Lecture # 03

Linear D.E

Ex: $\frac{dy}{dx} + \underbrace{\sec x}_{P(x)} \cdot y = \underbrace{\tan x}_{Q(x)}$

$x \leftrightarrow$ Independent Variable

$\rightarrow$ Trancendental functions

Step 1: Integrating factor.

$$I.f = e^{\int P(x)\,dx}$$

$$I.f = e^{\int \sec x\,dx}$$

$$I.f = e^{\int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx}$$

Let $u = \sec x + \tan x$

$$I.f = e^{\int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x}\,dx}$$

$du = \sec x \tan x + \sec^2 x \, dx$

$$I.f = e^{\int \frac{du}{u}}$$

$$I.f = e^{\ln u +}$$

$$I.f = e^{\ln(\sec x + \tan x)}.$$

$$I.f = \sec x + \tan x.$$

Multiple I.f on b/s.

$$y(I.f) = \int Q(x)[I.f]\,dx + c$$

$$y(\sec x + \tan x) = \int \tan(x)(\sec x + \tan x)\,dx + c.$$

$$y(\sec x + \tan x) = \int \tan(x)\sec x + \tan^2 x\,dx + c$$

$$y(\sec x + \tan x) = \int \tan x \sec x\,dx + \int \tan^2 x\,dx + c$$

$$y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1)\,dx + c$$

$$\because \tan^2 x = \sec^2 x - 1.$$

$$y(\sec x + \tan x) = \sec x + \int \sec^2 x\,dx - \int dx + c$$

$$\boxed{y(\sec x + \tan x) = \sec x + \tan x - x + c}$$ Ans.

Q 12 -

$$\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$$

Sol:

$$\frac{dy}{dx} + y\underbrace{\frac{1}{x}}_{P(x)} = \underbrace{x^3 - 3}_{Q(x)}.$$

Step 1:- I.F

$$I.f = e^{\int P(x)\,dx}.$$

$$I.f = e^{\int \frac{1}{x}\,dx}.$$

$$I.f = e^{\ln x} = x.$$

Step 2- Multiply I.f on b/s.

$$[I.f]y = \int Q(x)(I.f)\,dx + c$$

$$xy = \int x(x^3 - 3)\,dx + c$$

$$xy = \int x^4 - 3x\,dx + c$$

$$\boxed{xy = \frac{x^5}{5} - \frac{3x^2}{2} + c}$$

$$I.F = e^{\int \frac{du}{u}}$$

$$I.F = e^{\ln u +}$$

$$I.F = e^{\ln(\sec x + \tan x)}$$

$$I.F = \sec x + \tan x.$$

Multiple I.F on b/s.

$$y(I.F) = \int Q(x)[I.F]\,dx + c$$

$$y(\sec x + \tan x) = \int \tan(x)(\sec x + \tan x)\,dx + c$$

$$y(\sec x + \tan x) = \int \tan(x)\sec x + \tan^2 x\,dx + c$$

$$y(\sec x + \tan x) = \int \tan x \sec x\,dx + \int \tan^2 x\,dx + c$$

$$y(\sec x + \tan x) = \sec x + \int(\sec^2 x - 1)\,dx + c$$

$$\because \tan^2 x = \sec^2 x - 1.$$

$$y(\sec x + \tan x) = \sec x + \int \sec^2 x\,dx - \int dx + c$$

$$\boxed{y(\sec x + \tan x) = \sec x + \tan x - x + c}$$

Ans.

Q12:-

$$\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$$

Sol.

$$\frac{dy}{dx} + y\underbrace{\frac{1}{x}}_{P(x)} = \underbrace{x^3 - 3}_{Q(x)}.$$

Step 1:- I.F

$$I.F = e^{\int P(x)\,dx}.$$

$$I.F = e^{\int \frac{1}{x}\,dx}.$$

$$I.F = e^{\ln x} = x.$$

Step 2:- Multiply I.F on b/s.

$$[I.F]y = \int Q(x)(I.F)\,dx + c$$

$$xy = \int x(x^3 - 3)\,dx + c$$

$$xy = \int x^4 - 3x\,dx + c$$

$$\boxed{xy = \frac{x^5}{5} - \frac{3}{2}x^2 + c}$$

(2) $(2y-3x)dx + xdy = 0$

Sol:-

$\Rightarrow$ Multiply $\frac{1}{dx}$ on b/s.

$2y - 3x + x\frac{dy}{dx} = 0$

$\Rightarrow$ Multiply $\frac{1}{x}$ on b/s.

$$\frac{dy}{dx} + \underbrace{\frac{2y-3x}{x}}_{P(x)} = 0$$

Step 1:- I.F

$$I.F = e^{\int P(x)\,dx}$$

$$I.F = e^{\int \frac{1}{x} - 3x\,dx}$$

$$I.F = e^{\ln x - \frac{3}{2}x^2}$$

$$\frac{dy}{dx} + \underbrace{\frac{2y}{x}}_{P(x)} = \underbrace{3}_{Q(x)}$$

$$I.F = e^{\int \frac{2}{x}\,dx}$$

$$I.F = e^{2\ln x} = x^2$$

$$y(I.F) = \int (I.F)Q(x)\,dx$$

$$x^2 y = \int x^2 3\,dx$$

$$x^2 y = \frac{3x^3}{3} + c$$

$$\boxed{x^2 y = x^3 + c}$$

(3) $\frac{dy}{dx} + y\underbrace{\cot x}_{P(x)} = \underbrace{\cos x}_{Q(x)}$

Sol:-

Step 1:- Integrating factor.

$$I.F = e^{\int P(x)\,dx}$$

$$I.F = e^{\int \cot x\,dx}$$

$$I.F = e^{\int \frac{\cos x}{\sin x}\,dx}$$

$u = \sin x$
$du = \cos x\,dx$

$$I.F = e^{\int \frac{du}{u}}$$

$$I.F = e^{\ln|u|}$$

$$I.F = e^{\ln|\sin x|}$$

$$I.F = \sin x$$

Step 2:- Multiply I.F on b/s

$$y(I.F) = \int Q(x)\,I.F\,dx$$

$$y(\sin x) = \int \cos x \sin x\,dx$$

$$y\sin x = -\frac{1}{2}\cos^2 x + c$$

or

~~$y\sin x = \frac{1}{2}\sin^2 x + c$~~

$$\boxed{y\sin x = \frac{\sin^2 x}{2} + c}$$

④ $\frac{dy}{dx} + y\underbrace{\sec x}_{P(x)} = \underbrace{\tan x}_{Q(x)}$

Solⁿ

Step 1: - Integrating Factor

$I.F = e^{\int P(x)dx}$

$I.F = e^{\int \sec x \, dx}$

⑤. $\cos^2 x \frac{dy}{dx} + y = \tan x.$

Solⁿ

Multiply $\frac{1}{\cos^2 x}$ on b/s.

$\frac{dy}{dx} + y\underbrace{\frac{1}{\cos^2 x}}_{P(x)} = \underbrace{\frac{\tan x}{\cos^2 x}}_{Q(x)}.$

Step 1: Integrating Factor

$I.F = e^{\int P(x) dx}$

$I.F = e^{\int \frac{1}{\cos^2 x} dx}$

$\cancel{I.F = e^{\int}}$

$u = \tan x = \frac{\sin x}{\cos x}$

$\frac{du}{dx} = \frac{\cos(x).\cos(x) - \sin x.(-\sin x)}{\cos^2(x)}$

$du = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cancel{\sin^2 x} 1}{\cos^2 x} dx$

$du = \frac{1}{\cos^2 x} dx.$

$I.F = e^{\int du} = e^{\tan x}$

Multiply on b/s

$y(I.F) = \int I.F \, Q(x) dx.$

$y e^{\tan x} = \int e^{\tan x} . \frac{\tan x}{\cos^2 x} dx$

$y e^{\tan x} = \int \frac{e^{\tan x} \tan x}{\cos^2 x} dx$

$\because u = \tan x, \frac{du}{dx} = \frac{1}{\cos^2 x}.$

$dx = \cos^2(x) du.$

$y e^{\tan x} = \int e^{(u)} . \frac{u}{\cancel{\cos^2 x}} . \cancel{\cos^2(x)} du$

$y e^{\tan x} = \int \underbrace{e^u . u}_{I} du$

$\cancel{IBP} =$

$\int u dv = uv - \int v du$

$u = u \quad dv = e^u$

$du = 1 du \quad v = e^u$

$y e^{\tan x} = C\cdot e^{u} - \int e^{u}\cdot 1\,du$

$y e^{\tan x} = u\cdot e^{u} - \int e^{u}\,du$

$y e^{\tan x} = u\cdot e^{u} - e^{u} + c$

$\because u = \tan x$

$y e^{\tan x} = \tan(x)\cdot e^{\tan(x)} - e^{\tan(x)} + c$

$y = \cancel{\tan(x)}$

$y = \frac{\cancel{e^{\tan(x)}}(\tan(x) - 1)}{\cancel{e^{\tan(x)}}} + \frac{c}{e^{\tan(x)}}$

$$\boxed{y = \tan(x) - 1 + c\,e^{-\tan(x)}}$$

⑥ $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$

Sol:

Multiply $\frac{1}{(x+a)}$ on b/s

$\frac{dy}{dx} - 3y\frac{1}{x+a} = (x+a)^4$

Step 1:

$P(x) = \frac{3}{x+a}$

$I.F = e^{\int P(x)\,dx}$

$I.F = e^{\int \frac{3}{x+a}dx} = e^{3\ln(x+a)}$

$I.F = (x+a)^3$

I.F multiply with:

$I.F(y) = \int (I.F)\, Q(x)\,dx$

$(x+a)^3 y = \int (x+a)^3 (x+a)^4\,dx$

$(x+a)^3 y = \int (x+a)^7\,dx$

$(x+a)^3 y = \frac{(x+a)^8}{8} + c$

$$\boxed{y = \frac{(x+a)^5}{8} + c(x+a)^{-3}}$$

(7) $x\cos(x)\frac{dy}{dx}+y(x\sin x+\cos x)=1$

Sol:

Multiply $\frac{1}{x\cos(x)}$ on b/s.

$$\frac{dy}{dx}+\frac{y(x\sin x+\cos x)}{x\cos(x)}=\frac{1}{x\cos(x)}$$

Step 1: Find the Integrating Factor.

$$P(x)=\frac{\cancel{x}\sin x}{\cancel{x}\cos(x)}+\frac{\cancel{\cos x}}{x\cancel{\cos x}}=\frac{\sin(x)}{\cos(x)}+\frac{1}{x}.$$

$$I.F=e^{\int\frac{\sin(x)}{\cos(x)}+\frac{1}{x}}=e^{\int -\ln|\cos(x)|+\ln x}$$

$\tan x$.

$$I.F=e^{\ln\left(\frac{x}{\cos(x)}\right)}=\cancel{-\frac{x}{\sin(x)}}\quad x\sec(x).$$

Step 2: Multiply I.F on b/s

$$y(I.F)=\int(I.F)Q(x)\,dx+c$$

$$\boxed{\begin{aligned} y\left(\frac{-x}{\sin(x)}\right)&=\int\frac{-\cancel{x}}{\sin(x)}\cdot\left(\frac{1}{\cancel{x}\cos(x)}\right)dx+c\\ y\left(\frac{-x}{\sin(x)}\right)&=\int\frac{1}{\sin(x)\cos(x)}\,dx+c\end{aligned}}$$

$$y(x\sec(x))=\int(\cancel{x}\sec(x))\cdot\frac{1}{\cancel{x}\cos(x)}\,dx+c$$

$$\frac{yx}{\cos(x)}=\int\sec(x)\cdot\sec(x)\,dx+c$$

$$\frac{y(x)}{\cos(x)} = \int \sec^2(x)\,dx + c$$

$$\frac{yx}{\cos(x)} = \tan(x) + c$$

$$\frac{yx}{\cos(x)} = \frac{\sin(x)}{\cos(x)} + c$$

$$\boxed{yx = \sin(x) + c\cos(x)}$$

(8) $\sec x \dfrac{dy}{dx} = y + \sin x$.

<u>Sol:</u>

$\sec x \dfrac{dy}{dx} - y = \sin x$.

<u>Multiply</u> $\cancel{\sec(x)}\ \frac{1}{\sec(x)}$ on b/s

$$\frac{dy}{dx} - y\underbrace{(\cos(x))}_{P(x)} = \underbrace{\sin(x)\cos(x)}_{Q(x)}$$

<u>Step 1:- Integrating Factor.</u>

$$I.F = e^{\int P(x)\,dx}$$

$$I.F = e^{\int \cos(x)\,dx}$$

$$I.F\ e^{-\sin x}$$

<u>Step 2:- Multiply I.F on b/s</u>

$$y(I.F) = \int (I.F)\,Q(x)\,dx + c$$

$$y(e^{-\sin(x)}) = \int e^{-\sin x} \cdot \sin x \cos(x)\,dx + c$$

<u>Let:-</u> $u = \sin x$

$du = \cos(x)\,dx$

$dx = \dfrac{du}{\cos(x)}$

$$ye^{-u} = \int e^{-u} \cdot u\,\frac{\cancel{\cos(x)}\,du}{\cancel{\cos(x)}}$$

$$ye^{-u} = \int e^{-u} \cdot u\,du$$

<u>Integration by Parts</u>

$u = u$; $dv = e^{-u}\,du$

$du = 1$, $v = \dfrac{e^{-u}}{-1}$ or $-e^{-u}$

$$= u(-e^{-u}) - \int 1 \cdot (-e^{-u})\,du + c$$

$$= -ue^{-u} - e^{-u} + c$$

$$= -e^{-u}(u+1) + c$$

$\because u = \sin x$

$$\boxed{ye^{-\sin x} = -e^{-\sin x}(\sin x + 1) + c}$$

<u>Or</u>

$$\boxed{y = -(\sin x + 1) + c\,e^{\sin x}}$$

9) $(1+y^2)dx = (\tan^{-1}y - x)dy$

Sol:-

$$\frac{dx}{dy}(1+y^2) = \tan^{-1}y - x$$

$$\frac{dx}{dy}(1+y^2) - \tan^{-1}y = -x$$

Multiply $\frac{1}{1+y^2}$ on b/s

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

Now it is in form.

$$\frac{dx}{dy} + P(y) = Q(y)$$

Step 1:- Integrating Factor.

$$I.F = e^{\int \frac{1}{1+y^2}dy}$$

$$I.F = e^{\tan^{-1}y}$$

Step 2:- Multiply on b/s

$$x(I.F) = \int (I.F)Q(y)\,dy + c$$

$$x(e^{\tan^{-1}y}) = \int e^{\tan^{-1}y} \cdot \frac{\tan^{-1}y}{1+y^2}dy + c$$

Let $u = \tan^{-1}y$

$$du = \frac{1}{1+y^2}dy$$

$$= \int e^{u} \cdot u\, du + c$$

$$xe^{u} = \int \underbrace{u \cdot e^{u}du}_{\text{Integration by Parts:}} + c$$

$$xe^{u} = u \cdot e^{u} - \int 1 \cdot e^{u}du + c$$

$$xe^{u} = u \cdot e^{u} - e^{u} + c$$

$$xe^{u} = e^{u}(u-1) + c$$

$$x = u - 1 + ce^{-u}$$

$$u = \tan^{-1}y$$

$$\boxed{x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}}$$

10). $dr + (2r\cot\theta + \sin 2\theta)d\theta = 0$

$$\frac{dr}{d\theta} + 2r\cot\theta + \sin 2\theta = 0$$

$$\frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$$

$$\frac{dr}{d\theta} + P(\theta) = Q(\theta)$$

Step 1:- Integrating Factor.

$$I.F = e^{\int P(\theta)\, d\theta}$$

$$I.F = e^{\int 2\cot\theta \, d\theta}$$

$$I.F = e^{2\int \frac{\cos\theta}{\sin\theta} d\theta}$$

$$e^{2\ln|\sin\theta|}$$

$$I.F = e^{\ln|\sin\theta|^2}$$

$$I.F = \sin^2\theta.$$

Step 2:- Multiply on b/s

$$r(I.F) = \int (I.F)\, Q(\theta)\, d\theta + c$$

$$r(\sin^2\theta) = \int \sin^2\theta \cdot (-\sin 2\theta)\, d\theta + c$$

$$r(\sin^2\theta) = \int \sin^2\theta \cdot -(2\sin\theta \cdot \cos\theta)\, d\theta + c$$

$$= -2\int \sin^2\theta \cdot \sin\theta \cdot \cos\theta \, d\theta + c$$

$$= -2\int \sin^3\theta \cdot \cos\theta \, d\theta + c$$

let $u = \sin\theta$

$$du = \cos\theta\, d\theta$$

$$= -2\int u^3 \cdot du$$

$$= -2\frac{u^4}{4} = \frac{-u^4}{2}$$

$$\because u = \sin\theta$$

$$\boxed{r\sin^2\theta = -\frac{\sin^4\theta}{2} + c}$$

Bernoulli's D.E

(11) $x^2 dy + y(x+y)\,dx = 0$

Sol:

$$x^2\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$$

$$\frac{dy}{dx} + \frac{xy + y^2}{x^2} = 0$$

$$\frac{dy}{dx} + \frac{xy}{x^2} = -\frac{y^2}{x^2}$$

Multiply $-\frac{1}{y^2}$ on b/s

$$-\frac{1}{y^2}\frac{dy}{dx} - \frac{y}{xy^2} = \frac{1}{x^2}$$

$$-\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{xy} = \frac{1}{x^2}$$

Let $u = +\frac{1}{y}$

$$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$\frac{du}{dx} - \frac{u}{x} = \frac{1}{x^2}$$

$$\frac{du}{dx} - u \cdot \frac{1}{x} = \frac{1}{x^2}$$

$$P(x) = -\frac{1}{x}, \quad Q(x) = \frac{1}{x^2}$$

Integrating Factor

$$I.F = e^{\int P(x)\,dx}$$

$$I.F = e^{\int -\frac{1}{x}\,dx}$$

$$I.F = e^{\ln\left|\frac{1}{x}\right|}$$

$$I.F = \frac{1}{x}$$

Multiply I.F on b/s

$$U(I.F) = \int (I.F)\,Q(x)\,dx + c$$

$$U\left(\frac{1}{x}\right) = \int \frac{1}{x} \cdot \frac{1}{x^2}\,dx + c$$

$$U\left(\frac{1}{x}\right) = \int x^{-3}\,dx + c$$

$$U\left(\frac{1}{x}\right) = \frac{x^{-2}}{-2} + c$$

$$\boxed{\frac{1}{xy} = -\frac{1}{2x^2} + c}$$

~~(12) $x\frac{dy}{dx} - \frac{\ln y}{1+}$~~

(12) $x\frac{dy}{dx} + y\ln y = xye^{x}$

Sol:-

Multiply $\frac{1}{(x)}$ on b/s

$$\frac{dy}{dx} + \frac{y}{x}\ln y = ye^{x}$$

Multiply $\frac{1}{y}$ on b/s

$$\frac{1}{y}\frac{dy}{dx} + \frac{y}{x}\ln y = e^{x}$$

Let $U = \ln y$

$$\frac{du}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{du}{dx} + \frac{y}{x}u = e^{x}$$

$$P(x) = \frac{1}{x} \quad | \quad Q(x) = e^{x}$$

Integrating Factor

$$I.F = e^{\int P(x)\,dx}$$

$$I.F = e^{\int \frac{1}{x}\,dx}$$

$$I.F = e^{\ln(x)} = x$$

Multiply I.F on b/s

~~xtt~~

$$U(I.F) = \int (I.F)\, Q(x)\, dx + c$$

$$U(x) = \int \underbrace{x \cdot e^{x}}_{} dx + c$$

Integration by Parts

$$Ux = x \cdot e^{x} - e^{x} + c$$

$$\boxed{x \ln y = e^{x}(x-1) + c}$$

(13) $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^{x}\sec y.$

Sol:-

Multiply $\dfrac{1}{\sec y}$ on b/s.

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y \cdot 1}{(1+x)\sec y} = (1+x)e^{x}.$$

$$\cos y \frac{dy}{dx} - \frac{\tan y}{1+x} \cdot \frac{1}{\sec y} = (1+x)e^{x}.$$

Let $U = \sin y \Rightarrow \dfrac{dU}{dx} = \cos y \dfrac{dy}{dx}$

$$\frac{dU}{dx} - \frac{\tan y}{\sec y} \cdot \frac{1}{1+x} = (1+x)e^{x}$$

$P(x) = \dfrac{1}{1+x}$, $Q(x) = (1+x)e^{x}$

Integrating Factor

$$I.F = e^{\int P(x)dx} = e^{\int \frac{1}{1+x}dx}$$

$$I.F = e^{\ln(1+x)^{-1}} = \frac{1}{1+x}$$

Multiply on b/s I.F

$$U(I.F) = \int (I.F)\, Q(x)\, dx + c$$

$$\frac{U}{1+x} = \int \frac{1}{\cancel{1+x}} \cancel{(1+x)} e^{x}\, dx + c$$

$$\frac{U}{1+x} = \int e^{x}\, dx + c$$

$$\frac{U}{1+x} = e^{x} + c$$

$$\therefore U = \sin y$$

$$\boxed{\frac{\sin y}{1+x} = e^{x} + c}$$

(4) $\tan y \frac{dy}{dx} + \tan x = \cos y \cdot \cos^2 x$

Sol:

Multiply $\frac{1}{\cos y}$ on b/s

$$\frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^2 x.$$

~~Multiply~~ Let $u = \sec y \Rightarrow \frac{du}{dx} = \sec y \cdot \tan y \frac{dy}{dx}$

or $\frac{du}{dx} = \frac{\tan y}{\cos y}\frac{dy}{dx}$

$$\frac{du}{dx} + \tan x \underbrace{\frac{1}{\cos y}}_{\sec y} = \cos^2 x.$$

$P(x) \cdot u = \frac{1}{\cos y}$, $Q(x) = \cos^2 x.$

$$\frac{du}{dx} + \tan x \, u = \cos^2 x.$$

Integrating Factor.

$I.F = e^{\int P(x)\,dx}$

$I.F = e^{\int \tan x\,dx}$

$I.F = e^{\ln\left|\frac{1}{\cos x}\right|}$

$I.F = \frac{1}{\cos x}$

Multiply on b/s

$u(I.F) = \int (I.F)\,Q(x)\,dx$

$u\left(\frac{1}{\cos x}\right) = \int \frac{1}{\cancel{\cos x}} \cdot \cos^2 x\,dx$

$u\frac{1}{\cos x} = \sin x + c$

$u = (\sin x + c)\cos x$

$\therefore u = \sec y$

$$\boxed{\sec y = (\sin x + c)\cos x}$$

(15) $x\left[\frac{dx}{dy}+y\right] = 1-y$

$x\frac{dx}{dy} + xy \cancel{+y} = 1-y$

$\frac{dx}{dy}$ $u = x^2$, $\frac{du}{dy} = \cancel{2x\frac{dx}{dy}}$

$\frac{1}{2}\frac{du}{dy}$

$\frac{du}{dy} = 2x\frac{dx}{dy}$

$\frac{1}{2}\frac{du}{dy} = x\frac{dx}{dy}$

$\frac{1}{2}\frac{du}{dy} + \sqrt{u}\,y = 1-y$

$\frac{1}{2}\frac{du}{dy} + \sqrt{u}\,\underbrace{2y}_{P(y)} = \underbrace{2-2y}_{Q(y)}$

$I.F = e^{\int P(y)dy} = e^{\int 2y\,dy}$

$I.F = e^{y^2}$

multiply with y^n and $Q(x)$

$x^2 \cdot e^{y^2} = \int e^{y^2}(2-2y)\,dy$

$u = 2-2y$ $du = -y\,dy$

$x \cdot e^{y^2} = \int v\,dv = \frac{e^{y^2}}{2}$

$uv - \int du\,v\,dv$

$(1-y)\frac{e^{y^2}}{2} - \int -y\frac{e^{y^2}}{2}dy$

$(1-y)\frac{e^{y^2}}{2} - \frac{1}{2}\left[-y\frac{e^{y^2}}{4} - \right.$

$\int -1\frac{e^{y^2}}{4}dy$

$(1-y)\frac{e^{y^2}}{2} + \frac{y}{8}e^{y^2} - \frac{1}{8}e^{y^2} + c$

$x^2 \cdot e^{y^2} = (1-y)\frac{e^{y^2}}{2} + \frac{y e^{y^2}}{8} - \frac{1}{8}e^{y^2} + c$

$x^2 = \frac{4(1-y)}{2} + \frac{y}{8} - \frac{1}{8} + c$

$x^2 = \frac{4-4y+y-\cancel{y}}{8} + c$

$\boxed{x^2 = \frac{-3y+3}{8} + c}$

(16) $y \ln y \, dx + (x - \ln y)\, dy = 0$

Sol:

$\frac{dx}{dy} y \ln y = -x + \ln y$

$\frac{dx}{dy} y \ln y + x = \ln y$

Multiply $\frac{1}{y \ln y}$ on b/s

$\frac{dx}{dy} + x \frac{1}{y \ln y} = \frac{1}{y}$

$P(y) = \frac{1}{y \ln y}, \quad Q(y) = \frac{1}{y}$

Step 1: Integrating Factor

$I.F = e^{\int \frac{1}{y \ln y} dy}$

$u = \ln y \Rightarrow du = \frac{1}{y} dy$

$I.F = e^{\int \frac{1}{u} \cdot \frac{1}{y} dy}$

$I.F = e^{\int \frac{1}{u} du}$

$I.F = e^{\ln |u|}$

$I.F = e^{\ln |\ln y|}$

$\boxed{I.F = \ln y}$

Step 2:- Multiply I.F on b/s

$x (I.F) = \int (I.F)\, Q(y)\, dy$

$x (\ln y) = \int \ln y \cdot \frac{1}{y} dy$

let $u = \ln y \Rightarrow du = \frac{1}{y} dy$

$x \ln y = \int u \cdot du$

$x \ln y = \frac{u^2}{2} + c$

$$\boxed{x \ln y = \frac{(\ln y)^2}{2} + c}$$

(17) $(1+y^2) dx = (\tan^{-1} y - x) dy$

Sol:

$(1+y^2) \frac{dx}{dy} = \tan^{-1} y - x$

$(1+y^2) \frac{dx}{dy} + x = \tan^{-1} y$

Multiply $\frac{1}{(1+y^2)}$ on b/s

$\frac{dx}{dy} + x \cdot \frac{1}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$

Multiply

$P(y) = \frac{1}{1+y^2}, \ Q(y) = \frac{\tan^{-1}y}{1+y^2}$

Step 1: Integrating Factor

$I.F = e^{\int P(y)\,dy}$

$I.F = e^{\int \frac{1}{1+y^2} dy}$

$I.F = e^{\tan^{-1}y}$

Step 2: Multiply I.F on b/s

$x(I.F) = \int (I.F)\,Q(y)\,dy$

$x\,e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \cdot \frac{\tan^{-1}y}{1+y^2}\,dy$

Let $u = \tan^{-1}y$

$du = \frac{1}{1+y^2}\,dy$

$x\,e^{\tan^{-1}y} = \int e^u \cdot u \cdot \underbrace{\boxed{\frac{1}{1+y^2}\,dy}}_{du}$

$x \cdot e^{\tan^{-1}y} = \int e^u \cdot u\,du$

Let $\cancel{v} = u \Rightarrow d\cancel{v} = 1\,du$

$dw = e^u \Rightarrow w = e^u$

$x \cdot e^{\tan^{-1}y} = ue^u - \int 1 e^u\,du$

$x \cdot e^{\tan^{-1}y} = ue^u - e^u + c$

$x \cdot e^{\tan^{-1}y} = e^u(u-1) + c$

$\because u = \tan^{-1}y$

$x \cdot e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$

$$\boxed{x = (\tan^{-1}y - 1) + \frac{c}{e^{\tan^{-1}y}}}$$

(18) $r\sin\theta - \frac{dr}{d\theta}\cos\theta = r^2$

Sol:

$\frac{dr}{d\theta}\cos\theta + r^2 = r\sin\theta$

Multiply $\frac{1}{\cos\theta}$ on b/s

$\frac{dr}{d\theta} + \frac{r^2}{\cos\theta} = r\frac{\sin\theta}{\cos\theta}$

$\frac{dr}{d\theta} \cdot \frac{1}{r} + \frac{r^2}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$

Let $u = \ln r \Rightarrow \frac{du}{d\theta} = \frac{1}{r}\frac{dr}{d\theta}$

$\frac{du}{d\theta} + \frac{r}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$P(\theta) = \frac{1}{\cos\theta}$, $Q(\theta) = \frac{\sin\theta}{\cos\theta}$ or $\tan\theta$

Step 1:- Integrating Factor.

$I.F = e^{\int \frac{1}{\cos\theta} d\theta}$

$I.F = e^{\int \sec\theta d\theta}$.

$I.F = e^{\int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} d\theta}$

$I.F = e^{\int \frac{\sec^2\theta + \sec\theta\tan\theta d\theta}{\sec\theta + \tan\theta}}$.

$u = \sec\theta + \tan\theta$

$du = \sec\theta\tan\theta + \sec^2\theta d\theta$

$I.F = e^{\int \frac{1}{u} du}$.

$I.F = e^{\ln|u|} = u$

$\boxed{I.F = \sec\theta + \tan\theta}$

Step 2:- Multiply I.F on b/s.

$u(I.F) = \int (I.F) Q(\theta) d\theta$

$u(\sec\theta + \tan\theta) = \int (\sec\theta + \tan\theta) \tan\theta d\theta$

$= \int \sec\theta \cdot \tan\theta + \tan^2\theta d\theta$

$= \int \sec\theta \cdot \tan\theta d\theta + \int \tan^2\theta d\theta$.

$\therefore \tan^2\theta = \sec^2\theta - 1$

$= \sec\theta + \int \sec^2\theta - 1 d\theta$

$= \sec\theta + \int \sec^2\theta d\theta - \int 1 d\theta$

$= \sec\theta + \tan\theta - \theta + c$

$u(\sec\theta + \tan\theta) = \sec\theta + \tan\theta - \theta + c$

$\therefore u = \ln r$.

$$\boxed{\ln r(\sec\theta + \tan\theta) = \sec\theta + \tan\theta - \theta + c}$$

(19) $\cos x \frac{dy}{dx} + 4y \sin x = 4\sqrt{y} \sec x$

Sol:

Multiply $\frac{1}{\cos x}$ on b/s.

$\frac{dy}{dx} + 4y \tan x = 4\sqrt{y} \sec^2 x$

divide $\frac{1}{\sqrt{y}}$ on b/s

$\frac{1}{\sqrt{y}} \frac{dy}{dx} + \frac{4y}{\sqrt{y}} \tan x = 4\sec^2 x$

Let $u = \sqrt{y}$.

$\frac{du}{dx} = \frac{1}{2\sqrt{y}} \frac{dy}{dx}$.

$$\frac{du}{dx} + 2\sqrt{y}\tan x = 2\sec^2 x$$

$P(x) = 2\tan x,\ Q(x) = 2\sec^2 x$

Step 1:- Integrating Factor

$I.F = e^{\int P(x)\,dx}$

$I.F = e^{\int 4\tan x\,dx}$

$I.F = e^{-2\ln|\cos x|}$

$I.F = e^{\ln|\cos^{-2} x|}$

$I.F = \cos^{-2} x$ or $\sec^2 x$

Step 2:- Multiply I.F on b/s

$y(I.F) = \int (I.F)\,Q(x)\,dx$

$u(\sec^2 x) = \int \sec^2 x \cdot 2\sec^2 x\,dx$

$= 2\int \sec^4 x\,dx$

$= 2\int \tan^2 x + 1 \cdot \sec^2 x\,dx$

Let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$= 2\int \cancel{\tan^2 x}\,u^2 + 1\,du$

$= 2\int u^2 du + \int du$

$= 2\left[\frac{u^3}{3} + u\right] + c$

$\because u = \tan x$

$= 2\left[\frac{\tan^3 x}{3} + \tan x\right] + c$

$u\sec^2 x = 2\left[\frac{\tan^3 x}{3} + \tan x\right] + c$

$\because u = \sqrt{y}$

$$\boxed{\sqrt{y}\sec^2 x = 2\left[\frac{\tan^3 x}{3} + \tan x\right] + c}$$

(20) $\frac{dy}{dx} + \frac{y}{x}\ln y = \frac{y}{x^2}(\ln y)^2$

Sol.

$\frac{1}{y(\ln y)^2}\frac{dy}{dx} + \frac{\ln y}{x} = \frac{1}{x^2}$

Let $u = \frac{1}{\ln y}$ or $(\ln y)^{-1}$

$\frac{du}{dx} = -(\ln y)^{-2}\frac{d}{dx}(\ln(y))$

$\frac{du}{dx} = -(\ln y)^{-2}\frac{1}{y}\frac{dy}{dx}$

$$\frac{du}{dx} = -\frac{1}{y(\ln y)^2}\frac{dy}{dx}.$$

$$-\frac{du}{dx} + \frac{\ln y}{x} = \frac{1}{x^2}$$

$$\frac{du}{dx} - \frac{\ln y}{x} = -\frac{1}{x^2}$$

$$P(x) = -\frac{1}{x}, \quad Q(x) = -\frac{1}{x^2}$$

Step 1:- Find the Integrating Factor

$$I.F = e^{\int -\frac{1}{x}dx}.$$

$$I.F = e^{-\ln|x|}$$

$$I.F = e^{\ln x^{-1}} = x^{-1}$$

$$I.F = \frac{1}{x}.$$

Step 2:- Multiply I.F on b/s

$$u(I.F) = \int (I.F)\,Q(x)\,dx.$$

$$\frac{u.1}{x} = \int \frac{1}{x}\left(-\frac{1}{x^2}\right)dx$$

$$\frac{u.1}{x} = -\int x^{-3}\,dx$$

$$\frac{u}{x} = -\left[\frac{x^{-2}}{-2}\right] + c$$

$$\frac{u}{x} = \frac{1}{2x^2} + c$$

$$\therefore u = \frac{1}{\ln y}$$

$$\boxed{\frac{1}{x\ln y} = \frac{1}{2x^2} + c}$$