Worksheet # 02.

Homogenous D.E's.

① $(y^2 - xy)\,dx + x^2 dy = 0$

Sol:- $x^2 dy = -(y^2 - xy)\,dx$

$$\frac{dy}{dx} = \frac{-(y^2 - xy)}{x^2}$$

Let $y = ux \implies \frac{dy}{dx} = \frac{du}{dx} \cdot x + u.$

$$\frac{du}{dx} \cdot x + u = -\frac{((ux)^2 - x(ux))}{x^2}$$

$$\frac{du}{dx}\cdot x + u = -\frac{(u^2x^2 - ux^2)}{x^2} = -\frac{\cancel{x^2}(u^2-u)}{\cancel{x^2}}$$

$$\frac{du}{dx}\cdot x = -(u^2-\cancel{u}) - \cancel{u} = -u^2$$

Integrating on b/s

$$+\int \frac{1}{u^2}\,du = -\int \frac{1}{x}\,dx$$

$$\frac{1}{2}\int \frac{2}{u^2}\,du = -\int \frac{1}{x}\,dx$$

$$\int u^{-2}\,du = -\ln x + c$$

$$\cancel{-}u^{-1} = \cancel{-}\ln x + c$$

$\because u = y/x$

$$\boxed{\frac{x}{y} = \ln x + c}$$ Ans

~~(3) $(x^2-y^2)dx+2xy$~~

(3) $x(y-x)\frac{dy}{dx} = y(y+x)$

$$\frac{dy}{dx} = \frac{y(y+x)}{x(y-x)}$$

let $y = vx \rightarrow \frac{dy}{dx} = \frac{dv}{dx}\cdot x + v$

$$\frac{dv}{dx}\cdot x + v = \frac{vx(vx+x)}{x(vx-x)}$$

$$\frac{dv}{dx}\cdot x + v = \frac{v^2x^2+vx^2}{vx^2-x^2}$$

$$\frac{dv}{dx}\cdot x + v = \frac{\cancel{x^2}(v^2+v)}{\cancel{x^2}(v-1)}$$

$$\frac{dv}{dx}\cdot x = \frac{v^2+v}{v-1} - v$$

$$\frac{dv}{dx}\cdot x = \frac{\cancel{v^2}+v-\cancel{v^2}+v}{v-1}$$

$$\frac{dv}{dx}\cdot x = \frac{2v}{v-1}$$

Integration on b/s

$$\int \frac{v-1}{2v}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\int 1\,dv - \int \frac{1}{v}\,dv = -\int \frac{1}{x}\,dx$$

$$v - \ln(v) = -2\ln x + c$$

$$\frac{y}{x} = \cancel{\phi} - \ln x^2 + \ln\left(\frac{y}{x}\right) + c$$

$$\frac{y}{x} - \ln\left(\frac{y}{\cancel{x}}\cdot x^{\cancel{2}}\right) = c$$

$$\boxed{\frac{y}{x} - \ln yx = c} \text{ Ans.}$$

(4) $x(x-y)\,dy + y^2\,dx = 0$

Sol:-

$$\frac{dy}{\cancel{dx}}\,x(x-y) = -y^2\,dx.$$

$$\frac{dy}{dx} = \frac{-y^2}{x(x-y)}$$

Let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}\cdot x + u$

$$\frac{du}{dx}\cdot x + u = \frac{-(ux)^2}{x(x-ux)}$$

$$\frac{du}{dx}\cdot x + u = \frac{-u^2x^2}{x^2 - ux^2}$$

$$\frac{du}{dx}\cdot x + u = \frac{-u^2\cancel{x^2}}{\cancel{x^2}(1-u)}$$

$$\frac{du}{dx}\cdot x = \frac{-u^2}{1-u} - u$$

$$\frac{du}{dx}\cdot x = \frac{-\cancel{u^2} - u + \cancel{u^2}}{1-u}$$

$$\frac{du}{dx}\cdot x = \frac{-u}{1-u}$$

Integration on b/s

$$\int \frac{1-u}{u}\,du = -\int \frac{1}{x}\,dx$$

$$\int \frac{1}{u}\,du - \int 1\,du = -\int \frac{1}{x}\,dx$$

$$\ln|u| - u = -\ln(x) + c$$

$$\ln\left(\frac{y}{x}\right) - \frac{y}{x} = -\ln(x) + c$$

$$\ln(y) - \cancel{\ln(x)} - \frac{y}{x} = \cancel{-\ln(x)} + c$$

$$-\frac{y}{x} = -\ln y + c$$

$$\frac{y}{x} = \ln y - c$$

or $\ln c$

$$\boxed{\frac{y}{x} = \ln\frac{y}{c}} \text{ Ans}$$

$$\int \frac{-2}{1+u^2}\,du + \int \frac{u}{1+u^2}\,du = \int \frac{1}{x}\,dx$$

Let $v = 1+u^2$

$dv = 2u\,du$

$u\,du = \frac{dv}{2}$

$$-2\tan^{-1}(u) + \frac{1}{2}\ln|1+u^2| = \ln|x| + c$$

$$\ln|1+u^2|^{\frac{1}{2}} - 2\tan^{-1}(u) = \ln x + c$$

$\because u = \frac{y}{x}$

$$\ln\left|1+\frac{y^2}{x^2}\right|^{\frac{1}{2}} - 2\tan^{-1}\left(\frac{y}{x}\right) - \ln x = c$$

$$\boxed{\ln\left|\frac{x^2+y^2}{x^2}\right|^{\frac{1}{2}} - 2\tan^{-1}\left(\frac{y}{x}\right) - \ln x = c}$$

(5) $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$

Sol:

$$\frac{dy}{dx} = \frac{2y-x}{2x-y}$$

Let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}\cdot x + u$

$$\frac{du}{dx}\cdot x + u = \frac{2(ux)-x}{2x-ux}$$

$$\frac{du}{dx}\cdot x + u = \frac{\cancel{x}(2u-1)}{\cancel{x}(2-u)}$$

$$\frac{du}{dx}\cdot x = \frac{2u-1}{2-u} - u$$

$$\frac{du}{dx}\cdot x = \frac{\cancel{2u}-1-\cancel{2u}-u^2}{2-u}$$

$$\frac{du}{dx}\cdot x = \frac{-1-u^2}{2-u}$$

Integration on b/s

$$\int \frac{-2+u}{1+u^2}\,du = \int \frac{1}{x}\,dx$$

$$\cancel{\ln}(\sin(u)) = \cancel{\ln}(x \cdot c)$$
$$u = y/x$$

$$\boxed{\sin\left(\frac{y}{x}\right) = x \cdot c} \text{ Ans.}$$

⑥ $\frac{dy}{dx} = \tan\left(\frac{y}{x}\right) + \frac{y}{x}$

Sol:

Let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot x + u$

or $u = \frac{y}{x}$

$$\frac{du}{dx} \cdot x + u = \tan(u) + u$$

$$\frac{du}{dx} \cdot x = \tan(u) + \cancel{u} - \cancel{u}$$

Integrating on b/s

$$\int \frac{1}{\tan(u)} du = \int \frac{1}{x} dx$$

$$\int \frac{\cos u}{\sin u} du = \ln(x) + c$$

Let $u = \sin u \Rightarrow du = \cos u \, dx$

$$\int \frac{\cancel{\cos u}}{u} \cdot \frac{du}{\cancel{\cos u}} = \ln x + c$$

$$\int \frac{1}{u} du = \ln x + c$$

$$\ln(u) = \ln x + \ln c$$

⑦ $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$

Let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot x + u$

$$\frac{du}{dx} \cdot x + u = \frac{3x(ux) + (ux)^2}{3x^2}$$

$$\frac{du}{dx} \cdot x + u = \frac{3x^2 u + u^2 x^2}{3x^2}$$

$$\frac{du}{dx} \cdot x + u = \frac{\cancel{x^2}(3u + u^2)}{3\cancel{x^2}}$$

$$\frac{du}{dx} \cdot x = \frac{3u + u^2}{3} - u$$

$$\frac{du}{dx} \cdot x = \frac{\cancel{3u} + u^2 - \cancel{3u}}{3}$$

$$\frac{du}{dx} \cdot x = \frac{u^2}{3}$$

Integration on b/s

$$3\int \frac{1}{u^2} du = \int \frac{1}{x} dx$$

$$-3u^{-1} = \ln x + c$$

$$-3\frac{x}{y} = \ln x + c$$

$$-3x = y\ln x + yc$$

$$\boxed{3x + y\ln x + yc = 0}$$ Ans.

⑧ $$\frac{dy}{dx} = \frac{x^2 - 2y^2}{2xy}$$

Let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}.x + u$

$$\frac{du}{dx}.x + u = \frac{x^2 - 2(ux)^2}{2x(ux)}$$

$$\frac{du}{dx}.x + u = \frac{x^2 - 2u^2x^2}{2x^2u}$$

$$\frac{du}{dx}.x + u = \frac{\cancel{x^2}(1 - 2u^2)}{2u\cancel{x^2}}$$

$$\frac{du}{dx}.x = \frac{1 - 2u^2}{2u} - u$$

$$\frac{du}{dx}.x = \frac{1 - 4u^2}{2u}$$

Integration on b/s

$$\int \frac{2u}{1 - 4u^2} du = \int \frac{1}{x} dx$$

$$-\frac{1}{4}\int \frac{-8u}{1 - 4u^2} du = \int \frac{1}{x} dx$$

$$-\frac{1}{4} \ln|1 - 4u^2| = \ln|x| + \ln c$$

$$\ln|1 - 4\left(\frac{y^2}{x^2}\right) = -4\ln x + \ln c$$

$$\cancel{\ln}\left(\frac{x^2 - 4y^2}{x^2}\right) = \cancel{\ln}\left(\frac{c}{x^4}\right)$$

$$x^2 - 4y^2 = \frac{x^2 . c}{x^{4\,2}}$$

$$\boxed{x^2 - 4y^2 = \frac{c}{x^2}}$$ Ans

⑨ $(x^2 + y^2)dy = xy\,dx$

Let:

$$y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}.x + u$$

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

$$\frac{du}{dx}.x + u = \frac{x(ux)}{x^2 + (ux)^2}$$

$$-\frac{1}{2\left(\frac{y}{x}\right)^2}+\ln\left(\frac{y}{x}\right)=-\ln x+c$$

$$-\frac{x^2}{2y^2}+\ln y-\cancel{\ln x}=-\cancel{\ln x}+c$$

$$\boxed{-\frac{x^2}{2y^2}+\ln y=c}$$ Ans.

$$\frac{du}{dx}\cdot x+u=\frac{ux^2}{x^2+u^2x^2}$$

$$\frac{du}{dx}\cdot x+u=\frac{u\cancel{x^2}}{\cancel{x^2}(1+u^2)}$$

$$\frac{du}{dx}\cdot x+u=\frac{u}{1+u^2}$$

$$\frac{du}{dx}\cdot x=\frac{u}{1+u^2}-u$$

$$\frac{du}{dx}\cdot x=\frac{\cancel{u}-\cancel{u}-u^3}{1+u^2}$$

$$\frac{du}{dx}\cdot x=\frac{-u^3}{1+u^2}$$

Integration on b/s

$$\int\frac{1+u^2}{u^3}du=-\int\frac{1}{x}dx$$

or

$$\int\frac{1}{u^3}du+\int\frac{1}{u}du=-\int\frac{1}{x}dx$$

$$-\frac{1}{2u^2}+\ln|u|=-\ln x+c$$

(10) $\left[x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right]y-\left[y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right]x\frac{dy}{dx}=0$

Sol:

$$\frac{dy}{dx}=\frac{\left[x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right]y}{\left[y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right]x}$$

Let $y=ux \Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot x+u$

$$\frac{du}{dx}\cdot x+u=\frac{[x\cos(u)+ux\sin(u)]ux}{[ux\sin(u)-x\cos(u)]x}$$

$$\frac{du}{dx}\cdot x+u=\frac{\cancel{x}[\cos(u)+u\sin(u)]u}{\cancel{x}[u\sin(u)-\cos(u)]}$$

$$\frac{du}{dx}\cdot x=\frac{u\cos(u)+u^2\sin(u)}{u\sin(u)-\cos(u)}-u$$

$$\frac{dy}{dx}\cdot x = \frac{u\cos(u) + \cancel{u^2\sin(u)} - \cancel{u^2\sin(u)} + u\cos(u)}{u\sin(u) \mp \cos(u)}$$

$$\frac{du}{dx}\cdot x = \frac{\cancel{u\cos(u) + u\cos} \; 2u\cos(u)}{u\sin(u) \mp \cos(u)}$$

<u>Integration on b/s</u>

$$\frac{1}{2}\int \frac{u\sin(u) \mp \cos(u)}{u\cos(u)}\,du = \int \frac{1}{x}\,dx.$$

$$-\frac{1}{2}\int \frac{1}{v}\,dv = \int \frac{1}{x}\,dx$$

Let $v = u\cos(u)$

$dv = \cos(u) + u(-\sin(u))$

$-\frac{dv}{du} = -\cos(u) + u\sin(u)$

$$-\frac{1}{2}\ln|v| = \ln|x| + \ln|c|$$

$$\cancel{\ln}|v| = \cancel{\ln}\left(\frac{c}{x^2}\right)$$

$$u\cos(u) = \frac{c}{x^2} \qquad \because u = y/x$$

$$\frac{y}{\cancel{x}}\cos\left(\frac{y}{x}\right) = \frac{c}{x^{\cancel{2}}}$$

$$\boxed{xy\cos\left(\frac{y}{x}\right) = c}$$

Reducible to Homogeneous:

(1) $$\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10}$$

Solv

Step 1: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow \frac{2}{6} \neq \frac{9}{2} = \frac{1}{3} \neq \frac{9}{2}$

Reducible to homogenous:

$x = X+h \Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$

$y = Y+k$

$$\frac{dY}{dX} = \frac{2(X+h)+9(Y+k)-20}{6(X+h)+2(Y+k)-10}$$

$$\frac{dY}{dX} = \frac{2X+9Y+2h+9k-20}{6X+2Y+6h+2k-10}$$

Step 2:- choose Constant. then Solve h & k.

$2h+9k-20 \rightarrow i \qquad 6h+2k-10 \rightarrow ii$

$12h+54k = 120$

$\cancel{\pm 12h} \pm 4k = \pm 20$

$50k = 100$

$\boxed{k = 2}$

$6h+2(2) = 10$

$6h = 6$

$\boxed{h = 1}$

$$\frac{dY}{dX} = \frac{2X+9Y+\cancel{2}+\cancel{18}-\cancel{20}}{6X+2Y+\cancel{6}+\cancel{4}-\cancel{10}} \Rightarrow \frac{2X+9Y}{6X+2Y}$$

Step 3:- Let $y = v\cancel{x}$ $\Longrightarrow$ $\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$

$$\frac{dv}{dx} \cdot x + v = \frac{2x + 9(vx)}{6x + 2(vx)} = \frac{2+9v}{6+2v}$$

Step

$$\frac{dv}{dx} \cdot x = \frac{2+9v}{6+2v} - v$$

$$\frac{dv}{dx} \cdot x = \frac{2+9v-6v-2v^2}{6+2v} \Rightarrow \frac{2+3v-2v^2}{6+2v}$$

Integration on b/s

$$\int \frac{6+2v}{\underbrace{2+3v-2v^2}} dv = \int \frac{1}{x} dx.$$

use Partial fraction.

$$\frac{6+2v}{(-2v-1)(v-2)}$$

$$\frac{-2v-6}{(v-2)(2v+1)} = \frac{A}{v-2} + \frac{B}{2v+1}$$

$$\frac{-2v-6}{\cancel{(v-2)(2v+1)}} = \frac{2Av + A + Bv - 2B}{\cancel{(v-2)(2v+1)}}$$

$$-2v - 6 = 2Av + A + Bv - 2B$$

$A - 2B = -6 \rightarrow (i)$ $\quad -2 = 2A + B \rightarrow (ii)$

$$2A - 4B = -12$$
$$\pm 2A \pm B = \mp 2$$

$$-5B = -10$$
$$\boxed{B = 2}$$

$$A - 2(2) = -6$$
$$\boxed{A = -2}$$

$$\int \frac{-2}{v-2}\,dv + \int \frac{2}{2v+1}\,dv = \int \frac{1}{x}\,dx.$$

$$-2\int \frac{1}{v-2}\,dv + \int \frac{2}{2v+1}\,dv = \int \frac{1}{x}\,dx.$$

$$-2\ln|v-2| + \ln|2v+1| = \ln|x| + \ln C$$

$$\cancel{\ln}\left[\frac{2v+1}{(v-2)^2}\right] = \cancel{\ln}(x \cdot c)$$

$$\therefore v = \frac{y}{x}$$

$$\frac{2\left(\frac{y}{x}\right)+1}{\left(\frac{y}{x}-2\right)^2} = x \cdot c$$

$$\frac{2y+x}{x} = x \cdot c\left(\frac{y-2x}{x}\right)^2$$

$$2y+x = \cancel{x^2} \cdot c\,\frac{(y-2x)^2}{\cancel{x^2}}$$

$$\therefore X = x-1 \quad Y = y-2$$

$$2(y-2)+(x-1) = (y-\cancel{2}-2x+\cancel{2})^2 c$$

$$2y-4+x-1 = (y-2x)^2 c$$

$$x+2y-5 = (y-2x)^2 c$$

or

$$\boxed{x+2y-5 = (2x-y)^2 c}$$

(3) $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$ $\qquad \frac{a_1}{a_2} \neq \frac{b_1}{b_2} = \frac{-1}{1} \neq \frac{1}{1}$

$= -1 \neq 1$

Sol:-

Step 1:- ~~y = X+~~

$x = X+h$ $\qquad \frac{dy}{dx} = \frac{dY}{dX}$

$y = Y+k$

So:-

$$\frac{dY}{dX} = \frac{Y+k-X-h+1}{Y+k+X+h+5} = \frac{Y-X-h+k+1}{Y+X+h+k+5}$$

Step 2:- choose Constant to Solve h & k.

$-h+k+1$

$+h+k+5$

$2k = -6$

$\boxed{k=-3}$

$-h+(-3) = -1$

$-h = -1+3$

$\boxed{h=-2}$

$$\frac{dY}{dX} = \frac{Y-X+\cancel{2}-\cancel{3}+\cancel{1}}{Y+X-\cancel{2}-\cancel{3}+\cancel{5}} \Rightarrow \frac{Y-X}{Y+X}$$

Step 3:- let $Y = UX \Rightarrow \frac{dY}{dX} = \frac{dU}{dX}.X+U$

$$\frac{dU}{dX}.X+U = \frac{UX-X}{UX+X} = \frac{U-1}{U+1}$$

$$\frac{dU}{dX}.X = \frac{U-1}{U+1} - U = \frac{\cancel{U}-1-U^2-\cancel{U}}{U+1}$$

$$\frac{du}{dx}\cdot x = \frac{-1-u^2}{u+1}$$

Integration on b/s

$$\int \frac{u+1}{1+u^2}\,du = -\int \frac{1}{x}\,dx$$

$$\int \frac{u}{1+u^2}\,du + \int \frac{1}{1+u^2}\,du = -\int \frac{1}{x}\,dx$$

Set $v = 1+u^2$

$dv = 2u\,du$

$u\,du = \frac{dv}{2}$

$$\frac{1}{2}\int \frac{1}{v}\,dv + \int \frac{1}{1+u^2}\,du = -\int \frac{1}{x}\,dx$$

$$\frac{1}{2}\ln|1+u^2| + \tan^{-1}(u) = -\ln|x| + c$$

$\because u = \frac{y}{x}$

$$\frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + c$$

$$\frac{1}{2}\ln\left|\frac{x^2+y^2}{x^2}\right| + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + c$$

$$\frac{1}{2}\ln|x^2+y^2| - \ln|x^2| + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + c$$

$$\frac{\ln|x^2+y^2|}{\cancel{\ln|x|^2}} = -2\tan\left(\frac{y}{x}\right) - \cancel{\ln\left(\frac{1}{x^2}\right)} + c$$

$$\ln|x^2+y^2| + 2\tan^{-1}\left(\frac{y}{x}\right) = c$$

$\therefore x = x+2$

$y = y+3$

$$\boxed{\ln|(x+2)^2+(y+3)^2| + 2\tan^{-1}\left(\frac{y+3}{x+2}\right) = c}$$

(3) $\dfrac{dy}{dx} = \dfrac{x-y-2}{x+y+6}$. $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \Rightarrow \dfrac{1}{1} \neq \dfrac{-1}{1}$

$= 1 \neq -1$

Sol:-

Step 1:- $x = X + h$, $y = Y + k$ $\qquad \dfrac{dy}{dx} = \dfrac{dY}{dX}$

So

$$\frac{dY}{dX} = \frac{X+h-Y-k-2}{X+h+Y+k+6} = \frac{X-Y+h-k-2}{X+Y+h+k+6}$$

Step 2:- Choose the Constant to solve h & k.

$$\begin{array}{l} h-\cancel{k}-2 \\ h+\cancel{k}+6 \\ \hline \boxed{2h = -4} \\ \boxed{h = -2} \end{array} \qquad \begin{array}{l} h-k-2 \\ -2-k=2 \\ \boxed{k=-4} \end{array}$$

$$\frac{dY}{dX} = \frac{X-Y\cancel{-2}\cancel{+4}\cancel{-2}}{X+Y\cancel{-2}\cancel{-4}\cancel{+6}} = \frac{X-Y}{X+Y}$$

Step 3:- let $Y = UX \Rightarrow \dfrac{dY}{dX} = \dfrac{dU}{dX} \cdot X + U$

$$\frac{du}{dX} \cdot X + U = \frac{X - UX}{X + UX} = \frac{1-U}{1+U}$$

$$\frac{dU}{dX} \cdot X = \frac{1-u}{1+u} - u = \frac{1-u-u-u^2}{1+u}$$

$$\frac{du}{dX} \cdot X = \frac{1-2u-u^2}{1+u}$$

Integration on b/s

$$\int \frac{1+u}{1-2u-u^2}\,du = \int \frac{1}{x}\,dx$$

Let $v = 1-2u-u^2$

$dv = -2-2u\,du$

$-\frac{dv}{2} = 1+u\,du$

$$-\frac{1}{2}\int \frac{dv}{v} = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln|v| = \ln|x| + c$$

$$\ln|1-2u-u^2| = \ln\left(\frac{1}{x}\right)^2 + c$$

$\because u = \frac{y}{x}$

$$\ln\left|1-2\left(\frac{y}{x}\right)-\left(\frac{y}{x}\right)^2\right| = \ln\left|\frac{1}{x^2}\right| + c$$

$$\ln\left|\frac{x^2-2xy-y^2}{\cancel{x^2}}\right| = \ln\left|\frac{1}{\cancel{x^2}}\right| + c$$

$$\cancel{\ln}\,|x^2-2xy-y^2| = \cancel{\ln}\,c$$

$\because x = n+2$

$y = y+4$

$$\boxed{(n+2)^2 - 2(n+2)(y+4) - (y+4)^2 = c}$$

5) $\frac{dy}{dn} = \frac{2n-5y+3}{2n+4y-6}$, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow \frac{2}{2} \neq \frac{-5}{4}$

Sol:-

$n = x+h$

$y = y+k$

$\frac{dy}{dn} = \frac{dy}{dx}$

So

$$\frac{dy}{dx} = \frac{2(x+h) - 5(y+k) + 3}{2(x+h) + 4(y+k) - 6}.$$

$$\frac{dy}{dx} = \frac{2x+2h-5y-5k+3}{2x+2h+4y+4k-6}$$

$$\frac{dy}{dx} = \frac{2x-5y+2h-5k+3}{2x+4y+2h+4k-6}.$$

Step 2:- choose the Constant to Solve h & k.

$$2h - 5k + 3$$
$$\pm 2h \pm 4k \mp 6$$
$$-9k = -9$$
$$\boxed{k = 1}$$

$$2h = +5k - 3$$
$$2h = +5(1) - 3$$
$$\boxed{h = 1}$$

$$\frac{dy}{dx} = \frac{2x-5y+\cancel{2}-\cancel{5}+\cancel{3}}{2x+4y+\cancel{2}+\cancel{4}-\cancel{6}} = \frac{2x-5y}{2x+4y}$$

Step 3:- Let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}\cdot x + u$

$$\frac{du}{dx}\cdot x + u = \frac{2x-5ux}{2x+4ux} = \frac{2-5u}{2+4u}$$

$$\frac{du}{dx}\cdot x = \frac{2-5u}{2+4u} - u$$

$$\frac{du}{dx}\cdot x = \frac{2-5u-2u-4u^2}{2+4u}$$

$$\frac{du}{dx}\cdot x = \frac{2-7u-4u^2}{2+4u}$$

Integration on b/s

$$\int \frac{2+4u}{\underbrace{2-7u-4u^2}} du = \int \frac{1}{x} dx$$

using Partial fraction

$$\frac{-2\cancel{+}4u}{(2+\cancel{4}u)(-1+4u)} = \frac{A}{(2+u)} + \frac{B}{(1-4u)}$$

$$\frac{2+4u}{\cancel{(2+u)(1-4u)}} = \frac{\cancel{2A} A-4Au+2B+Bu}{\cancel{(2+u)(1-4u)}}$$

$$2+4u = A-4Au+2B+Bu$$

$A+2B=2 \rightarrow i$ $\qquad -4A+B=4 \rightarrow ii$

$$-4A\cancel{+}8B=-8$$

$$\cancel{+}4A\pm B=\cancel{4}$$

$$-\cancel{9}B=-\cancel{12}$$

$A=-2B+2$

$A=-2\left(\frac{4}{3}\right)+2$

$\boxed{B=\frac{-4}{3}}$ $\qquad \boxed{A=-2/3}$

$$\int \frac{-\cancel{4}/3}{2+u} du + \int \frac{4/3}{-1+4u} du = \int \frac{1}{x} dx$$

$$-\frac{2}{3}\int \frac{1}{2+u} du + \frac{1}{3}\int \frac{4}{-1+4u} du = \int \frac{1}{x} dx$$

$$-\frac{2}{3}\ln|\cancel{u}\,2+u| \cancel{+} \frac{1}{3}\ln|-1+4u| = \ln|x| + C$$

$$\ln|2+u|^{-\frac{2}{3}} + \ln|-1+4u|^{1/3} = \ln|x| + C$$

$$+\cancel{\ln}\left((2+u)^{-\frac{2}{3}}\cdot(4u-1)^{-\frac{1}{3}}\right)=\cancel{\ln}(x\cdot c)$$

$$(2+u)^{-\frac{2}{3}\times -3}\cdot(4u-1)^{-\frac{1}{3}\times -3}=x^{-3}\cdot c$$

$$\because u=\frac{y}{x}$$

$$\left(2+\frac{y}{x}\right)^2\cdot\left(4\left(\frac{y}{x}\right)-1\right)=\frac{1}{x^3}\cdot c$$

$$\frac{(2x+y)^2}{\cancel{x^2}}\cdot\frac{(4y-x)}{\cancel{x}}=\frac{1}{\cancel{x^3}}\cdot c$$

$$x=x-1\ ,\ y=y-1$$

$$(2(x-1)+(y-1))^2\cdot(4(y-1)-(x-1))=c$$

$$(2x-2+y-1)^2\cdot(4y-4-x+1)=c$$

$$\boxed{(-x+4y-3)\ (2x+y-3)^2=c}$$

(6) $(2x+y+1)dx+(4x+2y-1)dy=0$

Soln

$$\frac{dy}{dx}=\frac{-2x-y-1}{4x+2y-1}\ ,\ \because\ \frac{a_1}{a_2}\ \cancel{\neq}\ \frac{b_1}{b_2}\Rightarrow\frac{-2}{4}\ \cancel{\neq}\ \frac{-1}{2}$$

$$=\boxed{\frac{-1}{2}=-\frac{1}{2}}$$

Step 1:- $\frac{dy}{dx} = \frac{-(2x+y)-1}{2(2x+y)-1}$

Let $u = 2x+y \Rightarrow \frac{du}{dx} = 2 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 2$

Sol:

$\frac{du}{dx} - 2 = \frac{-u-1}{2u-1}$

$\frac{du}{dx} = \frac{-u-1}{2u-1} + 2 \Rightarrow \frac{-u-1+4u-2}{2u-1}$

$\frac{du}{dx} = \frac{3u-3}{2u-1}$

Step 2:- Integration on b/s

$\int \frac{2u-1}{3u-3}\, du = \int dx$

$\frac{1}{3}\int \frac{2u-1}{u-1}\, du = \int dx.$

L.D.M

$u-1 \overline{)\, 2u-1\,}\, (2$
$\underline{\pm 2u \mp 2}$
1

$\frac{1}{3}\int 2 + \frac{1}{u-1}\, du = \int dx$

$\frac{1}{3}\int 2\, du + \int \frac{1}{u-1}\, du = \int dx$

$\frac{1}{3}\left[2u + \ln|u-1|\right] = x + c$

Step 1:- $\frac{dy}{dx} = \frac{-(2x+y)-1}{2(2x+y)-1}$

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$$\cancel{\frac{du}{dx}} \quad \frac{du}{dx} = \frac{3u-3}{2u-1}.$$

Step 2:- Integration on b/s

$$\int \frac{2u-1}{3u-3}\,du = \int dx$$

$$\frac{1}{3}\int \frac{2u-1}{\boxed{u-1}}\,du = \int dx.$$

L.D.M

$$u-1 \,\big)\, 2u-1 \,\big(\, 2$$
$$\underline{\pm 2u \mp 2}$$
$$1$$

$$\frac{1}{3}\int 2 + \frac{1}{u-1}\,du = \int dx$$

$$\frac{1}{3}\int 2\,du + \int \frac{1}{u-1}\,du = \int dx$$

$$\frac{1}{3}\left[2u + \ln|u-1|\right] = x + c$$

$$2u + \ln|u-1| = 3x + 3c$$

$$2(2x+y) + \ln|2x+y-1| = 3x + 3c \quad \because u = 2x+y$$

$3c \to c_1$

$$\boxed{2(2x+y) + \ln|2x+y-1| = 3x + c_1}$$

(7) $(x-y-2)dx - (2x-2y-3)dy = 0$

Sol:-

$$\frac{dy}{dx} = \frac{x-y-2}{2x-2y-3} \quad \Bigg| \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \boxed{\frac{1}{2} = \frac{1}{2}}$$

Step 1:-

$$\frac{dy}{dx} = \frac{x-y-2}{2(x-y)-3}$$

Let $u = x-y \Rightarrow \frac{du}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{du}{dx} + 1$

So $-\frac{du}{dx} + 1 = \frac{u-2}{2(u)-3}$

$$-\frac{du}{dx} = \frac{u-2}{2u-3} - 1 \Rightarrow \frac{u-2-2u+3}{2u-3}$$

$$\frac{du}{dx} = \frac{-(-u+1)}{2u-3} = \frac{u-1}{2u-3}$$

Integration on b/s

$$\int \underbrace{\frac{2u-3}{u-1}}_{L\cdot D\cdot M} du = \int dx$$

$$u-1 \overline{\big)\, 2u-3 \,}\big(2$$
$$\underline{\pm 2u \mp 2}$$
$$-1$$

$$\int 2 - \frac{1}{u-1}\, du = \int dx$$

$$\int 2\, du - \int \frac{1}{u-1}\, du = \int dx$$

$$2u - \ln|u-1| = x + c$$

~~2(x+~~ $\quad \because u = x-y$

$$2(x-y) - \ln|x-y-1| = x + c$$

$$2x - 2y - \ln|x-y-1| = x + c$$

$$\boxed{x - 2y - \ln|x-y-1| = c} \quad \text{Ans.}$$

(8) $(6x - 4y + 1)dy - (3x - 2y + 1)dx = 0$

Sol:

$$\frac{dy}{dx} = \frac{3x-2y+1}{6x-4y+1} \quad \Big| \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{3}{6} = \frac{-2}{-4}$$

$$\Rightarrow \boxed{\frac{1}{2} = \frac{1}{2}}$$

Step 1:

$$\frac{dy}{dx} = \frac{\cancel{3(x+}\, 3x - 2y + 1}{2(3x-2y)+1}$$

Let $u = 3x - 2y \Rightarrow \frac{du}{dx} = 3 - 2\frac{dy}{dx}$

$$2\frac{dy}{dx} = -\left[\frac{du}{dx} - 3\right]\frac{1}{2}$$

Sol:

$$-\frac{du}{dx} + 3 = \frac{(u+1)2}{2u+1} = \frac{2u+2}{2u+1}.$$

(9) $(7y-3x+3)\,dy = -(3y-2x+7)\,dx$

Sol:-

$$\frac{dy}{dx} = -\frac{(3y-2x+7)}{7y-3x+3}$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{3}{7} = \frac{-2}{3} \Rightarrow \frac{3}{7} \neq \frac{2}{3}$$

Step 1:- let $x = X+h$, $y = Y+k$, $\frac{dy}{dx} = \frac{dY}{dX}$

so $$\frac{dy}{dx} = \frac{3y-2x+7}{-7y+3x-3} \quad \text{or} \quad \frac{-2x+3y+7}{3x-7y-3}$$

$$\frac{dY}{dX} = \frac{-2(X+h)+3(Y+k)+7}{3(X+h)-7(Y+k)-3} \Rightarrow \cancel{-2X-2h}$$

$$\frac{dY}{dX} = \frac{-2X-2h+3Y+3k+7}{3X+3h-7Y-7k-3} \Rightarrow \frac{-2X+3Y-2h+3k+7}{3X-7Y+3h-7k-3}$$

Step 2:- Choose the constant to solve h & k.

$-2h+3k+7 \to i$ $\qquad$ $3h-7k-3 \to ii$

$-6h+9k+21$

$\mp 6h \pm 14k \pm 6$

$-5k = -15$

$\boxed{k=3}$

$3h - 7(3) = 3$

$3h = 24$

$\boxed{h=8}$

$$\frac{dY}{dX} = \frac{-2X+3Y\cancel{-16+9+7}}{3X-7Y+\cancel{24-21-3}} \Rightarrow \frac{-2X+3Y}{3X-7Y}$$

Step 3:- let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot x + u$

So $\frac{du}{dx} \cdot x + u = \frac{-2x + 3ux}{3x - 7ux} = \frac{-2 + 3u}{3 - 7u}$

$$\frac{du}{dx} \cdot x = \frac{-2+3u}{3-7u} - u \Rightarrow \frac{-2+\cancel{3u}-\cancel{3u}+7u^2}{3-7u}$$

$$\frac{du}{dx} \cdot x = \frac{-2+7u^2}{3-7u}$$

Step 4:- Integration on b/s

$$\int \frac{3-7u}{\underbrace{-2+7u^2}} du = \int \frac{1}{x} dx$$

Partial ~~dif~~ fraction.

$$\frac{3-7u}{-2+7u^2} = \frac{3-7u}{(-\sqrt{2}+\sqrt{7}u)(-\sqrt{2}+\sqrt{7}u)}$$

$$\frac{3-7u}{(-\sqrt{2}+\sqrt{7}u)(\sqrt{2}+\sqrt{7}u)} = \frac{A}{(-\sqrt{2}+\sqrt{7}u)} + \frac{B}{(\sqrt{2}+\sqrt{7}u)}$$

$$3-7u = +A\sqrt{2} + A\sqrt{7}u - B\sqrt{2} + B\sqrt{7}u$$

$+A\sqrt{2} - B\sqrt{2} = 3 \rightarrow (i)$ $\quad A\sqrt{7} + B\sqrt{7} = -7 \rightarrow (ii)$

(10) $(y-3x+3)\,dy = (2y-x-4)\,dx$.

Sol: $\frac{dy}{dx} = \frac{2y-x-4}{y-3x+3}$ | $\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{2}{1} = \frac{-1}{-3}$

$= 2 \neq \frac{1}{3}$

Step 1: Let: $y = Y+k$, $x = X+h$ $\qquad \frac{dy}{dx} = \frac{dY}{dX}$

So $\frac{dY}{dX} = \frac{2(Y+k)-(X+h)-4}{(Y+k)-3(X+h)+3}$

$\frac{dY}{dX} = \frac{2Y+2k-X-h-4}{Y+k-3X-3h+3} \Rightarrow \frac{-X-2Y+h-2k+4}{-(3X-Y+3h-k-3)}$

Step 2: Choose the constant to solve h & k.

$-h+2k-4 \rightarrow$ i $\qquad -3h+k+3 \rightarrow$ ii

$-3h+6k = +12$

$-3h+k = -3$

$+5k = +15$

$\boxed{k = 3}$

$-h+2(3)-4=0$

$-h+2=0$

$-h = -2$

$\boxed{h=2}$

$\frac{dY}{dX} = \frac{-X+2Y-2+2(3)-4}{-3X+Y-3(2)+3+3}$

$\frac{dY}{dX} = \frac{-X+2Y-\cancel{2}+\cancel{6}-\cancel{4}}{-3X+Y-\cancel{6}+\cancel{3}+\cancel{3}} = \frac{-X+2Y}{-3X+Y}$

Step 3: Let: $Y = uX \Rightarrow \frac{dY}{dX} = \frac{du}{dX}.X + u$

so $\frac{du}{dx} \cdot x + u = \frac{-x + 2ux}{-3x + ux} = \frac{1-2u}{3-u}$

$$\frac{du}{dx} \cdot x = \frac{1-2u}{3-u} - u \Rightarrow \frac{1-2u-3u+u^2}{3-u}$$

$$\frac{du}{dx} \cdot x = \frac{1-5u+u^2}{3-u}$$

Step 4: Integration on b/s

$$\int \frac{3-u}{1-5u+u^2} du = \int \frac{1}{x} dx$$

Partial fraction.

$$\int \frac{3-u}{1-5u+u^2} du = \int \frac{-u+3}{\left(u-\frac{5}{2}+\frac{1}{2}\sqrt{21}\right)\left(u-\frac{5}{2}-\frac{1}{2}\sqrt{21}\right)} du$$

$$\int \left[\frac{-\frac{1}{2}-\frac{1}{42}\sqrt{21}}{u-\frac{5}{2}+\frac{1}{2}\sqrt{21}} + \frac{-\frac{1}{2}+\frac{1}{42}\sqrt{21}}{u-\frac{5}{2}-\frac{1}{2}\sqrt{21}}\right] du$$

$$\int \frac{-\frac{1}{2}-\frac{1}{42}\sqrt{21}}{u-\frac{5}{2}+\frac{1}{2}\sqrt{21}} du + \int \frac{-\frac{1}{2}+\frac{1}{42}\sqrt{21}}{u-\frac{5}{2}-\frac{1}{2}\sqrt{21}} du$$

$$\left(-\frac{1}{2}-\frac{1}{42}\sqrt{21}\right) \ln\left|u-\frac{5}{2}+\frac{1}{2}\sqrt{21}\right| + \left(-\frac{1}{2}+\frac{1}{42}\sqrt{21}\right) \ln\left|u-\frac{5}{2}-\frac{1}{2}\sqrt{21}\right|$$

$$\cancel{\ln}\left[\left(-\frac{1}{2}-\frac{\sqrt{21}}{42}\right)\left(u-\frac{5}{2}+\frac{\sqrt{21}}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{21}}{42}\right)\left(u-\frac{5}{2}-\frac{\sqrt{21}}{2}\right)\right]=\cancel{\ln}(x\cdot c)$$

$$\therefore u=\frac{Y}{X}$$

$$\left(-\frac{1}{2}-\frac{\sqrt{21}}{42}\right)\left(\frac{Y}{X}-\frac{5}{2}+\frac{\sqrt{21}}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{21}}{2}\right)\left(\frac{Y}{X}-\frac{5}{2}-\frac{\sqrt{21}}{2}\right)=X\cdot c$$

$$\therefore Y=y-3$$

$$X=x-2$$

$$\left(-\frac{1}{2}-\frac{\sqrt{21}}{42}\right)\left(\frac{y-3}{x-2}-\frac{5}{2}+\frac{\sqrt{21}}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{21}}{2}\right)\left(\frac{y-3}{x-2}-\frac{5}{2}-\frac{\sqrt{21}}{2}\right)=(x-2)c$$

(4) $\dfrac{dy}{dx} = \dfrac{y+x-2}{y-x-4}$

Sol

Step 1:- $x = X+h$, $y = Y+k$, $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

So

$$\frac{dY}{dX} = \frac{Y+k+X+h-2}{Y+k-X-h-4} = \frac{(X+Y)+(k+h-2)}{(-X+Y)+(k-h-4)}$$

Step 2:- Now choose constant then solve h & k

$$k+h-2=0$$
$$k-h-4=0$$
$$2k = 6$$

$$\boxed{k=3} \quad \boxed{h=-1}$$

So $\dfrac{dY}{dX} = \dfrac{X+Y+\cancel{3}-\cancel{1}-\cancel{2}}{-X+Y+\cancel{3}+\cancel{1}-\cancel{4}} = \dfrac{X+Y}{-X+Y}$

Step 2: let $y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}.x + v$

$$\frac{dv}{dx}.x + v = \frac{x + v\cancel{x}}{-x + vx}$$

$$= \frac{1+v}{-1+v}$$

$$\frac{dv}{dx}.x = \frac{1+v}{-1+v} - v$$

$$\frac{dv}{dx}.x = \frac{v+1-v^2+v}{v-1} = \frac{-v^2+2v+1}{v-1}$$

Integration on b/s

$$\int \frac{v-1}{-v^2+2v+1} dv = \int \frac{1}{x} dx.$$

let $U = -v^2 + 2v + 1$

$$\frac{du}{dv} = -2v + 2$$

$$-\frac{1}{2} du = (v-1) dv$$

$$-\frac{1}{2} \int \frac{du}{u} = \ln x + c$$

$$-\frac{1}{2} \ln u - \ln x = c$$

$$-(\ln u^{\frac{1}{2}} + \ln x) = c$$

$$\cancel{\ln}(u^{\frac{1}{2}} x) = -c \quad \text{or} \quad \cancel{\ln} e^{-1} = c$$

$$x\sqrt{-v^2+2v+1} = c$$

$$x\sqrt{-\frac{y^2}{x^2} + \frac{2y}{x} + 1} = c$$

$$\cancel{x}\sqrt{-y^2 + 2x + x^2} = c$$

$$= \sqrt{-(y \mp k)^2 + 2(y \mp k)(x-h) + (x-h)^2}$$

$\boxed{k=3}$ $\boxed{h=-1}$

$$= \sqrt{-y^2 + 6y \mp 9 + 2[xy + y - 3x - 3] + x^2 + 2x + 1} = c$$

$$= (-y^2 + 6y - 9 + 2xy + 2y - 6x - 6 + x^2 + 2x + 1)^{\frac{1}{2}} = c$$

$$= \boxed{(x^2 - y^2 + 2xy - 4x + 8y - 14)^{\frac{1}{2}} = c}$$

Example ①: ② $(x^2-y^2)dx + 2xy\,dy = 0$

Solution $2xy\,dy = -(x^2-y^2)dx$

$$\frac{dy}{dx} = \frac{-(x^2-y^2)}{2xy}$$

Let $Ux = y \Rightarrow \frac{dy}{dx} = U + x\frac{dU}{dx}$ $\quad \because U = \frac{y}{x}$

$$U + x\frac{dU}{dx} = \frac{-(x^2-(Ux)^2)}{2x(Ux)}$$

$$U + x\frac{dU}{dx} = \frac{-\cancel{x^2}(1-U^2)}{2U\cancel{x^2}}$$

$$x\frac{dU}{dx} = \frac{U^2-1}{2U} - U = \frac{U^2-1-2U^2}{2U} = \frac{-1-U^2}{2U}$$

Integration on b/s

$$\int \frac{2U}{1+U^2}dU = -\int \frac{1}{x}dx$$

$$\ln(1+U^2) = -\ln x + c$$

$$\ln(1+U^2) + \ln x = c \text{ or } \ln c$$

$$\cancel{\ln}[(1+U^2)(x)] = \cancel{\ln} c$$

$$\left(1+\frac{y^2}{x^2}\right)x = C \qquad \therefore u = \frac{y}{x}$$

$$\boxed{\frac{x^2+y^2}{x} = C} \quad \underline{\text{or}} \quad \boxed{x^2+y^2 = Cx}$$ Ans