Homogeneous Equations

Definition:- A function $f(x,y)$ is said to be homogeneous function of degree "n" if it can be expressed in the form $f(x,y) = x^n f(y/x)$ or $f(tx,ty) = t^n f(x,y)$, $t \neq 0$

$\Rightarrow$ D.E in x and y is said to be homogeneous if it can be in the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$, where $f_1(x,y)$ and $f_2(x,y)$ are homogeneous function of the same degree in x and y

Ex. $f(x,y) = x^2 + xy$

Sol:

$$f(tx,ty) = (tx)^2 + (tx)(ty)$$
$$= t^2x^2 + t^2xy$$
$$= t^2(x^2+xy)$$
$$\boxed{= t^2 f(x,y)}$$

Hence Homogeneous function degree 2.

another Definition

A D.E $Mdx + Ndy = 0$ is homogeneous if both M and N are homo's of the same degree.

Working Rule; 1- $y = Ux$ $\quad \frac{dy}{dx} = \frac{dU}{dx} \cdot x + U$

put the values in D.E

2- Seperate the variables and Integrate

3- Replace the values which you let.

Ex:- $(x^2-y^2)dx + 2xy\,dy = 0$

Sol:- $2xy\,dy = -(x^2-y^2)dx$

$$\frac{dy}{dx} = -\frac{(x^2-y^2)}{2xy} \longrightarrow$$

$\therefore v = \frac{y}{x}$

Let $vx = y \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = -\frac{(x^2-(vx)^2)}{2x(vx)}$$

$$v + x\frac{dv}{dx} = \frac{-\cancel{x^2}(1-v^2)}{2v\cancel{x^2}}$$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v} = \frac{-1-v^2}{2v}$$

$$\int \frac{2v}{1+v^2}\,dv = -\int \frac{1}{x}\,dx$$

$$\ln(1+v^2) = -\ln x + c$$

$$\ln(1+v^2) + \ln x = c \text{ or } \ln c$$

$$\cancel{\ln}[(1+v^2)(x)] = \cancel{\ln} c$$

$$\left(1+\frac{y^2}{x^2}\right)x = c$$

$$\frac{x^2+y^2}{x} = c \quad \text{Ans} \quad \text{or} \quad x^2+y^2 = cx$$

Equations Reducible to Homogeneous

The D.E of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is not homogeneous

Such a D.E depends on the coefficients a_1, b_1, a_2 and b_2

We shall consider two cases:

Case-I when $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then reduce to Homogeneous D.E

Working Rule:- $x = X+h \rightarrow dx = dX$, h and k are constant
$y = Y+k \rightarrow dy = dY$

Ex: $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$

Sol:

Step 1 $x = X+h$, $\frac{dy}{dx} = \frac{dY}{dX}$
$y = Y+k$

So $\frac{dY}{dX} = \frac{Y+k+X+h-2}{Y+k-X-h-4} = \frac{(X+Y)+(k+h-2)}{(-X+Y)+(k-h-4)}$

Step 2 Now choose constant then solve h & k

$$\begin{array}{l} k+h-2=0 \\ k-h-4=0 \\ \hline 2k = 6 \end{array}$$

$\boxed{k=3}$ $\boxed{h=-1}$

so $\frac{dY}{dX} = \frac{X+Y+\cancel{3}-\cancel{1}-\cancel{2}}{-X+Y+\cancel{3}+\cancel{1}-\cancel{4}} = \frac{X+Y}{-X+Y}$

Step 3 let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + X\frac{dv}{dX} = \frac{X+vX}{-X+vX}$

$= \frac{1+v}{-1+v}$

$X\frac{dv}{dX} = \frac{v+1}{v-1} - v$

$X\frac{dv}{dX} = \frac{v+1-v^2+v}{v-1} = \frac{-v^2+2v+1}{v-1}$

$\int \frac{v-1}{-v^2+2v+1}\,dv = \int \frac{1}{X}\,dX$

$u = -v^2+2v+1$

$\frac{du}{dv} = -2v+2$

$-\frac{1}{2}du = (v-1)dv$

$-\frac{1}{2}\int \frac{du}{u} = \ln X + c$

$-\frac{1}{2}\ln u - \ln X = c$

$-(\ln u^{1/2} + \ln X) = c$

$\ln(u^{1/2}X) = -c \Rightarrow u^{1/2}X = e^{-c} = C$

$X\sqrt{-v^2+2v+1} = C$

$X\sqrt{-\frac{Y^2}{X^2}+\frac{2Y}{X}+1} = C$

$\cancel{X}\sqrt{\frac{-Y^2+2YX+X^2}{\cancel{X^2}}} = C$

$\Rightarrow \sqrt{-(y-k)^2+2(y-k)(x-h)+(x-h)^2} = C$

Ex:- $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Sol:

$a_1 = 1 \quad b_1 = 2 \quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow \frac{1}{2} \neq \frac{2}{1}$

$a_2 = 2 \quad b_2 = 1$

So $x = X + h \qquad \frac{dy}{dx} = \frac{dY}{dX}$

$y = Y + k$

$$\frac{dY}{dX} = \frac{X+h+2Y+2k-3}{2X+2h+Y+k-3}$$

$$= \frac{X+2Y+h+2k-3}{2X+Y+2h+k-3}$$

$h+2k-3 = 0 \rightarrow i$

$2h+k-3 = 0 \rightarrow ii$

$\cancel{2h}+4k-6=0$

$-\cancel{2h} \mp k \mp 3 = 0$

$\cancel{3}k = \cancel{3} \quad \boxed{k=1}, \boxed{h=1}$

$$\frac{dY}{dX} = \frac{X+2Y+1+2-3}{2X+Y+2+1-3} = \frac{X+2Y}{2X+Y}$$

let $Y = VX$

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX} = \frac{1+2V}{2+V}$$

$$X\frac{dV}{dX} = \frac{1+2V}{2+V} - V = \frac{1+\cancel{2V}-\cancel{2V}-V^2}{2+V}$$

$$X\frac{dV}{dX} = \frac{1-V^2}{2+V} \Rightarrow \int \frac{2+V}{1-V^2}dV = \int \frac{1}{X}dX$$

use partial fraction

$$\frac{2+V}{1-V^2} = \frac{2+V}{(1-V)(1+V)} = \frac{A}{1-V} + \frac{B}{1+V}$$

$$\frac{2+V}{(1-V)(1+V)} = \frac{A+AV+B-BV}{(1-V)(1+V)}$$

$2+V = AV - BV + A + B$

$1 = A - \cancel{B} \rightarrow i$

$2 = A + \cancel{B} \rightarrow ii$

$3 = 2A \Rightarrow A = 3/2$

$B = 2 - \frac{3}{2} = \frac{1}{2}$

$$\int \frac{3/2}{1-V}dV + \int \frac{1/2}{1+V}dV = \int \frac{1}{X}dX$$

$$-\frac{3}{2}\int \frac{-1}{1-V}dV + \frac{1}{2}\int \frac{1}{1+V}dV = \int \frac{1}{X}dX$$

$$-\frac{3}{2}\ln(1-V) + \frac{1}{2}\ln(1+V) = \ln X + \ln C$$

$$\cancel{\ln}\left[\frac{(1+V)^{1/2}}{(1-V)^{3/2}}\right] = \cancel{\ln} CX$$

$$(1+V)^{\cancel{1/2}\times\cancel{2}} = CX^2(1-V)^{3/2 \times 2}$$

$$1 + \frac{Y}{X} = CX^2\left(1-\frac{Y}{X}\right)^3$$

$$X+Y = CX^{\cancel{2}}\left(\frac{X-Y}{X}\right)^3$$

$$X+Y = CX^{\cancel{3}}\frac{(X-Y)^3}{X^{\cancel{3}}}$$

$$x-1+y-1 = c(x\cancel{-1}-y\cancel{+1})^3$$

$$(x+y-2) = c(x-y)^3$$

$$x+y-2 = c(x-y)^3$$

Ans

Equations Reducible to Homogeneous

Case-II: when $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ then reduce to Homo:

Ex: $\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3}$ $a_1 = 1$ $b_1 = 2$ $a_2 = 2$ $b_2 = 4$

Soln

$\frac{1}{2} = \frac{2}{4}$ $\frac{dy}{dx} = \frac{x+2y+1}{2(x+2y)+3}$

Steps

$u = x+2y$

$\frac{du}{dx} = 1 + 2\frac{dy}{dx} \Rightarrow \frac{1}{2}\left[\frac{du}{dx} - 1\right] = \frac{dy}{dx}$

$\Rightarrow \frac{1}{2}\left[\frac{du}{dx} - 1\right] = \frac{u+1}{2u+3}$

$\Rightarrow \frac{du}{dx} = \frac{2u+2}{2u+3} + 1 = \frac{2u+2+2u+3}{2u+3} = \frac{4u+5}{2u+3}$

$\Rightarrow \int \frac{2u+3}{4u+5}\,du = \int dx$

$4u+5 \,)\, 2u+3 \,(\, \frac{1}{2}$
$\quad 2u + \frac{5}{2}$
$\quad \frac{1}{2}$

$\Rightarrow \int \left(\frac{1}{2} + \frac{1}{2(4u+5)}\right) du = \int dx$

$\Rightarrow \frac{1}{2}\int du + \frac{1}{2}\int \frac{1}{4u+5}\,du = \int dx$

$\Rightarrow \frac{1}{2}u + \frac{1}{8}\ln(4u+5) = x + c$

$\Rightarrow 4u + \ln(4u+5) = 8x + 8c$

$\Rightarrow 4(x+2y) + \ln[4(x+2y)+5] = 8x + 8c$

$\Rightarrow 4(x+2y) + \ln(4x+8y+5) = 8x + c_1$ $\boxed{8c = c_1}$

Ans

Ex: $\frac{dy}{dx} = \frac{x-2y+3}{2x-4y+5}$

Solu

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{1}{2} = \frac{+2'}{+4_2}$

$$\frac{dy}{dx} = \frac{x-2y+3}{2(x-2y)+5}$$

Let $v = x-2y$

$\frac{dv}{dx} = 1 - 2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\left[\frac{dv}{dx} - 1\right]$

$\Rightarrow -\frac{1}{2}\left[\frac{dv}{dx} - 1\right] = \frac{v+3}{2v+5}$

$\Rightarrow -\frac{dv}{dx} = \frac{2v+6}{2v+5} - 1 \Rightarrow \frac{\cancel{2v}+6-\cancel{2v}-5}{2v+5} = \frac{1}{2v+5}$

$$\frac{dv}{dx} = -\frac{1}{2v+5}$$

$\int(2v+5)dv = \int -1 dx$

$2\int v dv + 5\int dv = -\int dx$

$\cancel{2}\frac{v^2}{\cancel{2}} + 5v = -x + c$

$(x-2y)^2 + 5(x-2y) = -x + c$ Ans

or $(x-2y)[x-2y+5] = -x+c$ Ans

Worksheet #02

Solve the Following Homogeneous D.E's.

1. $(y^2 - xy)dx + x^2 dy = 0$ Ans: $x/y = \ln x + c$
2. $(x^2 - y^2)dx + 2xy\,dy = 0$ Ans: $x^2 + y^2 = ax$
3. $x(y - x)\frac{dy}{dx} = y(y + x)$ Ans: $y/x - \ln xy = c$
4. $x(x - y)dy + y^2 dx = 0$ Ans: $y = xy\ln c$
5. $\frac{dy}{dx} + \frac{x - 2y}{2x - y} = 0$ Ans: $y - x = c(x + y)^3$
6. $\frac{dy}{dx} = \tan(y/x) + \frac{y}{x}$ Ans: $\sin(y/x) = cx$
7. $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$ Ans: $3x + y\ln x + cy =$
8. $\frac{dy}{dx} = \frac{x^2 - 2y^2}{2xy}$ Ans: $4y^2 - x^2 = c/x^2$
9. $(x^2 + y^2)dy = xy\,dx$ Ans: $-\frac{x^2}{2y^2} + \ln y = c$
10. $\left[x\cos(y/x) + y\sin(y/x)\right]y - \left[y\sin(y/x) - x\cos(y/x)\right]x\frac{dy}{dx} = 0$ Ans: $xy\cos y/x = a$

W

Reducible to Homogeneous

last question put the value of $x = X + h \ldots$

1. $\frac{dy}{dx} = \frac{2x + 9y - 20}{6x + 2y - 10}$ Ans: $(2x - y)^2 = c(x + 2y - 5)$
2. $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$ Ans: $\ln\left[(y+3)^2 + (x+2)^2\right] + 2\tan^{-1}\left(\frac{y+3}{x+2}\right) = c$
3. $\frac{dy}{dx} = \frac{x - y - 2}{x + y + 6}$ Ans: $(y+4)^2 + 2(x+2)(y+4) - (x+2)^2 = c$
4. $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$ Ans: $-(y-3)^2 + 2(x+1)(y-3) + (x+1)^2 = c$
5. $\frac{dy}{dx} = \frac{2x - 5y + 3}{2x + 4y - 6}$ Ans: $(x - 4y + 3)(2x + y - 3) = c$
6. $(2x + y + 1)dx + (4x + 2y - 1)dy = 0$ Ans: $2(2x + y) + \ln(2x + y - 1) = 3x + c$
7. $(x - y - 2)dx - (2x - 2y - 3)dy = 0$ Ans: $\ln(x - y - 1) = x - 2y + c$
8. $(6x - 4y + 1)dy - (3x - 2y + 1)dx = 0$ Ans: $4x - 8y - \ln(12x - 8y + 1) = c$
9. $(7y - 3x + 3)dy = -(3y - 2x + 7)dx$ Ans: $(x + y - 1)^5(x - y - 1)^2 = 1$
10. $(y - 3x + 3)dy = (2y - x - 4)dx$ Ans: $X^2 - 5XY + Y^2 = c\left[\frac{2Y + (-5 + \sqrt{21})X}{2Y - (5 + \sqrt{21})X}\right]^{\frac{1}{\sqrt{21}}}$