Hall's Marriage Theorem

Matching:- A matching is a set of edges that don't share any common vertices. This means that no two edges in the set are "touching" each other.

Types of Matching

① Maximum Matching ② Perfect Matching
③ Maximal Matching

Ex- Think of a group of People at a Party. A matching is like set of handshakes where no Person is shaking two hands at the same time.

1. A "Maximum Matching" is the largest number of handshakes you can have at once.

2. A "Perfect matching" is when everyone at the Party is shaking hands with exactly one other Person.

3. A "maximal matching" is a situation where you can't add any more hanshakes without someone needing to drop a hand they're already holding.

Theorem:-

Hall's Marriage Theorem also known as the Hall's Conditions is a fundamental result in Combinatoric that Provides a necessary and sufficient condition for the existance of a Perfect matching in a bipartite graph.

* In simple terms, it tells us when it's Possible to Pair up all the elements of one set with distinct elements from another set.

The Formal Condition

In a bipartite graph $G = (U \cup V), E)$, where U and V are disjoint sets of vertices and E is the set of edges, there exists a matching that covers every vertex in U if and only if for every subset $A \subseteq U$, the size of its neighborhood is at least the size of the set itself.

Mathematically, this is expressed as:

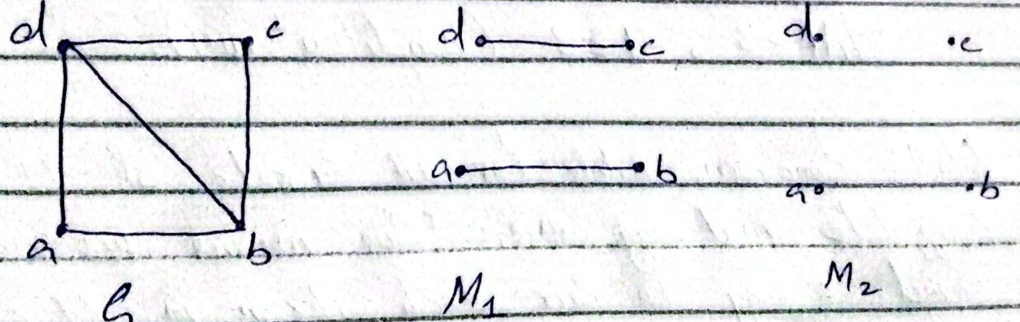
$$|N(A)| \geq |A|$$

* Where N(A) is the neighborhood of A, which is the set of all vertices in V that are connected to at least one vertex in A.

Problem

Subgraph is a matching?

Subgraph is a matching if and only if it graph have all vertices of orignal graph and each vertex has atleast one degree. Then this subgraph is matching of orignal graph.

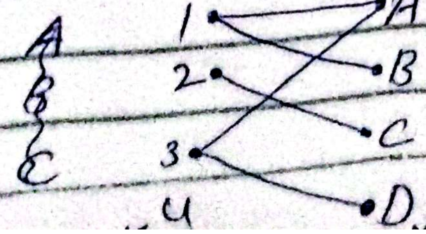


* M_1 and M_2 matching subgraph of G.
* M_2 is a maximum matching and matching number is equal to edges "2".

Applying the Theorem

Bipartite graph with two sets U and V.

U = {1, 2, 3} and V = {A, B, C, D}



Bipartite graph.

Check Hall's Condition:

We need to check the condition $|N(A)| \geq |A|$ for every possible subset A of U.

① Subset of size 1:-

vertex 1 : $N(A) = \{A, B\}$. $|N(A)| = 2$, $|A| = 1$, $2 \geq 1$ (condition met)

vertex 2 : $N(A) = \{C\}$. $|N(A)| = 1$, $|A| = 1$, $1 \geq 1$ (condition met)

vertex 3 : $N(A) = \{A, D\}$. $|N(A)| = 2$, $|A| = 1$, $2 \geq 1$ (condition met)

② Subset of size 2:-

vertex $\{1, 2\}$: $N(A) = \{A, B, C\}$, $N(A) = 3$, $|A| = 2$, $3 \geq 2$ (✓)

vertex $\{1, 3\}$: $N(A) = \{A, B, D\}$, $N(A) = 3$, $|A| = 2$, $3 \geq 2$ (✓)

* vertex $\{2, 3\}$: $N(A) = \{A, C, D\}$, $N(A) = 3$. $|A| = 2$, $3 \geq 2$ (✓)

③ Subset of size 3:-

vertex = $\{1, 2, 3\}$, $N(A) = \{A, B, C, D\}$. $N(A) = 4$

↳ $|A| = 3$, $4 \geq 3$ (condition met).

* Hall's condition holds for all subsets of U, a complete matching exists: This means we can find a way to pair each person in U with a unique person in V.

Ex:- $(1, B)$, $(2, C)$, $(3, A)$.

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