Worksheet Lecture #1

Q2. obtain the D.E :-

(i) $y^2 = 4a(x+a)$.

↳ $y^2 = 4ax + 4a^2$

* Differentiate w.r.t x.

$$2y\frac{dy}{dx} = 4a + 0$$

$$\boxed{2\frac{dy}{dx} = 4a - 2y}$$

↳ Diffr equation.

(ii) $Ax^2 + By^2 = 1$

* Differentiate w.r.t x

$$\boxed{2Ax + 2By\frac{dy}{dx} = 0}$$

$$2A + 2B\frac{d^2y}{dx^2} = 0$$

Q3 By eliminating Constant find D.E.

$y = e^x(A\cos x + B\sin x) \rightarrow y = e^x A\cos x + Be^x \cos x$

* Differentiate w.r.t x.

$$\frac{dy}{dx} = \underbrace{e^x(A\cos x + B\sin x)}_{} + e^x(-A\sin x + B\cos x)$$

↳ Substitute y

$$\frac{dy}{dx} = y + e^x(-A\sin x + B\cos x).$$

* Differentiate again w.r.t x.

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \underbrace{e^x(-A\sin x + B\cos x)}_{\text{Substitute } \frac{dy}{dx} - y =} +$$

$e^x(-A\cos x - B\sin x)$.

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} - y - \underbrace{e^x(A\cos x + B\sin x)}_{\text{Substitute } y}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} - y - y$$

$$\boxed{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0}$$

Q4 Variable Seperable

(i) $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

1) Separating the Variable.

$$(\sin y + y\cos y)dy = x(2\log x + 1)dx.$$

2. Integrating both sides

$$\int \sin y + y\cos y\, dy = \int 2x\log x + x\, dx.$$

$$\int \sin y\, dy + \underbrace{\int y\cos y\, dy}_{\text{ILATE.}} = 2\int \underbrace{x\log x}_{\text{ILATE}}\, dx + \int x\, dx.$$

~~$-\cos y + [$~~ $\therefore \int u\cdot v = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx.$

$\int y\cos y\, dy.$

$u = y$, $V = \cos y\, dy$

$\frac{du}{dy} = \cancel{dy}\,1$, $\int v = \sin y$

$\int x\log x\, dx.$

$U = \cancel{x}\log x$, $V = x\, dx$

$\frac{du}{dx} = \frac{1}{x}$, $\int V = \frac{x^2}{2}.$

Arrange the equation.

$$-\cos y + y\sin y - \int 1\cdot\sin y\, dy = 2\left[\log x\cdot\frac{x^2}{2} - \int\frac{1}{x}\cdot\frac{x^2}{2}\,dx\right] + \frac{x^2}{2} + c$$

$$\cancel{-\cos y} + y\sin y + \cancel{\cos y} = 2\left[\log x\cdot\frac{x^2}{2} - \frac{1}{2}\int x\, dx\right] + \frac{x^2}{2} + c$$

$$y\sin y = 2\left[\log x\cdot\frac{x^2}{2} - \frac{x^2}{4}\right] + \frac{x^2}{2} + c$$

$$y \sin y = \cancel{2} \cdot \frac{1}{\cancel{2}} \left[x^2 \cdot \log x - \cancel{\frac{x^2}{2}} \right] + \cancel{\frac{x^2}{2}} + c$$

$$\boxed{y \sin y = x^2 \cdot \log x + c}$$

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(iii) $\cos(x+y)\,dy = dx.$

Separating the Variable

$\cos(x+y)\,dy = dx.$ $\therefore$ let $u = x+y.$

$du = dx + dy$

$dx = du - dy$

$\cos(u)\,dy = du - dy$

$\cos(u)\,dy + dy = du$

~~Common~~

$dy\,(\cos(u)+1) = du$

$\cos(u) = 2\cos^2\frac{u}{2} - 1.$

(i) $\cos x = 2\cos^2\frac{x}{2} - 1$

(ii) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$dy = \frac{du}{2\cos^2\frac{u}{2} - \cancel{1} + \cancel{1}}$$

$$dy = \frac{du}{2\cos^2\frac{u}{2}}$$

Integrating on b/s

$$\int dy = \frac{1}{2}\int \frac{1}{\cos^2\frac{u}{2}}\, du.$$

$$y = \frac{1}{2}\int \sec^2\frac{u}{2}\, du.$$

$$y = \frac{1}{2}\tan\frac{u}{\cancel{2}}\cdot\cancel{2} + c$$

$$y = \frac{\tan(u)}{2} + c.$$

$u = x + y$

$$\boxed{y = \frac{\tan(x+y)}{2} + c}$$

Q iv) $(2x^2+3y^2-7)x\,dx - (3x^2+2y^2-8)y\,dy = 0$

Let $X = x^2$, $Y = y^2$

$dX = 2x\,dx$, $dY = 2y\,dy$

$2x^3 + 3xy^3 - 7x - 3x^2y -$

$(2x+3y-7)\,dx = (3x+2y-8)\,dy.$

$$\frac{dy}{dx} = \frac{2x+3y-7}{3x+2y-8.} \Bigg] \rightarrow \text{homogenaus equation.}$$

let. $x = s+a.$

$y = t+b.$

$$\frac{dt}{ds} = \frac{2s+3t+\overbrace{2a+3b-7}}{3s+2t+\underbrace{3a+2b-8}_{\text{Costent.}}} \Rightarrow \frac{2s+3t}{3s+2t.}$$

$$\frac{dt}{ds} = \frac{2s+3t}{3s+2t.}$$

$\therefore\ 2a+3b-7=0$

$3a+2b-8=0$

$\downarrow$

$a=2$

$b=1.$

$t(s) = s\,u(s)$

$$\frac{dt}{ds} = u+s\frac{du}{ds}$$

$$u+s\frac{du}{ds} = \frac{2+3u}{3+2u.} -$$

$$s\frac{du}{ds} = \frac{2+3u}{3+2u} - u = \frac{2-2u^2}{3+2u}$$

$$\frac{ds}{s} = \frac{3+2u}{2-2u^2}\,du.$$

<u>Integrating on b/s:</u>

$$\int \frac{ds}{s} = \int \frac{3+2u}{2-2u^2}\, du$$

$$\frac{1}{4}\ln(u+1) - \frac{5}{4}\ln(u-1) = \ln|s| + c$$

$$\ln\left(\frac{(u+1)^{\frac{1}{4}}}{(u-1)^{\frac{5}{4}}}\right) = \ln|s| + c$$

$$s = \frac{(u+1)^{\frac{1}{4}\times 4}}{(u-1)^{\frac{5}{4}\times 4}}$$

$$c(t-s)^5 = (t+s)$$

$\therefore t = y - b$

$$c(y-1-x+2)^5 = (y-1+x-2) \qquad t = y-1$$

$$s = x - a$$

$$c(y-x+1)^5 = (y+x-3) \qquad s = x-2$$

$$\boxed{c(y^2-x^2+1)^5 = (y^2+x^2-3)} \qquad \therefore Y = y^2$$

$$\therefore X = x^2$$

(vi) $\dfrac{dx}{x} = \tan y \, dy$

Integrating on b/s.

$$\int \frac{1}{x}\,dx = \int \tan y\, dy$$

$\because \tan y = \frac{\sin y}{\cos y}$

$$\ln|x| = \int \frac{\sin y}{\cos y}\, dy.$$

$\therefore$ Let $u = \cos y$

$du = -\sin y\, dy$

$dy = \frac{du}{-\sin y}$

$$\ln|x| = -\int \frac{\cancel{\sin y}}{u} \cdot \frac{du}{\cancel{\sin y}}$$

$$\ln|x| = -\int \frac{1}{u}\, du.$$

$$\ln|x| = -\ln|u| + \ln|c|$$

$$\ln|x| + \ln|u| = \ln|c|$$

$$\cancel{\ln}(x \cdot u) = \cancel{\ln}|c|$$

$\because u = \cos y$

$$\boxed{x \cdot \cos y = c}$$

(vi) $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1+x^2}}$

(i) Separating Variables.

$$\frac{1\,dy}{\sqrt{1-y^2}} = \frac{1}{\sqrt{1+x^2}}\, dx.$$

(ii) Integrating on b/s:

$$\int \sqrt{\frac{1}{1-y^2}}\,dy = \int \frac{1}{\sqrt{1+x^2}}\,dx.$$

$$\int \frac{1}{\sqrt{1-\sin^2 y}}\,dy = \int \frac{1}{\sqrt{1+\cancel{\sin^2} x}}\,dx$$

let $y = \sin u$, $dy = \cos u\,du$

$x = \cancel{\sin}\ \tan u$

$dx = \sec^2 u\,du$

$$\int \frac{1}{\sqrt{\cos^2 u}} \cdot \cos u\,du = \int \frac{1}{\sqrt{\sec^2 u}} \cdot \sec^2 u\,du$$

$$\int \frac{1}{\cancel{\cos u}} \cdot \cancel{\cos u}\,du = \int \frac{1}{\cancel{\sec u}} \cdot \sec^2 u\,du$$

$$\int 1\,du = \int \sec u\,du$$

$$\cancel{S}\,u = \cancel{\sec u + \tan u + c}$$

$$\sin^{-1} y = \int \sec u \cdot \frac{\sec u + \tan u}{\sec u + \tan u}\,du$$

$$\sin^{-1} y = \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u}\,du$$

$t = \sec u + \tan u$

$\frac{dt}{du} = \sec u \tan u + \sec^2 u$

$$\sin^{-1} y = \int \frac{dt}{t}$$

$$\sin^{-1} y = \ln|t| + c$$

$$\sin^{-1} y = \ln|\sec u + \tan u|$$

$$\boxed{\sin^{-1} y = \ln|\sec^{-1} x + \tan^{-1} x| + c}$$

(vii) $y(1+x^2)^{\frac{1}{2}}\,dy + x\sqrt{1+y^2}\,dx = 0$

or

$$\frac{y}{\sqrt{1+y^2}}\,dy + \frac{x}{\sqrt{1+x^2}}\,dx = 0$$

Integrating on b/s

$$\int \frac{y}{\sqrt{1+y^2}}\,dy + \int \frac{x}{\sqrt{1+x^2}}\,dx = 0.$$

$$\int \frac{y}{(1+y^2)^{\frac{1}{2}}}\,dy + \int \frac{x}{(1+x^2)^{\frac{1}{2}}}\,dx = 0$$

Let $u = 1+y^2$

$du = 2y\,dy$

$$\int \frac{\cancel{y}}{u^{\frac{1}{2}}}\,\frac{du}{2\cancel{y}} + \int \frac{\cancel{x}}{v^{\frac{1}{2}}}\,\frac{dv}{2\cancel{x}} = 0$$

• $v = 1+x^2$

$dv = 2x\,dx.$

$$\frac{1}{2}\int \frac{1}{u^{\frac{1}{2}}}\,du + \frac{1}{2}\int \frac{1}{v^{\frac{1}{2}}}\,dv = 0$$

$$\frac{\cancel{1}}{2}\cdot\frac{u^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + \frac{\cancel{1}}{2}\cdot\frac{v^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} = 0$$

$$u^{\frac{1}{2}} + v^{\frac{1}{2}} + c = 0$$

$$\boxed{\sqrt{1+y^2} + \sqrt{1+x^2} = c}$$

(viii) $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

$$\frac{\cos x}{\sin x}\,dx + \frac{e^y}{e^y+1}\,dy = 0$$

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Integrating

$$\int \frac{\cos x}{\sin x}\,dx + \int \frac{e^y}{e^y+1}\,dy = 0$$

Let $u = \sin x$.

$du = \cos x\,dx$

$$\int \frac{\cos x}{u} \cdot \frac{du}{\cancel{\cos x}} + \ln|e^y+1| + c = 0$$

$$\int \frac{1}{u} \cdot du + \ln|e^y+1| = \ln|c|$$

$$\ln|\sin x| + \ln|e^y+1| = \ln|c|$$

$$\cancel{\ln}(\sin x \cdot (e^y+1)) = \cancel{\ln}|c|$$

$$\boxed{\sin x \cdot (e^y+1) = c}$$

(ix) $(e^y+2)\sin x\,dx - e^y\cos x\,dy = 0$

Seperating Variables:

$$\frac{\sin x\,dx}{\cos x} - \frac{e^y}{e^y+2}\,dy = 0$$

Integrating

$$\int \frac{\sin x}{\cos x}\,dx - \int \frac{e^y}{e^y+2}\,dy = 0$$

Let $u = \cos x$

$du = -\sin x\,dx$

$$\int \frac{\sin x}{u} \cdot \frac{du}{\sin x} - \ln|e^y+2| = \ln|c|$$

$$-\int \frac{1}{u}\, du - \ln|e^y+2| = \ln|c|$$

$$-\ln|\cos x| - \ln|e^y+2| = \ln|c|$$

$$-\left[\cancel{\ln}\left(\cos x \cdot (e^y+2)\right)\right] = \cancel{\ln}|c|$$

$$\boxed{-\left(\cos x \cdot (e^y+2)\right) = c}$$

(x) $\frac{dy}{dx} = 1+\tan(y-x)$ hint [Put $y-x=z$]

$y-x=z$

$\frac{dy}{dx} - 1 = \frac{dz}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} + 1$

$$\frac{dz}{dx} + \cancel{1} = \cancel{1} + \tan(z)$$

$$\cancel{\tan(z)}\ \frac{dz}{\tan(z)} = dx$$

$$\frac{\cos z\, dz}{\sin z} = dx$$

Integrating on b/s

$$\int \frac{\cos z\, dz}{\sin z} = \int 1\, dx$$

Let $u = \sin z$

$du = \cos z\, dz$

$dz = \frac{du}{\cos z}$

$$\int \frac{\cancel{\cos z}}{u} \cdot \frac{du}{\cancel{\cos z}} = x + c$$

$$\int \frac{1}{u} du = x + c$$

$$\ln|\sin z| = x + c \qquad \because z = y - x.$$

$$\ln|\sin(y-x)| = x + c$$

<u>taking exponential on b/s</u>

$$e^{\ln|\sin(y-x)|} = e^{x+c}$$

$$\boxed{\sin(y-x) = e^{x+c}}$$