example of homology is called simplicial homology. Its natural domain is Delta-complex.

## 3.1 Delta-complex

Delta-complex is a set of maps that constructs topological spaces from the generalized n-dimensional triangle called n-simplex.

**Definition 3.1** (Hatcher[6])**.** A **n-simplex** is defined as:

$$\Delta^n = \{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1 \; and \; t_i \geq 0 \; for \; all \; i\}$$

An n-simplex can be represented by its vertices, $[v_0, ..., v_n]$

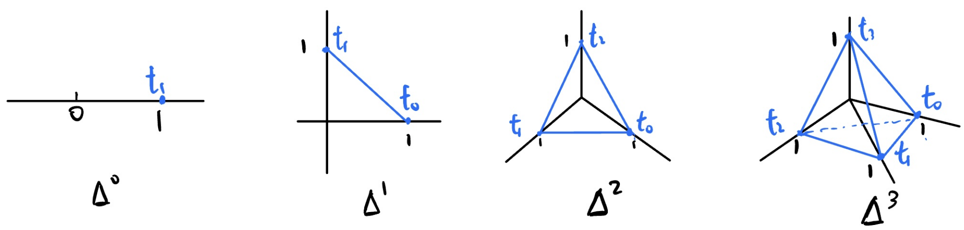


**Figure 3.1** n-simplices for $n \leq 3$

If we delete one of the n+1 vertices of an n-simplex, the remaining vertices form an (n-1)-simplex, called the **face** of the n-simplex, written as $[v_0, ..., \hat{v}_i, ..., v_n]$, where $\hat{v}_i$ is the vertex being taken out. Then the union of all faces of an n-simplex becomes its boundary $\partial\Delta^n$. The **open simplex** $\mathring{\Delta}^n$ is $\Delta^n - \partial\Delta^n$. For example, consider the 3-simplex illustration above. The boundaries of a tetrahedron are its four triangular faces. The boundaries of a line segment are $t_0, t_1$, and the open 3-simplex is the interior of the tetrahedron. The boundary of $\Delta^0$ is 0, i.e. it has no boundary.

**Definition 3.2** (Hatcher[6])**.** A $\Delta$-complex structure on a space X is a collection of maps $\sigma_\alpha$ : $\Delta^n \to X$, with n depending on the index $\alpha$, such that:

(i) The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X is in the image of one such restriction.

(ii) Each restriction of $\sigma_\alpha$ to a face of $\Delta^n$ is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$.

(iii) A set $A \in X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in $\Delta^n$ for each $\sigma_\alpha$.

After defining what a Delta-complex is, we can compute its homology group.

## 3.2 Homology Group

Let $\Delta_n(X)$ be the free abelian group with basis $\mathring{\Delta}^n$ of X. Elements of $\Delta^n(X)$, called **n-chains**, can be written as the sum $\sum_\alpha n_\alpha \mathring{\Delta}^n$ with coefficients $n_\alpha \in \mathbb{Z}$.

**Definition 3.3** (Hatcher[6])**.** For a Delta-complex in X the **boundary homomorphism** $\partial_n$ : $\Delta^n(X) \to \Delta^{n-1}(X)$ is:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|[v_0, ..., \hat{v}_i, ..., v_n]$$

**Lemma 3.1.** *The composition $\partial_{n-1}\partial_n$ is zero.*

*Proof: We have*

$$\partial_{n-1}\partial_n = \sum_{i<j}(-1)^i(-1)^{j-1}\sigma_\alpha|[v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_n] + \sum_{i>j}(-1)^i(-1)^j\sigma_\alpha|[v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_n]$$

*Notice that the two terms cancel out because, after switching the i and j in the first term, they differed by a factor of -1.* □

Thus we have free abelian groups linked by boundary homomorphism:

$$... \xrightarrow{\partial_{n+1}} C_{n+1} \xrightarrow{\partial_n} C_n \xrightarrow{\partial_{n-1}} C_{n-1} \to ... \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n\partial_{n+1} = 0$. This is called **chain complex**. The equation $\partial_n\partial_{n+1} = 0$ implies the inclusion $\operatorname{im}\partial_{n+1} \subset \ker\partial_n$.

**Definition 3.4** (Hatcher[6])**.** The nth **homology group** of the chain complex is the quotient group

$$H_n = \ker\partial_n / \operatorname{im}\partial_{n+1}$$

Elements of $\ker\partial_n$ are called **cycles** and elements of $\operatorname{im}\partial_{n+1}$ are called **boundaries**. Two cycles are **homologous** if they are in the same homology group, i.e. they differ only by the boundaries.

In the case of simplicial homology, we are interested in computing the **simplicial homology group**. Then the free abelian group $C_n$ becomes $\Delta_n(X)$.

## 3.3 Examples

**Example 3.5.** $X = S^1$.

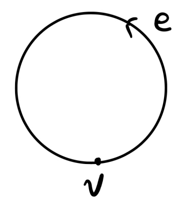


**Figure 3.2** 1-sphere.

The 1-sphere has 1 vertex $v$ and 1 edge $e$, so $\Delta_0(X) = \Delta_1(X) = \mathbb{Z}$. The boundary map $\partial_1$ is 0 since following the definition 3.3,

$$\partial e = v_1 - v_0 = v - v = 0$$

The group $\Delta_n(S^1)$ is 0 for $n \geq 2$ because there is no complex in dimension higher than 1. Thus, $\partial_n, n \geq 2$ is not defined. We have the homology group of a circle:

$$\begin{aligned} H_n(S^1) &= \ker \partial_n / \operatorname{im} \partial_{n+1} \\ &\approx \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2 \end{cases} \end{aligned}$$

Generally, if all boundary maps in a chain complex are 0, then the homology group equals the chain group itself.

**Example 3.6.** $X = T$. The torus has 1 vertex, 3 edges a,b, and c, and 2 2-simplices U and L (Fig.3.3). Imagine a continuous map from a square to a torus: first, connect the opposite sides a and a, then connect b and b. Notice in both cases, 2 edges have the same orientation. As a result, the torus has the same n-simplices as the square does. Since we can identify the opposite sides as one and all 4 vertices as one in the mapping, we say a torus has 1 vertex and 3 edges.

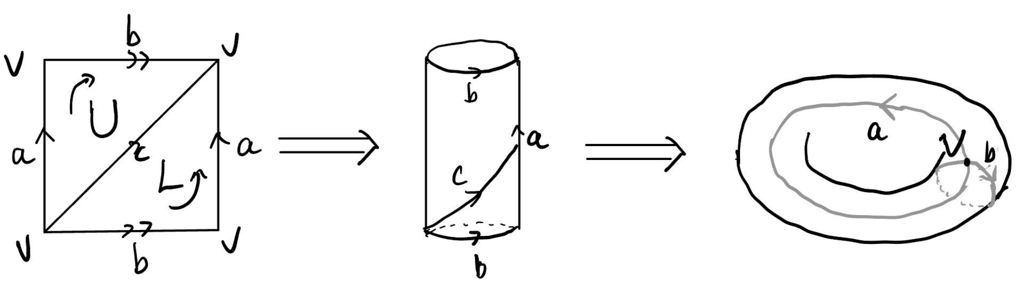


**Figure 3.3** n-simplices on a torus.

For all three edges, $\partial_1 = v - v = 0$. Together with the fact that the torus has 1 vertex, its homology group $H_0(T) \approx \mathbb{Z}/\emptyset \approx \mathbb{Z}$. Since $\partial_2 U = [a+b-c] = \partial_2 L$ and $[a, b, a+b-c]$ is a basis for $\Delta_1(T)$, $H_1(T) \approx \mathbb{Z} \times \mathbb{Z}$. We have $\ker \partial_2 = U - L$, an infinite cyclic group generated by $U - L$. Since there are no 3-simplices, $H_2(T)$ is isomorphic to $\ker \partial_2$. Thus,

$$\begin{aligned} H_n(T) &= \ker \partial_n / \operatorname{im} \partial_{n+1} \\ &\approx \begin{cases} \mathbb{Z} \times \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases} \end{aligned}$$

This example illustrates the utility of homology groups to calculate n-dimensional holes. The torus has two 1-d holes, traversed respectively by edges a and b, which correspond to its first homology group $\mathbb{Z} \times \mathbb{Z}$. It has a 2-d hole, i.e. its interior volume, and no holes in dimension 3 or greater.

**Example 3.7.** $X = \mathbb{R}P^2$, the real projective plane, with 2 vertices v and w, 3 edges a,b, and c, and 2 w-simplices U and L (Fig.3.4). Various representations of the projective plane exist, but here we adopt the most common construction: the space of lines in $\mathbb{R}^3$ which pass through the origin. The projective plane cannot be embedded in 3-dimensional Euclidean space, but we can use the techniques of simplicial homology to calculate its "holes" regardless.

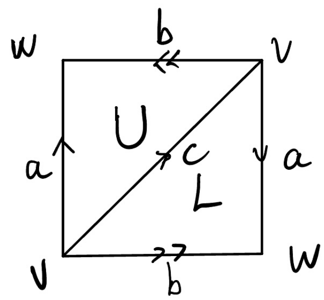


**Figure 3.4** triangulation of the projective plane.

We have $\ker \partial_0 = [v, w]$, and $\operatorname{im} \partial_1$ generated by $[w - v]$. Then we calculate $H_1(\mathbb{R}P^2)$: $\ker \partial_1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ with basis [c] and [a-b]. $\partial_2(U) = [a - b - c]$, $\partial_2(L) = [a - b + c]$. Alternatively, we can write the basis of $\ker \partial_1$ as [a-b+c] and [c], and the basis of $\operatorname{im} \partial_2$ as [a-b+c] and $[2c] = [a - b + c] - [a - b - c]$. Then $\operatorname{im} \partial_2$ is an index-2 subgroup of $\ker \partial_1$, and $H_1(\mathbb{R}P^2) \approx \mathbb{Z}_2$.

Finally, we see that $\partial_2$ is injective, and therefore $\ker \partial_2 = 0$. Thus,

$$H_n(\mathbb{R}P^2) \approx \begin{cases} \mathbb{Z}_2 & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \geq 2 \end{cases}$$

# 4 Singular Homology

Indeed, not all topological spaces can be triangulated, so the simplicial homology cannot be defined for all general topological spaces. Therefore, we need to find a more general homology over the topological category extending the simplicial homology.

The key idea behind singular homology is that any topological space can be approximated arbitrarily well by simplicial complexes. By considering all possible continuous maps, singular homology takes into account a larger class of spaces and allows for a more flexible and powerful approach to studying their properties.

## 4.1 Singular chain complex

In a simplicial complex, geometric simplices of various dimensions are glued together according to specific rules. The complex is built by considering a finite set of vertices and forming simplices by taking subsets of these vertices. The simplices are then combined to form higher-dimensional simplices.

On the other hand, the singular complex does not rely on a predefined combinatorial structure. Instead, it considers all continuous maps from standard geometric simplices to the topological space under study. These continuous maps are called singular simplices.

**Definition 4.1.** A **singular $n$-simplex** in $X$ is a continuous map $\sigma : \Delta^n \to X$

**Definition 4.2.** A **singular chain complex** of $X$ is formed by a pair $(C_\bullet(X), \partial_\bullet)$ where

the **singular $n$-chains** $C_n(X)$ denote the free abelian group generated by all singular $n$-simplex in $X$,

and the **boundary maps** $\partial_n : C_n(X) \to C_{n-1}(X)$ are defined by linear extension of $\partial_n(\sigma) := \sum_{i=0}^{n}(-1)^i \sigma \circ d_i$ where the **face maps** $d_i : \Delta^{n-1} \to \Delta^n$ are defined by $d_i(t_0, \ldots, t_{n-1}) := (t_0, t_1 \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$

Then a singular chain complex can be shown as a diagram of listed singular $n$-chains chained by their boundary maps:

$$\ldots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

## 4.2 Singular homology group

**Lemma 4.1.** $\partial_n \circ \partial_{n+1} = 0$

*Proof.* For $\sigma : \Delta^n \to X$,

$$\begin{aligned}
\partial_n(\partial_{n+1}\sigma) = \partial_n\left(\sum_{i=0}^{n+1}(-1)^i\sigma\circ d_i\right) &= \sum_{i=0}^{n+1}\sum_{j\neq i}(-1)^j(-1)^i\sigma\circ d_i\circ d_j \\
&= \sum_{i=0}^{n+1}\sum_{j=0}^{i-1}(-1)^{i+j}\sigma\circ d_i\circ d_j + \sum_{i=0}^{n+1}\sum_{j=i+1}^{n+1}(-1)^{i+j}\sigma\circ d_i\circ d_j \\
&= \sum_{i=0}^{n+1}\sum_{j=0}^{i-1}(-1)^{i+j}\sigma\circ d_i\circ d_j + \sum_{i=0}^{n+1}\sum_{j=i+1}^{n}(-1)^{i+j-1}\sigma\circ d_j\circ d_i
\end{aligned}$$

Similar to Lemma 3.1, each two terms can be cancelled, implying $\partial_n(\partial_{n+1}\sigma) = 0$ □

**Lemma 4.2.** $\operatorname{im}\partial_{n+1} \trianglelefteq \ker\partial_n$

*Proof.* For any $\partial_{n+1}\sigma \in \operatorname{im}\partial_{n+1}$, we know $\partial_n\partial_{n+1}\sigma = 0\sigma = 0$, so $\partial_{n+1}\sigma \in \ker\partial_n$, implying $\operatorname{im}\partial_{n+1} \leq \ker\partial_n$.
Also, since $C_n(X)$ is abelian, all its subgroups are normal trivially. □

**Definition 4.3.** The $n$-th homology group is defined as $H_n(X) := \ker\partial_n/\operatorname{im}\partial_{n+1}$

In fact, if a space is a simplicial complex, then its singular homology is isomorphic to its simplicial homology. Therefore, singular homology can be regard as an extension of simplicial homology while usually the simplicial homology is easier to compute.

## 4.3 Induced homomorphism

**Definition 4.4.** Given continuous map $f : X \to Y$, a chain map induced by $f$ is a family of homomorphisms $f_\# : C_n(X) \to C_n(Y)$, $f_\#(\sigma) := f\circ\sigma$

**Lemma 4.3.** *$f_\#$ are homomorphisms.*

*Proof.* For $\sigma, \tau \in C_n(X)$, we have $f_\#(\sigma+\tau)(x) = (f\circ(\sigma+\tau))(x) = f((\sigma+\tau)(x)) = f(\sigma(x)+\tau(x)) = f(\sigma(x)) + f(\tau(x)) = (f\circ\sigma + f\circ\tau)(x) = (f_\#(\sigma) + f_\#(\tau))(x)$, so $f_\#(\sigma+\tau) = f_\#(\sigma) + f_\#(\tau)$.
Thus, $f_\#$ is an abelian group homomorphism. □

**Lemma 4.4.** *The following diagram commutes:*

$$\begin{array}{ccccccccccccc}
\dots \xrightarrow{\partial_{X,n+2}} & C_{n+1}(X) & \xrightarrow{\partial_{X,n+1}} & C_n(X) & \xrightarrow{\partial_{X,n}} & C_{n-1}(X) & \xrightarrow{\partial_{X,n-1}} \dots \xrightarrow{\partial_{X,2}} & C_1(X) & \xrightarrow{\partial_{X,1}} & C_0(X) & \xrightarrow{\partial_{X,0}} 0 \\
& \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} & \\
\dots \xrightarrow{\partial_{Y,n+2}} & C_{n+1}(Y) & \xrightarrow{\partial_{Y,n+1}} & C_n(Y) & \xrightarrow{\partial_{Y,n}} & C_{n-1}(Y) & \xrightarrow{\partial_{Y,n-1}} \dots \xrightarrow{\partial_{Y,2}} & C_1(Y) & \xrightarrow{\partial_{Y,1}} & C_0(Y) & \xrightarrow{\partial_{Y,0}} 0
\end{array}$$

*Proof.* For each $n \geq 0$, for $\sigma \in C_n(X)$, $(\partial_{Y,n} \circ f_{\#})(\sigma) = \partial_{Y,n}(f_{\#}(\sigma)) = \partial_{Y,n}(f \circ \sigma) = \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ d_i = \sum_{i=0}^{n}(-1)^i f_{\#}(\sigma \circ d_i) = f_{\#}(\sum_{i=0}^{n}(-1)^i \sigma \circ d_i) = f_{\#}(\partial_{X,n}(\sigma)) = (f_{\#} \circ \partial_{X,n})(\sigma)$,
so $\partial_{Y,n} \circ f_{\#} = f_{\#} \circ \partial_{X,n}$ and the diagram commutes. $\square$

Here, a chain map preserves the boundary relations between the chains in the two complexes. It ensures that if a singular chain $\sigma$ in $X$ has a boundary in $X$, then its image under $f_{\#}$ will have a corresponding boundary in $Y$, i.e. $f_{\#}\partial = \partial f_{\#}$. This property allows the chain map to induce homomorphisms between the homology groups of the two complexes.

**Definition 4.5.** Given continuous map $f : X \to Y$, the induced homomorphism between the homology groups are defined by $f_* : H_n(X) \to H_n(Y)$ linearly extending $f_*([\sigma]) := [f_{\#}(\sigma)]$.

**Example 4.6.** Consider the injective continuous map $f : S^1 \hookrightarrow S^2$. We have

By employing the induced homomorphism, we gain valuable insights into the relationship between the topological properties of spaces and their algebraic invariants (which is homology group here). It enables us to compare and contrast the homology groups of different spaces, analyze the impact of continuous maps on the algebraic structure, and establish connections between topological characteristics and algebraic properties.

## 4.4 Relative homology groups

Relative homology allows us to study the homology of a topological space relative to a subspace. It provides a way to capture the topological properties of a space in relation to another space.

**Definition 4.7.** Given topological space $X$ with $A \subseteq X$, define $C_n(X, A) := C_n(X)/C_n(A)$.

Intuitively, this means we consider chains in $X$ that can be freely deformed within $A$ without changing their homology class.

**Lemma 4.5.** *$(C_\bullet(X,A),\partial_\bullet)$ is a chain complex where $\partial_n : C_n(X,A) \to C_{n-1}(X,A)$ linearly extending $\partial_n[\sigma] = [\partial_{X,n}\sigma]$*

*Proof.* Since $C_n(A)$ is a normal subgroup of $C_n(X)$, we know $\partial_{X,n}C_n(A) \subseteq C_{n-1}(A)$. Then for $\sigma' \in [\sigma]$, we have $\sigma' - \sigma \in C_{n-1}(A)$, so $\partial_n[\sigma'] = [\partial_{X,n}\sigma'] = [\partial_{X,n}\sigma + \partial_{X,n}(\sigma' - \sigma)] = [\partial_{X,n}\sigma] = \partial_n[\sigma]$. Therefore, $\partial_n$ are well-defined.

For $[\sigma],[\tau] \in C_n(X,A)$, we have $\partial_n([\sigma]+[\tau]) = \partial_n([\sigma+\tau]) = [\partial_{X,n}(\sigma+\tau)] = [\partial_{X,n}\sigma + \partial_{X,n}\tau] = [\partial_{X,n}\sigma] + [\partial_{X,n}\tau] = \partial_n([\sigma]) + \partial_n([\tau])$. Therefore, $\partial_n$ are abelian group homomorphisms.

For $[\sigma] \in C_n(X,A)$, we know $\partial_{n+1}\partial_n[\sigma] = \partial_{n+1}[\partial_{x,n}\sigma] = [\partial_{X,n+1}\partial_{x,n}\sigma] = [0]$.

Thus, $(C_\bullet(X,A),\partial_\bullet)$ is a chain complex. □

**Definition 4.8.** Given topological space $X$ with $A \subseteq X$, the relative homology groups $H_n(X,A)$ on the pair $(X,A)$ are the homology groups induced by the chain complex $(C_\bullet(X,A),\partial_\bullet)$.

One way to think about relative homology is that it "ignores" the subspace $A$, focusing instead on the "new" structure in $X$ not already in $A$. This is particularly useful when $A$ is a boundary of $X$, as relative homology can then provide information about the "interior" of $X$.

**Example 4.9.** Let $D^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$ be an $n$-dimensional disk, and let $\partial D^n = S^{n-1}$ be the boundary of $D^n$. Then, we have

$$H_i(D^n, \partial D^n) \approx \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{o.w.} \end{cases}$$

# 5 Conclusion

In conclusion, this project has provided a detailed exploration of various concepts in algebraic topology, starting with fundamental mathematical theories such as topological space and group theory. The project then progressed to more complex ideas such as simplicial and singular homology, making use of examples to ensure clarity. The introduction of the simplicial complex and the homology group has helped to shed light on the concept of simplicial homology. Additionally, the project has delved into the definition of the singular chain complex, singular homology group, induced homology, and relative homology groups. This comprehensive study serves as a stepping stone into the vast field of algebraic topology. Overall, the project has fulfilled its aim of making these complex concepts more understandable for students and individuals interested in the field, laying a firm groundwork for further studies in algebraic topology.

# 6 Bibliography

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