~~Lecture # 04~~

Work Sheet # 04.

Exact & Reducible to Exact O.E's

Def: D.E $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, that is when $M_y = N_x$.

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

(1) $(y^2 - x^2)\,dx + 2xy\,dy = 0$.

Sol: Step 1: Analyze the equation.

$M = y^2 - x^2 \qquad N = 2xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2y$

$\left.\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right] \rightarrow$ Satisfy the Exact D.E.

Step 2: Working Rule

$F(x,y) = \int M\,dx + \int (\text{term } \cancel{\text{only}} \text{ not Contain } x)\,dy = c$

$= \int y^2 - x^2\,dx + \int 0\,dy = c$

$= \cancel{y^2 \int - x^2\,dx = c}\ \int y^2\,dx - \int x^2\,dx = c$

$$= \boxed{xy^2 - \frac{x^3}{3} + c}$$

② $\left(1+3e^{x/y}\right)dx + 3e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol:

Step 1: Analyze the Equation

$M = \left(1+3e^{x/y}\right) \qquad N = 3e^{x/y}\left(1-\frac{x}{y}\right)$

$\frac{dM}{dy} = -\frac{3x}{y^2}e^{x/y} \qquad \frac{\partial N}{\partial x} = \frac{3}{y}e^{x/y}\left(1-\frac{x}{y}\right) - \frac{3e^{x/y}}{y}$

$\frac{\partial N}{\partial x} = \cancel{\frac{3}{y}e^{x/y}} - \frac{3x}{y^2}e^{x/y} - \cancel{\frac{3}{y}e^{x/y}}$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \qquad \frac{\partial N}{\partial x} = -\frac{3x}{y^2}e^{x/y}$

↳ Satisfy the Exact D.E.

Step 2: Integration

$\int M dx + \int (\text{only term } \cancel{\text{cont of}}\ y)\, dy = c$

$\int 1+3e^{x/y}dx + \int 0\, dy = c$

$\int 1dx + 3\int e^{x/y}dx = c \qquad \text{let } u = x/y$

$x + 3\int e^{u}y\, du = c \qquad du = \frac{1}{y}dx.$

$x+3ye^{u} = c \Rightarrow \boxed{x+3ye^{x/y} = c}$

③ $(2x-y)dx = (x-y)dy \Rightarrow (2x-y)dx-(x-y)dy=0$

Sol: $M = (2x-y) \qquad N = -(x-y)$

$\frac{dM}{dy} = -1 \qquad \frac{dN}{dx} = -1 \qquad \boxed{\frac{dM}{dy} = \frac{dN}{dx}}$

• It Satisfy the Exact D.E.

Integration

$\int M\,dx + \int (\text{Not Containing } x \text{ term})\,dy = 0$

$\int 2x - y\,dx + \int y\,dy = 0$

$\int 2x\,dx - \int y\,dx + \int y\,dy = 0$

$$\boxed{x - xy + \frac{y^2}{2} = 0}$$

or

$$\boxed{x + \frac{y^2}{2} + c = xy}$$

(4) $(y\sec^2 x + \sec x \tan x)\,dx + (\tan x + 2y)\,dy = 0$

Sol:- Step 1:- $M = y\sec^2 x + \sec x \tan x$, $N = \tan x + 2y$

$\frac{dM}{dy} = \sec^2 x$ $\qquad \frac{dN}{dx} = \sec^2 x$

$$\boxed{\frac{dM}{dy} = \frac{dN}{dx}} \longrightarrow \text{Satisfy Exact D.E.}$$

Step 2:- Integration

$\int M\,dx + \int \text{only } y \text{ term (Not Containing } x\text{)}\,dy = 0$

$\int y\sec^2 x + \sec x \tan x\,dx + \int 2y\,dy = 0$

$y\int \sec^2 x\,dx + \int \sec x \tan x\,dx + 2\int y\,dy = 0$

$$\boxed{y\tan x + \sec x + y^2 = c}$$

⑤ $(ax+hy+g)dx + (hx+by+f)dy = 0$

Sol:-

Step 1:- Analyze the equation

$M = (ax+hy-g)$ $\quad$ $N = hx+by+f$

$\frac{dM}{dy} = h$ $\quad$ $\frac{dN}{dx} = h$

$\boxed{\frac{dM}{dy} = \frac{dN}{dx}}$ Satisfy the Exact D.E.

Step 2:- Integration of M and N.

$\int M dx + \int (\text{only } y \text{ term}) dy = 0$

not containg x.

$\int (ax+hy-y)dx + \cancel{\int (hx+by+f)d} \int by\, dy \cancel{+0} \int f dy = 0$

$a\int \cancel{xdx} \int hy dx - \int y dx + b\int y\, dy \cancel{+0} \int f\, dy = 0$

$$\boxed{\frac{ax^2}{2} + xhy - xy + \frac{by^2}{2} \cancel{+0}\, fy = c}$$

or

$$\boxed{ax^2 + 2hxy - 2xy + by^2 + 2fy = 0}$$

⑥ $(x^2 + 2ye^{2x})dy + (2xy + 2y^2e^{2x})dx = 0$

Sol:-

Step 1:- Analyze the Equation.

$M = x^2 + 2ye^{2x}$ $\quad$ $N = 2xy + 2y^2e^{2x}$

$\frac{dM}{dy} = 2x + 4ye^{2x}$, $\frac{\partial N}{\partial y} = 2x + 4ye^{2x}$

$\boxed{\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}}$ Satisfy Exact D.E.

Step 2:- Integration of M(x,y) and N(x,y).

$\int M\,dy + \cancel{\int}$

$\int (\text{N of } \cancel{\text{only}} \text{ Containing } x \text{ term})\,dy + \int N\,dx = 0$

$\int 0\,dy + \int 2xy + 2y^2 e^{2x}\,dx = 0$

$0 + 2\int xy\,dx + 2y^2 \int e^{2x}\,dx = 0$

$x^2 y + 2y^2 \int e^u \frac{1}{2}\,du = 0$

$\therefore u = 2x \Rightarrow du = \frac{1}{2}dx$

$x^2 y + y^2 e^u = C \Longrightarrow \boxed{x^2 y + y^2 e^{2x} = C}$

(7) $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x \cdot \sin y)\,dy = 0$

Sol:-

Step 1:- Analyze the Equation

$M = y + y/x + \cos y$, $N = x + \log x - x \cdot \sin y$.

$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$ $\quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$.

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ Satisfy Exact D.E.

Step 2:- Integration of $M(x,y)$ and $N(x,y)$

$$\int M\,dx + \int (\text{not Containing } x \text{ in term})\,dy = 0$$

$$\int y + \frac{y}{x} + \cos y\,dx + \int \cancel{x} \, dy = 0$$

$$\int y\,dx + \int \frac{y}{x}\,dx + \int \cos y\,dx = 0$$

$$\boxed{xy + y\ln x + x\cos y = c}$$

or

$$\boxed{y(yx + \ln x) + x\cos y = c}$$

(8) $(x^3 - 3xy^2)\,dx + (y^3 - 3x^2y)\,dy = 0 \qquad y(0) = 1$

Sol:

Step 1:- Analyze the Equation.

$M = x^3 - 3xy^2 \quad , \quad N = y^3 - 3x^2y$

$\frac{dM}{dy} = -6xy \quad , \quad \frac{dN}{dx} = -6xy$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \quad \text{Satisfy Exact D.E.}$$

Step 2:- Integration of $M(x,y)$ and $N(x,y)$.

$$\int M\,dx + \int (\text{terms in } N \text{ not Containing } x)\,dy = c$$

$$\int x^3 - 3xy^2\,dx + \int y^3\,dy = c$$

$$\int x^3\,dx - 3\int xy^2\,dx + \int y^3\,dy = c$$

$$\boxed{\frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} = c}$$

or

$\therefore y(0) = 1$

$\cancel{(0)^4 - 6(0)(1) + (1)^4 = 4c}$

$\cancel{0 - 0 + 1 = 4c}$

$$\frac{(0)^2}{4} - \frac{3}{2}(0)^2(1)^2 + \frac{(1)^4}{4} = c$$

$$0 - 0 + \frac{1}{4} = c$$

$$\boxed{c = \frac{1}{4}}$$

$$\frac{x^2}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} = \frac{1}{4}$$

$$\boxed{x^2 - 6x^2y^2 + y^4 = 1}$$

(10) $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}(1+y^2)x\,dy=0$

Sol:

Step 1: Analyze the equation

$M=\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)$. $N=\frac{1}{4}(1+y^2)x$

$\frac{\partial M}{\partial y}=1+y^2$ $\qquad$ $\frac{\partial N}{\partial x}=\frac{1}{4}(1+y^2)$

$\boxed{\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}}$ → It doesn't satisfy Exact D.E
→ Now apply the rule of Exact D.E.

Rule 1: If $M(x,y)dx+N(x,y)dy=0$. is not exact

then $\boxed{\frac{M_y-N_x}{N}=P(x)}$ → $I.F=e^{\int P(x)dx}$

Step 1:

$P(x)=\frac{1+y^2-\left(\frac{1}{4}(1+y^2)\right)}{\frac{1}{4}(1+y^2)}=\frac{3}{x}\frac{\cancel{(1+y^2)}}{\cancel{(1+y^2)}}=\frac{3}{x}$

$I.F=e^{\int P(x)dx}=e^{\int\frac{3}{x}dx}=e^{\ln x^3}=\boxed{x^3}$

Step 2: Multiply I.F with M and N.

$M_1=x^3y+\frac{x^3}{3}y^3+\frac{1}{2}x^5$ $\qquad$ $N_1=\frac{x^4}{4}(1+y^2)$

$\frac{\partial M_1}{\partial y}=x^3+x^3y^2$ $\qquad$ $\frac{\partial N_1}{\partial x}=\cancel{4}\frac{x^3}{\cancel{4}}(1+y^2)$

$$\frac{\partial M_1}{\partial y} = x^3(1+y^2) \qquad \frac{\partial N_1}{\partial x} = x^3(1+y^2)$$

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}} \Rightarrow \text{Now it is Exact D.E.}$$

Step 3:- Integration of $M_1(x,y)$ and $N_1(x,y)$.

$$\int M_1\,dx + \int (N_1 \text{ not Containing } x \text{ term})\,dy = 0$$

$$\int x^3y + \frac{x^3}{3}y^3 + \frac{1}{2}x^5\,dx + \int 0\,dy = 0$$

$$\int x^3y\,dx + \int \frac{x^3}{3}y^3\,dx + \frac{1}{2}\int x^5\,dx + 0 = 0$$

$$\boxed{\frac{x^4}{4}y + \frac{x^4y^3}{12} + \frac{x^6}{12} = c.}$$