Simplicial Homology and Singular Homology

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Topology

Topology

Definition A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

- $\varnothing$ and X are in $\mathcal{T}$.
- Union of arbitrary subcollections of $\mathcal{T}$ is in $\mathcal{T}$.
- Intersection of finite subcollections of $\mathcal{T}$ are in $\mathcal{T}$.

A set X with a topology $\mathcal{T}$ specified, $(X, \mathcal{T})$, is called a topological space.

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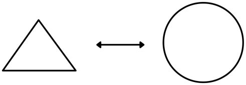
Example Let $X = \{a, b, c\}$, let $\mathcal{T} := \{\emptyset, X\}$ be the trivial topology on X.

Topology

Definition Let $(X, \mathcal{T})$, $(X', \mathcal{T}')$ be two topological spaces. A function $f : X \to X'$ is *continuous* if for any $U \in \mathcal{T}'$, $f^{-1}(U) \in \mathcal{T}$. Moreover, if there exists a function $g : X' \to X$ such that f, g are inverse of each other and g is continuous, then f, g are called *homeomorphisms*, and we say X, X' are *homeomorphic*.

Topology

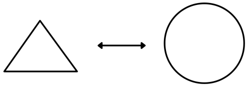
Examples of homeomorphic spaces



A triangle is homeomorphic to a circle.

Topology

Examples of homeomorphic spaces



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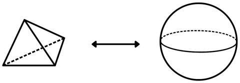


A tetrahedron is homeomorphic to a sphere.

CW Complex

Definition

An n-dimensional cell is homeomorphic to an n-dimensional disk
$D^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$
and its boundary is $(n-1)$-dimensional sphere
$S^{n-1} := \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : x_1^2 + \cdots + x_{n-1}^2 = 1\}$

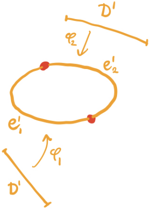
Definition

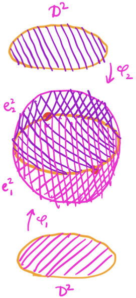
A topological space X is called a CW complex if:

1. We begin with a discrete set X^0. The space X is the union of an increasing sequence of subspaces X^n, where each X^n is built from X^{n-1} by attaching n-dimensional cells. X^n is called the n-th skeleton of X.
2. Each n-dimensional cell e^n_α in the space X is attached by a map $\phi_\alpha : D^n \to X^n$ from the closed n-dimensional ball to the space X in such a way that the image of the interior of D^n is the interior of the cell and the image of the boundary of D^n is contained in X^{n-1}.

CW Complex

$X = S^2$





$X^0 = \{e_1^0, e_2^0\}$

$$X^1 = \frac{\{e_1^1, e_2^1\} \cup X^0}{\forall i.\ x \sim \varphi_i(x)}$$

$$X^2 = \frac{\{e_1^2, e_2^2\} \cup X^1}{\forall i.\ x \sim \varphi_i(x)}$$

two 0-cells

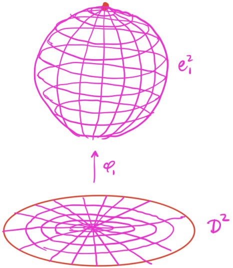
two 1-cells

two 2-cells

CW Complex

$X = S^2$

$\cdot\, e_1^0$



$X^0 = \{e_1^0\}$

one 0-cells

$X^1 = X^0$

no 1-cells

$$X^2 = \frac{\{e_1^2\} \cup X^1}{x \sim f(x)}$$

one 2-cells

Simplicial Homology

Simplicial Homology

How many holes are there in a pair of pants?

Motivation: The homology group of a topological space provides a rigorous definition of its "holes". Intuitively, an n-dimensional hole represents a fundamental n-dimensional cycle that can't be filled in with (n+1)-dimensional "simplices".

What is a simplex?

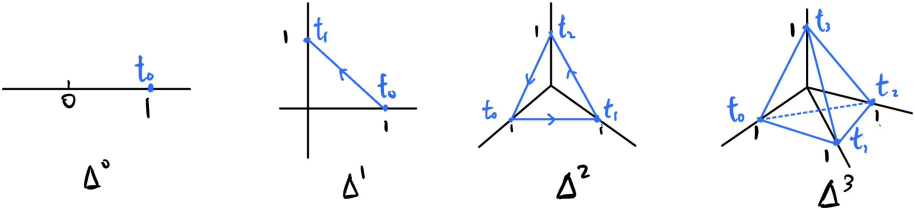
Simplicial Homology

Definition An n-simplex is the set of all points of the form:

$$\Delta^n = \{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1 \; and \; 0 \leq t_i \leq 1 \; for \; all \; i\}$$

To keep track of the ordering of its vertices, we write $[v_0, ..., v_n]$.

Example



Simplicial Homology

If we delete one of the vertices of an n-simplex, the remaining vertices span an (n-1)-simplex, called the **face** of the n-simplex: $[v_0, ..., \hat{v}_i, ..., v_n]$.

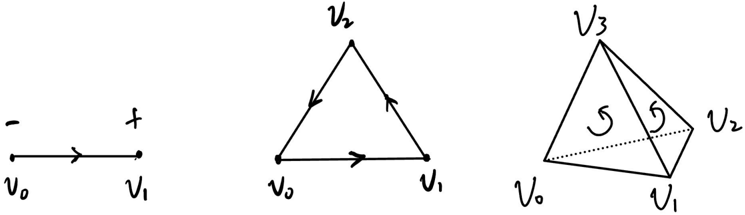
The union of all faces of an n-simplex becomes its **boundary** $\partial\Delta^n$.

Let $C_n(X)$ be the free abelian group with basis the open n-simplices $\mathring{\Delta}^n$ of X. Elements of $C_n(X)$, called **n-chains**, can be written as the sum $\sum_\alpha n_\alpha \mathring{\Delta}^n$ with coefficients $n_\alpha \in \mathbb{Z}$.

Simplicial Homology

Definition: The **boundary homomorphism** $\partial_n : C_n(X) \to C_{n-1}(X)$ is:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n]$$



Simplicial Homology

Intuitively, a closed loop forms a hole, except when the loop is the boundary of a higher-dimensional simplex, i.e. its interior is not hollow.

The nth **homology group** of the chain complex is the quotient group

$$H_n = \ker \partial_n / im\, \partial_{n+1}$$

Elements of $\ker \partial_n$ are called cycles and elements of $im\, \partial_{n+1}$ are called boundaries.

Simplicial Homology

Example: ...

Singular Homology

Singular Simplex

Definition

A **singular** n-**simplex** in X is a continuous map $\sigma : \Delta^n \to X$

- "singular" means σ is quite flexible

Singular Chain Complex

Definition

A **singular chain complex** of X is formed by a pair $(C_\bullet(X), \partial_\bullet)$ where

- the **singular** n**-chains** $C_n(X)$ is the free abelian group generated by all singular n-simplex in X
- **face maps** $d_i : \Delta^{n-1} \to \Delta^n$,
  $d_i(t_0, \ldots, t_{n-1}) := (t_0, t_1 \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$
- **boundary maps** $\partial_n : C_n(X) \to C_{n-1}(X)$, $\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma \circ d_i$

- $C_n(X)$ can be really large!

$$\ldots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

Singular Homology

- $\partial_n \circ \partial_{n+1} = 0$
- $\mathrm{im}\partial_{n+1} \trianglelefteq \ker \partial_n$

Definition

The n-th singular homology group is $H_n(X) := \ker \partial_n / \mathrm{im}\partial_{n+1}$

Induced Homomorphisms

Definition

Given continuous map $f : X \to Y$, a chain map induced by f is a family of homomorphisms $f_\# : C_n(X) \to C_n(Y)$, $f_\#(\sigma) := f \circ \sigma$

The following diagram commutes:

$$\begin{array}{ccccccccccccc} \dots & \xrightarrow{\partial_{X,n+1}} & C_n(X) & \xrightarrow{\partial_{X,n}} & C_{n-1}(X) & \xrightarrow{\partial_{X,n-1}} & \dots & \xrightarrow{\partial_{X,2}} & C_1(X) & \xrightarrow{\partial_{X,1}} & C_0(X) & \xrightarrow{\partial_{X,0}} & 0 \\ & & \downarrow f_\# & & \downarrow f_\# & & & & \downarrow f_\# & & \downarrow f_\# & & \\ \dots & \xrightarrow{\partial_{Y,n+1}} & C_n(Y) & \xrightarrow{\partial_{Y,n}} & C_{n-1}(Y) & \xrightarrow{\partial_{Y,n-1}} & \dots & \xrightarrow{\partial_{Y,2}} & C_1(Y) & \xrightarrow{\partial_{Y,1}} & C_0(Y) & \xrightarrow{\partial_{Y,0}} & 0 \end{array}$$

Definition

Given continuous map $f : X \to Y$, the induced homomorphisms between the homology groups are defined by $f_* : H_n(X) \to H_n(Y)$, $f_*([\sigma]) := [f_\#(\sigma)]$.

Relative Homology

Definition

Given topological space X with $A \subseteq X$, define $C_n(X, A) := C_n(X)/C_n(A)$.

- $(C_\bullet(X, A), \partial_\bullet)$ is a chain complex where $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$ linearly extending $\partial_n[\sigma] = [\partial_{X,n}\sigma]$

Definition

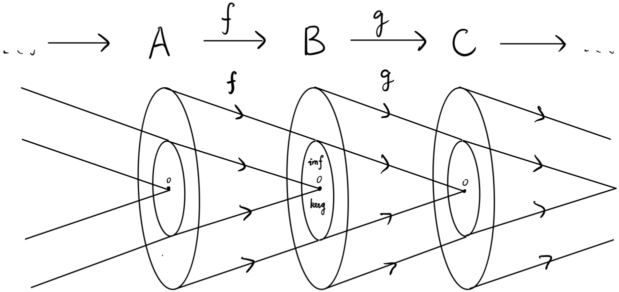
Given topological space X with $A \subseteq X$, the relative homology groups $H_n(X, A)$ on the pair (X, A) are the homology groups induced by the chain complex $(C_\bullet(X, A), \partial_\bullet)$.

Exact Sequence

Definition

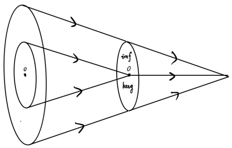
A sequence of homomorphism $\ldots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \ldots$ is **exact** if $\ker f_n = \mathrm{im} f_{n+1}$ for each n.

- An exact sequence is also a chain complex.
- The homology groups of an exact sequence is trivial.



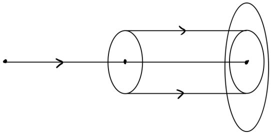
Exact Sequence

$$A \xrightarrow{f} B \xrightarrow{g} 0$$



$$\ker g = B$$
$$\Downarrow$$
$$\operatorname{im} f = \ker g = B$$
$$\Downarrow$$
f is surjective

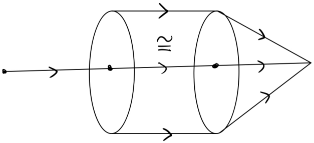
$$0 \xrightarrow{f} A \xrightarrow{g} B$$



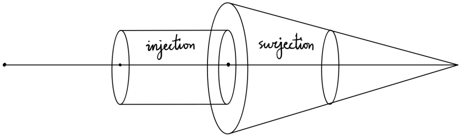
$$\operatorname{im} f = 0$$
$$\Downarrow$$
$$\ker g = \operatorname{im} f = 0$$
$$\Downarrow$$
g is injective

Exact Sequence

$$0 \longrightarrow A \xrightarrow{\sim} B \longrightarrow 0$$



$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \longrightarrow 0$$



Long Exact Sequence

Theorem

Given a CW complex X, if $A \subseteq X$ is also a CW complex, then we have a long exact sequence:

$$\ldots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \ldots \quad \textit{where}$$

- i_* *is induced by the embedding* $i : C_n(A) \to C_n(X)$
- j_* *is induced by the canonical projection* $j : C_n(X) \to C_n(X,A)$
- $\partial : C_n(X,A) \to C_{n-1}(A)$ *is the boundary map induced from the following diagram:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r] & C_{n+1}(A) \arrow[r, "\partial_{n+1}"] \arrow[d, "i"] & C_n(A) \arrow[r, "\partial_n"] \arrow[d, "i"] & C_{n-1}(A) \arrow[r] \arrow[d, "i"] & \ldots \\
\ldots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "j"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "j"] & C_{n-1}(X) \arrow[r] \arrow[d, "j"] & \ldots \\
\ldots \arrow[r] & C_{n+1}(X,A) \arrow[r, "\partial_{n+1}"] \arrow[d] & C_n(X,A) \arrow[r, "\partial_n"] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & \ldots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Long Exact Sequence

$$
\begin{tikzcd}
 & 0 \arrow[d, hook] & 0 \arrow[d, hook] & 0 \arrow[d, hook] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, hook, "i"] & A_n \arrow[r, "\partial"] \arrow[d, hook, "i"] & A_{n-1} \arrow[r] \arrow[d, hook, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

a

$b \mapsto \partial b$

c

$$\cdots \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{j_*} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} \cdots$$

$$\ker\partial_{n+1}/\operatorname{im}\partial_{n+2} \to \ker\partial_n/\operatorname{im}\partial_{n+1}$$

$\forall c \in \ker\partial_{n+1}$. $\exists b \in B_{n+1}$. $c = j(b)$

$j\partial(b) = \partial j(b) = \partial c = 0$

$\Rightarrow \partial b \in \ker j = \operatorname{im} i$

$\Rightarrow \exists a \in A_n$. $i(a) = \partial b$

$\Rightarrow \partial[c] := \partial[a]$

Let $b' \in B_{n+1}$ s.t. $j(b') = j(b) = c$

$\Rightarrow j(b' - b) = j(b') - j(b) = 0$

$\Rightarrow b' - b \in \ker j = \operatorname{im} i$

$\Rightarrow \exists a' \in A_n. b' - b = i(a')$

Cellular Chain Complex

$$
\begin{tikzcd}
 & & & & & H_n(X^{n+1},X^n) & & & & H_{n-2}(X^{n-1},X^{n-2}) \\
 & & H_n(X^{n-1}) \arrow[dr, "i"] & & H_n(X^{n+1}) \arrow[ur, "j"] & & & & H_{n-2}(X^{n-1}) \arrow[ur, "j"] & \\
 & & & H_n(X^n) \arrow[ur, "i"] \arrow[dr, "j"] & & & & H_{n-2}(X^{n-2}) \arrow[ur, "i"] & & \\
\dots \arrow[rr, "d_{n+2}"] & & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[dr, "\partial_n"] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1},X^{n-2}) \arrow[ur, "\partial_{n-1}"] \arrow[rr, "d_{n-1}"] & & \dots & \\
 & H_{n+1}(X^{n+1}) \arrow[ur, "j"] & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j"] \arrow[dr, "i"] & & & & \\
H_{n+1}(X^n) \arrow[ur, "i"] & & & & H_{n-1}(X^{n-2}) \arrow[ur, "i"] & & H_{n-1}(X^n) \arrow[dr, "j"] & & & \\
 & & & & & & & H_{n-1}(X^n,X^{n-1}) & &
\end{tikzcd}
$$

Thank you!