Lecture #04

Non-Exact D.E's

If the equation $M(x,y)dx + N(x,y)dy = 0$ is not exact it may be possible to multiple it by a function $\mu(x,y)$ so that the resulting equation $\mu M(x,y) + \mu N(x,y)dy = 0$ becomes exact.

Rule#01:- If $M(x,y)\,dx + N(x,y)dy = 0$ is not exact then

$$\boxed{\frac{M_y - N_x}{N} = P(x)} \rightarrow \quad I.F = e^{\int P(x)dx}$$

Rule#02:- If rule #01 fail then

$$\boxed{\frac{N_x - M_y}{M} = Q(y)} \rightarrow \quad I.F\ e^{\int Q(y)dy}$$

Rule#03:- If $Mdx + Ndy = 0$ is a homogeneous and $xM + yN \neq 0$ then $\boxed{I.F = \frac{1}{xM + yN}}$

Rule#04:- If $M(x,y)dx + N(x,y)dy = 0$ is of the form $y f(x,y)dx + x g(x,y)dy = 0$ and $xM - Ny \neq 0$ then $\boxed{I.F = \frac{1}{xM - yN}}$

Ex:- $(x^2+y^2+2x)dx + 2y\,dy = 0 \longrightarrow i$

Soln

$M = x^2+y^2+2x \qquad N = 2y$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 0$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

check rule #01 $\frac{M_y - N_x}{N} = P(x)$

$$\frac{2y-0}{2y} = \frac{\cancel{2y}}{\cancel{2y}} = 1 \rightarrow P(x) = 1$$

$I.F = e^{\int dx} = e^{x} \longrightarrow$ multiple with eq(i)

$e^{x}(x^2+y^2+2x)dx + e^{x}(2y)dy = 0$

$(x^2e^{x}+y^2e^{x}+2xe^{x})dx + 2ye^{x}dy = 0$

$M = x^2e^{x}+y^2e^{x}+2xe^{x} \qquad N = 2ye^{x}$

$\frac{\partial M}{\partial y} = 0+2ye^{x} \qquad \frac{\partial N}{\partial x} = 2ye^{x}$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ satisfied

$\int_{y\text{-const}} M\,dx + \int N\,dy = c$

$\int (x^2e^{x}+y^2e^{x}+2xe^{x})dx + \int 0\,dy = c$

$\int x^2e^{x}dx - y^2\int e^{x}dx + 2\int xe^{x}dx = c$

$x^2e^{x} - \cancel{2\int xe^{x}dx} - y^2e^{x} + \cancel{2\int xe^{x}dx} = c$

$e^{x}(x^2-y^2) = c$ Ans

lecture # 04

Exact Differential Equations

Def: D.E $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, that is when $M_y = N_x$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex: $(2x^3 - 6x^2y + 3xy^2)dx - (2x^3 - 3x^2y + y^3)dy = 0$

Sol:

$M = 2x^3 - 6x^2y + 3xy^2$ $\qquad$ $N = -(2x^3 - 3x^2y + y^3)$

$\frac{\partial M}{\partial y} = 0 - 6x^2(1) + 3x\cdot(2y)$ $\qquad$ $\frac{\partial N}{\partial x} = -(6x^2 - 6xy)$

$\frac{\partial M}{\partial y} = -6x^2 + 6xy$ $\qquad$ $\frac{\partial N}{\partial x} = -6x^2 + 6xy$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then

$$f(x,y) = \int M\,dx + \int (\text{term in } N \text{ not containing } x)\,dy = c$$

$$\Rightarrow \int (2x^3 - 6x^2y + 3xy^2)dx + \int -(y^3)dy = c$$

$$\Rightarrow \frac{\cancel{2}x^4}{\cancel{4}_2} - \frac{{}^2\cancel{6}x^3y}{\cancel{3}} + \frac{3x^2y^2}{2} - \frac{y^4}{4} = c$$

$$\Rightarrow \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 - \frac{y^4}{4} = c \quad \text{Ans}$$

Ex #2 $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol:

$M = (e^y + 1)\cos x$ $\qquad$ $N = e^y \sin x$

$\frac{\partial M}{\partial y} = \cos x(e^y + 0)$ $\qquad$ $\frac{\partial N}{\partial x} = e^y \cos x$

$= e^y \cos x$ $\qquad$ $\frac{\partial y}{\partial x} = e^y \cos x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$f(x,y) = \int M\,dx + \int (\text{term in } N \text{ not containing } x)\,dy = c$$

$$\Rightarrow \int (e^y + 1)\cos x\,dx + \int (0)\,dy = c$$

$$\Rightarrow \int e^y \cos x\,dx + \int \cos x\,dx = c$$

$$\Rightarrow e^y \sin x + \sin x = c \quad \text{Ans}$$

or $\quad (e^y + 1)\sin x = c \quad$ Ans

Ex #03 $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Solu

$M = \sec^2 x \tan y$

$\frac{\partial M}{\partial y} = \sec^2 x \cdot \sec^2 y$

$N = \sec^2 y \tan x$

$\frac{\partial N}{\partial x} = \sec^2 y \cdot \sec^2 x$

CamScanner

Worksheet # 04

Exact & Reducible to Exact D.E's

1. $(y^2-x^2)\,dx + 2xy\,dy = 0$ Ans: $\frac{x^3}{3} = xy^2 + c$

2. $(1+3e^{x/y})\,dx + 3e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$ Ans: $x + 3ye^{x/y} = c$

3. $(2x-y)\,dx = (x-y)\,dy$ Ans: $xy = x^2 + \frac{y^2}{2} + c$

4. $(y\sec^2 x + \sec x\cdot\tan x)\,dx + (\tan x + 2y)\,dy = 0$ Ans: $y\cdot\tan x + \sec x + y^2 = c$

5. $(ax+hy+g)\,dx + (hx+by+f)\,dy = 0$ Ans: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

6. $(x^2 + 2ye^{2x})\,dy + (2xy + 2y^2e^{2x})\,dx = 0$ Ans: $x^2y + y^2e^{2x} = c$

7. $\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\cdot\sin y)\,dy = 0$ Ans: $y(x + \log x) + x\cos y = c$

8. $(x^3 - 3xy^2)\,dx + (y^3 - 3x^2y)\,dy = 0$, $y(0) = 1$ Ans: $x^4 - 6x^2y^2 + y^4 = 1$

9. $(y\cdot\log y)\,dx + (x - \log y)\,dy = 0$ Ans: $2x\log y = c + (\log y)^2$

10. $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(1+y^2)\,x\,dy = 0$ Ans: $\frac{yx^4}{4} + \frac{y^3x^4}{12} + \frac{x^6}{12} = c$

11. $(x\cdot\sec^2 y - x^2\cos y)\,dy = (\tan y - 3x^4)\,dx$ Ans: $-\frac{1}{x}\tan y - x^3 + \sin y = c$

12. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$ Ans: $x^3y^2 + \frac{x^2}{y} = c$

13. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ Ans: $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$

14. $(2x^4y^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$ Ans: $x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$

15. $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$ Ans: $-\frac{1}{xy} + 2\log x - \log y = b$

16. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ Ans: $\log\left(\frac{x}{y}\right) - xy = c$

17. $(xy\cdot\sin(xy) + \cos(xy))\cdot y\,dx + (xy\cdot\sin(xy) - \cos(xy))\,x\,dy = 0$ Ans: $y\cos xy = cx$

18. $y(1+xy)\,dx + x(1 + xy + x^2y^2)\,dy = 0$ Ans: $\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = c$

19. $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$ Ans: $\frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = c$

20. $(3y - 2xy^3)\,dx + (4x - 3x^2y^2)\,dy = 0$ Ans: $x^3y^4 - \frac{x^4}{2}y^6 = c$

21. $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$ Ans: $\frac{7}{5}x^{17/7}y^{-5/7} - \frac{7}{4}x^{-4/7}y^{-12/7} = c$

22. $(y^2 + 2yx^2)\,dx + (2x^3 - xy)\,dy = 0$ Ans: $4(xy)^{1/2} - \frac{2}{3}\left(\frac{y}{x}\right)^{3/2} = c$