CHAPTER FOUR

LAPLACE TRANSFORMS

4.1 INTRODUCTION

Laplace transform is a mathematical tool or mechanism that can be used to solve several problems in science and engineering. This was first introduced by French mathematician "Laplace" in the year 1790, in his work on probability theory. This technique became popular when Heaviside (An Electrical Engineer) applied it to the solution of ordinary differential equations in electrical engineering. It has become an essential part of mathematical background of engineers, physicists, mathematicians and many scientists.

Laplace Transform has got wider applications in physics, engineering, mechanics, heat flow etc. The methods of Laplace transforms are very simple, and they give solutions of differential equations satisfying given initial/boundary conditions without the use of the general solutions. Since these particular solutions are usually required in physics, mechanics, chemistry and various fields of practical research, Laplace transform is highly important.

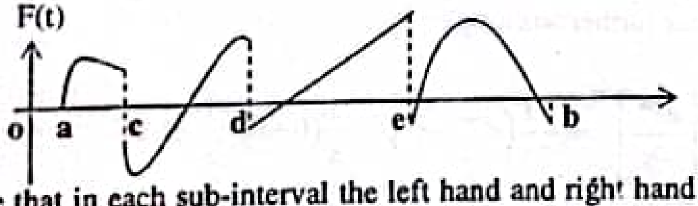
The basic question as to why one should learn Laplace transforms techniques when other techniques are available, the answer is very simple .Transforms are used to accomplish the solution of certain problems with less effort and in a simple routine way.

For illustration, consider the problem of finding the value of x from the equation $x^{1.85}=3$. It is extremely difficult task to solve this problem algebraically. However, taking logarithms on both sides, we have the transformed equation as $1.85 \ln x = \ln 3$. In this transformed equation, the algebraic operation and exponentiation have been changed to multiplication which immediately gives $\ln x = \ln 3/(1.85)$. To get the required result, it is enough if we take antilogarithm on both sides of the above equation, which yields:

$$x = \ln^{-1}[\ln 3/(1.85)] = 1.3785.$$

Piece-wise or Sectionally Continuous Function

A function F(t) is said to be piecewise or sectionally continuous in any interval [a, b] if it is continuous in every sub-intervals $a < c < d < e < \dots < b$. This is show as under.



You may observe that in each sub-interval the left hand and right hand limits of F(t) are finite. Thus piecewise function is continuous function except for a finite number of jump discontinuities.

Function of Exponential Order

A function F(t) defined on $[0, \infty)$ is said to be of exponential order a as $t \to \infty$ if

$$\lim_{t\to\infty} e^{-at} F(t) = \text{finite quantity}$$

This means for given integer $m > 0$ there exists a real number $M > 0$ such that

$$|e^{-at} F(t)| < M \quad \text{OR} \quad |F(t)| \le M e^{at} \text{ whenever, } t \ge m$$

Example 01: Show that function $F(t) = t^n$ is of exponential order.

Solution: Consider: $\lim_{t\to\infty} e^{-at} t^n = \lim_{t\to\infty} \frac{t^n}{e^{at}} = \lim_{t\to\infty} \frac{n!}{a^n e^{at}} = 0$ [Use L'Hospital Rule]

Function of Class 'A' and Integral Transform

A function F(t) is said to be of class A if it is piecewise continuous and it is of exponential order.

An improper integral of the form

$$\int_{-\infty}^{\infty} K(s,t)\, F(t)\,dt = f(s) \qquad (1)$$

is called **integral transform** of F(t) provided that the integral exists/converges. It is usually denoted by T[(F(t)] of f(s). The function K(s, t) appearing in the integrand is called Kernel of the transform. It may be noted that s is parameter which may be real or complex. It may further be noted that "s" is known as "Frequency Domain" variable. Throughout this book, we shall consider s as a real number. If we take

$$K(s,t) = \begin{cases} e^{-st} & ,t \ge 0 \\ 0 & ,t < 0 \end{cases}$$

Then (1) becomes: $$L[F(t)] = f(s) = \int_{0}^{\infty} e^{-st}\, F(t)\,dt$$

This is known as **Laplace's Transform.**

If we take

$$K(s,t) = e^{-ist} \text{ for } -\infty < t < \infty$$

Then (1) becomes: $$L[F(t)] = f(s) = \int_{-\infty}^{\infty} e^{-ist}\, F(t)\,dt$$

This is known as **Fourier Transform.**

REMARK: It may be noted that Laplace transformation is very much used in **"Control Engineering Theory"** where F(t) is considered as **"Input Signal"** and f(s) as an **"Output signal"**. Thus we have to observe the effect of Laplace transform on the input signal F(t). F(t) is also known as object function.

Laplace Transforms of Some Elementary Functions

In this section we shall represent the Laplace transforms of some elementary functions that would help us in our further working.

I. L[1]: By definition,

$$L[1] = \int_0^{\infty} e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = -\frac{1}{s}\left(e^{-\infty} - e^0\right) = -\frac{1}{s}(0-1) = \frac{1}{s}, s>0.$$

II. L[t]: By definition,

$L[t] = \int_0^{\infty} e^{-st} t\,dt$. Now integrating by parts taking $u = t$ and $v = e^{-st}$

$$= \lim_{p\to\infty}\left[t\,\frac{1}{-se^{st}}\right]_0^p - \int_0^{\infty} 1.\frac{e^{-st}}{-s}dt = 0 + \frac{1}{s}\int_0^{\infty}(1)\,e^{-st}dt = \frac{1}{s}L(1) = \frac{1}{s}.\frac{1}{s} = \frac{1}{s^2}, s>0.$$

[NOTE: $\lim_{p\to\infty} \frac{1}{e^{sp}} = 0$]

III. $L[t^n]$: By definition,

$L[t^n]=\int_0^\infty t^n e^{-st}\,dt$. Now integrating by parts taking $u=t^n$ and $v=e^{-st}$, we get

$$L\left[t^n\right]=\lim_{p\to\infty}\left[t^n\,\frac{e^{-st}}{-s}\right]_0^p-\int_0^\infty n\,t^{n-1}.\frac{e^{-st}}{-s}\,dt$$

$$=\lim_{p\to\infty}\left[t^n\,\frac{1}{-se^{st}}\right]_0^p+\frac{n}{s}\int_0^\infty t^{n-1}\,e^{-st}dt=0+\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

Thus, $$L\left[t^n\right]=\frac{n}{s}L\left(t^{n-1}\right)$$

This is recurrence formula. If we repeatedly apply this formula, we get

$$L\left[t^n\right]=\frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}...\frac{3}{s}.\frac{2}{s}.\frac{1}{s}L\left[t^0\right]=\frac{n!}{s^n}L(1)=\frac{n!}{s^n}.\frac{1}{s}=\frac{n!}{s^{n+1}}$$

Thus, $$L[t^n]\ =\ \frac{n!}{s^{n+1}},s>0.$$

We know that gamma function for a positive integer n is given by:

$$\Gamma(n+1)=\int_0^\infty e^{-u}u^n du\ =\ n!.$$

Hence above formula could also be expressed as:

$$L[t^n]\ =\ \frac{\Gamma(n+1)}{s^{n+1}},\ s>0.\qquad \textbf{NOTE: }\left[\Gamma(1/2)=\sqrt{\pi}\right]$$

EXAMPLE 02: Find the Laplace transforms of $t^2$and $\sqrt{t}$.

Solution: By definition

$$L[t^2]=\frac{2!}{s^3}\text{ and }L\left[\left(\sqrt{t}\right)\right]=\frac{\Gamma(1/2+1)}{s^{1/2+1}}=\frac{\Gamma(3/2)}{s^{3/2}}=\frac{1}{2s^{3/2}}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2s^{3/2}}$$

IV. $L(e^{at})$: By definition

$$L\left[e^{at}\right]=\int_0^\infty e^{-st}e^{at}dt=\int_0^\infty e^{-st+at}dt=\int_0^\infty e^{-t(s-a)}dt=\left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty=\frac{-1}{s-a}\left(e^{-\infty}-e^0\right)=\frac{-1}{s-a}\Rightarrow$$

$$L[e^{at}]\ =\ \frac{1}{(s-a)},s>a\text{ and }L[e^{-at}]\ =\ \frac{1}{(s+a)},s>-a.$$

EXAMPLE 03: Find $L[t^{-1/2}]$

Solution: $$L\left[t^{-1/2}\right]=\left[\frac{\Gamma(-1/2+1)}{s^{-1/2+1}}\right]=\Gamma\left(\frac{1}{2}\right)\frac{1}{\sqrt{s}}=\sqrt{\frac{\pi}{s}},s>0$$

EXAMPLE 04: Find the Laplace transform of e^{3t}.

Solution: By definition $$L\left[e^{at}\right]=\frac{1}{(s-a)}\Rightarrow L\left[e^{3t}\right]=\frac{1}{(s-3)}$$

V. L[sin at]: Before we find the Laplace transform of sin at and cos at, the following formulas may kindly be noted:

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)\text{ and}$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$$

Now, $L[\sin at]=\int_0^\infty e^{-st}\sin at\,dt=\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at-a\cos at)\right]_0^\infty$

$$=\frac{1}{s^2+a^2}\left(e^{-\infty}-e^0(0-a)\right)=\frac{a}{s^2+a^2}\qquad(\because \sin 0=0,\cos 0=1,\text{and } e^{-\infty}=0)$$

Thus, $$L[\sin at]=a/\left(s^2+a^2\right)$$

VI. L[cos at]: By definition

$$L[\cos at]=\int_0^\infty e^{-st}\cos at\,dt=\frac{1}{s^2+a^2}\left[e^{-st}(-s\cos at+a\sin at)\right]_0^\infty$$

$$=\frac{1}{s^2+a^2}\left(e^{-\infty}+e^0(s-0)\right)=\frac{s}{s^2+a^2}\qquad(\because \sin 0=0,\cos 0=1,\text{and } e^{-\infty}=0)$$

Thus, $$L[\cos at]=s/\left(s^2+a^2\right)$$

EXAMPLE 05: Find the Laplace transform of sin 3t and cos 4t.

Solution: By definition, $L[\sin at]=\frac{a}{s^2+a^2}\Rightarrow L[\sin 3t]=\frac{3}{s^2+9}$

Also by definition, $L[\cos at]=s/\left(s^2+a^2\right)\Rightarrow L[\cos 4t]=s/\left(s^2+16\right)$

VII. L[sinh at]: We know that sinh at = $(e^{at}-e^{-at})/2$, therefore

$$L[\sinh at]=½\,L[e^{at}-e^{-at}]=½\,[L(e^{at})-L(e^{-at})]$$

Using formula (IV), we get

$$L[\sinh at]=\frac{1}{2}\left[\frac{s+a-(s-a)}{(s-a)(s+a)}\right]=\frac{a}{s^2-a^2}$$

Thus, $$L[\sinh at]=a/\left(s^2-a^2\right)\quad, s>|a|$$

VIII. L[cosh at]: We know that cosh at = $(e^{at}+e^{-at})/2$, therefore

$$L[\cosh at]=½\,L[e^{at}+e^{-at}]=½\,[L(e^{at})+L(e^{-at})].$$

Using formula (IV), we get

$$L[\cosh at]=\frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]=\frac{1}{2}\left[\frac{s+a+(s-a)}{(s-a)(s+a)}\right]=\frac{s}{s^2-a^2}$$

Thus, $$L[\cosh at]=s/\left(s^2-a^2\right),\ s>|a|$$

EXAMPLE 06: Find the Laplace transform of sinh 3t and cosh 4t.

Solution: By definition, $L[\sinh at]=a/\left(s^2-a^2\right)\Rightarrow L[\sinh 3t]=3/\left(s^2-9\right)$

and : $L[\cosh at]=s/\left(s^2-a^2\right)\Rightarrow L[\cosh 4t]=s/\left(s^2-16\right)$

The following table summarizes the Laplace transform of these elementary formulas:

F(t)	L[F(t)]
1	1/s, $s>0$
T	$1/s^2$, $s>0$
t^n	$n!/(s^{n+1})$, $s>0$, n is natural number
t^n	$\Gamma(n+1)/s^{n+1}$, $s>0$ and n is real.

e^{at}	$1/(s-a),\ s>a$
e^{-at}	$1/(s+a),\ s>a$
sin at	$a/(s^2+a^2),\ s>a$
cos at	$s/(s^2+a^2),\ s>a$
sinh t	$a/(s^2-a^2),\ s>\lvert a\rvert$
cosh t	$s/(s^2-a^2),\ s>\lvert a\rvert$

4.2 PROPERTIES OF LAPLACE TRANSFORMS

In this section, we shall learn the most important properties of Laplace transforms which, are frequently used in solving many engineering problems in an easy way.

Linearity Property

If $c_1, c_2, \ldots, c_n$ are constants and $F_1, F_2, \ldots, F_n$ are functions of t, whose Laplace transforms exist, then

$$L[c_1F_1(t)+c_2F_2(t)+\ldots+c_nF_n(t)]=c_1f_1(s)+c_2f_2(s)]+\ldots+c_nf_n(s)$$

Proof: By definition,

$$L[c_1F_1(t)+c_2F_2(t)+\ldots+c_nF_n(t)]=\int_0^\infty e^{-st}[c_1F_1(t)+c_2F_2(t)+\ldots+c_nF_n(t)]dt$$

$$=c_1\int_0^\infty e^{-st}F_1(t)dt+c_2\int_0^\infty e^{-st}F_2(t)dt+\ldots+c_n\int_0^\infty e^{-st}F_n(t)dt$$

$$=c_1LF_1(t)+c_2LF_2(t)+\ldots+c_nLF_n(t)=c_1f_1(s)+c_2f_2(s)+\ldots+c_nf_n(s)$$

EXAMPLE 01: Evaluate $L[3e^{-4t}+\cosh 2t-2\sin 3t+t^3]$

Solution: By linearity property.

$$L\left[3e^{-4t}+\cosh 2t-2\sin 3t+t^3\right]=3L[e^{-4t}]+L[\cosh 2t]-2L[\sin 3t]+L[t^3]$$

$$=\frac{3}{(s+4)}+\frac{s}{(s^2-4)}-2\frac{3}{(s^2+9)}+\frac{3!}{s^4}=\frac{3}{(s+4)}+\frac{s}{(s^2-4)}-\frac{6}{(s^2+9)}+\frac{6}{s^4}$$

EXAMPLE 02: Evaluate $L[e^{-4t}\cosh 2t]$

Solution: We know that

$$\text{Cosh } 2t=\frac{e^{2t}+e^{-2t}}{2}\ \rightarrow\ \left[e^{-4t}\cosh 2t\right]=e^{-4t}\left(\frac{e^{2t}+e^{-2t}}{2}\right)=\frac{e^{-2t}+e^{-6t}}{2}.$$

$$\text{Thus, } L\left[e^{-4t}\cosh 2t\right]=L\left[\frac{e^{-2t}+e^{-6t}}{2}\right]=\frac{1}{2}L[e^{-2t}]+\frac{1}{2}L[e^{-6t}]=\frac{1}{2}\left[\frac{1}{s+2}+\frac{1}{s+6}\right]$$

$$=\left[\frac{s+4}{(s+2)(s+6)}\right]$$

EXAMPLE 03: Evaluate L[sin 2t sin 3t]

Solution: (i) We know that $\sin\alpha\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$

$$\therefore \sin 2t\sin 3t=\frac{1}{2}(\cos(2t-3t)-\cos(2t+3t))=\frac{1}{2}(\cos t-\cos 5t)\quad [\cos -a=\cos a]$$

$$\therefore L[\sin 2t\sin 3t]=L\left[\frac{1}{2}(\cos t-\cos 5t)\right]=\frac{1}{2}[L(\cos t)-L(\cos 5t)]=\frac{1}{2}\left[\frac{s}{s^2+1}-\frac{s}{s^2+25}\right]$$

$$=\frac{1}{2}\left[\frac{s(s^2+25)-s(s^2+1)}{(s^2+1)(s^2+25)}\right]=\frac{1}{2}\left[\frac{s^3+25s-s^3-s}{(s^2+1)(s^2+25)}\right]$$

EXAMPLE 04: Find $L\{\sin\sqrt{t}\}$

Solution: We know that $\sin x = x - x^3/3! + x^5/5! - \dots$

$$\therefore \sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \dots = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{3/2}}{5!} - \dots$$

Taking Laplace transform on both sides, we have

$$L\{\sin\sqrt{t}\} = L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{3/2}}{5!} - \dots\right\} = L\{t^{1/2}\} - L\left\{\frac{t^{3/2}}{3!}\right\} + L\left\{\frac{t^{5/2}}{5!}\right\}\dots$$

Using the formula: $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, we obtain

$$L\{\sin\sqrt{t}\} = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \dots$$

Now using the recurrence relation of gamma function, that is, $\Gamma(n+1) = n\Gamma(n)$ and notice that:

$$\Gamma(1/2) = \sqrt{\pi} \Rightarrow \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}, \Gamma(5/2) = \frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2) = \frac{1.3}{2.2}\sqrt{\pi},$$

$$\Gamma(7/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2) = \frac{1.3.5}{2.2.2}\sqrt{\pi}, \text{ etc;}$$

$$\sqrt{\sin t} = \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{3\sqrt{\pi}}{24s^{5/2}} + \frac{15\sqrt{\pi}}{120.8s^{5/2}} - \dots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{(1/2^2s)}{1!} + \frac{(1/2^2s)^2}{2!} - \frac{(1/2^2s)^3}{3!}\dots\right] \qquad (1)$$

Now, we know that $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$ (2)

Comparing (1) and (2), we see that

$$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/2^2s} = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$$

EXAMPLE 05: Find L[$\cos^2 3t$] and L[$\cosh^2 3t$]

Solution: (i) We know that $\cos^2 a = (1 + \cos 2a)/2$. Therefore,

$L[\cos^2 3t] = L[(1 + \cos 6t)/2] = 0.5\ \{L(1) + L(\cos 6t)\}$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

NOTE: $\sin^2 t = (1 - \cos 2t)/2$

(ii) We know that $\cosh^2 a = (1 + \cosh 2a)/2$. Therefore,

$L[\cosh^2 3t] = L[(1 + \cosh 6t)/2] = 0.5\ \{L(1) + L(\cosh 6t)\}$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-36}\right]$$

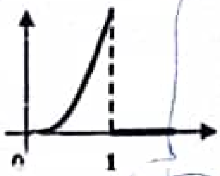
NOTE: $\sinh^2 t = (\cosh 2t - 1)/2$

Laplace Transform of Discontinuous Functions

The Laplace transform of object function F(t) also exists if it is piecewise discontinuous and provided it is of class 'A'.

EXAMPLE 06: Find the Laplace transform of a function defined as:

$$F(t)=\begin{cases} t^2 & 0<t<1 \\ 0 & \text{Elsewhere} \end{cases}$$



Solution: The graph of the given function is shown. By definition,

$$L[F(t)]=\int_0^{\infty} e^{-st}F(t)\,dt=\int_0^1 e^{-st}t^2\,dt+\int_1^{\infty} e^{-st}(0)\,dt=\int_0^1 e^{-st}t^2\,dt$$

Now integrating by parts, we get

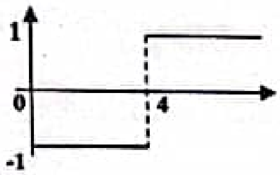
$$L[F(t)]=\left[t^2.\frac{e^{-st}}{-s}\right]_0^1-2\int_0^1 t\frac{e^{-st}}{-s}dt=-\frac{e^{-s}}{s}+\frac{2}{s}\int_0^1 te^{-st}\,dt$$

Integrating by parts once again, we obtain

$$L[F(t)]=-\frac{e^{-s}}{s}+\frac{2}{s}\left\{\left[t\frac{e^{-st}}{-s}\right]_0^1-\int_0^1 1.\frac{e^{-st}}{-s}dt\right\}=-\frac{e^{-s}}{s}-\frac{2e^{-s}}{s^2}+\frac{2}{s^2}\int_0^1 e^{-st}\,dt$$

$$=-\frac{e^{-s}}{s}-\frac{2e^{-s}}{s^2}+\frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^1=-\frac{e^{-s}}{s}-\frac{2e^{-s}}{s^2}-\frac{2}{s^3}[e^{-s}-1]$$

EXAMPLE 07: Find the Laplace transform of a function whose graph is shown here.



Solution: The function of the graph will be

$$F(t)=-1 \quad t\le 4$$
$$=1 \quad t>4$$

Now $L\left[F(t)\right]=\int_0^{\infty} e^{-st}F(t)dt=f(s)$

$$L\left[F(t)\right]=\int_0^{\infty} e^{-st}F(t)dt=\int_0^4 e^{-st}F(t)dt+\int_4^{\infty} e^{-st}F(t)dt$$

$$=\int_0^4 e^{-st}(-1)dt+\int_4^{\infty} e^{-st}(1)dt=-\left[\frac{e^{-st}}{-s}\right]_0^4+\int_4^{\infty} e^{-st}dt$$

$$=\frac{1}{s}\left(e^{-4s}-e^{0}\right)+\lim_{p\to\infty}\left[\frac{e^{-st}}{-s}\right]_4^p=\frac{1}{s}\left(e^{-4s}-1\right)+\lim_{p\to\infty}\left(\frac{e^{-ps}}{-s}+\frac{e^{-4s}}{s}\right)$$

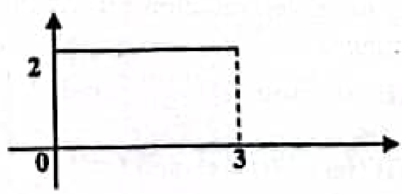
$$=\frac{1}{s}\left(e^{-4s}-1\right)+0+\frac{e^{-4s}}{s} \qquad \left(\text{Note: When } p\to\infty \Rightarrow e^{-ps}\to 0\right)$$

$$=\left(\frac{e^{-4s}}{s}-\frac{1}{s}+\frac{e^{-4s}}{s}\right)=\left(\frac{2e^{-4s}}{s}-\frac{1}{s}\right),\quad s>0$$

EXAMPLE 08: Find the Laplace transform of the function shown below.

Solution: The function shown in the graph is:

$$F(t)=2 \quad 0\le t\le 3$$
$$=0 \quad t>3$$



Now, $L\{F(t)\}=\int_0^{\infty} e^{-st}F(t)dt$

$$=\int_0^3 e^{-st}F(t)dt+\int_3^\infty e^{-st}F(t)dt=\int_0^3 e^{-st}(2)dt+\int_3^\infty e^{-st}(0)dt=2\int_0^3 e^{-st}dt$$

$$=2\left|\frac{e^{-st}}{-s}\right|_0^3=\frac{-2}{s}\left(e^{-3s}-e^{0}\right)=\frac{-2}{s}\left(e^{-3s}-1\right)=\frac{2}{s}\left(1-e^{-3s}\right)$$

$$\Rightarrow \quad L\{F(t)\}=\frac{2}{s}\left(1-e^{-3s}\right)$$

EXAMPLE 09: Find Laplace transform of function $F(t) = 1/t, t > 0$.

Solution: By definition,

$$L\left\{\frac{1}{t}\right\}=\int_0^\infty \frac{e^{-st}}{t}dt=\int_0^1 \frac{e^{-st}}{t}dt+\int_1^\infty \frac{e^{-st}}{t}dt$$

For $0\le t\le 1$ and $s>0$, we have $e^{-st}\ge e^{-s}$.

Therefore, $L\left\{\frac{1}{t}\right\}=\int_0^\infty \frac{e^{-st}}{t}dt \ge \int_0^1 \frac{e^{-st}}{t}dt+\int_1^\infty \frac{e^{-st}}{t}dt$

But, $\int_0^1 \frac{e^{-st}}{t}dt=e^{-s}\int_0^1 \frac{1}{t}dt=e^{-s}\lim_{p\to 0}\int_p^1 \frac{1}{t}dt=e^{-s}\lim_{p\to 0}[\ln t]_p^1$

$$=e^{-s}\lim_{p\to 0}(\ln 1-\ln p)=e^{-s}(\infty)=\infty$$

Hence, $\int_0^1 \frac{e^{-st}}{t}dt$ diverges and consequently $\int_0^\infty \frac{e^{-st}}{t}dt$ diverges.

Therefore, $L\left\{\frac{1}{t}\right\}$ does not exist.

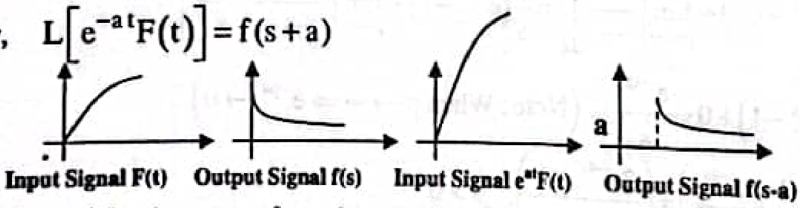
First Shifting Property of Laplace Transform

Statement: If $L[F(t) = f(s)$,then $L\left[e^{at}F(t)\right]=f(s-a)$

Proof: By definition, $L\left[F(t)\right]=\int_0^\infty e^{-st}F(t)dt=f(s)$

Then, $L\left[e^{at}F(t)\right]=\int_0^\infty e^{-st}e^{at}F(t)dt=\int_0^\infty e^{-st+at}F(t)dt=\int_0^\infty e^{-(s-a)t}F(t)dt=f(s-a)$

Similarly, $L\left[e^{-at}F(t)\right]=f(s+a)$



This property of Laplace transform has an important physical implication. It helps to delay an output signal and to release it after the required destination. This is done by multiplying the given function F(t) by the factor e^{at} or e^{-at} respectively. This is shown in the above figures.

EXAMPLE 10: Find $L\left[t^3e^{-3t}\right]$ and $L[\sinh 2t\sin 3t]$

Solution: (i) Here $F(t)=t^3$,and $L[t^3]=\frac{3!}{s^{3+1}}=\frac{6}{s^4}$

Thus by first shifting property, $L\left[t^3e^{-3t}\right]=\dfrac{6}{(s+3)^4}$

(ii) Here $L[\sinh 2t\sin 3t]=L\left[\dfrac{e^{2t}-e^{-2t}}{2}\right]\sin 3t$. Now $L(\sin 3t)=\dfrac{3}{s^2+9}$. Thus

$L[\sinh 2t\sin 3t]=\dfrac{1}{2}\left\{L\left(e^{2t}\sin 3t\right)-L\left(e^{-2t}\sin 3t\right)\right\}$. By shift property, we have

$$L[\sinh 2t\sin 3t]=\frac{1}{2}\left\{\frac{3}{(s-2)^2+9}-\frac{3}{(s+2)^2+9}\right\}.$$

EXAMPLE 11: Find $L\left[(t+2)^2e^t\right]$

Solution: $L\left[(t+2)^2e^t\right]=L\left[\left(t^2+4t+4\right)e^t\right]=L\left(t^2e^t\right)+4L\left(te^t\right)+4L(1)$ (1)

Now $L[t]=\dfrac{1}{s^2}$, $L[t^2]=\dfrac{2}{s^3}$ and $L[1]=\dfrac{1}{s}$

Using the first shifting property, we get: $L[te^t]=\dfrac{1}{(s-1)^2}$ and $L[t^2e^t]=\dfrac{2}{(s-1)^3}$

Thus (1) becomes: $$L\left[(t+2)^2e^t\right]=\frac{2}{(s-1)^3}+\frac{4}{(s-1)^2}+\frac{4}{s-1}$$

EXAMPLE 12: Find $L\left[e^{-3t}(2\cos 5t-3\sin 5t)\right]$

Solution: $L\left[e^{-3t}(2\cos 5t-3\sin 5t)\right]=2L\left[e^{-3t}\cos 5t\right]-3L\left[e^{-3t}\sin 5t\right]$ (1)

Now, $L(\cos 5t)=\dfrac{s}{s^2+25}$ and $L(\sin 5t)=\dfrac{5}{s^2+25}$

Thus using first shifting property, we get

$L\left[e^{-3t}\cos 5t\right]=\dfrac{s+3}{(s+3)^2+25}$ and $L\left[e^{-3t}\sin 5t\right]=\dfrac{5}{(s+3)^2+25}$.

Thus (1) becomes, $L\left[e^{-3t}(2\cos 5t-3\sin 5t)\right]=\dfrac{2s-9}{s^2+6s+34}$

EXAMPLE 13: Evaluate $L(1+t\,e^{-t})^3$

Solution: $L\left(1+t\,e^{-t}\right)^3=L\left[\left(1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t}\right)\right]$

$$=L[1]+3L\left[te^{-t}\right]+3L\left[t^2e^{-2t}\right]+L\left[t^3e^{-3t}\right] \quad (1)$$

Now, $L[1]=\dfrac{1}{s}$, $L[t]=\dfrac{1}{s^2}$, $L[t^2]=\dfrac{2}{s^3}$ and $L[t^3]=\dfrac{6}{s^4}$. Now by Ist shifting property:

$L\left[te^{-t}\right]=\dfrac{1}{(s+1)^2}$, $L\left[t^2e^{-2t}\right]=\dfrac{2}{(s+2)^3}$ and $L\left[t^3e^{-3t}\right]=\dfrac{6}{(s+3)^4}$

Hence equation (1) becomes:

$$L\left[\left(1+te^{-t}\right)^3\right]=\frac{1}{s}+3.\frac{1}{(s+1)^2}+3.\frac{2}{(s+2)^3}+\frac{6}{(s+3)^4}=\frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}$$

Second Shifting Property of Laplace transform

Statement: If $L[F(t)] = f(s)$ and

$$G_a(t) = F(t-a) \qquad t > a$$
$$= 0 \qquad t < a,$$

then $L[G_a(t)] = e^{-as} f(s)$

Proof: By definition, $L[G_a(t)] = \int_0^\infty e^{-st} G_a(t)\,dt$

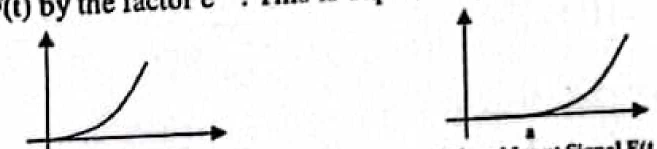
$$= \int_0^a e^{-st} G_a(t)\,dt + \int_a^\infty e^{-st} G_a(t)\,dt = \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st} F(t-a)\,dt = \int_a^\infty e^{-st} F(t-a)\,dt$$

Substituting $t - a = z \therefore dt = dz$. Now as $t \to a,\ z \to 0$ and as $t \to \infty, z \to \infty$

Thus $L[G_a(t)] = \int_a^\infty e^{-s(z+a)} F(z)\,dz = \int_a^\infty e^{-sz} e^{-as} F(z)\,dz$

$$\Rightarrow \quad L[G_a(t)] = e^{-as} \int_a^\infty e^{-sz} F(z)\,dz = e^{-as} L[F(z)] = e^{-as} f(s)$$

This property has also an important physical implication. It states that if an input signal is to be delayed for some time $t = a$, then the effect of this delay on the output signal is to multiply LT of F(t) by the factor e^{-as}. This is depicted in the following figure.



Input Signal F(t) Delayed Input Signal F(t – a)

EXAMPLE 14: Evaluate L[cos(t – 2)]

Solution: We know that $L \cos t = s/(s^2 + a^2)$. Using second shifting property, we get:

$$L[\cos(t-2)] = e^{-2s}\, s/(s^2 + a^2)$$

Third Shifting Property or Change of Scale Property

Statement: If $L[F(t)] = f(s)$, then $L[F(at)] = 1/a\, f(s/a)$ for some constant 'a'.

Proof: By definition, $L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt \Rightarrow L[F(at)] = \int_0^\infty e^{-st} F(at)\,dt$

Let $at = z \Rightarrow t = z/a \therefore dt = dz/a$. Now as $t \to 0 \ \therefore z \to 0$, $t \to \infty \ \therefore z \to \infty$.

Hence,

$$L[F(at)] = \int_0^\infty e^{-s(z/a)} F(z) \frac{1}{a}\,dz = \frac{1}{a}\int_0^\infty e^{-sz/a} F(z)\,dz = \frac{1}{a}\int_0^\infty e^{-(s/a)z} F(z)\,dz = \frac{1}{a} f\left(\frac{s}{a}\right)$$

EXAMPLE 15: If $L[F(t)] = \dfrac{s^2 - s + 1}{(2s+1)^2 (s-1)}$, find $L[e^{-t}F(2t)]$

Solution: In this problem, we have to use two properties. First property that we shall use is the change of scale property and the second one is first shifting property. According to change of scale property, if

$$L[F(t)] = f(s), \text{ then } L[F(at)] = 1/a\, f(s/a).$$

Now given that $L[F(t)] = \dfrac{s^2 - s + 1}{(2s+1)^2 (s-1)} = f(s).$

Using the above formula with $a = 2$, we get

$$L[F(2t)]=\frac{1}{2}\left\{\frac{\left(\frac{s}{2}\right)^2-\frac{s}{2}+1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}\right\}=\frac{1}{2}\left\{\frac{\frac{s^2}{4}-\frac{s}{2}+1}{(s+1)^2\left(\frac{s-2}{2}\right)}\right\}$$

$$=\frac{1}{2}\left\{\frac{s^2-2s+4}{4}\cdot\frac{2}{(s+1)^2(s-2)}\right\}=\frac{1}{4}\left\{\frac{s^2-2s+4}{(s+1)^2(s-2)}\right\}$$

Now using the first shifting property, we have

$$L\left[e^{-t}F(2t)\right]=\frac{1}{4}\left\{\frac{(s+1)^2-2(s+1)+4}{(s+1+1)^2(s+1-2)}\right\}=\frac{1}{4}\frac{s^2+3}{(s+2)^2(s-1)}$$

Multiplication by t^n Property

Statement: If L[F(t) = f(s) ,then L[t^n F(t)] = $(-1)^n f^n(s)$, where $f^n(s)$ is the n^{th} derivative of f (s).

Proof: By definition, $L[F(t)]=f(s)=\int_0^\infty e^{-st}F(t)dt.$

Differentiate both sides w.r.t s and using Leibniz Theorem of integration under differentiation, that is:

$$\frac{d}{dx}\int f(x,y)dy=\int\frac{\partial}{\partial x}f(x,y)dy,$$

We get:

$$\frac{d}{ds}f(s)=\frac{d}{ds}\int_0^\infty e^{-st}F(t)\,dt=\int_0^\infty\frac{\partial}{\partial s}\left(e^{-st}\right)F(t)dt=\int_0^\infty -te^{-st}F(t)dt$$

$$\Rightarrow (-1)\ \frac{d}{ds}f(s)\ =\int_0^\infty e^{-st}\,tF(t)dt=L[tF(t)]$$

Differentiating again w.r.t s and using Leibniz Theorem, we obtain

$(-1)^2\ \frac{d^2}{ds^2}f(s)\ =\int_0^\infty e^{-st}\,t^2F(t)dt=L\left[t^2F(t)\right]$. Generalizing, we get

$$(-1)^n\ \frac{d^n}{ds^n}f(s)\ =\int_0^\infty e^{-st}\,t^nF(t)dt\ =L\left[t^nF(t)dt\right]$$

Hence $L\left[t^nF(t)dt\right]=\ (-1)^n\ \frac{d^n}{ds^n}f(s)\ =\ (-1)^n f^n(s)$

EXAMPLE 16: Find L [$t^2 e^{2t}$].

Solution: We know that $L\left[e^{2t}\right]=\frac{1}{s-2}=f(s)$. Now using multiplication by t^n property,

we get: $$L\left[t^2e^{at}\right]=\ (-1)^2\ \frac{d^2}{ds^2}\cdot\frac{1}{s-2}$$

Now differentiate R.H.S twice w.r.t s, we have

$$L\left[t^2e^{at}\right]=\ (-1)^2\ \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s-2}\right)\right]=\frac{d}{ds}\left[\frac{-1}{(s-2)^2}\right]=\frac{d}{ds}\left[-(s-2)^{-2}\right]=2(s-2)^{-3}=\frac{2}{(s-2)^3}$$

EXAMPLE 17: Find L(t e^{-4t} sin 3t)

Solution: $L\left[te^{-4t}\sin 3t\right]=L\left[e^{-4t}(t\sin 3t)\right]$ (1)

Now using multiplication by t property, we obtain

$$L[t\sin 3t] = -\frac{d}{ds}L(\sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

Now using first shifting property on (1), we have

$$L\left[e^{-4t}(t\sin 3t)\right] = \frac{6(s+4)}{\left[(s+4)^2+9\right]^2} = \frac{6(s+4)}{\left[s^2+8s+25\right]^2}$$

EXAMPLE 18: Use multiplication by t^n property to evaluate $\int_0^\infty e^{-3t} t\sin 3t\,dt$

Solution: We have from example 17 that, $L[t\sin 3t] = 6s/(s^2+9)^2$.

This implies that: $L[t\sin 3t] = \int_0^\infty e^{-st} t\sin 3t\,dt = \frac{6s}{(s^2+9)^2}$

Substituting s = 3, we obtain $\int_0^\infty e^{-3t} t\sin 3t\,dt = \frac{18}{(18)^2} = \frac{1}{18}$.

Laplace Transform of Derivatives

Statement: If $L[F(t) = f(s)$, then $L[F'(t)] = s\,f(s) - F(0)$. Hence show that
$L[F^{(n)}(t)] = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \ldots - s\,F^{n-2}(0) - F^{n-1}(0)$

Proof: We know that, $L[F(t)] = f(s) = \int_0^\infty e^{-st}F(t)dt \Rightarrow L[F'(t)] = \int_0^\infty e^{-st}F'(t)dt$

Integrating by parts, taking $u = e^{-st}$ and $v = F'(t)$ we get,

$$L[F'(t)] = \left[e^{-st}F(t)\right]_0^\infty - \int_0^\infty -se^{-st}F(t)dt = \left[e^{-\infty} - e^0F(0)\right] + s\int_0^\infty e^{-st}F(t)dt$$

$$= \left[0 - F(0) + s\int_0^\infty e^{-st}F(t)dt\right] = -F(0) + sf(s)$$

Hence, $L[F'(t)] = sf(s) - F(0)$. Similarly,

$$L[F''(t)] = \left[e^{-st}F'(t)\right]_0^\infty - \int_0^\infty -se^{-st}F'(t)dt = \left[e^{-\infty} - e^0F'(0)\right] + s\int_0^\infty e^{-st}F'(t)dt$$

$$= -F'(0) + sL(F'(t)) = -F'(0) + s[sf(s) - F(0)] = s^2f(s) - sF(0) - F'(0)$$

In general, $L\left[F^{(n)}(t)\right] = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) \ldots sF^{n-2}(0) - F^{n-1}(0)$

This property is useful for solving differential equations.

EXAMPLE 19: (a) Evaluate $L\{\sin\sqrt{t}\}$, hence find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

(b) Show that $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = 1/s^{3/2}$. Hence show that $L\left\{\sqrt{\frac{1}{\pi t}}\right\} = 1/\sqrt{s}$

Solution: (a) Let $F(t) = \sin\sqrt{t}$ ➔ $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ and $F(0) = \sin(0) = 0$

Now using property of Laplace transform of derivative, that is,

$L\{F'(t)\} = sL\{F(t)\} - F(0)$, we have

$L\{F'(t)\} = L\{\cos\sqrt{t}/2\sqrt{t}\} = sL\{\sin\sqrt{t}\} - 0 \Rightarrow L\{\cos\sqrt{t}/2\sqrt{t}\} = \sqrt{\pi}se^{-1/4s}/2s^{3/2}$

since, $L(\sin\sqrt{t}) = \sqrt{\pi}e^{-1/4s}/2s^{3/2}$. See example 04.

$$\therefore L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = \frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{2s^{1/2}}e^{-1/4s} \Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{s}\right)}e^{-1/4s}$$

(b) $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{2}{\sqrt{\pi}}L\sqrt{t} = \frac{2}{\sqrt{\pi}}\left(\frac{\Gamma(3/2)}{s^{3/2}}\right) = \frac{2}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2}.\frac{1}{s^{3/2}} = \frac{1}{s^{3/2}}$

Let $F(t) = \left\{2\sqrt{\frac{t}{\pi}}\right\}$. Then $F'(t) = \frac{1}{\sqrt{\pi t}}$ and $F(0) = 0$.

Now $LF'(t) = sf(s) - F(0) = s\,L[F(t)] - 0 = s\,L[F(t)]$.
Substituting the values of $F'(t)$ and $F(t)$, we obtain:

$$L\left\{\sqrt{\frac{1}{\pi t}}\right\} = s.\frac{1}{s^{3/2}} = 1/\sqrt{s}.$$

Division by t Property

Statement: If $L[F(t)] = f(s)$, then $L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(u)du$, provided $\lim\limits_{t\to 0}\frac{F(t)}{t}$ exists.

Proof: We know that $L[F(t)] = f(s) = \int_0^\infty e^{-st}F(t)dt$

Let $G(t) = F(t)/t$ ➔ $F(t) = t\,G(t)$. Taking Laplace transform on both sides, we have

$$L[F(t)] = L[tG(t)]$$

Using the multiplication by t property, we get

$$f(s) = -g`(s) \quad ➔ \quad g`(s) = -f(s)$$

Now integrating both sides from ∞ to s, we have

$$g(s) = -\int_\infty^s f(u)du \quad ➔ \quad g(s) = \int_s^\infty f(u)du$$

[Note: Here we have changed the variable from s to u]

But $g(s) = L[G(t)]$ ➔ $L[G(t)] = \int_s^\infty f(u)du$. Put $G(t) = F(t)/t$, we get

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(u)du$$

EXAMPLE 20: Evaluate (i) $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$ (ii) $L\left[\frac{\cos 4t - \cos 6t}{t}\right]$. **Use the result to evaluate** $\int_0^\infty e^{-2t}\frac{\cos 4t - \cos 6t}{t}dt$ and $\int_0^\infty \frac{\cos 4t - \cos 6t}{t}dt$

Solution: We know that $L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(u)du$. Here $F(t) = e^{-at} - e^{-bt}$

(i) Thus, $L[F(t)] = L\left[e^{-at} - e^{-bt}\right] = \frac{1}{s+a} - \frac{1}{s+b}$.

Hence, $L\left[\frac{F(t)}{t}\right] = L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{u+a} - \frac{1}{u+b}\right) du$

Using division by t property. Integrating R.H.S from s to ∞, we have

$$L\left[\frac{F(t)}{t}\right] = \left[\ln(u+a) - \ln(u+b)\right]_s^\infty = \left[\ln\frac{(u+a)}{(u+b)}\right]_s^\infty = \left[\ln\frac{u(1+a/u)}{u(1+b/u)}\right]_s^\infty$$

$$= \left[\ln\frac{(1+a/u)}{(1+b/u)}\right]_s^\infty = \ln\frac{1-0}{1-0} - \ln\frac{(1+a/s)}{(1+b/s)} = \ln 1 - \ln\frac{(s+a)}{(s+b)} = 0 + \ln\frac{(s+b)}{(s+a)}$$

Therefore, $L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \ln\frac{(s+b)}{(s+a)}$

(ii) $L(\cos 4t) = \left[\frac{s}{s^2+16}\right] \Rightarrow L\left(\frac{\cos 4t}{t}\right) = \int_s^\infty \frac{u}{u^2+16} du = \frac{1}{2}\int_s^\infty \frac{2u}{u^2+16} du$

$$= \frac{1}{2}\ln\left[s^2+16\right]_s^\infty = \frac{1}{2}\lim_{P\to\infty}\ln\left[s^2+16\right]_s^P = \frac{1}{2}\left[\lim_{P\to\infty}\ln\left(P^2+16\right) - \ln\left(s^2+16\right)\right]$$

Similarly, $L\left(\frac{\cos 6t}{t}\right) = \frac{1}{2}\left[\lim_{P\to\infty}\ln\left(P^2+36\right) - \ln\left(s^2+36\right)\right]$. Subtracting, we get

$$L\left[\frac{\cos 4t - \cos 6t}{t}\right] = L\left(\frac{\cos 6t}{t}\right) = \frac{1}{2}\left[\lim_{P\to\infty}\ln\left(\frac{P^2+16}{P^2+36}\right) + \ln\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{P\to\infty}\ln\left(\frac{P^2(1+16/P^2)}{P^2(1+36/P^2)}\right) + \ln\left(\frac{s^2+36}{s^2+16}\right)\right] = \frac{1}{2}\left[\lim_{P\to\infty}\ln\left(\frac{\left(1+\frac{16}{P^2}\right)}{\left(1+\frac{36}{P^2}\right)}\right) + \ln\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{P\to\infty}\ln 1 + \ln\left(\frac{s^2+36}{s^2+16}\right)\right] = \frac{1}{2}\left[0 + \ln\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$\therefore L\left[\frac{\cos 4t - \cos 6t}{t}\right] = \frac{1}{2}\ln\left(\frac{s^2+36}{s^2+16}\right).$$

Now by definition:

$$L[F(t)] = \int_0^\infty e^{-st}F(t)dt \Rightarrow L\left[\frac{\cos 4t - \cos 6t}{t}\right] = \int_0^\infty e^{-st}\frac{\cos 4t - \cos 6t}{t}dt = \frac{1}{2}\ln\left(\frac{s^2+36}{s^2+16}\right)$$

(a) Put s = 2, we get, $\int_0^\infty e^{-2t}\frac{\cos 4t - \cos 6t}{t}dt = \frac{1}{2}\ln\left(\frac{4+36}{4+16}\right) = \frac{1}{2}\ln 2 = \ln 2^{1/2} = \ln\sqrt{2}$

(b) Put s = 0, we get $\int_0^\infty \frac{\cos 4t - \cos 6t}{t}dt = \frac{1}{2}\ln\left(\frac{36}{16}\right) = \ln\sqrt{\frac{36}{16}} = \ln\left(\frac{6}{4}\right) = \ln\left(\frac{3}{2}\right)$

EXAMPLE 21: Show that $L\left(\frac{\sin^2 t}{t}\right) = \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$. **Hence evaluate** $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}dt$

Solution: Let $F(t) = \sin^2 t = \frac{1-\cos 2t}{2}$

$$\Rightarrow L\left[\sin^2 t\right]=L\left[\frac{1-\cos 2t}{2}\right]=\frac{1}{2}\left[L[1]-L[\cos 2t]\right]=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

Using the division by t property of Laplace transform, we get

$$L\left[\frac{F(t)}{t}\right]=L\left[\frac{\sin^2 t}{t}\right]=\frac{1}{2}\int_s^\infty\left[\frac{1}{u}-\frac{u}{u^2+4}\right]du=\frac{1}{2}\int_s^\infty\left[\frac{1}{u}-\frac{2u}{2(u^2+4)}\right]du.$$

$$=\frac{1}{2}\left[\ln u-\frac{1}{2}\ln\left(u^2+4\right)\right]_0^\infty=\frac{1}{2}\left[\ln u-\ln\left(u^2+4\right)^{\frac{1}{2}}\right]_s^\infty=\frac{1}{2}\left[\ln\left(\frac{u}{\left(u^2+4\right)^{\frac{1}{2}}}\right)\right]_0^\infty$$

$$=\frac{1}{2}\left[\ln 1-\log\frac{s}{\left(s^2+4\right)^{\frac{1}{2}}}\right]=\frac{1}{2}\left[\ln\left(\frac{\left(s^2+4\right)^{\frac{1}{2}}}{s}\right)\right]=\frac{1}{4}\ln\left(s^2+4\right)-\frac{1}{2}\ln s=\frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$$

Since $L\left(\frac{\sin^2 t}{t}\right)=\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$. Then by definition

$L\left(\frac{\sin^2 t}{t}\right)=\int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t}\right)=\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$. Substituting $s=1$,we get

$$\int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)=\frac{1}{4}\ln\left(\frac{1^2+4}{1^2}\right)=\frac{1}{4}\ln 5$$

EXAMPLE 22: Show that $\int_0^\infty \frac{e^{-t}\sin t}{t}dt=\frac{\pi}{4}$

Solution: We know that $L[\sin t] = 1/(s^2 + 1)$. Using division by t property, we get

$$L\left[\frac{\sin t}{t}\right]=\int_s^\infty\frac{1}{u^2+1}du=\left[\tan^{-1}u\right]_s^\infty=\tan^{-1}\infty-\tan^{-1}s=\frac{\pi}{2}-\tan^{-1}s$$

$$\int_s^\infty\frac{1}{u^2+1}du=\left[\tan^{-1}u\right]_s^\infty=\tan^{-1}\infty-\tan^{-1}s=\frac{\pi}{2}-\tan^{-1}s$$

Thus, $\int_0^\infty e^{-st}\frac{\sin t}{t}dt=\frac{\pi}{2}-\tan^{-1}s$

Now substituting $s = 1$, we get: $\int_0^\infty e^{-t}\frac{\sin t}{t}dt=\frac{\pi}{2}-\tan^{-1}1=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

EXAMPLE 23: Show that $\int_0^\infty\frac{\sin^2 t}{t^2}dt=\frac{\pi}{2}$

Proof: Let $F(t)=\sin^2 t=\left(\frac{1-\cos 2t}{2}\right)$,then

$$L[F(t)]=\frac{1}{2}\left[L\{1\}-L\{\cos 2t\}\right]=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)=f(s).\text{ Now}$$

$$L\left\{\frac{F(t)}{t}\right\}=\int_s^\infty\frac{\sin^2 t}{t}dt=\frac{1}{2}\int_s^\infty\left(\frac{1}{u}-\frac{u}{u^2+4}\right)du=\frac{1}{2}\lim_{t\to\infty}\left(\ln u-\frac{1}{2}\ln\left(u^2+4\right)\right)_s^t$$

$$= \frac{1}{4}\lim_{t\to\infty}\left(\ln\frac{u^2}{u^2+4}\right)_s^t = \frac{1}{4}\lim_{t\to\infty}\left(\ln\frac{t^2}{t^2+4} - \ln\frac{s^2}{s^2+4}\right)$$

$$= \frac{1}{4}\lim_{t\to\infty}\left(\ln\frac{t^2}{t^2(1+4/t^2)} - \lim_{t\to\infty}\ln\frac{s^2}{s^2+4}\right)$$

$$= \frac{1}{4}\lim_{t\to\infty}\left(\ln\frac{1}{1+0} - \ln\frac{s^2}{s^2+4}\right) = -\frac{1}{4}\ln\left(\frac{s^2}{s^2+4}\right) \quad [\ln 1 = 0]$$

Thus, $L\left\{\frac{F(t)}{t}\right\} = -\frac{1}{4}\ln\left(\frac{s^2}{s^2+4}\right) = f(s)$.

Now using division by t property once again, we obtain

$$L\left\{\frac{F(t)}{t^2}\right\} = -\frac{1}{4}\int_s^\infty \ln\left(\frac{u^2}{u^2+4}\right)du = -\frac{1}{4}\int_s^\infty \ln\left(\frac{u^2}{u^2+4}\right).1\,du$$

Now integrate by parts, gives

$$L\left\{\frac{F(t)}{t^2}\right\} = -\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right) - \int\frac{u^2+4}{u^2}.\left(\frac{2u.(u^2+4)-2u.u^2}{(u^2+4)^2}\right)u\,du\right]_s^\infty$$

$$= -\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right) - \int\frac{u^2+4}{u}.\left(\frac{2u^3+8u-2u^3}{(u^2+4)^2}\right)du\right]_s^\infty$$

$$= -\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right) - \int\frac{8u(u^2+4)}{u(u^2+4)^2}du\right]_s^\infty = -\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right) - 8\int\frac{1}{(u^2+4)}du\right]_s^\infty$$

$$= -\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right) - \frac{8}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_s^\infty = -\frac{1}{4}\left[0 - 4\frac{\pi}{2} - s\ln\left(\frac{s^2}{s^2+4}\right) + 4\tan^{-1}\left(\frac{s}{2}\right)\right]$$

Thus $\int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t^2}\right)dt = -\frac{1}{4}\left[-2\pi - s\ln\left(\frac{s^2}{s^2+4}\right) + 4\tan^{-1}\left(\frac{s}{2}\right)\right]$

Now substituting $s = 0$, we get $\int_0^\infty\left(\frac{\sin^2 t}{t^2}\right)dt = \frac{\pi}{2}$ [Note: $\tan^{-1} 0 = 0$ & $\ln 1 = 0$]

Laplace Transform of Integrals

Statement: If $L[F(t)] = f(s)$, then $L\left[\int_0^t F(u)du\right] = \frac{f(s)}{s}$

Proof: Let $G(t) = \int_0^t F(u)\,du \Rightarrow G(0) = \int_0^0 F(u)\,du = 0$

Differentiate both sides w.r.t t, we have

$$G'(t) = \frac{d}{dt}\int_0^t F(u)\,du = F(t)$$

Taking Laplace transform on both sides, we obtain

$$L(F(t)) = L(G'(t)) = s\,g(s) - G(0) = s\,g(s) \quad [\because G(0) = 0]$$

→ $f(s) = sg(s) \Rightarrow g(s) = f(s)/s$. But $g(s) = L[G(t)]$.

Thus, $$L[G(t)] = \frac{f(s)}{s}$$

EXAMPLE 24: Prove that $L\left[\int_0^t \sin u\,du\right] = \frac{1}{s(s^2+1)}$

Proof: First method:

$\int_0^t \sin u\,du = -[\cos u]_0^t = -\{\cos t - \cos 0\} = 1 - \cos t$. Thus

$$L\left[\int_0^t \sin u\,du\right] = L[1-\cos t] = L[1] - L[\cos t] = \frac{1}{s} - \frac{s}{s^2+1} = \frac{s^2+1-s^2}{s(s^2+1)} = \frac{1}{s(s^2+1)}$$

Second Method: $L[\sin t] = 1/(s^2+1)$, hence by Laplace transforms of integral property,

$$L\left[\int_0^t \sin u\,du\right] = \frac{1}{s(s^2+1)}$$

EXAMPLE 25: Prove that $L\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$

Solution: Consider $\left[\int_0^t \frac{\sin u}{u}\,du\right]$. Let $u = tv$ → $du = t\,dv$.

Now, if $u \to 0$ then $v \to 0$ and similarly if $u \to t$ then $v \to 1$.

Hence $\left[\int_0^t \frac{\sin u}{u}\,du\right] = \int_0^1 \frac{\sin tv}{v}\,dv$

Now taking the Laplace transform on both sides, we have

$$L\left[\int_0^t \frac{\sin u}{u}\,du\right] = L\left[\int_0^1 \frac{\sin tv}{v}\,dv\right] = \int_0^\infty e^{-st}\left\{\int_0^1 \frac{\sin tv}{v}\,dv\right\}dt = \int_0^1 \frac{1}{v}\left\{\int_0^\infty e^{-st}\sin vt\,dt\right\}dv$$

$$= \int_0^1 \frac{1}{v}\{L[\sin vt]\}dv = \int_0^1 \frac{1}{v}\left\{\frac{v}{v^2+s^2}\right\}dv$$

$$= \int_0^1 \left\{\frac{1}{v^2+s^2}\right\}dv = \frac{1}{s}\tan^{-1}\left[\frac{v}{s}\right]_0^1 = \frac{1}{s}\left[\tan^{-1}\left(\frac{1}{s}\right) - \tan^{-1}\left(\frac{0}{s}\right)\right]$$

$$= \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right) \qquad [\text{Note}: \tan^{-1}0 = 0]$$

Laplace Transform of Periodic Functions

A function F(t) is said to be periodic of period T if it satisfies the condition:

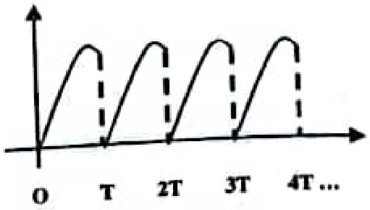
$$F(t \pm T) = F(t) \text{ for all } T > 0.$$

Theorem: If F(t) is periodic function of period T, then

$$L[F(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\,dt$$

Proof: Since F(t) is a periodic function of period T, let its graph be as under:

Now by definition: $L[F(t)] = \int_0^{\infty} e^{-st}F(t)dt$



$$= \int_0^T e^{-st}F(t)dt + \int_T^{2T} e^{-st}F(t)dt + \int_{2T}^{3T} e^{-st}F(t)dt + \ldots$$

Substituting:
$t = u$ in the 1st integral $\rightarrow$ $dt = du$. Also if $t = 0 \rightarrow u = 0$ and if $t = T \rightarrow u = T$
$t = u + T$ in the 2nd integral $\rightarrow$ $dt = du$. Also if $t = T \rightarrow u = 0$ and if $t = 2T \rightarrow u = T$
$t = u + 2T$ in the 3rd integral $\rightarrow$ $dt = du$. Also if $t = 2T \rightarrow u = 0$ and if $t = 3T \rightarrow u = T$
Thus,

$$L[F(t)] = \int_0^T e^{-su}F(u)du + \int_0^T e^{-s(u+T)}F(u+t)du + \int_0^T e^{-s(u+2T)}F(u+2t)du + \ldots$$

$$= \int_0^T e^{-su}F(u)du + e^{-sT}\int_0^T e^{-su}F(u)du + e^{-2sT}\int_0^T e^{-su}F(u)du + \ldots$$

Since, F(T) is periodic function hence $F(u) = F(U+T) = F(u+2T) = \ldots$ Thus,

$$L[F(t)] = (1 + e^{-sT} + e^{-2sT} + \ldots)\int_0^T e^{-su}F(u)du = (1 - e^{-sT})^{-1}\int_0^T e^{-su}F(u)du$$

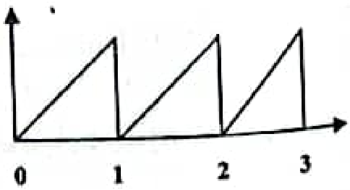
$$= \frac{1}{(1 - e^{-sT})}\int_0^T e^{-st}F(t)dt$$

REMARKS: (i) It may be noted that here we have used an infinite binomial series: $1 + x + x^2 + \ldots = (1 - x)^{-1}$
(ii) Variable u is changed with variable s in the last integral. Such change has no effect on the result.

EXAMPLE 26: Obtain the Laplace transform of the periodic saw-tooth wave function given by: $F(t) = \frac{t}{k}$, $0 < t < 1$

Solution: The graph of the periodic saw-tooth wave function is shown below. Here F(t) is periodic function with period T = 1. Hence, by definition

$$L[F(t)] = \frac{1}{1 - e^{-sT}}\int_0^T e^{-st}F(t)dt = \frac{1}{1 - e^{-s}}\int_0^1 e^{-st}\frac{t}{k}dt$$



Integration by parts, gives

$$L[F(t)] = \frac{1}{k(1 - e^{-s})}\left\{\left[t\frac{e^{-st}}{-s}\right]_0^1 - \int_0^1 1.\frac{e^{-st}}{-s}dt\right\}$$

$$= \frac{1}{k(1 - e^{-s})}\left\{\left[\frac{e^{-s}}{-s} - 0\right] + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^1\right\} = \frac{1}{k(1 - e^{-s})}\left\{\left[\frac{e^{-s}}{-s}\right] - \frac{1}{s^2}\left[e^{-s} - e^0\right]\right\}$$

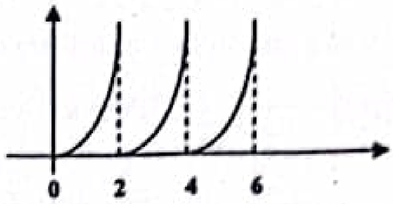
$$= \frac{1}{k(1 - e^{-s})}\left\{\left[\frac{e^{-s}}{-s}\right] + \frac{1}{s^2}\left[1 - e^{-s}\right]\right\} = -\frac{e^{-s}}{sk(1 - e^{-s})} + \frac{1}{ks^2} = \frac{1}{sk}\left[\frac{1}{s} - \frac{e^{-s}}{(1 - e^{-s})}\right]$$

EXAMPLE 27: Find the Laplace transform of the function F(t) defined as: $F(t) = t^2, 0 < t < 2$ and $F(t+2) = F(t)$

Solution: The function F(t) is periodic function of period 2 hence,

$$L\{F(t)\}=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}F(t)dt.$$

$$\therefore L\{F(t)\}=\frac{\int_0^2 e^{-st}F(t)dt}{1-e^{-2s}}=\frac{\int_0^2 e^{-st}t^2dt}{1-e^{-2s}}.$$



Integrating by parts gives:

$$\therefore L[F(t)]=\frac{1}{(1-e^{2s})}\left[\left[t^2\frac{e^{-st}}{-s}\right]_0^2-2\int_0^2 t\frac{e^{-st}}{-s}dt\right]=\frac{1}{(1-e^{2s})}\left[\left[\frac{4e^{-2s}-0}{-s}\right]+\frac{2}{s}\int_0^2 te^{-st}dt\right]$$

$$=\frac{1}{(1-e^{2s})}\left[-\frac{4e^{-2s}}{s}+\frac{2}{s}\left\{\left[\frac{te^{-st}}{-s}\right]_0^2-\int_0^2\frac{e^{-st}}{-s}dt\right\}\right]$$

$$=\frac{1}{(1-e^{2s})}\left[-\frac{4e^{-2s}}{s}+\frac{2}{s}\left\{\left[\frac{2e^{-2s}}{-s}\right]+\frac{1}{s}\int_0^2 e^{-st}dt\right\}\right]$$

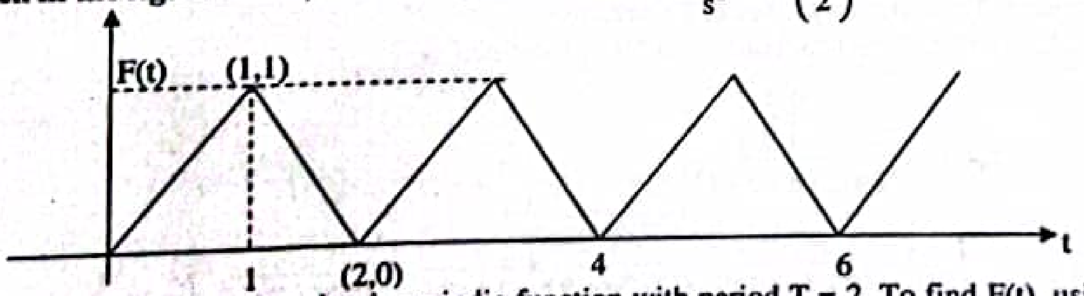
$$=\frac{1}{(1-e^{2s})}\left[-\frac{4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}+\frac{2}{s^2}\int_0^2 e^{-st}dt\right]$$

$$=\frac{1}{(1-e^{2s})}\left[-\frac{4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}+\frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^2\right]$$

$$=\frac{1}{(1-e^{2s})}\left[-\frac{4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}-\frac{2}{s^3}\left[e^{-2s}-1\right]\right]$$

$$=\frac{1}{(1-e^{2s})}\left[-\frac{4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}-\frac{2e^{-2s}}{s^3}+\frac{2}{s^3}\right]$$

EXAMPLE 28: Show that the function F(t) whose graph is the triangular wave given in the figure below, has the Laplace transform $\frac{1}{s^2}\tanh\left(\frac{s}{2}\right)$



Solution: Here given function is periodic function with period $T = 2$. To find F(t), using the formula of a straight line passing through two points, we have:
Equation of line through (1, 1) and (2, 0) is by using formula

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-1=\frac{0-1}{2-1}(x-1)\rightarrow y-1=-1(x-1) \text{ or } y=2-x.$$

Similarly, equation of line through two points (0, 0) and (1, 1) is $y = t$.

$\therefore F(t)=t \quad 0\le t\le 1$

$\quad\; =2-t \quad 1\le t\le 2$

Now using the Laplace transform of periodic function:

$L[F(t)]=\frac{1}{1-e^{sT}}\int_0^T e^{-st}F(t)dt$, we have

$$=\frac{1}{1-e^{2s}}\left[\int_0^1 e^{-st}F(t)dt+\int_1^2 e^{-st}F(t)dt\right]=\frac{1}{1-e^{2s}}\left[\int_0^1 e^{-st}tdt+\int_1^2 e^{-st}(2-t)dt\right]$$

$$=\frac{1}{1-e^{2s}}\left[\left(\frac{te^{-st}}{-s}-\frac{e^{-st}}{(-s)(-s)}\right)_0^1+\left(\frac{e^{-st}(2-t)}{-s}-\frac{e^{-st}.-1}{(-s)(-s)}\right)_1^2\right]$$

$$=\frac{1}{1-e^{2s}}\left[\left(\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}\right)-\left(0-\frac{1}{s^2}\right)+\left(0+\frac{e^{-2s}}{s^2}-\left(\frac{e^{-s}}{-s}+\frac{e^{-s}}{s^2}\right)\right)\right]$$

$$=\frac{1}{1-e^{2s}}\left[\left(\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right)+\left(\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}\right)\right]$$

$$=\frac{1}{1-e^{2s}}\left[\left(\frac{-se^{-s}-e^{-s}+1+e^{-2s}+se^{-s}-e^{-s}}{s^2}\right)\right]$$

$$=\frac{1}{s^2\left(1-e^{2s}\right)}\left[e^{-2s}-2e^{-s}+1\right]=\frac{\left(1-e^{-s}\right)^2}{s^2\left(1-e^{-s}\right)\left(1+e^{-s}\right)}=\frac{\left(1-e^{-s}\right)}{s^2\left(1+e^{-s}\right)}$$

Now multiply and divide by $e^{s/2}$, we get

$$L[F(t)]=\frac{\left(1-e^{-s}\right)}{s^2\left(1+e^{-s}\right)}.\frac{e^{s/2}}{e^{s/2}}=\frac{\left(e^{s/2}-e^{-s/2}\right)}{s^2\left(e^{s/2}+e^{-s/2}\right)}=\frac{1}{s^2}\tanh\left(\frac{s}{2}\right)$$

4.3 LAPLACE TRANSFIRMS OF SPECIAL FUNCTIONS

In this section we shall study the Laplace transforms of some special functions that are frequently used in engineering applications.

Laplace Transforms of Bessel Functions

We know that Bessel function of order n is defined as

$$J_n(t)=\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{t}{2}\right)^{n+2r}}{r!\Gamma(n+r+1)}$$

Or $$J_n(t)=\frac{t^n}{2^n\Gamma(n+1)}\left\{1-\frac{t^2}{2(2n+2)}+\frac{t^4}{2.4(2n+2)(2n+4)}+\ldots\right\} \qquad (1)$$

As an special case if we put $n=0$ in (1) ,we get

$$J_0(t)=\frac{t^0}{2^0\Gamma(0+1)}\left\{1-\frac{t^2}{2(2(0)+2)}+\frac{t^4}{2.4(2(0)+2)(2(0)+4)}+\ldots\right\}$$

$$=\frac{1}{\Gamma(1)}\left\{1-\frac{t^2}{2(2)}+\frac{t^4}{2.4(2)(4)}+\ldots\right\}=\left\{1-\frac{t^2}{2^2}+\frac{t^4}{2^2.4^2}+\ldots\right\}(\because\Gamma(1)=1)$$

Taking Laplace transform on both sides, we get

$$L\{J_0(t)\}=L\left\{1-\frac{t^2}{2^2}+\frac{t^4}{2^2.4^2}+\ldots\right\}=L\{J_{0(t)}\}=L\{1\}-\frac{1}{2^2}L\{t^2\}+\frac{1}{2^24^2}L\{t^4\}+\ldots$$

$$=\frac{1}{s}-\frac{1}{2^2}\frac{2!}{s^3}+\frac{1}{2^24^2}\frac{4!}{s^5}-\ldots \qquad \left(\because L\{t^n\}=\frac{n!}{s^{n+1}},s>0\right)$$

$$=\frac{1}{s}-\frac{1}{2s^3}+\frac{3}{8s^5}-\ldots=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1.3}{2.4}\left(\frac{1}{s^4}\right)-\frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right)+\ldots\right]$$

$$\left(\text{Use the formula}:(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots\right)$$

$$=\frac{1}{s}\left\{1+\frac{1}{s^2}\right\}^{-1/2}=\frac{1}{s}\left\{\frac{s^2+1}{s^2}\right\}^{-1/2}=\frac{1}{s}\frac{(s^2+1)^{-1/2}}{(s^2)^{-1/2}}$$

$$=\frac{1}{s}\frac{(s^2+1)^{-1/2}}{s^{-1}}=\frac{1}{\sqrt{s^2+1}} \qquad \Rightarrow L\{J_0(t)\}=\frac{1}{\sqrt{s^2+1}}=f(s)$$

$$\therefore L\{J_0(at)\}=\frac{1}{a}\frac{1}{\sqrt{(s/a)^2}} \qquad \left(\because L\{J_0(at)\}=\frac{1}{a}f\left(\frac{s}{a}\right)\text{by change of scale property}\right)$$

$$=\frac{1}{a}\frac{1}{\frac{\sqrt{s^2+a^2}}{a}}=\frac{1}{a}\frac{a}{\sqrt{s^2+a^2}}=\frac{1}{\sqrt{s^2+a^2}}$$

Hence $L\{J_0(at)\}=\dfrac{1}{\sqrt{s^2+a^2}}$

EXAMPLE 01: Find $L\{J_1(t)\}$, where $J_1(t)$ is the Bessel function of order 1.

Solution: We know from recurrence formula of Bessel function that

$$\frac{d}{dx}\{J_0(t)\}=-J_1(t) \qquad \Rightarrow J(t)_1=-J'_0(t)$$

Taking Laplace transform on both sides, we have: $L\{J_1(t)\}=-L\{J'_0(t)\}$

Using Laplace transform of derivatives property, that is,

$$L\{F'(t)\}=sf(s)-F(0)$$

we have: $L\{J_1(t)\}=-[sL\{J_0(t)\}-J_0(0)]$

$$\rightarrow L\{J_1(t)\}=-\left[s\frac{1}{\sqrt{s^2+1}}-1\right]=\frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}} \qquad \left(\because L\{J_0(t)\}=\frac{1}{\sqrt{s^2+1}},J_0(0)=1\right)$$

EXAMPLE 02: Evaluate $L\{t\,J_0(t)\}$. Hence or otherwise find $\int_0^\infty e^{-2t}t\,J_0(t)dt$

Solution: We know that

$$L\{J_0(t)\}=\frac{1}{\sqrt{s^2+1}}\Rightarrow L\{tJ_0(t)\}=(-1)\frac{d}{ds}\frac{1}{\sqrt{s^2+1}}=(-1)\left[-\frac{1}{2}\right](s^2+1)^{-3/2}(2s)=\frac{s}{(s^2+1)^{3/2}}$$

Thus by definition,

$$L\{tJ_0(t)\}=\int_0^{\infty} e^{-st}tJ_0(t)\,dt=\frac{s}{(s^2+1)^{3/2}} \Rightarrow \int_0^{\infty} e^{-2t}tJ_0(t)=\frac{2}{(4+1)^{3/2}}=\frac{2}{5\sqrt{5}}$$

Unit Step Function and Its Laplace Transform

Sometimes we come across functions whose inverse Laplace transforms cannot be determined from the formulas so for derived in the previous sections. In order to cover such cases, we introduce the unit step function also known as Heaviside's unit function named after the British Electrical Engineer Oliver Heaviside (1850-1925) who widely used this function in his work .

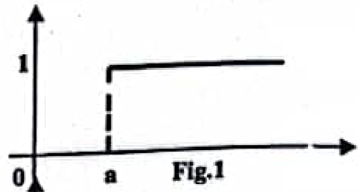


Fig.1

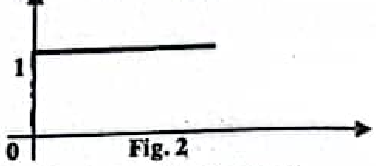


Fig. 2

The unit step function is denoted by U(t – a) and is defined as:

$$U(t-a)=0, \quad 0\le t\le a$$
$$=1, \quad t>a,$$

where 'a' is always positive. The graph of unit step function is shown in figure 1.

By definition,

$$L\{U(t-a)\}=\int_a^{\infty} e^{-st}dt=\frac{-1}{s}\left[e^{-st}\right]_a^{\infty}=\frac{-1}{s}(e^{-\infty}-e^{-as})=\frac{-1}{s}(0-e^{-as})=\frac{e^{-as}}{s}$$

If a = 0, we get

$$L\{U(t)\}=\int_0^{\infty} e^{-st}dt=\frac{-1}{s}\left[e^{-st}\right]_0^{\infty}=\frac{-1}{s}(e^{-\infty}-e^{0})=\frac{-1}{s}(0-1)=\frac{1}{s}=L(1)$$

The figure 2 shows the unit step function with a = 0.

Readers have studied second shifting property. We now, use unit step function to study second shift property and connect them to solve the problems that otherwise would be solved with great efforts.

It may be noted that second shifting property can be expressed as:

$$L\{G_a(t)\}=L[F(t-a)\,U(t-a)]=e^{-as}L[F(t)] \qquad (1)$$

EXAMPLE 03: Find the Laplace transform of:

(i) $F(t)=\begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$ (ii) $F(t)=e^{-t}[1-U(t-2)]$

Solution: (i) Given function can be expressed using unit step function as under:

$$\begin{aligned}F(t)&=(t-1)[U(t-1)-U(t-2)]+(3-t)[U(t-2)-U(t-3)]\\
&=(t-1)U(t-1)-(t-1)U(t-2)-(t-3)U(t-2)+(t-3)U(t-3)\\
&=(t-1)U(t-1)-(t-2+1)U(t-2)-(t-2-1)U(t-2)+(t-3)U(t-3)\\
&=(t-1)U(t-1)-(t-2)U(t-2)+U(t-2)-(t-2)U(t-2)-U(t-2)\\
&\quad+(t-3)U(t-3)\\
&=(t-1)U(t-1)-(t-2)U(t-2)-(t-2)U(t-2)+(t-3)U(t-3)\\
&=(t-1)U(t-1)-2(t-2)U(t-2)+(t-3)U(t-3)\end{aligned}$$

Thus using equation (1) and note that $L(t)=1/s^2$, we see that:

$$L\{F(t)\}=L(t-1)U(t-1)-2L(t-2)U(t-2)+L(t-3)U(t-3)$$
$$=\frac{e^{-s}}{s^2}-2\frac{e^{-2s}}{s^2}+\frac{e^{-3s}}{s^2}=\frac{1}{s^2}\left[e^{-s}-2e^{-2s}+e^{-3s}\right]$$

The readers must have realized that how we have used unit step function and second shifting property to compute the Laplace transform without involving the integration.

(ii) $L\{e^{-t}[1-U(t-2)]\} = L\{e^{-t}\} - L\{e^{-t}U(t-2)\} = L\{e^{-t}\} - L\{e^{-t+2-2}U(t-2)\}$

$$= L\{e^{-t}\} - e^{-2}L\{e^{-(t-2)}U(t-2)\} = L\{e^{-t}\} - e^{-2}e^{-2s}L\{e^{-t}\}$$
$$= [1-e^{-2(s+1)}]L\{e^{-t}\} = [1-e^{-2(s+1)}]/(s+1)$$

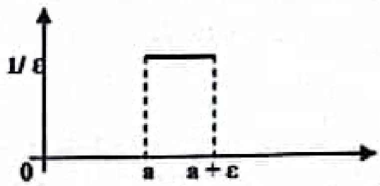
Impulse Function and Its Laplace Transform

In many engineering applications such as "Electrical and Mechanical Engineering" an idea of very large force acting for a very short time is of frequent occurrence. To deal with such and similar problems, we introduce the Unit Impulse Function (also known as Dirac Delta function, after the English physicist Paul Dirac (1902 – 1984) who was awarded the Nobel prize in 1933 for his work in Quantum Mechanics.

The unit impulse function is considered as limiting form of the function

$\delta(t-a) = 1/\varepsilon, \ a \le t \le a+\varepsilon$

$\quad\quad\quad = 0, \quad$ elsewhere



The graph of Dirac delta function is shown here.
One may observe that as ε approaches zero, δ indefinitely getting large and the width decreases in such a way that area under the rectangle is always unity. This shows that $\delta(t-a)$ tends to infinity as t approaches a and that $\delta(t-a) = 0$ for $t \ne a$ such that:

$$\int_0^\infty \delta(t-a)\,dt = 1 \quad (a \ge 0)$$

Now let us find the Laplace transform of unit impulse function. By definition,

$$L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)\,dt = \int_a^{a+\varepsilon} e^{-st}\frac{1}{\varepsilon}\,dt = \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon} = \frac{-1}{\varepsilon s}\left[e^{-as-\varepsilon s} - e^{-as}\right] = \frac{-e^{-as}}{s}\left[\frac{e^{-\varepsilon s}-1}{\varepsilon}\right]$$

Now, if $\varepsilon \to 0$ then by L'Hopital rule,

$$\lim_{\varepsilon\to 0}\frac{(e^{-\varepsilon s}-1)}{\varepsilon} = -s. \text{ Thus, } L[\delta(t-a)] = \frac{se^{-as}}{s} = e^{-as}$$

WORKSHEET 04

Find the Laplace transforms of the following functions:

1. $e^{3t}+t^4-2\sin 4t$ 2. $2+2\sqrt{t}+1/\sqrt{t}$ 3. $\cos(2t+b)$

4. $(\sin t-\cos t)^2$ 5. $\sin 2t\cos 4t$ 6. $\sin^2 4t+3\cosh 4t$

7. $\sin 2t\cosh 2t$ 8. t^2e^{-4t} 9. $e^{-2t}\cos^2 t$

10. (a) $F(t)=\begin{cases} e^t, & 0<t<1 \\ 0, & t>1 \end{cases}$ (b) $F(t)=\begin{cases} 2t, & 0<t<3 \\ 1, & t>3 \end{cases}$ 11. $F(t)=\begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$

12. $F(t)=|t-1|+|t+1|, \ t\ge 0$

[NOTE: $L[F(t)]=\int_0^1 -(t-1)e^{-st}dt+\int_1^\infty (t-1)e^{-st}dt+\int_0^\infty (t+1)e^{-st}dt$

13. (a) $F(t)=\begin{cases} \sin(t-\pi/3), & t>\pi/3 \\ 0, & t<\pi/3 \end{cases}$ (b) $F(t)=\begin{cases} (t-2), & t>2 \\ 0, & t<2 \end{cases}$

14. If $L(\cos^2 t) = \dfrac{s^2+2}{s(s^2+4)}$, find $L(\cos^2 4t)$

15. $\int_0^t e^{-t}\cos t\,dt$ 16. $t\sin^2 t$ 17. $t^2\cos at$ 18. $te^{-2t}\sin 2t$

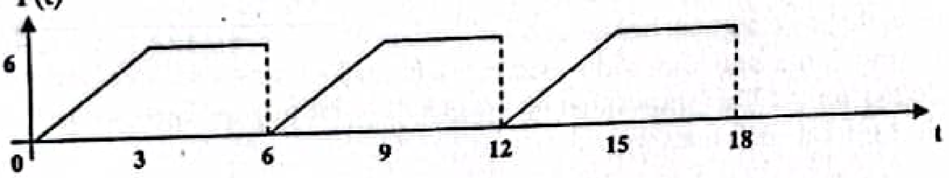
19. $(e^{-4t}-e^{-6t})/t$ 20. $(\cos 2t - \cos 3t)/t$ 21. $(1-\cos 2t)/t$ 22. $(1-\cos t)/t^2$

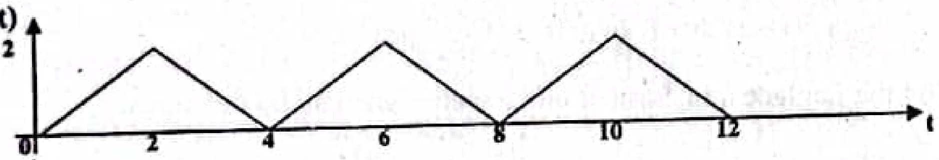
Using an appropriate property of Laplace transform, evaluate the following:

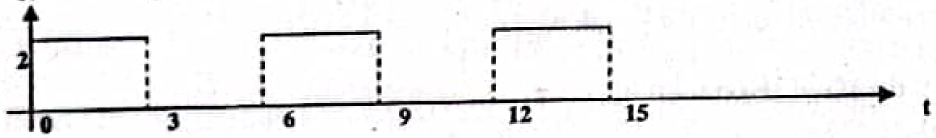
23. $\int_0^\infty te^{-3t}\sin t\,dt$ 24. $\int_0^\infty \dfrac{e^{-t}-e^{-3t}}{t}dt$ 25. $L\left(\int_0^t \dfrac{\sin u}{u}du\right)$ 26. $L\left(\int_0^t e^{-t}\cos t\,dt\right)$

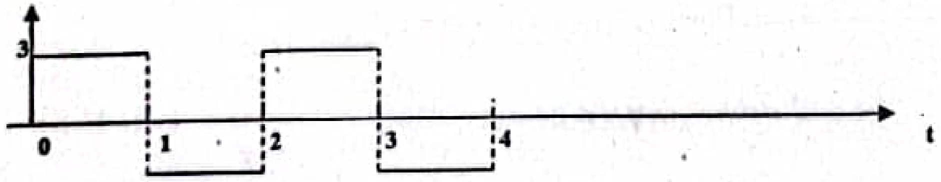
Find the Laplace transform of the following periodic functions:

27. $F(t)=\begin{cases}1, & \text{when } 0<t<2\\ -1, & \text{when } 2<t<4\end{cases}$

28.


29.


30.


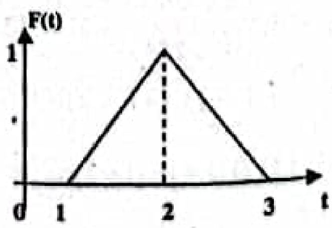
31.


32. Show that $\int_0^\infty e^{-\sqrt{2}t}\dfrac{\sinh t\sin t}{t}dt = \pi/8$

33. Find the Laplace transforms of (i) $\int_0^t e^{-t}\cos t\,dt$ (ii) $\int_0^t e^{t}\dfrac{\sin t}{t}dt$

34. Find the Laplace transforms of the following functions:
(i) $(t-1)^2 U(t-1)$ (ii) $\sin t\, U(t-\pi)$ (iii) $e^{-3t}U(t-2)$

35. Express the function shown in the figure below in terms of unit step function and then find its Laplace transform.



36. Evaluate $\int_0^\infty e^{-4t}\delta(t-3)\,dt$

37. Find the Laplace transforms of function $t^3\delta(t-5)$

CHAPTER FIVE

INVERSE LAPLACE TRANSFORMS

5.1 INTRODUCTION

In chapter one we discussed the Laplace transforms of elementary function and some of important properties used to solve certain improper integrals which could not be evaluated otherwise by well known methods of integration. In this chapter we shall study the inverse Laplace transform and some of its important properties as well. These properties will also help to solve some of typical integrals.

Definition: If the Laplace transform of function F(t) is f(s) that is; $L\{F(t)\} = f(s)$ then F(t) is called an inverse Laplace transform of f(s). Symbolically we may express it as: $L^{-1}\{f(s)\} = F(t)$ where L^{-1} denotes the inverse Laplace transformation operator.

Inverse Laplace Transforms of Some Elementary Functions

The following table presents a list of inverse Laplace transforms of some elementary functions.

S. No.	F(t)	L F(t) = f(s)	$L^{-1}f(s) = F(t)$
1	t	$\frac{1}{s}$	$L^{-1}\left\{\frac{1}{s}\right\}=t$
2.	t^n n being a positive integer	$\frac{n!}{s^{n+1}}$	$L^{-1}\left\{\frac{1}{s^{n+1}}\right\}=\frac{t^n}{n!}$
3.	t^n n being a real number	$\frac{\Gamma(n+1)}{s^{n+1}}$	$L^{-1}\left\{\frac{1}{s^{n+1}}\right\}=\frac{t^n}{\Gamma(n+1)}$
4.	e^{at}	$1/(s-a), s>a$	$L^{-1}\{1/(s-a)\}=e^{at}$
5.	sin at	$a/(s^2+a^2)$	$L^{-1}\{1/(s^2+a^2)\}=\frac{\sin at}{a}$
6.	cos at	$s/(s^2+a^2)$	$L^{-1}\{s/(s^2+a^2)\}=\cos at.$
7.	sinh at	$a/(s^2-a^2)$	$L^{-1}\{1/(s^2-a^2)\}=\frac{\sinh at}{a}$
8.	cosh at	$s/(s^2-a^2)$	$L^{-1}\{s/(s^2-a^2)\}=\cosh at$

5.2 PROPERTIES OF INVERSE LAPLACE TRANSFORMS

This section will introduce you some of the important properties of "Inverse Laplace Transforms" with proofs.

Linearity Property

If $c_1, c_2, \ldots c_n$ are any constants while $f_1(s), f_2(s), \ldots f_n(s)$ are the inverse Laplace transforms of $F_1(t), F_2(t), \ldots F_n(t)$ respectively, then

$L^{-1}\{c_1f_1(s)+c_2f_2(s)+\ldots+c_nf_n(s)\}=c_1L^{-1}\{f_1(s)\}+c_2L^{-1}\{f_2(s)\}+\ldots+c_nL^{-1}\{f_n(s)\}$ Proo

f: According to linearity property of Laplace transform

$L\{c_1F_1(t)+c_2F_2(t)+\ldots+c_nF_n(t)\}=c_1L\{F_1(t)\}+c_2L\{F_2(t)\}+\ldots+c_nL\{F_n(t)\}$

$$= c_1 f_1(s) + c_2 f_2(s) + \ldots + c_n f_n(s)$$

$$\Rightarrow L^{-1}\{c_1 f_1(s) + c_2 f_2(s) + \ldots + c_n f_n(s)\} = c_1 F_1(t) + c_2 F_2(t) + \ldots + c_n F_n(t)$$

$$= c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\} + \ldots + c_n L^{-1}\{f_n(s)\}$$

EXAMPLE 01: Find $L^{-1}\left\{\left(\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right)\right\}$

Solution: By using linearity property of inverse Laplace transform, we have

$$L^{-1}\left\{\left(\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right)\right\} = 4L^{-1}\left\{\frac{1}{s-2}\right\} - 3L^{-1}\left\{\frac{s}{s^2+16}\right\} + 5L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 4L^{-1}\left\{\frac{1}{s-2}\right\} - 3L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + \frac{5}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

EXAMPLE 02: Find $L^{-1}\left\{\left(\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right)\right\}$

Solution: Using the linearity property of inverse Laplace transform, we have

$$L^{-1}\left\{\left(\frac{5s}{s^3} + \frac{4}{s^3} - \frac{2s}{s^2+9} + \frac{18}{s^2+9} + \frac{24}{s^4} - \frac{30\sqrt{s}}{s^4}\right)\right\}$$

$$= L^{-1}\left\{\left(\frac{5}{s^2} + \frac{4}{s^3} - \frac{2s}{s^2+9} + \frac{18}{s^2+9} + \frac{24}{s^4} - \frac{30}{s^{\frac{7}{2}}}\right)\right\}.$$

$$= 5L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\} - 2L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + 6L^{-1}\left\{\frac{3}{s^2+3^2}\right\} + 24L^{-1}\left\{\frac{1}{s^4}\right\} - 30L^{-1}\left\{\frac{1}{s^{7/2}}\right\}$$

$$= 5t + 4\frac{t^2}{2!} - 2\cos 3t + 6\sin 3t + 4t^3 - 30\frac{t^{5/2}}{\Gamma(7/2)}$$

$$= 5t + 2t^2 - 2\cos 3t + 6\sin 3t + 4t^3 - \frac{16}{\sqrt{\pi}}t^{5/2}$$

NOTE: $\Gamma(7/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2) = \frac{15}{8}\sqrt{\pi}$

EXAMPLE 03: Find $L^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\}$

Solution: Using the linearity property of inverse Laplace transforms, we have

$$L^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\} = L^{-1}\left\{\frac{s-2\sqrt{s}+1}{s^2}\right\} = L^{-1}\left\{\frac{1}{s}\right\} - 2L^{-1}\left\{\frac{1}{s^{3/2}}\right\} + L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \left(1 - 2\frac{t^{\frac{1}{2}}}{\Gamma(3/2)} + t\right) = \left(1 + t - \frac{4\sqrt{t}}{\sqrt{\pi}}\right)$$

First Shifting Property

If $L^{-1}f(s)=F(t)$, then $L^{-1}f(s-a)=e^{at}F(t)$

Proof: According to the first shifting property of Laplace transformation, we know that if

$$L\{F(t)\}=f(s),\text{ then } L\{e^{at}F(t)\}=f(s-a)$$

Now taking inverse Laplace transform on each side, we have

$$L^{-1}\left[L\{e^{at}F(t)\}\right]=L^{-1}\{f(s-a)\}\Rightarrow L^{-1}\{f(s-a)\}=e^{at}F(t)$$

REMARK: The following Inverse Laplace transform results will frequently be used in our problems. These results are obtained by using shifting properties of ILT.

$$L\{e^{at}\sin bt\}=\frac{b}{(s-a)^2+b^2}\Rightarrow L^{-1}\frac{b}{(s-a)^2+b^2}=e^{at}\sin bt$$

$$L\{e^{at}\cos bt\}=\frac{s}{(s-a)^2+b^2}\Rightarrow L^{-1}\frac{s}{(s-a)^2+b^2}=e^{at}\cos bt$$

$$L\{e^{at}\sinh bt\}=\frac{b}{(s-a)^2-b^2}\Rightarrow L^{-1}\frac{b}{(s-a)^2-b^2}=e^{at}\sinh bt$$

$$L\{e^{at}\cosh bt\}=\frac{s}{(s-a)^2-b^2}\Rightarrow L^{-1}\frac{s}{(s-a)^2-b^2}=e^{at}\cosh bt$$

EXAMPLE 04: Find $L^{-1}\left\{\frac{8s-6}{s^2-4s+20}\right\}$

Solution: $L^{-1}\left\{\frac{8s-6}{s^2-4s+20}\right\}=L^{-1}\left\{\frac{8s-6}{s^2-4s+20+4-4}\right\}=L^{-1}\left\{\frac{8s-6}{(s-2)^2+16}\right\}$

$$=L^{-1}\left\{\frac{8s-16+10}{(s-2)^2+16}\right\}=L^{-1}\left\{\frac{8(s-2)+10}{(s-2)^2+16}\right\}$$

Now using the linearity property, we have

$$L^{-1}\left\{\frac{8s-6}{s^2-4s+20}\right\}=8L^{-1}\left\{\frac{(s-2)}{(s-2)^2+4^2}\right\}+\frac{10}{4}L^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

Now using the first shifting property of inverse Laplace transform, we have

$$L^{-1}\left\{\frac{8s-6}{s^2-4s+20}\right\}=8e^{2t}\cos 4t+\frac{5}{2}e^{2t}\sin 4t$$

REMARK: Above method of finding ILT is known as **"Completing the Squares Method"**.

EXAMPLE 05: Find $L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$

Solution: $L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}=L^{-1}\left\{\frac{1}{\sqrt{2(s+3/2)}}\right\}=\frac{1}{\sqrt{2}}L^{-1}\left\{\frac{1}{\sqrt{s+3/2}}\right\}=\frac{1}{\sqrt{2}}L^{-1}\left\{\frac{1}{(s+3/2)^{\frac{1}{2}}}\right\}$

Using the shifting property of inverse Laplace transforms, we have

$$L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}=\frac{1}{\sqrt{2}}e^{\frac{-3t}{2}}\frac{t^{-1/2}}{\Gamma(1/2)}=\frac{e^{-3t/2}}{\sqrt{2\pi t}}\qquad\left(\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\}=\frac{t^n}{\Gamma(n+1)},\Gamma(1/2)=\sqrt{\pi}\right)$$

Second Shifting Property

If $L^{-1}\{f(s)\}=F(t)$, then $L^{-1}\{e^{-as}f(s)\}=\begin{cases}F(t-a) & \text{if } t>a\\ 0 & t<a\end{cases}$

Proof : We know that

$L\{F(t)\}=\int_0^\infty e^{-st}F(t)dt=f(s)$. Multiplying each side by e^{-as}, we obtain

$\int_0^\infty e^{-as}e^{-st}F(t)dt=\int_0^\infty e^{-(t+a)s}F(t)dt=e^{-as}f(s)$. Substituting $u=t+a \Rightarrow dt=du$.

Now if $t\to 0$ then $u\to a$ and if $t\to\infty$ then $u\to\infty$. Thus,

$\int_a^\infty e^{-us}F(u-a)du=e^{-as}f(s)$. Replacing u by t, we obtain

$$\int_a^\infty e^{-st}F(t-a)dt=e^{-as}f(s) \qquad (1)$$

Now, $\int_a^\infty e^{-st}F(t-a)dt=\int_0^a e^{-su}(0)dt+\int_a^\infty e^{-st}F(t-a)dt=L\{F(t-a)\}$ provided $t>a$. (2)

From (1) and (2) we deduce that $L\{F(t-a)\}=e^{-as}f(s)$, provided $t>a$. Thus,

$$L^{-1}\{e^{-as}f(s)\}=\begin{cases}F(t-a) & \text{if } t>a\\ 0 & t<a\end{cases}$$

REMARK: Above function may be expressed in terms of Heaviside unit step function as F(t) U(t-a). It may further be noted that this property is used for finding the inverse Laplace transform of product of two functions where one function must be multiple of e^{-as}.

EXAMPLE 06: Find $L^{-1}\left\{\dfrac{se^{\frac{-4\pi s}{5}}}{s^2+25}\right\}$

Solution: $L^{-1}\left\{\dfrac{se^{\frac{-4\pi s}{5}}}{s^2+25}\right\}=L^{-1}\left\{\dfrac{s}{s^2+25}e^{\frac{-4\pi s}{5}}\right\}$

We see that given function is product of two functions where one of the function is $e^{\frac{-4\pi s}{5}}$.

Now $L^{-1}\left\{\dfrac{s}{s^2+5^2}\right\}=\cos 5t=F(t)$

Thus by second shift property,

$$L^{-1}\{e^{-as}f(s)\}=F(t-a), t>a$$
$$=0, \quad t<a$$

Therefore, $L^{-1}\left\{\dfrac{se^{\frac{-4\pi s}{5}}}{s^2+5^2}\right\}=\begin{cases}\cos 5\left(t-\dfrac{4\pi}{5}\right), & t>4\pi/5\\ 0, & t<4\pi/5\end{cases}$

Thus, $L^{-1}\left\{\dfrac{se^{-4\pi s/5}}{s^2+5^2}\right\}=\cos 5\left(t-\dfrac{4\pi}{5}\right)=\cos(5t-4\pi)$ provided $t>4\pi/5$

$=\cos 5t \quad (\because \cos(\theta-4\pi)=\cos\theta)$

EXAMPLE 07: Find $L^{-1}\left\{\dfrac{e^{-3s}}{(s-2)^4}\right\}$

Solution: $L^{-1}\left\{\dfrac{e^{-3s}}{(s-2)^4}\right\}=L^{-1}\left\{\dfrac{1}{(s-2)^4}e^{-3s}\right\}$

We know that $L^{-1}\left\{\dfrac{1}{s^4}\right\}=\dfrac{t^3}{3!}=\dfrac{t^3}{6}$. Therefore, using first shifting property of inverse Laplace, we have $L^{-1}\left\{\dfrac{1}{(s-2)^4}\right\}=\dfrac{1}{6}e^{2t}t^3$

Now using the 2nd shifting property of Laplace transform, we have

$$L^{-1}\left\{\frac{e^{-3s}}{(s-2)^4}\right\}=\frac{1}{6}(t-3)^3e^{2(t-3)},\quad t>3$$
$$=0,\qquad t<3$$

Change of Scale Property

If $L^{-1}\{f(s)\}=F(t)$, then $L^{-1}\{f(ks)\}=\dfrac{1}{k}F\left(\dfrac{t}{k}\right)$

Proof: We know that $L\{F(t)\}=\int_0^\infty e^{-st}F(t)\,dt=f(s)$.

Therefore, $$f(ks)=\int_0^\infty e^{-skt}F(t)\,dt=\int_0^\infty e^{-(kt)s}F(t)\,dt \qquad (1)$$

Let $u=kt \Rightarrow t=u/k \Rightarrow dt=du/k$ (2)

Now from (2) we see that if $t\to 0$ then $u\to 0$ and if $t\to\infty \Rightarrow u\to\infty$.

Thus equation (1) becomes,

$$f(ks)=\int_0^\infty e^{-su}F\left(\frac{u}{k}\right)\frac{du}{k}=\frac{1}{k}\int_0^\infty e^{-su}F\left(\frac{u}{k}\right)du=\frac{1}{k}L\left\{F\left(\frac{t}{k}\right)\right\}$$ (Replacing u by t)

$$\Rightarrow \qquad L^{-1}\{f(ks)\}=\frac{1}{k}F\left(\frac{t}{k}\right)$$

EXAMPLE 08: Find $L^{-1}\left\{\dfrac{s}{4s^2+16}\right\}$

Solution: $L^{-1}\left\{\dfrac{s}{4s^2+16}\right\}=L^{-1}\left\{\dfrac{s}{(2s)^2+4^2}\right\}=\dfrac{1}{2}L^{-1}\left\{\dfrac{2s}{(2s)^2+4^2}\right\}$

We know that $L^{-1}\left\{\dfrac{s}{s^2+16}\right\}=\cos 4t=F(t)$.

Using change of scale property of ILT, we obtain

$$\frac{1}{2}L^{-1}\left\{\frac{2s}{(2s)^2+4^2}\right\}=\frac{1}{2}\cos\left(\frac{4t}{2}\right).\text{ Thus, } L^{-1}\left\{\frac{s}{4s^2+16}\right\}=\frac{1}{2}\cos 2t$$

Inverse Laplace Transform of Derivatives

If $L^{-1}\{f(s)\}=F(t)$, then $L^{-1}\{f^{(n)}(s)\}=L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=(-1)^n t^n F(t)$

Proof: We know that by derivatives property of Laplace transforms that if $L\{F(t)\}=f(s)$, then $L\{t^nF(t)\}=(-1)^n\frac{d^n}{ds^n}f(s)$

Taking inverse Laplace transform on both sides, we have

$$L^{-1}L\{t^nF(t)\}=L^{-1}\left\{(-1)^n\frac{d^n}{ds^n}f(s)\right\}$$

$$\Rightarrow\{t^nF(t)\}=L^{-1}\left\{(-1)^n\frac{d^n}{ds^n}f(s)\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=L^{-1}\{f^{(n)}(s)\}=(-1)^n t^nF(t)$$

EXAMPLE 09: Find $L^{-1}\left\{s/(s^2+1)^2\right\}$

Solution: Consider, $\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=\frac{-2s}{(s^2+1)^2}\quad\Rightarrow\frac{s}{(s^2+1)^2}=\frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

Taking inverse Laplace transform on both sides, and using the inverse Laplace transform property of derivatives, we get

$$L^{-1}\left\{s/(s^2+1)^2\right\}=\frac{-1}{2}L^{-1}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=\frac{-1}{2}(-1)t\sin t\qquad\left(\because L^{-1}\left(\frac{1}{s^2+1}\right)=\sin t\right)$$

Thus, $$L^{-1}\left\{s/(s^2+1)^2\right\}=\frac{1}{2}t\sin t$$

EXAMPLE 10: Find $L^{-1}\left\{\ln\left(\frac{s+2}{s+1}\right)\right\}$

Solution: Let $f(s)=\ln\left(\frac{s+2}{s+1}\right)=\ln(s+2)-\ln(s+1)$

Differentiate w.r.t s, we get: $f'(s)=\frac{1}{s+2}-\frac{1}{s+1}$

Now taking inverse Laplace transform on both sides, we obtain

$$L^{-1}\{f'(s)\}=L^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+1}\right\}$$

Using the derivative property, that is, $L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=(-1)^n t^nF(t)$, we get

$$(-1)tL^{-1}\{f(s)\}=L^{-1}\left(\frac{1}{s+2}\right)-L^{-1}\left(\frac{1}{s+1}\right)\ \rightarrow\ -tF(t)=e^{-2t}-e^{-t}$$

$\Rightarrow \quad F(t)=-\left(\frac{e^{-2t}-e^{-t}}{t}\right)=L^{-1}f(s) \qquad \Rightarrow \quad L^{-1}\left\{\ln\left(\frac{s+2}{s+1}\right)\right\}=\left(\frac{e^{-t}-e^{-2t}}{t}\right)$

EXAMPLE 11: Find $L^{-1}\left\{\ln\left(\frac{s^2+1}{(s-1)^2}\right)\right\}$

Solution: Let $f(s)=\ln\frac{s^2+1}{(s-1)^2}$

We know that $L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=L^{-1}\left\{f^{(n)}(s)\right\}=(-1)^n t^n F(t)$. Substituting $n=1$, we get,

$L^{-1}\{f'(s)\}=-tF(t) \Rightarrow F(t)=-\frac{1}{t}L^{-1}f'(s)=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\left(\ln\frac{s^2+1}{(s-1)^2}\right)\right\}$

$=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\left(\ln(s^2+1)-2\ln\ln(s-1)\right)\right\} \quad (\because \ln a^b=b\ln a)$

$=\frac{-1}{t}L^{-1}\left\{\frac{2s}{s^2+1}-\frac{2}{s-1}\right\} \quad =\frac{-1}{t}\left[2L^{-1}\left\{\frac{s}{s^2+1}\right\}-2L^{-1}\left\{\frac{1}{s-1}\right\}\right] \quad =\frac{-1}{t}\left[2\cos t-2e^t\right]$

Hence, $L^{-1}\left\{\ln\left(\frac{s^2+1}{(s-1)^2}\right)\right\}=\frac{-2\cos t}{t}+\frac{2e^t}{t}$

EXAMPLE 12: Evaluate $L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}$

Solution: Let $f(s)=\tan^{-1}\left(\frac{a}{s}\right)$

We know that $L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=L^{-1}\left\{f^{(n)}(s)\right\}=(-1)^n t^n F(t)$. Substituting $n=1$, we get

$L^{-1}\{f'(s)\}=-tF(t) \Rightarrow F(t)=-\frac{1}{t}L^{-1}f'(s)=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\left(\tan^{-1}\frac{a}{s}\right)\right\}$

Thus $F(t)=\frac{-1}{t}L^{-1}\left\{\frac{1}{1+\frac{a^2}{s^2}}\cdot\frac{-a}{s^2}\right\}=\frac{-1}{t}L^{-1}\left\{\frac{s^2}{s^2+a^2}\cdot\frac{-a}{s^2}\right\}$

$=\frac{-1}{t}L^{-1}\left\{\frac{-a}{s^2+a^2}\right\}=\frac{1}{t}L^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\frac{1}{t}(\sin at)$

$F(t)=L^{-1}\{f(s)\}=\frac{\sin at}{t} \quad \Rightarrow \quad L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}=\frac{\sin at}{t}$

Division by s^n Property

If $L^{-1}\{f(s)\}=F(t)$, then $L^{-1}\left\{\frac{f(s)}{s}\right\}=\int_0^t F(u)\,du$

Proof: By Laplace transform of integrals property, we know that

$L\left\{\int_0^t F(u)du\right\}=\frac{f(s)}{s}$. Taking Inverse Laplace Transform on each side, we get

$$L^{-1}\left\{\frac{f(s)}{s}\right\}=\int_0^t F(u)du.$$

EXAMPLE: 13 **Find** $L^{-1}\left\{\frac{1}{s^2(s+2)}\right\}$

Solution: We know that $L\left\{\frac{1}{s+2}\right\}=e^{2t}$

Now using division by s property, we get

$$L^{-1}\left\{\frac{1}{s(s+2)}\right\}=L^{-1}\left\{\frac{1/(s+2)}{s}\right\}=\int_0^t e^{-2u}du$$

$$=\left[\frac{e^{-2u}}{-2}\right]_0^t=\frac{-1}{2}\left(e^{-2t}-e^0\right)=\frac{-1}{2}\left(e^{-2t}-1\right)=\frac{(1-e^{-2t})}{2}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^2(s+2)}\right\}=\frac{1}{2}\int_0^t\left(1-e^{-2u}\right)du=\frac{1}{2}\left[u-\frac{e^{-2u}}{-2}\right]_0^t=\frac{1}{2}\left[u+\frac{e^{-2u}}{2}\right]_0^t$$

$$=\frac{1}{2}\left(t+\frac{e^{-2t}}{2}-\left(0+\frac{e^0}{2}\right)\right)=\frac{1}{2}\left(t+\frac{e^{-2t}}{2}-\frac{1}{2}\right)$$

$$=\frac{1}{2}\left(\frac{e^{-2t}+2t-1}{2}\right)=\left(\frac{e^{-2t}+2t-1}{4}\right)$$

Thus, $L^{-1}\left\{\frac{1}{s^2(s+2)}\right\}=\left(\frac{e^{-2t}+2t-1}{4}\right)$

EXAMPLE 14: Find $L^{-1}\left\{\frac{1}{s}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

Solution: Let us first evaluate $L^{-1}\left\{\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

Let, $f(s)=\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)=\ln\left(s^2+a^2\right)-\ln\left(s^2+b^2\right)$

Differentiate both sides w.r.t s, we have

$$f'(s)=\frac{2s}{s^2+a^2}-\frac{2s}{s^2+b^2}$$

Taking inverse Laplace transform on both sides

$$L^{-1}\{f'(s)\}=L^{-1}\left\{\frac{2s}{s^2+a^2}\right\}-L^{-1}\left\{\frac{2s}{s^2+b^2}\right\}$$

Now by inverse Laplace transform of derivative property, we have

$$(-1)tL^{-1}\{f(s)\}=2L^{-1}\left\{\frac{s}{s^2+a^2}\right\}-2L^{-1}\left\{\frac{s}{s^2+b^2}\right\}$$

Therefore, $$L^{-1}\{f(s)\} = 2\left(\frac{\cos bt - \cos at}{t}\right)$$

Now by division by s property i-e $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)du$

$$\therefore L^{-1}\left\{\frac{1}{s}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = 2\int_0^t\left(\frac{\cos bu - \cos au}{u}\right)du$$

Inverse Laplace Transform by Partial Fractions

Inverse Laplace transform of a function of form f(s) = P(s)/Q(s), where P(s) and Q(s) are polynomials such that degree of P(s) is lower than the degree of Q(s) may easily be evaluated by using the techniques of "Partial Fractions". There are three cases of partial fractions.

CASE I: When Q(s) contains linear and non-repeated factors.

CASE II: When Q(s) contains linear and repeated factors.

CASE III: When Q(s) contains non-factorizable quadratic factors.

The following examples will help you to understand the method of finding inverse Laplace transforms by this method.

EXAMPLE 15: Use partial fraction method to find $L^{-1}\left\{\left(\frac{3s+16}{(s-3)(s+2)}\right)\right\}$

Solution: Here we see that Q(s) contains the factors that are linear and non-repeated. Hence we are having Case I.. Consider

$$\frac{3s+16}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} \qquad (1)$$

$$\Rightarrow \quad 3s+16 = A(s+2) + B(s-3) \qquad (2)$$

To find A and B, we proceed as under:

Put s - 3 = 0 or s = 3 in (2) we get

$$3(3)+16 = A(3+2) + B(3-3) \qquad \Rightarrow A = 5$$

Now put s + 2 = 0 or s = -2 in (2) we get

$$3(-2)+16 = A(-2+2) + B(-2-3) \qquad \Rightarrow B = -2$$

Thus equation (1) becomes: $$\frac{3s+16}{(s-3)(s+2)} = \frac{5}{s-3} + \frac{-2}{s+2} \qquad (3)$$

Now taking the inverse Laplace transform on both sides, we get

$$L^{-1}\left\{\frac{3s+16}{(s-3)(s+2)}\right\} = 5L^{-1}\left\{\frac{1}{s-3}\right\} - 2L^{-1}\left\{\frac{1}{s+2}\right\} = 5e^{3t} - 2e^{-2t}$$

2nd Method: To resolve $\left\{\frac{3s+16}{(s-3)(s+2)}\right\}$ into partial fractions, we observe that in the denominator there exists two factors (s – 3) and (s + 2). Now first we put s = 3 in the above expression neglecting the factor (s – 3). Then put s = 2 in the given expression neglecting the factor (s + 2). It may be noted that this method is only applicable when all factors of denominator are linear.

Thus, $$\left\{\frac{3s+16}{(s-3)(s+2)}\right\} = \frac{3(3)+16}{(s-3)(3+2)} + \frac{3(-2)+16}{(-2-3)(s+2)} = \frac{5}{(s-3)} - \frac{2}{(s+2)}$$

This is same as equation (3). Taking the inverse Laplace transform on both sides, we get:

$$L^{-1}\left\{\frac{3s+16}{(s-3)(s+2)}\right\}=5L^{-1}\left\{\frac{1}{s-3}\right\}-2L^{-1}\left\{\frac{1}{s+2}\right\}=5e^{3t}-2e^{-2t}$$

EXAMPLE 16: Find $L^{-1}\left\{\frac{3s+16}{(s^2-s-6)}\right\}$

Solution: We see that $\frac{3s+16}{(s^2-s-6)}=\frac{3s+16}{(s-3)(s+2)}$. This function is same as in EXAMPLE 15.

Thus

$$L^{-1}\left\{\frac{3s+16}{(s^2-s-6)}\right\}=L^{-1}\left\{\frac{3s+16}{(s-3)(s+2)}\right\}=5e^{3t}-2e^{-2t}$$

EXAMPLE 17: Evaluate $L^{-1}\left\{\left(\frac{4s+5}{(s-1)^2(s+2)}\right)\right\}$

Solution: Since Q(s) contains linear factors but one of these factors is repeated, hence this is an example of partial fractions Case II. The method of resolving into partial fractions is shown as under. Consider,

$$\frac{4s+5}{(s-1)^2(s+2)}=\frac{A}{(s-1)^2}+\frac{B}{(s-1)}+\frac{C}{(s+2)} \quad (1)$$

$$\Rightarrow \quad 4s+5=A(s+2)+B(s-1)(s+2)+C(s-1)^2 \quad (2)$$

Now put $s-1=0 \Rightarrow s=1$ in (2), we get: $9=3A \Rightarrow A=3$.

Now put $s+2=0 \Rightarrow s=-2$ in (2), we get: $-3=9C \Rightarrow C=-1/3$.

To find B, we re-write equation (2) to get:

$$4s+5=A(s+2)+B(s^2+s-2)+C(s^2-2s+1)$$

Now comparing the coefficients of:

s^2: $0=B+C \Rightarrow B=-C=-(-1/3)=1/3$

Thus equation (1) becomes: $\frac{4s+5}{(s-1)^2(s+2)}=\frac{3}{(s-1)^2}+\frac{1/3}{(s-1)}+\frac{-1/3}{(s+2)}$.

Taking the inverse Laplace transform on each side to get

$$L^{-1}\left\{\left(\frac{4s+5}{(s-1)^2(s+2)}\right)\right\}=3L^{-1}\frac{1}{(s-1)^2}+\frac{1}{3}L^{-1}\frac{1}{(s-1)}-\frac{1}{3}L^{-1}\frac{1}{(s+2)}=3te^t+\frac{1}{3}e^t-\frac{1}{3}e^{-2t}$$

EXAMPLE 18: Find $L^{-1}\left\{\frac{1}{(s-1)(s^2+4)}\right\}$

Solution: Here Q(s) contains a non-factorizable quadratic factor hence this is an exampleof partial fractions case III. Thus

$$\frac{1}{(s-1)(s^2+4)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+4} \quad (1)$$

$$\Rightarrow 1=A(s^2+4)+(s-1)(Bs+C) \quad (2)$$

To find A, put $s=1$ in (2), we get

$1=A(1^2+4)+(1-1)(B.1+C) \quad \Rightarrow A=1/5$

To compute B and C, we rewrite equation (2) in simplified form as

$1=As^2+4A+Bs^2-Bs+Cs-C$

Now comparing the coefficients of same powers of

s^2: $A+B=0 \rightarrow B=-A \rightarrow B=-1/5.$

s: $C-B=0 \rightarrow C=B \rightarrow C=-1/5$

Thus equation (1) becomes: $\dfrac{1}{(s-1)(s^2+4)}=\dfrac{1}{5(s-1)}-\dfrac{s+1}{5(s^2+4)}$

Taking inverse Laplace transform on both sides, we have

$$L^{-1}\left\{\frac{1}{(s-1)(s^2+4)}\right\}=\frac{1}{5}L^{-1}\left\{\frac{1}{(s-1)}\right\}-\frac{1}{5}L^{-1}\left\{\frac{s+1}{(s^2+4)}\right\}$$

$$=\frac{1}{5}\left\{\frac{1}{(s-1)}\right\}-\frac{1}{5}L^{-1}\left\{\frac{s}{s^2+4}\right\}-\frac{1}{5}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=\frac{1}{5}\left\{\frac{1}{(s-1)}\right\}-\frac{1}{5}L^{-1}\left\{\frac{s}{s^2+2^2}\right\}-\frac{1}{10}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$=\frac{1}{5}e^t-\frac{1}{5}\cos 2t-\frac{1}{10}\sin 2t$$

EXAMPLE 19: Evaluate $L^{-1}\left\{\dfrac{s^3+16s-24}{(s^4+20s^2+64)}\right\}$

Solution: We see that: $s^4+20s^2+64=(s^2+16)(s^2+4)$. Hence

$L^{-1}\left\{\dfrac{s^3+16s-24}{(s^4+20s^2+64)}\right\}=L^{-1}\left\{\dfrac{s^3+16s-24}{(s^2+16)(s^2+4)}\right\}$. Now consider,

$$\frac{s^3+16s-24}{(s^2+16)(s^2+4)}=\frac{As+B}{(s^2+16)}+\frac{Cs+D}{(s^2+4)} \qquad (1)$$

$\rightarrow s^3+16s-24=(As+B)(s^2+4)+(Cs+D)(s^2+4)$

Simplifying we get

$s^3+16s-24 = As^3+4As+Bs^2+4B+Cs^3+16Cs+Ds^2+16D$

Or $\quad s^3+16s-24=(A+C)s^3+(B+D)s^2+(4A+16C)s+(4B+16D)$

Now comparing the coefficients of same powers of s, we have

$A+C=1$ (i) $\quad B+D=0$ (ii) $\quad A+4C=4$ (iii) and $B+4D=-6$ (iv)

Solving equations (i) through (iv) simultaneously, we obtain:

$A=0, B=2, C=1$ and $D=-2$. Substituting these values in equation (1) and taking the inverse Laplace transform, we get

$$L^{-1}\left\{\frac{s^3+16s-24}{(s^4+20s^2+64)}\right\} = 2L^{-1}\left\{\frac{1}{s^2+16}\right\}+L^{-1}\left\{\frac{s}{(s^2+4)}\right\}-2L^{-1}\left\{\frac{1}{(s^2+4)}\right\}$$

$$= 2\sin 4t+\cos 2t-\sin 2t$$

Inverse Laplace Transform by Convolution Method

Here we present another method of finding inverse Laplace transforms. Although this method involves simple integration of well known functions like:
$e^{at}\cos(b+ct)$, $e^{at}\sin(b+ct)$, $\sin at.\sin bt$, $\sin at\cos bt$ or $\cos at\cos bt$ it is limited to certain functions. The method is described as under:

Let $L^{-1}\{f(s)\}=F(t)$ and $L^{-1}\{g(s)\}=G(t)$, then

$$L^{-1}\{f(s)g(s)\}=\int_0^t F(u)G(t-u)du$$

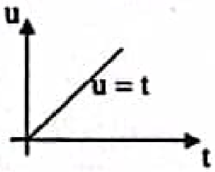
This property is known as convolution property and is very much useful in finding the Inverse Laplace Transform of product of two functions whose inverse Laplace transforms are known.

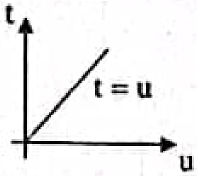
Proof: Since $L^{-1}\{f(s)g(s)\}=\int_0^t F(u)G(t-u)du$

$$\Rightarrow \quad \{f(s)g(s)\}=L\int_0^t F(u)G(t-u)du \qquad (1)$$

Now, $$L\int_0^t F(u)G(t-u)du=\int_0^\infty\int_0^t e^{-st}F(u)\,G(t-u)\,du\,dt \qquad (2)$$

Now to evaluate (2) we have to change its order. This is done as under:
Limits of u are from 0 to t, that is $u=0$ and $u=t$. $u=t$ is a straight line in ut-plane where t is dependent variable. This is shown in fig. 1. Now if $u=t$ then $t=u$. Here t depends on u and the equation $t=u$ is a straight line in tu-plane. See fig2.





Thus if we change the order of integration the limits of u will be from 0 to ∞ and limits of t will be from u to ∞. Thus above integration becomes:

$$\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st}F(u)\,G(t-u)\,du\,dt=\int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st}F(u)\,G(t-u)\,dt\,du$$

$$=\int_{u=0}^{\infty} F(u)\,du\int_{t=u}^{\infty} e^{-st}\,G(t-u)\,dt$$

Substituting $t-u=z \Rightarrow dt=dz$ (u being constant for the second integral). Also the limits of z will become 0 to ∞. Thus,

$$\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st}F(u)\,G(t-u)\,du\,dt=\int_{u=0}^{\infty} F(u)\,du\int_{z=0}^{\infty} e^{-s(z+u)}\,G(z)\,dz$$

$$=\int_{u=0}^{\infty} e^{-su}F(u)\,du\int_{z=0}^{\infty} e^{-sz}\,G(z)\,dz=L[F(u)].L[G(z)]=f(s).g(s)$$

Thus from (2), we have: $L\int_0^t F(u)G(t-u)du=\int_0^\infty\int_0^t e^{-st}F(u)\,G(t-u)\,du\,dt=f(s).g(s)$

Taking inverse Laplace transform on each side, we get

$$L^{-1}[f(s)g(s)]=\int_0^t F(u)\,G(t-u)\,du$$

EXAMPLE 20: Find $L^{-1}\left\{\dfrac{1}{(s+3)(s-1)}\right\}$ using convolution property.

Solution: Let f (s) = 1/(s + 3) and g(s) = 1/(s − 1)

➔ $L^{-1}\{f(s)\}=L^{-1}\left(\dfrac{1}{s+3}\right)=e^{3t}=F(t)$ and $L^{-1}\{g(s)\}=L^{-1}\left(\dfrac{1}{s-1}\right)=e^{t}=G(t)$

Now by using convolution property, we obtain

$$L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}=\int_0^t e^{-3u}e^{t-u}\,du=\int_0^t e^{-3u}e^{t}e^{-u}\,du=e^{t}\int_0^t e^{-4u}\,du=e^{t}\left[\frac{e^{-4u}}{-4}\right]_0^t$$

$$=-\frac{e^{t}}{4}\left(e^{-4t}-e^{0}\right)=-\frac{e^{t}}{4}\left(e^{-4t}-1\right)$$

$$\therefore L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}=-\frac{1}{4}\left(e^{-3t}-e^{t}\right)=\frac{e^{t}-e^{-3t}}{4}$$

EXAMPLE 21: Find $L^{-1}\left\{\dfrac{1}{(s+2)^2(s-2)}\right\}$

Solution: Let us assume that $f(s)=\dfrac{1}{s-2}$ and $g(s)=\dfrac{1}{(s+2)^2}$

➔ $L^{-1}\{f(s)\}=L^{-1}\left\{\dfrac{1}{s-2}\right\}=e^{2t}=F(t)$ and $L^{-1}\{g(s)\}=L^{-1}\left\{\dfrac{1}{(s+2)^2}\right\}=te^{-2t}$

$\left(\because L^{-1}\dfrac{1}{s^2}=t \Rightarrow L^{-1}\left\{\dfrac{1}{(s+2)^2}\right\}=te^{-2t}\text{ (by first shifting property)}\right)$

Now by using convolution property, we have

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}=\int_0^t e^{2u}(t-u)e^{-2(t-u)}du=\int_0^t e^{2u}e^{-2t}e^{2u}(t-u)du=e^{-2t}\int_0^t e^{4u}(t-u)du$$

Integration by parts, we get

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}=e^{-2t}\left[\frac{(t-u)e^{4u}}{4}+\frac{4e^{u}}{16}\right]_0^t$$

$$=e^{-2t}\left[0+\frac{e^{4t}}{16}+\frac{4e^{u}}{16}-\left(\frac{t}{4}+\frac{1}{16}\right)\right]=e^{-2t}\left[\frac{e^{4t}}{16}-\frac{t}{4}-\frac{1}{16}\right]$$

$$=\frac{1}{16}e^{-2t}\left[e^{4t}-4t-1\right]=\frac{1}{16}\left[e^{2t}-4te^{-2t}-e^{-2t}\right]$$

EXAMPLE 22: Using Convolution theorem verify that

$$\int_0^t \sin u\cos(t-u)du=\frac{1}{2}t\sin t$$

Proof: By convolution theorem, $L^{-1}\{f(s)g(s)\}=\int_0^t F(u)G(t-u)du$ (1)

Now given integral is: $\int_0^t \sin u\cos(t-u)du$ (2)

Comparing the two integrals, we see that: $F(u) = \sin u$, $G(t - u) = \cos(t - u)$

$\Rightarrow F(t)=\sin t$ and $G(t)=\cos t$

Now $L^{-1}\{f(s)\}=F(t)$ and $L^{-1}\{g(s)\}=G(t)$

$\Rightarrow f(s)=L\{F(t)\}$ and $g(s)=L\{G(t)\}$

$\Rightarrow L\{\sin t\}=\dfrac{1}{s^2+1}=f(s)$, $L\{\cos t\}=\dfrac{s}{s^2+1}=g(s)$

$$\text{Now } L^{-1}\{f(s)g(s)\}=L^{-1}\left\{\left(\frac{1}{s^2+1}\right)\left(\frac{s}{s^2+1}\right)\right\}=L^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\}$$

$$=\frac{-1}{2}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} \qquad \left(\because \frac{d}{ds}\left(\frac{-1}{2\left(s^2+1\right)}=\frac{s}{\left(s^2+1\right)^2}\right)\right)$$

Now using inverse Laplace transform property of derivatives, we get

$$L^{-1}\{f(s)g(s)\}=\frac{-1}{2}\left(-t\,L^{-1}\left(\frac{1}{s^2+1}\right)\right)=\frac{t}{2}\sin t$$

Thus,
$$\int_0^t \sin u\cos(t-u)\,du=\frac{t}{2}\sin t$$

EXAMPLE 23: Evaluate $L^{-1}\left\{\dfrac{1}{(s+3)\left(s^2+4\right)}\right\}$

Solution: Let $f(s)=\dfrac{1}{(s+3)}$ and $g(s)=\dfrac{1}{\left(s^2+4\right)}\Rightarrow F(t)=e^{-3t}$, $G(t)=\sin 2t/2$

$$\therefore L^{-1}\left\{\frac{1}{(s+3)\left(s^2+4\right)}\right\}=\frac{1}{2}\int_0^t e^{-3u}\sin(2t-2u)\,du$$

Using the formula: $\int e^{ax}\sin(bx+c)dx=\dfrac{e^{ax}}{\left(a^2+b^2\right)}[a\sin(bx+c)-b\cos(bx+c)]$

REMARK: Integration is w.r.t u thus taking a = -3, b = -2 and using the above formula, we get:

$$L^{-1}\left\{\frac{1}{(s+3)\left(s^2+4\right)}\right\}=\frac{1}{2}\left[\frac{e^{-3u}}{(9+4)}\{-3\sin(2t-2u)+2\cos(2t-2u)\}\right]_0^t$$

$$=\frac{1}{2.13}\left[e^{-3t}\{-3\sin(2t-2t)+2\cos(2t-2t)\}-e^0\{-3\sin(2t-0)+2\cos(2t-0)\}\right]$$

$$=\frac{1}{26}\left[2e^{-3t}+3\sin 2t-2\cos 2t\right]$$ [Note: sin 0 = 0, cos 0 = 1, $e^0 = 1$]

EXAMPLE 24: Find $L^{-1}\left\{\dfrac{s^2}{\left(s^2+9\right)\left(s^2+1\right)}\right\}$

Solution: Let $f(s)=\dfrac{s}{\left(s^2+9\right)}$, $g(s)=\dfrac{s}{\left(s^2+4\right)}\Rightarrow F(t)=\cos 3t$, $G(t)=\cos 2t$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}=\int_0^t \cos 3u\ \cos(2t-2u)\,du=\frac{1}{2}\int_0^t 2\cos 3u\ \cos(2t-2u)\,du$$

Using the formula: $2\cos a\ \cos b=\cos(a+b)+\cos(a-b)$

$$\text{we get: } L^{-1}\left\{\frac{1}{(s^2+4)(s^2+9)}\right\}=\frac{1}{2}\int_0^t\left[\cos(u+2t)+\cos(5u-2t)\right]du$$

$$=\frac{1}{2}\left[\sin(u+2t)+\sin(5u-2t)/5\right]_0^t$$

$$=\frac{1}{2}\left[\sin 3t-\sin 2t+\sin 3t/5-\sin(-2t)/5\right]$$

$$=\frac{1}{10}\left[6\sin 3t-4\sin 2t\right]=\frac{1}{5}\left[3\sin 3t-2\sin 2t\right]$$

Inverse Laplace Transform by Heaviside Expansion Formula

Let P(s) and Q(s) be polynomials, where P(s) has degree less than that of Q(s) and suppose that Q(s) is a polynomial with n distinct zeros $\alpha_1,\alpha_2,\alpha_3,\ldots$ then

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right]=\frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t}+\frac{P(\alpha_2)}{Q'(\alpha_2)}e^{\alpha_2 t}+\frac{P(\alpha_3)}{Q'(\alpha_3)}e^{\alpha_3 t}+\ldots\frac{P(\alpha_n)}{Q'(\alpha_n)}e^{\alpha_n t}$$

$$=\sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$$

This formula is known as Heaviside Expansion Formula.

EXAMPLE 25: Use Heaviside expansion to find $L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}$

Solution: Here P(s) = 1 and Q(s) = (s + 3) (s - 1). Since Q(s) has two zeros, that is, $\alpha_1=-3$ and $\alpha_2=1$, hence $P(\alpha_1)=P(-3)=1$ and $P(\alpha_2)=P(1)=1$.

Also $Q(s)=(s+3)(s-1)=s^2+2s+2 \rightarrow Q'(s)=2s+2$

Hence, $Q'(\alpha_1)=Q'(-3)=2(-3)+2=-4$ and $Q'(\alpha_2)=Q'(1)=2(1)+2=4$.

Now using Heaviside expansion formula, we get

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right]=L^{-1}\left[\frac{1}{(s+3)(s-1)}\right]=\frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t}+\frac{P(\alpha_2)}{Q'(\alpha_2)}e^{\alpha_2 t}$$

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right]=L^{-1}\left[\frac{1}{(s+3)(s-1)}\right]=-\frac{1}{4}e^{-3t}+\frac{1}{4}e^{t}$$

EXAMPLE 26: Use Heaviside expansion to find $L^{-1}\left[\frac{s+5}{(s+1)(s^2+1)}\right]$

Solution: Here $P(s)=s+5$ and $Q(s)=(s+1)(s^2+1)=s^3+s^2+s+1$ and

$Q'(s)=3s^2+2s+1$

Since $s^2+1=(s+i)(s-i)$, hence zeros of Q(s) are $\alpha_1=-1$, $\alpha_2=-i$ and $\alpha_3=i$

Now using Heaviside's expansion formula

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right]=\frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t}+\frac{P(\alpha_2)}{Q'(\alpha_2)}e^{\alpha_2 t}+\frac{P(\alpha_3)}{Q'(\alpha_3)}e^{\alpha_3 t}=\frac{P(-1)}{Q'(-1)}e^{-1t}+\frac{P(-i)}{Q'(-i)}e^{-it}+\frac{P(i)}{Q'(i)}e^{it}$$

$$=\frac{4}{2}e^{-t}+\frac{5-i}{-2-2i}e^{-it}+\frac{5+i}{-2+2i}e^{it}$$

$$=\frac{4}{2}e^{-t}+\frac{5-i}{-2-2i}\cdot\frac{-2+2i}{-2+2i}e^{-it}+\frac{5+i}{-2+2i}\cdot\frac{-2-2i}{-2-2i}e^{it} \text{ (by rationalization)}$$

$$=2e^{-t}+\frac{12i-8}{8}.e^{-it}+\frac{-8-12i}{8}.e^{it}=2e^{-t}+\left(-1+\frac{3i}{2}\right)e^{-it}+\left(-1-\frac{3i}{2}\right)e^{it}$$

$$=2e^{-t}+\left(-1+\frac{3i}{2}\right)(\cos t-i\sin t)+\left(-1-\frac{3i}{2}\right)(\cos t+i\sin t)$$

$$\therefore L^{-1}\left[\frac{s+5}{(s+1)(s^2+1)}\right] \qquad \left(\text{NOTE: } e^{i\theta}=\cos\theta+i\sin\theta \text{ is called Eulers formula}\right)$$

$$=\left(2e^{-t}-\cos t+\frac{3}{2}i\cos t+i\sin t+\frac{3}{2}\sin t-\cos t-\frac{3}{2}i\cos t-i\sin t+\frac{3}{2}\sin t\right)$$

$$=\left(2e^{-t}-2\cos t+\frac{6\sin t}{2}\right)=\left(2e^{-t}-2\cos t+3\sin t\right)$$

REMARK: Heaviside expansion formula is preferable when denominator possesses linear factors, otherwise calculation is little difficult.

Evaluation of Improper Integrals

Laplace transform technique is often useful in evaluating some definite integrals. The method is procedure is illustrated in the following examples.

Example 27: Evaluate $\int_0^\infty \frac{\sin x}{x}dx$

Solution: Consider, $G(t)=\int_0^\infty \frac{\sin tx}{x}dx$.

Taking Laplace transform we get: $L[G(t)]=L\left[\int_0^\infty \frac{\sin tx}{x}dx\right]=\int_0^\infty e^{-st}\left[\int_0^\infty \frac{\sin tx}{x}dx\right]dt$.

Changing the order of integration, we get

$$L[G(t)]=\int_0^\infty \frac{1}{x}dx\left[\int_0^\infty e^{-st}\sin tx\, dt\right]=\int_0^\infty \frac{1}{x}dx\,[L(\sin tx)]=\int_0^\infty \frac{1}{x}\frac{x}{s^2+x^2}dx$$

$$L[G(t)]=\int_0^\infty \frac{1}{s^2+x^2}dx=\frac{1}{s}\tan^{-1}\left[\frac{x}{s}\right]_0^\infty=\frac{1}{s}\left(\tan^{-1}(\infty)-\tan^{-1}(0)\right)=\frac{\pi}{2s}$$

Taking the inverse Laplace transform on both sides, we obtain

$[G(t)]=L^{-1}\left(\frac{\pi}{2s}\right)=\frac{\pi}{2}L^{-1}\left(\frac{1}{s}\right)=\frac{\pi}{2}(1)=\frac{\pi}{2}$. Now,

$G(t)=\int_0^\infty \frac{\sin tx}{x}dx=\frac{\pi}{2}$. Substituting $t=1$, we get $\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$.

Example 28: Evaluate $\int_0^\infty \cos x^2\, dx$

Solution: Consider, $G(t)=\int_0^\infty \cos tx^2\, dx$.

Taking Laplace transform we get:

$$L[G(t)]=L\left[\int_0^\infty \cos tx^2\,dx\right]=\int_0^\infty e^{-st}\left[\int_0^\infty \cos tx^2\,dx\right]dt$$

Changing the order of integration, we get

$$L[G(t)]=\int_0^\infty dx\left[\int_0^\infty e^{-st}\cos tx^2\,dt\right]=\int_0^\infty dx\left[L\left(\cos tx^2\right)\right]=\int_0^\infty \frac{s}{s^2+x^4}dx=s\int_0^\infty \frac{1}{x^4+s^2}dx$$

Substituting $x^2=s\tan\theta \Rightarrow 2x\,dx=s\sec^2\theta\,d\theta \Rightarrow dx=s\sec^2\theta\,d\theta/2x$.
Also if $x=0$ then $\theta=0$ and if $x=\infty$ then $x=\pi/2$. Thus

$$L[G(t)]=s\int_0^{\pi/2}\frac{s.\sec^2\theta}{s^2\tan^2\theta+s^2}\frac{d\theta}{2x}=\int_0^{\pi/2}\frac{\sec^2\theta}{\tan^2\theta+1}.\frac{d\theta}{2\sqrt{s}\tan\theta}=\int_0^{\pi/2}\frac{\sec^2\theta}{\sec^2\theta}.\frac{d\theta}{2\sqrt{s}\tan\theta}$$

$$=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\frac{1}{\sqrt{\tan\theta}}d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\cos^{1/2}\theta.\sin^{-1/2}\theta d\theta \qquad (1)$$

Notice that **Beta function** is define as: $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta.\cos^{2n-1}\theta d\theta$.

Also relation between **Beta** and **Gamma Functions** is:
Comparing with (1), we see that $2m-1=-1/2 \Rightarrow m=1/4$ and
$2n-1=1/2 \Rightarrow n=3/4$

Thus equation (1) becomes:

$$L[G(t)]=\frac{1}{2\sqrt{s}}\frac{\beta\left(\frac{1}{4},\frac{3}{4}\right)}{2}=\frac{1}{4\sqrt{s}}\frac{\Gamma(1/4).\Gamma(3/4)}{\Gamma(1/4+3/4)}=\frac{1}{4\sqrt{s}}\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)}$$

Now $\Gamma(1)=1$ and $\Gamma(1/4)\Gamma(3/4)=\sqrt{2}\pi$.

NOTE: This identity can be found in advance calculus book.

Thus, $L[G(t)]=\frac{\pi\sqrt{2}}{4\sqrt{s}}$

Taking the inverse Laplace transform on both sides, we obtain

$$[G(t)]=L^{-1}\frac{\pi\sqrt{2}}{4\sqrt{s}}=\frac{\sqrt{2}\pi}{4}L^{-1}\left(\frac{1}{\sqrt{s}}\right)=\frac{\sqrt{2}\pi}{4}\frac{t^{-1/2}}{\Gamma(1/2)}=\frac{\sqrt{2}\pi}{4\sqrt{\pi t}}.\text{ Thus,}$$

$G(t)=\int_0^\infty \cos^2 tx\,dx=\frac{\sqrt{2}\pi}{4\sqrt{\pi t}}$. Substituting $t=1$, we get $\int_0^\infty \cos x^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}$

WORKSHEET C5

1. Use linear property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{\frac{6}{2s-3}-\frac{3+4s}{9s^2-16}+\frac{8-6s}{16s^2+9}\right\}$ (ii) $L^{-1}\left\{\frac{3\left(s^2-1\right)^2}{2s^5}+-\frac{4s-18}{9-s^2}+\frac{(s+1)\left(2-\sqrt{s}\right)}{s^{5/2}}\right\}$

(iii) $L^{-1}\left\{\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right\}$ (iv) $L^{-1}\left\{\frac{s+1}{s^{4/3}}\right\}$ (v) $L^{-1}\left\{\frac{5s+10}{9s^2-16}\right\}$

2. Use first shifting property of inverse Laplace evaluate the following:

(i) $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$ (ii) $L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$ (iii) $L^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$ (iv) $L^{-1}\left\{\frac{s}{(s+1)^{5/2}}\right\}$

(v) $L^{-1}\left\{\frac{5s-12}{3s^2+4s+8}\right\}$ (vi) $L^{-1}\left\{\frac{1}{\sqrt[3]{8s-27}}\right\}$ (vii) $L^{-1}\left\{\frac{3s+1}{s^2-6s+18}\right\}$

3. Use second shifting property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{\frac{e^{-3s}}{s^2-2s+5}\right\}$ (ii) $L^{-1}\left\{\frac{se^{-2s}}{s^2+3s+2}\right\}$ (iii) $L^{-1}\left\{\frac{e^{-\pi s/3}}{s^2+1}\right\}$ (iv) $L^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$

4. Use change of scale property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{\frac{s}{4s^2+16}\right\}$ (ii)If $L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\}=\cos\frac{2\sqrt{t}}{\sqrt{\pi t}}$ find $L^{-1}\left\{\frac{e^{-2/s}}{s^{1/2}}\right\}$

5. Using derivative property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{s/\left(s^2+1\right)^2\right\}$ (ii) $L^{-1}\left\{s/\left(s^2-1\right)^2\right\}$ (iii) $L^{-1}\left\{\ln\left(\frac{s+2}{s+1}\right)\right\}$

6. If $F(t)=L^{-1}\{f(s)\}$ prove that:

(a) $L^{-1}\{sf'(s)\}=-tF'(t)-F(t)$ (b) $L^{-1}\{sf''(s)\}=t^2F'(t)+2tF(t)$

(c) $L^{-1}\{s^2f''(s)\}=t^2F''(t)+4tF'(t)+2F(t)$

7. Use division by s property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{\frac{1}{s\left(s^2+4\right)}\right\}$ (ii) $L^{-1}\left\{\frac{1}{s^3\left(s^2+1\right)}\right\}$ (iii) $L^{-1}\left\{\frac{1}{s(s+1)^3}\right\}$ (iv) $L^{-1}\left\{\frac{1}{s}\ln\left(1+\frac{1}{s^2}\right)\right\}$

(v) $L^{-1}\left\{\ln\left(\frac{s+2}{s+1}\right)\right\}$

8. Use partial fraction method to evaluate the following:

(i) $L^{-1}\left\{\frac{3s+7}{\left(s^2-2s-3\right)}\right\}$ (ii) $L^{-1}\left\{\frac{2s^2-4}{(s-2)(s-3)(s+1)}\right\}$ (iii) $L^{-1}\left\{\frac{2s-1}{\left(s^3-s\right)}\right\}$

(iv) $L^{-1}\left\{\frac{s^2-2s+3}{(s-1)^2(s+1)}\right\}$ (v) $L^{-1}\left\{\frac{s^3+16s-24}{s^4+20s^2+64}\right\}$ (vi) $L^{-1}\left\{\frac{2s^3+10s^2+8s+40}{s^2\left(s^2+9\right)}\right\}$

(vii) $L^{-1}\left\{\frac{2s-3}{\left(2s^3+3s^2-2s-3\right)}\right\}$ (viii) $L^{-1}\left\{\frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}\right\}$

9. Use convolution theorem evaluate the following:

(i) $L^{-1}\left\{1/(s+2)^2(s-2)\right\}$ (ii) $L^{-1}\left\{1/(s+1)\left(s^2+1\right)\right\}$ (iii) $L^{-1}\left\{s^2/\left(s^2+4\right)^2\right\}$

(iv) $L^{-1}\left\{1/s^2(s+1)^2\right\}$ (v) $L^{-1}\left\{s/\left(s^2+a^2\right)^2\right\}$

10. Using Heaviside Expansion evaluate the following:

(i) $L^{-1}\{(19s+37)/(s-2)(s+1)(s+3)\}$ (ii) $L^{-1}\left\{(s+5)/(s+1)\left(s^2+1\right)\right\}$

(iii) $L^{-1}\left\{(3s+16)/\left(s^2-s-6\right)\right\}$ (iv) $L^{-1}\left\{(2s-1)/\left(s^3-s\right)\right\}$ (v) $L^{-1}\left\{(s+1)/\left(6s^2+7s+2\right)\right\}$

11. Show that (i) $\int_0^\infty \sin x^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}$ (ii) $\int_0^\infty \frac{x\sin tx}{1+x^2}\,dx=\frac{\pi}{2}e^{-t}$ $t>0$