Part 3

1) $$\frac{dg(t)}{dt} = -k_4 g(t) - k_6 i(t) + A(t) \quad \text{——— (1)}$$

$$\frac{di(t)}{dt} = k_3 g(t) - k_1 i(t) + B(t) \quad \text{——— (2)}$$

Differentiating (1) w.r.t. t;

$$\frac{d^2 g(t)}{dt} = -k_4 \frac{d[g(t)]}{dt} - k_6 \frac{d[i(t)]}{dt} + \frac{d[A(t)]}{dt}$$

From Glucose Tolerance Test, it can be assumed that $A(t) = a\,u(t)$ & $B(t) = 0$

$$\frac{d^2[g(t)]}{dt} = -k_4 \frac{d[g(t)]}{dt} - k_6 \frac{d[i(t)]}{dt} + \frac{d[a\,u(t)]}{dt} \quad \text{——— (3)}$$

From (2); $$\frac{d[i(t)]}{dt} = k_3 g(t) - k_1 i(t) + 0 \quad \text{— (4)}$$

Substituting (4) to (3);

$$\frac{d^2[g(t)]}{dt} = -k_4 \frac{d[g(t)]}{dt} - k_6 [k_3 g(t) - k_1 i(t)] + a \frac{d[u(t)]}{dt}$$

$$\frac{d^2 g}{dt} = -k_4 \frac{dg}{dt} - k_3 k_6 g(t) + k_1 k_6 i(t) + a \frac{du}{dt} \quad \text{— (5)}$$

From (1); $$k_6 i(t) = -\frac{dg}{dt} - k_4 g(t) + \underbrace{A(t)}_{= a u(t)}$$

Substituting (6) to (5);

$$\frac{d^2g}{dt} = -k_4\frac{dg}{dt} - k_3k_6g(t) + k_1\left[\frac{-dg}{dt} - k_4g(t) + au(t)\right] + \frac{adu}{dt}$$

$$\frac{d^2g}{dt} + (k_1+k_4)\frac{dg}{dt} + (k_1k_4 + k_3k_6)g = k_1au(t) + \frac{adu}{dt}$$

$$\frac{d^2g}{dt} + (k_1+k_4)\frac{dg}{dt} + (k_1k_4 + k_3k_6)g = k_1a + \frac{ad[u(t)]}{dt}$$

Typical values;

$k_1 = 0.8\ h^{-1}$, $k_3 = 0.2\ IU/h/g$, $k_4 = 2h^{-1}$

$k_6 = 5g/h/IU$, $a = 1g/l/h$

$$\frac{d^2g}{dt} + 2.8\frac{dg}{dt} + 2.6g = 0.8 + \cancel{1.\frac{d[1]}{dt}}^{0}$$

$$\frac{d^2g}{dt} + 2.8\frac{dg}{dt} + 2.6g = 0.8$$

Solution is in the form; $g(t) = g_c(t) + g_p(t)$

Complementary Function

$m^2 + 2.8m + 2.6 = 0$

$m = \dfrac{-7+4i}{5}$ $\qquad$ $m = \dfrac{-7-4i}{5}$

$= -1.4 + 0.8i$ $\qquad$ $= -1.4 - 0.8i$

$$g_c(t) = e^{-1.4t}[M\cos(0.8t) + N\sin(0.8t)] \quad \text{—— (I)}$$

Assume $g_p(t) = k$

$0 + 0 + 2.6k = 0.8 \Rightarrow k = \dfrac{4}{13}$

$$g(t) = e^{-1.4t}\left[M\cos(0.8t) + N\sin(0.8t)\right] + 4/13$$

from initial conditions; $g(0)=0$

$$0 = M + 0 + \frac{4}{13} \Rightarrow M = -\frac{4}{13}.$$

$$g'(t) = e^{-1.4t}\left[-0.8M\sin(0.8t) + 0.8N\cos(0.8t)\right] - 1.4e^{-1.4t}\left[M\cos(0.8t) + N\sin(0.8t)\right] + 0$$

$g'(0)=1$; $1 = 0.8N(1) - 1.4M(1)$

$$N = \frac{37}{52}$$

$$g(t) = e^{-1.4t}\left[\frac{-4}{13}\cos(0.8t) + \frac{37}{52}\sin(0.8t)\right] + \frac{4}{13}$$

$$i(t) = \frac{1}{k_6}\left[-k_4 g - \frac{dg}{dt} + a u(t)\right].$$

$$= e^{-1.4t}\left[-\frac{1}{13}\cos(0.8t) - \frac{7}{52}\sin(0.8t)\right] + \frac{1}{13}u(t)$$

2) $\frac{dG}{dt} = k_5 + A(t) - k_4 G - k_6 I + k_{10} G_n(t)$ — ① (Glucose)

$\frac{dI}{dt} = k_2 + k_3 G + B(t) - k_1 I$ — ② (Insulin)

$\frac{dG_n}{dt} = k_8 + C(t) + k_9 G - k_7 G_n$ — ③ (Glucogen)

In equlibrium state ; $\frac{dG}{dt} = 0$, $\frac{dI}{dt} = 0$, $\frac{dG_n}{dt} = 0$

$$k_5 = k_4 G_0 + k_6 I_{10} - k_{10} G_{n0}$$
$$k_2 = k_1 I_0 - k_3 G_0$$
$$k_8 = k_7 G_{n0} - k_4 G_0$$

Let $i = I - I_0$, $g = G - G_0$, $g_n = G_n - G_{n0}$

Assume;

$A(t) = a\,u(t)$

$B(t) = 0$

$C(t) = 0$

$$\frac{dg}{dt} = \frac{dG}{dt} \quad [\because g = G - G_0]$$
$$\frac{di}{dt} = \frac{dI}{dt} \quad [\because i = I - I_0]$$
$$\frac{dg_n}{dt} = \frac{dG_n}{dt} \quad [\because g_n = G_n - G_{n0}]$$

$$\frac{dg}{dt} = k_5 + a u(t) - k_4 G - k_6 I + k_{10} G_n(t)$$
$$= k_5 + a u(t) - k_4[g + G_0] - k_6[i + I_0] + k_{10}[g_n + G_{n0}]$$
$$= k_5 + a u(t) - k_4 g - k_4 G_0 - k_6 i - k_6 I_0 + k_{10} g_n + k_{10} G_{n0}$$
$$= -k_4 g - k_6 i + k_{10} g_n + a u(t) + k_5 - \underbrace{[k_4 G_0 + k_6 I_0 - k_{10} G_{n0}]}_{= k_5}$$

$$\frac{dg}{dt} = -k_4 g(t) - k_6 i(t) + k_{10} g_n(t) + a u(t) \quad \text{—— (I)}$$

Similarly we can derive that;

$$\frac{di(t)}{dt} = k_3 g(t) - k_1(t) \quad \text{—— (II)}$$

$$\frac{dg_n(t)}{dt} = k_9 g(t) - k_7 g_n(t) \quad \text{—— (III)}$$

Initial conditions; $g(0) = 0$, $i(0) = 0$, $g_n(0) = 0$

$$\begin{bmatrix} \frac{dg(t)}{dt} \\ \frac{di(t)}{dt} \\ \frac{dg_n(t)}{dt} \end{bmatrix} = \begin{bmatrix} -k_4 & -k_6 & k_{10} \\ k_3 & -k_1 & 0 \\ k_9 & 0 & -k_8 \end{bmatrix} \begin{bmatrix} g(t) \\ i(t) \\ g_n(t) \end{bmatrix} + \begin{bmatrix} au(t) \\ 0 \\ 0 \end{bmatrix}$$