

# Tensor decomposition and entanglement entropy

Author: [Seung-Sup Lee](#)

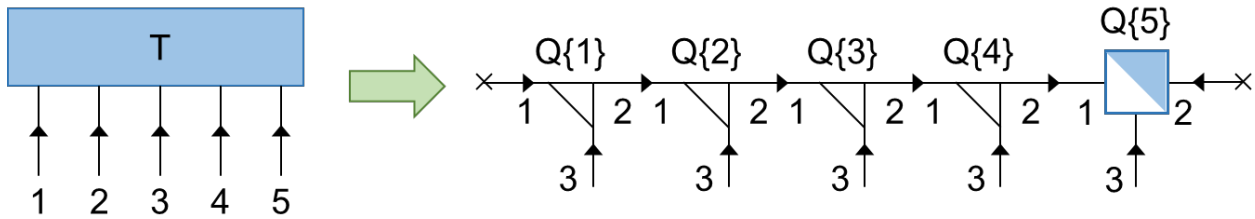
In this tutorial, we will decompose a high-rank tensor into a matrix product state (MPS) that consists of rank-2 and -3 tensors, by using the QR decomposition or the singular value decomposition (SVD).

Let's define a rank-5 tensor acting onto five subsystems, whose dimensions are 2, 3, 2, 3, and 4, respectively.

```
clear
sz = [2 3 2 3 4]; % local space dimensions
T = reshape((1:prod(sz)),sz); % rank-5 tensor
T = T/norm(T(:)); % normalize
```

Here  $T$  is normalized by its norm. Note that, for the computation of the norm, it's necessary to take a linearized form of the tensor by using  $(:)$ .

I will demonstrate the decomposition of the tensor "from left to right", i.e., decomposing a tensor for the first leg, then for the second, and so on, by using the QR decomposition:



The numbers next to the tensor legs indicate the order of the legs. A left-unitary matrix obtained after the  $n$ -th QR decomposition are reshaped and stored in the  $n$ -th cell  $Q\{n\}$ . At the last iteration (i.e., the fourth iteration), we dump the remaining tensor into  $Q\{5\}$ . For  $Q\{1\}$  and  $Q\{5\}$ , we assign the dummy legs of dimension 1, indicated by the  $x$  symbols at the ends. By introducing the dummy legs, we can treat all the tensors  $Q\{n\}$  as rank-3, which simplifies the code writing.

```
Q = cell(1,numel(sz));
R = T; % temporary tensor to be QR-decomposed
szl = 1; % the bond dimension of the left leg of Q{n} to be obtained after
% the QR decomposition at iteration n; for n = 1, szl = 1 for the dummy leg
for it = (1:(numel(sz)-1))
    R = reshape(R,[szl*sz(it), prod(sz(it+1:end))]);
    [Q{it},R] = qr(R,0);
    Q{it} = reshape(Q{it},[szl, sz(it), numel(Q{it})/szl/sz(it)]);
    Q{it} = permute(Q{it},[1 3 2]); % permute to the left-right-bottom order
    szl = size(Q{it},2); % update the bond dimension
    R = reshape(R,[szl,sz(it+1:end)]);
end
Q{end} = R;
```

Note we use the thin QR decomposition by setting the second input argument to `qr` as  $\emptyset$ .

Check the dimensions of tensors in the MPS.

Q

Q = 1x5 cell

|   | 1            | 2            | 3             | 4             | 5          |
|---|--------------|--------------|---------------|---------------|------------|
| 1 | 1x2x2 double | 2x6x3 double | 6x12x2 double | 12x4x3 double | 4x4 double |

Note that the MATLAB automatically truncates the trailing singleton dimension, so the dummy leg dimension of  $Q\{5\}$  is not displayed.

### Exercise (a): Check the integrity of the tensor decomposition

Contract the tensors  $Q\{1\}, \dots, Q\{5\}$  to make a rank-5 tensor again. Check whether the contraction result is the same as the original tensor T.

### Exercise (b): Entanglement entropies for different bipartitions

Compute the entanglement entropy  $S_{A/B}$  for the following three ways of bipartitioning the tensor T's five legs into two sets, A and B:

- (i)  $A = \{1, 2\}, B = \{3, 4, 5\}$ ;
- (ii)  $A = \{1, 3\}, B = \{2, 4, 5\}$ ;
- (iii)  $A = \{1, 5\}, B = \{2, 3, 4\}$ .

### Exercise (c): Use the SVD for the tensor decomposition and compute the entanglement entropy

Let's consider the same tensor T defined above. Apply the series of the SVD from left to right to decompose T into an MPS, represented by a cell array M.

- At each step  $n$  of the SVD, compute the entanglement entropy  $S_n = -\sum_i s_i^2 \log_2(s_i^2)$  by using the singular values  $\{s_i\}$ . What are the values of  $S_n$  for different iterations?
- After the SVD, truncate the decomposed components associated with the singular values smaller than the double precision accuracy limit eps. How do the size of tensors  $M\{n\}$  differ from the above results  $Q\{n\}$ ?
- Similarly as in Exercise (a), check whether the contraction of  $M\{n\}$ 's reproduce the original tensor T.