1. Substitution Method

divide & C

$$T(n) \rightarrow T(n/2),\ T(n/2)$$
$$T(n/2) \rightarrow T(n/4),\ T(n/4) \qquad T(n/2) \rightarrow T(n/4),\ T(n/4)$$
$$\vdots \qquad \cdots$$

$$\left.\begin{array}{c} T(n) \\ \downarrow \\ T(n/2) \\ \downarrow \\ T(n/4) \end{array}\right\} \rightarrow T(n/2) + \underset{\text{constant time complexity}}{c}$$

Ex: $T(n/2)+c$, if $n>1$
1, if $n=1$

$T(n) = T(n/2) + c$ —(1)
$T(n/2) = T(n/4) + c$ —(2)
$T(n/4) = T(n/8) + c$ —(3)

Substitute eq (2) into eq (1)

$T(n) = T(n/2) + c$
$T(n) = [T(n/4) + c] + c$
$T(n) = T(n/4) + 2c$

Substitute eq (3) into

$T(n) = [T(n/8) + c] + 2c$
$T(n) = T(n/8) + 3c$
$T(n) = T(n/2^3) + 3c$
$= T(n/2^k) + kc$

$\boxed{\text{let } 2^k = n}$

$T(n) = T(n/2^k) + kc,$
$2^k = n$
$T(n) = T(n/n) + kc$
$T(n) = T(1) + kc$
$T(n) = 1 + kc$
$T(n) = kc,\ k = ?$
$2^k = n \rightarrow \log_2 2^k = \log_2 n$
$k = \log n$

$\boxed{T(n) = k = \log n}$

↳ $T(n) = k \cdot c$
$= c\log(n)$

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2. Master Theorem

$T(n) = aT(n/b) + f(n)$
$a \geq 1,\ b > 1$

Sol: $T(n) = n^{\log_b a} \cdot U(n)$

where $U(n)$ depends on $h(n)$

$h(n) = \frac{f(n)}{n^{\log_b a}}$

if $h(n)$	$U(n)$
$n^r, r>0$	$O(n^r)$
$n^r, r<0$	$O(1)$
$(\log n)^i, i\geq 0$	$\frac{(\log n)^{i+1}}{i+1}$

$T(n) = T(n/2) + c$

$\overset{1}{a} \geq 1$ ✓ $\overset{2}{b} > 1$ ✓ $f(n) = c$

$T(n) = n^{\log_b a} \cdot U(n)$
$= n^{\log_2 1} \cdot U(n)$
$= n^0 \cdot U(n)$

$h(n) = \frac{f(n)}{n^{\log_b a}} = \frac{c}{1}$

$T(n) = n^0 \cdot U(n)$
$= 1 \cdot U(n)$

$h(n) = c = (\log n)^0 \cdot c$
$= \frac{1 \cdot (\log n)^{0+1}}{0+1} \cdot c$

$\boxed{T(n) = (\log n)c}$

$T(n) = 8T(n/2) + n^2$
$a = 8,\ b = 2,\ f(n) = n^2$

$T(n) = n^{\log_b a} \cdot U(n)$
$= n^{\log_2 8} \cdot U(n)$
$= n^{\log_2 2^3} U(n)$
$= n^{3\log_2 2} \cdot U(n)$
$= n^3 \cdot U(n)$

$h(n) = \frac{f(n)}{n^{\log_b a}} = \frac{n^2}{n^3} = n^{-1}$

$h(n) = n^{-1},\ n^r, r<0 \rightarrow O(1) = 1$

$T(n) = n^3 \cdot U(n) = n^3 O(1) = n^3$

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3. Recurrence Tree

$T(n) = 2T(n/2) + Cn$

Sum

n
$n/2 \quad n/2$: cn
$n/4 \quad n/4 \quad n/4 \quad n/4$: cn

Divide by 2

Level 1 16
Level 2 8
Level 3 4
Level 4 2

→ $\log_2 16 = \log_2 2^4 = 4$ Levels

$T(n) = \underbrace{2T(n/2)}_{} + cn$

$\log n \cdot cn$

$O(n\log n) \cdot c$

$\boxed{O(n\log n)}$