Laplace Transform

1749 - 1827

:→ French mathematician and stronomer " Pierre-Simon Marquis"

:→ Laplace transforms help in solving the D.E with boundary values without finding the general solution and values of arbitrary constant.

Definition:- Let $f(t)$ be a function defined for all positive values of "t" then $F(s)=\int_0^\infty e^{-st}f(t)\,dt$ provide the general integral exists, is called laplace.

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Some elementary functions

$f(t)$	$L\{f(t)\}=F(s)$	
1⇒ c	c/s	$s>0$
2⇒ t	$\frac{1}{s^2}$	$s>0$
3⇒ t^n	$\frac{n!}{s^{n+1}}$	$n \in W$
4⇒ e^{at}	$\frac{1}{s-a}$	$s>a$
5⇒ $\sin at$	$\frac{a}{s^2+a^2}$	
6⇒ $\cos at$	$\frac{s}{s^2+a^2}$	
7⇒ $\sinh at$	$\frac{a}{s^2-a^2}$	
8⇒ $\cosh at$	$\frac{s}{s^2-a^2}$	

Prove that: $L\{c\}=\frac{c}{s}$ and $L\{t\}=\frac{1}{s^2}$, $L\{t^n\}=\frac{n!}{s^{n+1}}$

Sol:

$$L\{f(t)\}=\int_0^\infty e^{-st}(c)\,dt$$

$$= c\int_0^\infty e^{-st}dt$$

$$= c\lim_{b\to\infty}\int_0^b e^{-st}dt$$

$$= c\lim_{b\to\infty}\left[e^{-st}\left(\frac{1}{-s}\right)\right]_0^b$$

$$= -\frac{c}{s}\lim_{b\to\infty}\left[e^{-bs}-e^{0}\right]$$

$$= \frac{c}{s} - e^{-\infty} = \frac{c}{s} - 0$$

$$= c/s \text{ Ans}$$

$$L\{t\} = \int_0^\infty e^{-st}\cdot t\,dt$$

$$= \lim_{b\to\infty}\int_0^b t\cdot e^{-st}dt$$

$$= \lim_{b\to\infty}\left[t\frac{e^{-st}}{-s} - \int_0^b \frac{e^{-st}}{-s}dt\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{te^{-st}}{-s} + \frac{1}{s}\frac{e^{-st}}{-s}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{be^{-bs}}{-s} - \frac{1}{s^2}e^{-sb}\right] - \lim_{b\to\infty}\left[0 - \frac{1}{s^2}e^{0}\right]$$

$$= \lim_{b\to\infty}\left\{\frac{b}{-s}\cdot\frac{1}{e^{bs}} - \frac{1}{s^2}\frac{1}{e^{sb}}\right\} + \frac{1}{s^2}$$

$$= \infty\cdot 0 - \frac{1}{s^2}\cdot 0 + \frac{1}{s^2} = \frac{1}{s^2} \text{ Ans}$$

$L\{t^n\} = ?$

$L\{t^n\} = \int_0^\infty e^{-st} t^n dt \Rightarrow t^n \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s} dt = \left[\infty \frac{e^{-\infty}}{-s} - 0 \cdot \frac{e^{-0}}{-s}\right] + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$

$= (\infty \cdot 0 - 0) + \frac{n}{s}\int_0^\infty e^{-st} \cdot \underbrace{t^{n-1}}_{f(t)} dt$ on definition $\int_0^\infty e^{-st} f(t) dt = F(s) = L\{f(t)\}$

$= 0 + \frac{n}{s} L\{t^{n-1}\}$

$= \frac{n!}{s^{n+1}}$

$\therefore L\{t\} = \frac{1}{s^2}$

$\therefore L\{t^2\} = \frac{2}{s^3}, \Rightarrow \frac{n}{s} L\{t^{n-1}\}$

$\therefore L\{t^3\} = \frac{3!}{s^4} = \frac{3}{s} \cdot L\{t^2\} = \frac{3}{s} \cdot \frac{2}{s^3}$

$\Rightarrow L\{e^{at}\}$

$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{at-st} dt = \int_0^\infty e^{(a-s)t} dt = e^{(a-s)t} \cdot \frac{1}{a-s}\Big|_0^\infty$

$= \frac{1}{a-s}\left[e^{(a-s)\infty} - e^{(a-s)0}\right] = \frac{1}{a-s}\left[e^{-\infty} - e^{0}\right] = \frac{1}{a-s}[0-1] = \frac{1}{s-a}$ Ans

$\Rightarrow f(t) = e^{-3t}$

$L\{e^{-3t}\} = \int_0^\infty e^{-3t} \cdot e^{-st} dt$

$= \int_0^\infty e^{(-3-s)t} dt = e^{(-3-s)t} \cdot \frac{1}{-3-s}\Big|_0^\infty = \frac{1}{-(3+s)}\left[e^{-\infty} - e^{0}\right] = \frac{1}{s+3}$ Ans

$\Rightarrow$ **Gamma Function:** $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$

$\Gamma n = (n-1)! \Rightarrow \Gamma 3 = (3-1)! = 2!$

$\Gamma n+1 = n \Gamma n \Rightarrow$ Fraction

$\Gamma n+1 = n! \Rightarrow$ +ve Inoger

$\Gamma n = \frac{\Gamma(n+1)}{n} \Rightarrow$ -ve Fraction

$\Gamma 0 = \infty$

$\Gamma \frac{1}{2} = \sqrt{\pi}$

$\therefore \Gamma -\frac{5}{3} = \frac{\Gamma -\frac{5}{3}+1}{-\frac{5}{3}} = -\frac{3}{5}\Gamma \frac{-2}{3} = \frac{-3}{5} \frac{\Gamma \frac{-2}{3}+1}{\frac{-2}{3}} = -\frac{3}{5}\left(-\frac{3}{2}\right)\Gamma \frac{1}{3} = \frac{9}{10}\Gamma \frac{1}{3}$

$\therefore \Gamma \frac{5}{2} = \Gamma n+1 \quad \therefore \frac{5}{2} - 1 = \frac{3}{2}$

$= \Gamma \frac{3}{2}+1 = \frac{3}{2}\Gamma \frac{3}{2}$ again

$= \frac{3}{2}\Gamma \frac{1}{2}+1 = \frac{3}{2}\left(\frac{1}{2}\right)\Gamma \frac{1}{2} = \frac{3}{4}\sqrt{\pi}$

$L\{\sqrt{t}\} = L\{t^{1/2}\}$

$L\{\sqrt{t}\} = \int_0^\infty e^{-st} t^{1/2} dt \Rightarrow L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

$= \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} = \frac{1\sqrt{\pi}}{2 s^{3/2}}$ Ans $\therefore \Gamma(1/2) = \sqrt{\pi}$

$L\{t^{-1/2}\} = \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$ Ans $\quad n\Gamma(n) = \Gamma(n+1)$

$L\{\sin at\} = \frac{a}{s^2+a^2}$ prove it:

$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt = e^{-st}\int \sin at\, dt - \int \left(\frac{d}{dt} e^{-st} \int \sin at\, dt\right) dt$

$= -e^{-st} \cos at \cdot \frac{1}{a}\Big|_0^\infty - \int_0^\infty e^{-st} (-s) \cdot \left(\frac{-\cos at}{a}\right) dt$

$= -\frac{1}{a}[\infty \cdot 0 - 1] - \frac{s}{a}\int_0^\infty e^{-st} \cos at\, dt$ again by parts

$= \frac{1}{a} - \frac{s}{a}\left[e^{-st}\frac{\sin at}{a}\right]_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}(-s)\frac{\sin at}{a} dt$

$= \frac{1}{a} - \frac{s}{a}[0-0] - \frac{s^2}{a^2}\int_0^\infty e^{-st} \sin at\, dt$

$= \frac{1}{a} - \frac{s^2}{a^2} L\{\sin at\}$

$L\{\sin at\} + \frac{s^2}{a^2} L\{\sin at\} = \frac{1}{a}$

$L\{\sin at\}\left(1 + \frac{s^2}{a^2}\right) = \frac{1}{a}$

$L\{\sin at\} = \frac{a^{\cancel{2}}}{\cancel{a}(a^2+s^2)}$

$= \frac{a}{s^2+a^2}$ Ans

$\therefore L\{\sin 2t\} = \frac{2}{s^2+4}$

Similarly solve for

$L\{\cos at\} = \frac{s}{s^2+a^2}$

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Laplace Transform

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- Named after its inventor Pierre-Simon laplace, French mathematician.

Start with a D.E → Use laplace Transform → This operator Convert DE into Linear or algebraic equation → Solve Algebraic equation → Use Inverse Laplace Transform → Solution of DE

Definition:- Let $f(t)$ be a function of t specified for $t>0$ then

- the laplace transform of $F(t)$ denoted by $\mathcal{L}\{f(t)\}$ is

$$\mathcal{L}\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$f(t)$	$\mathcal{L}\{f(t)\} = f(s)$
c	c/s
t	$1/s^2$
t^n	$\frac{n!}{s^{n+1}}$, $n \in N$, $t^n \Rightarrow \Gamma(n+1)/s^{n+1}$, $n \in \mathbb{R}$
e^{at}	$1/(s-a)$
e^{-at}	$1/(s+a)$
$\sin at$	$\frac{1}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh t$	$a/(s^2-a^2)$
$\cosh t$	$\frac{s}{s^2-a^2}$

$\therefore$ 1st shifting property

$\mathcal{L}\{f(t)\} = f(s)$

$\mathcal{L}\{e^{at} f(t)\} = f(s-a)$

Ex: $f(t) = 1$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{e^{-sb} - e^{-s0}}{-s}\right]$$

$$= \lim_{b\to\infty}\left[\frac{e^{-sb}}{-s}\right] + \lim_{b\to\infty}\left[\frac{1}{s}\right]$$

$$= 0 + 1/s = 1/s \text{ Ans}$$

Ex: $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\,dt$$

$$= \lim_{b\to\infty}\left[t\frac{e^{-st}}{-s} + \frac{1}{s}\int_0 e^{-st}dt\right]_0^b$$

$$= 0 + \lim_{b\to\infty}\frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^b$$

$$= -\frac{1}{s^2}\lim_{b\to\infty}\left[e^{-sb} - e^{-s0}\right]$$

$$= -\frac{1}{s^2}(-1) = 1/s^2 \text{ Ans}$$

Ex $f(t) = e^{-3t}$

$$\mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-st}.e^{-3t}dt$$

$$= \lim_{b\to\infty}\int_0^b e^{-t(s+3)}dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-t(s+3)}}{-(s+3)}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{e^{-b(s+3)} - e^0}{-(s+3)}\right]$$

$$= \frac{1}{s+3} \text{ Ans}$$

$\Gamma n = n!$ $n \in N$

$\Gamma n = n\Gamma n$ $n \in R^+$

$\Gamma n = \frac{\Gamma(n+1)}{n}$ $n \in R^-$

$n = -5/3$

so

$$\Gamma{-5/3} = \frac{\Gamma{-5/3+1}}{-5/3} = -\frac{3}{5}\Gamma{-2/3}$$

$$= -\frac{3}{5}\left[\frac{\Gamma{\frac{-2}{3}+1}}{\frac{-2}{3}}\right] = \left(\frac{-3}{5}\right)\left(\frac{-3}{2}\right)\Gamma{1/3}$$

$$= \frac{9}{10}\Gamma{1/3} \text{ Ans}$$

$\because \Gamma{1/2} = \sqrt{\pi}$

$$\because \Gamma{3/2} = \Gamma{1/2+1} = \frac{1}{2}\Gamma{1/2} = \frac{1}{2}\sqrt{\pi}$$

Ex $f(t) = \sqrt{t}$

$$\mathcal{L}\{t^{1/2}\} = \int_0^\infty e^{-st} t^{1/2} dt$$

$$= \frac{\Gamma{\frac{1}{2}+1}}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\Gamma{1/2}}{s^{3/2}}$$

$$= \frac{1\sqrt{\pi}}{2s^{3/2}} \text{ Ans}$$

Ex: $f(t) = \sin 2t$

$$\mathcal{L}\{\sin 2t\} = \int_0^\infty e^{-st}\sin 2t\,dt$$

$$= \lim_{b\to\infty}\left[\sin 2t\frac{e^{-st}}{-s} + \int_0^b \cos 2t.2\frac{e^{-st}}{s}dt\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{\sin 2t\, e^{-st}}{-s}\right]_0^b + \lim_{b\to\infty}\frac{2}{s}\int_0^b \cos 2t\, e^{-st}dt$$

$$= 0 + \frac{2}{s}\lim_{b\to\infty}\left[\frac{\cos 2t\, e^{-st}}{-s} - \frac{2}{s}\int_0 \sin 2t\, e^{-st}dt\right]$$

$$= \frac{2}{s}\lim_{b\to\infty}\left[\frac{\cos 2t\, e^{-st}}{-s}\right]_0^b - \frac{4}{s^2}\lim_{b\to\infty}\int_0^b \sin 2t\, e^{-st}dt$$

$$I = \frac{2}{s}\left[0 + \frac{1}{s}\right] - \frac{4}{s^2}I$$

$$I\left(\frac{s^2+4}{s^2}\right) = \frac{2}{s^2} \Rightarrow I = \frac{2}{s^2+4} \text{ Ans}$$

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Linearity property

If $c_1, c_2, \dots c_n$ are constants and $f_1, f_2, \dots f_n$ are functions of t whose laplace is

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \dots c_n f_n(t)\} = c_1 F_1(s) + c_2 F_2(s) + \dots + c_n F_n(s)$$

Ex:- $\mathcal{L}\{3e^{-4t} + \cosh 2t - 2\sin 3t + t^3\}$

Solu $= 3\mathcal{L}\{e^{-4t}\} + \mathcal{L}\{\cosh 2t\} - 2\mathcal{L}\{\sin 3t\} + \mathcal{L}\{t^3\}$

$$= 3\frac{1}{s+4} + \frac{s}{s^2-4} - 2\frac{3}{s+9} + \frac{3!}{s^4}$$

$$= \frac{3}{s+4} + \frac{s}{s^2-4} - \frac{6}{s+9} + \frac{6}{s^4} \quad \text{Ans}$$

Ex:- $\mathcal{L}\{e^{-4t}\cosh 2t\}$

Solu $= \mathcal{L}\left\{e^{-4t}\left(\frac{e^{2t}+e^{-2t}}{2}\right)\right\} = \frac{1}{2}\mathcal{L}\{e^{-2t} + e^{-6t}\} = \frac{1}{2}\left[\mathcal{L}\{e^{-2t}\} + \mathcal{L}\{e^{-6t}\}\right]$

$$= \frac{1}{2}\left[\frac{1}{s+2} + \frac{1}{s+6}\right] \text{ Ans or } \frac{1}{2}\left[\frac{s+6+s+2}{s^2+6s+2s+12}\right] = \frac{1}{2}\left(\frac{2(s+4)}{s^2+2(s)(4)+(4)^2-4}\right) = \frac{s+4}{(s+4)^2-4} \text{ Ans}$$

First Shifting Theorem

$$\boxed{\text{If } \mathcal{L}\{f(t)\} = F(s) \text{ then } \mathcal{L}\{e^{at} f(t)\} = F(s-a)}$$

Ex: $\mathcal{L}\{e^{2t}\sin 3t\}$

Solu $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

$$\mathcal{L}\{e^{2t}\sin 3t\} = \frac{3}{(s-2)^2+9} \text{ Ans}$$

Ex:- $\mathcal{L}\{e^{-3t}\cos 2t\}$

$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

$$\mathcal{L}\{e^{-3t}\cos 2t\} = \frac{s+3}{(s+3)^2+4} \text{ Ans}$$

Ex:- $\mathcal{L}\{e^{-4t}\cosh 2t\}$

Solu $\mathcal{L}\{\cosh 2t\} = \frac{s}{s^2-4}$

$$\mathcal{L}\{e^{-4t}\cosh 2t\} = \frac{(s+4)}{(s+4)^2-4} \text{ Ans}$$

Ex: $\mathcal{L}\{t^4 e^{7t}\}$

Laplace transform of derivatives.

$$\boxed{\begin{aligned} &\mathcal{L}\{F(t)\} = f(s) \\ \text{then } &\mathcal{L}\{f'(t)\} = sf(s) - F(0) \end{aligned}}$$

Ex:- $f(t) = \cos 3t$ $\quad f'(t) = -\sin 3t \cdot 3$, $f(0) = \cos(0) = 1$

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+9}$$

$$\mathcal{L}\{f'(t)\} = s\left[\frac{s}{s^2+9}\right] - 1 = \frac{\cancel{s^2} - \cancel{s^2} - 9}{s^2+9} = \frac{-9}{s^2+9} \text{ Ans}$$

$$\boxed{\begin{aligned} &\mathcal{L}\{F(t)\} = f(s) \\ &\mathcal{L}\{f''(t)\} = s^2 f(s) - s f(0) - F'(0) \end{aligned}}$$

Ex: $f(t) = \cos 3t$ $\quad f'(t) = -\sin 3t \cdot 3$ $\quad f''(t) = -3\cos 3t \cdot 3$
$= -9\cos 3t.$

$\mathcal{L}\{f''(t)\} = ?$

$$\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}, \quad f(0) = \cos(0) = 1$$
$$f'(0) = -9$$

$$\mathcal{L}\{f''(t)\} = s^2\left[\frac{s}{s^2+9}\right] - s(1) - (-9)$$

$$= \frac{s^3}{s^2+9} - s + 9 = \frac{\cancel{s^3} - \cancel{s^3} - 9s + 9s^2 + 81}{s^2+9}$$

$$= \frac{9[s^2 - s + 9]}{s^2+9} \text{ Ans}$$

So General form of

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - F^{(n-1)}(0)}$$

Laplace transform of Integrals.

$$\boxed{\mathcal{L}\{F(t)\} = f(s) \text{ then } \mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}}$$

Ex: $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

$$\mathcal{L}\left\{\int_0^t \sin 2u\,du\right\} = \left(\frac{2}{s^2+4}\right)\frac{1}{s} = \frac{2}{s(s^2+4)} \text{ Ans}$$

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Multiplication by t^n

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)}$$

ex:- $\mathcal{L}\{t^2 e^{2t}\} \Rightarrow \mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$

- $\mathcal{L}\{t \cdot e^{2t}\} = (-1)\frac{d}{ds}\left(\frac{1}{s-2}\right) = -\frac{d}{ds}(s-2)^{-1} = (s-2)^{-2} = \frac{1}{(s-2)^2}$

$\mathcal{L}\{t^2 e^{2t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{d^2}{ds^2}(s-2)^{-1} = \frac{d}{ds}\left[-(s-2)^{-2}\right] = +2(s-2)^{-3} = \frac{2}{(s-2)^3}$

OR

$\mathcal{L}\{t^2 e^{2t}\} \Rightarrow \mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

$\mathcal{L}\{e^{2t} \cdot t^2\} = \frac{2}{(s-2)^3}$ Ans

1st shifting property

- ## Division by t

$$\boxed{\mathcal{L}\{f(t)\} = F(s) \quad \text{then} \quad \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du} \quad \lim_{t\to 0} \frac{f(t)}{t}$$

Ex:- $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ $\quad \lim_{t\to 0} \frac{\sin t}{t} = 1$

So $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du = \tan^{-1}(u)\Big|_s^\infty = \boxed{\tan^{-1}\left(\frac{1}{\infty}\right) - \tan^{-1}\left(\frac{1}{s}\right)}$

$= \tan^{-1}\left(\frac{1}{s}\right)$ Ans

Ex (mult: $\mathcal{L}\{t \sin at\}$

$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

So $\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{0 - a(2s)}{(s^2+a^2)^2} = \frac{-2as}{(s^2+a^2)^2}$

$\mathcal{L}\{t \cdot \sin at\} = \frac{-2as}{(s^2+a^2)^2}$ Ans