Convex Optimization Homework Solutions

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Contents

1 Homework 1

1.1 Exercise 2.1

Showing the definition of convexity for arbitrary k. $\theta_1 x_1 + \cdots + \theta_k x_k \in C$.

$$\theta_1 x_1 + \cdots + \theta_k x_k = \theta_1 x_1 + (1 - \theta_1)(\mu_2 x_2 + \mu_3 x_3 + \cdots + \mu_k x_k) \tag{1}$$

Where each $\mu_n = \frac{\theta_n}{1-\theta_1}$

$$\sum_i \mu_i = \frac{\sum_i \theta_i}{1 - \theta_1} = \frac{1 - \theta_1}{1 - \theta_1} = 1 \tag{2}$$

1.2 Exercise 2.2

The intersection between any two convex sets is convex. A line segment by definition is convex because it contains itself. Therefore, if there is any line segment L that intersects C, for C to be convex, the intersection must also be convex.

Similarly, for an affine set, the definition is that the set must contain hte line through any distinct points in the set. Therefore if there is a line that line segment L that intersects an affine set A, its intersection is also affine.

1.3 Exercise 2.5

Distance between two parallel hyperplanes $\{x \in \mathbb{R}^n | a^T x = b_1\}, \{x \in \mathbb{R}^n | a^T x = b_2\}$:

$$a^T(x_1 + at) = b_2 \tag{3}$$

$$a^T x_1 + a^T at = b_2 \tag{4}$$

$$t = \frac{b_2 - a^T x_1}{a^T a} \tag{5}$$

$$t = \frac{b_2 - b_1}{a^T a} \tag{6}$$

$$\frac{|b_1 - b_2|}{||a||_2} \tag{7}$$

1.4 Exercise 2.7

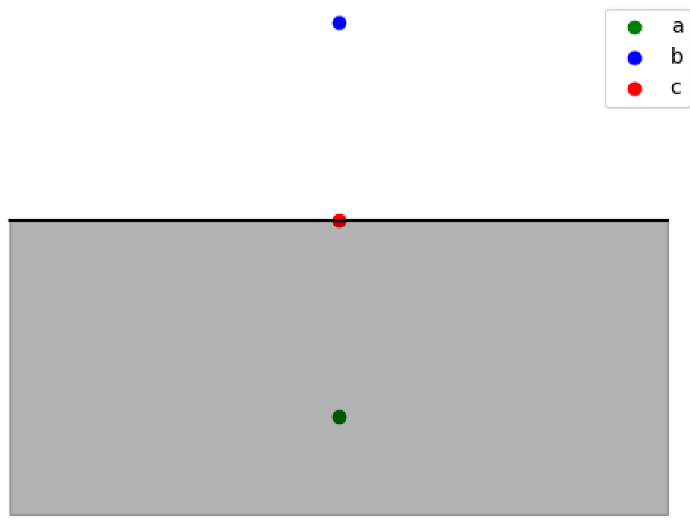
The image below illustrates the intuition behind this in $\mathbb{R}^2$. The set of all points that are closer to a than b $\{x | ||x - a||_2 \leq ||x - b||_2\}$, is a halfspace of the form

$$\{x | c^T x \leq d\} \tag{8}$$

where c is the normal vector that points from a to b, $c = \frac{a-b}{||a-b||_2}$ and $d = (\frac{a-b}{||a-b||_2})^T \frac{a+b}{2}$

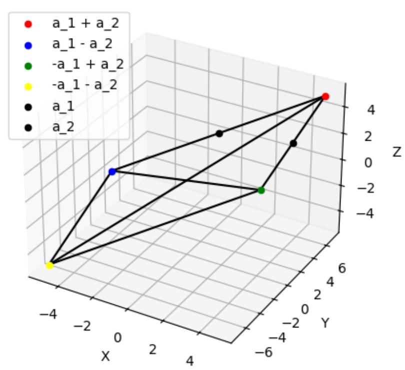
1.5 Exercise 2.8

Finding which of the sets are polyhedra



1.5.1 a

S $= \{y_1a_1 + y_2a_2| -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1\}, a_1, a_2 \in \mathbb{R}^n$ This set is a 2D parallelogram in any $\mathbb{R}^n$. The image shown is one that is generated with $\mathbb{R}^3$ coordinates $a_1 = (1, 2, 3)$ and $a_2 = (4, 5, 2)$. It can be defined as the intersection between four halfspaces with their normals pointing from the origin to and centered on $a_1, a_2, -a_1, -a_2$.



The normal vectors for the halfspaces are:

$$A = \begin{bmatrix} \frac{a_1}{||a_1||_2} \\ \frac{a_2}{||a_2||_2} \\ \frac{-a_1}{||a_1||_2} \\ \frac{-a_2}{||a_2||_2} \end{bmatrix} \tag{9}$$

and the b vector is:

$$b = \begin{bmatrix} ||a_1||_2 \\ ||a_2||_2 \\ ||a_1||_2 \\ ||a_2||_2 \end{bmatrix} \tag{10}$$

Therefore, the set S can be expressed as $S = \{x|Ax \preceq b\}$

1.5.2 b

This is a polyhedra where:

$$A = I^n \tag{11}$$

$$b = 0 \tag{12}$$

$$C = \begin{bmatrix} \mathbf{1} \\ a \\ a^2 \end{bmatrix} \tag{13}$$

$$d = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix} \tag{14}$$

1.5.3 c

Not attempted

1.5.4 d

Not attempted

1.6 Exercise 2.11

Showing that the hyperbolic set $\{x \in \mathbb{R}^2_{++}|x_1x_2 \geq 1\}$ is convex. As a generalization, show that $\{x \in \mathbb{R}^2_+|\Pi_{i=1}^n x_i \geq 1\}$

1.7 Exercise 2.12

Finding which of the following sets are convex:

1.7.1 a. slab

A set of the form $\{x \in \mathbb{R}^n|\alpha \leq a^T x \leq \beta\}$, A slab is the intersection of two halfspaces $\mathbb{C}$ and $\mathbb{D}$ where $\mathbb{C} = \{x|a^T x \leq \beta\}$ and $\mathbb{D} = \{x| - a^T x \leq -\alpha\}$

1.7.2 b. rectangle

A set of the form $\{x \in \mathbb{R}^n|\alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$. This is also the intersection of halfspaces, but this time it is the intersection of four halfspaces (in $\mathbb{R}^2$). But this is a polyhedra.

1.7.3 c. wedge

A set of the form $\{x \in \mathbb{R}^n | a_1^T x \leq b_1, a_2^T x \leq b_2\}$. Intersection of two halfspaces

1.7.4 d. point-set distance

A set of the form $\{x | \|x - x_0\|_2 \leq \|x - y\|_2 \forall y \in S\}$ where $S \subseteq \mathbb{R}^n$. This can be viewed as the intersection of many euclidean balls. The intersection of convex spaces is convex so the here, the set can be turned into $\cap_{y \in S}\{x | \|x - x_0\|_2 \leq \|x - y\|_2\}$

1.7.5 e. set-set distance

A set of the form $\{x | \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\}$ This set is not convex. Consider two crescent-moon shaped sets that lie next to each other so the tip of one moon is in the center of the other moon. The set that is closer to either of them is not convex because the crescent shape would stay in the set and is not convex.

1.7.6 f. affine set combo

A set of the form $\{x | x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ Yes, this is essentially the section of S_2 that can be translated to the convex S_1

1.7.7 g. fraction distance

Not attempted

1.8 Exercise 2.15

Finding which conditions sets are convex that are conditions on the probability simplex $\{p | \mathbf{1}^T p = 1, p \succeq 0\}$:

1.8.1 a. bounded expectation

This is a linear bound, so this set is convex.

1.8.2 b. prob a and b

Not attempted

2 Homework 2

2.1 Exercise 2.28

A matrix is positive semi-definite if we define two matrices symmetric A and Q, such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, Q and A are positive semidefinite if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$.
In other words, for a matrix A to be positive semidefinite, these two conditions must be satisfied:

- A is symmetric
- $x^T Ax \geq 0$ for all x

The second condition can be broadened to multiple equivalent definitions:

- $x^T Ax \geq 0$ for all x
- All eigenvalues are non-negative
- There exists a matrix B s.t. $B^T B = A$
- All principal minors are non-negative

2.1.1 n=1

For $n = 1$, the cone is defined by the inequality $x_1 \geq 0$

2.1.2 n=2

$x_1x_3 - x_2^2 \geq 0$ and $x_1, x_3 \geq 0$

2.1.3 n=3

All principal minors must be non-negative, so blocking off each row and column of the matrix.

Diagonals: $x_1, x_4, x_6 \geq 0$

Full matrix determinant: $x_1(x_4x_6 - x_5^2) - x_2(x_2x_6 - x_3x_5) + x_3(x_2x_5 - x_3x_4) \geq 0$ 2x2 determinants: $x_4x_6 - x_5^2 \geq 0$, $x_1x_6 - x_3^2 \geq 0$ $x_1x_4 - x_2^2 \geq 0$

2.2 Exercise 2.33

2.2.1 part a

A convex cone $K \subseteq \mathbb{R}^n$ is a proper cone if

- K is closed (contains its boundary)
- K is solid (nonempty interior)
- K is pointed (contains no line)

For this problem, we must show that the monotone non-negative cone defined as

$$K_{m+} = \{\mathbf{x} \in \mathbb{R}^n | x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\} \tag{15}$$

is a proper cone.

Proving the first, where K_{m+} is closed and contains its boundary, we first find the boundary of K_{m+}. The boundaries are the two vectors in the direction of

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{16}$$

With magnitude x_1. These boundaries clearly satisfy the $\geq$ requirements of the set. If one of the conditions was a strict equality, then the convex cone would be open.

Proving the second, since all vectors are contained in between the boundaries, the set is solid.

Proving the third, because of the ≥ 0 requirement, the $\mathbf{x} \in \mathbb{R}^n$ of the set also have the $\mathbf{x} \succeq 0$ applicable. Therefore the cone is pointed.

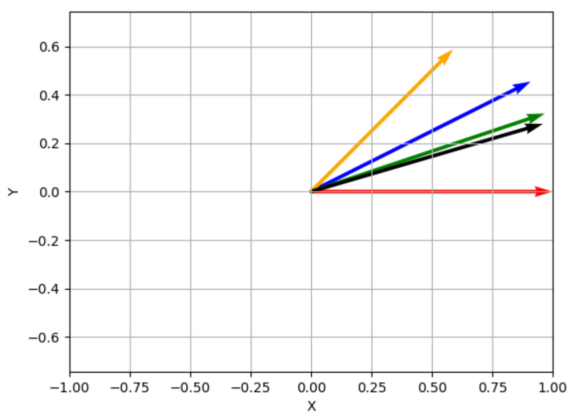


Figure 1: Image of a monotone non-negative cone in $\mathbb{R}^2$

2.2.2 part b

Finding the dual cone K^*_{m+}. A dual cone is defined as

$$K^* = \{y | x^T y \geq 0 \text{ for all } x \in K\} \tag{17}$$

$$x^T y \tag{18}$$

$$= \sum_{i=1}^{n} x_i y_i \tag{19}$$

$$= (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + \cdots + (x_{n-1} - x_n)(y_1 + \cdots + y_n - 1) + x_n(y_1 + \cdots + y_n) \tag{20}$$

Since each term of y is getting multiplied by $x_i - x_{i+1}$, and $x_i \geq x_{i+1}$, each y term is getting multiplied by a non-negative x term, so if we want $x^T y \geq 0$ for all x, then we will need in the worst case scenario, for each one of the y terms to be positive.

$$y_1 \geq 0, y_1 + y_2 \geq 0, y_1 + y_2 + y_3 \geq 0, \ldots, \sum_{i=1}^{n} y_i \geq 0 \tag{21}$$

These conditions are all satisfied when $y_n \geq y_{n-1} \geq \cdots \geq y_2 \geq y_1 \geq 0$. Therefore, the dual cone is

$$K^* = \{y \in \mathbb{R}^n | y_n \geq y_{n-1} \geq \cdots \geq y_2 \geq y_1 \geq 0\} \tag{22}$$

2.3 Exercise 3.2

The first function is potentially quasiconvex because the sublevel sets shown are convex. The second function does not seem to have convex sublevel sets. So it is not quasiconvex or quasiconcave.

2.4 Exercise 3.5

Showing that the running average of a convex function is also convex. Since f is differentiable, we can attempt to differentiate F twice and evaluate it on the domain of F

$$F(x) = \frac{1}{x} \int_0^x f(t)dt \tag{23}$$

$$F'(x) = -\frac{1}{x^2} \int_0^x f(t)dt + \frac{1}{x} \frac{d}{dx} \int_0^x f(t)dt \tag{24}$$

$$= -\frac{1}{x^2} \int_0^x f(t)dt + \frac{1}{x}(f(x) + \int_0^x \frac{d}{dx} f(t)dt) \tag{25}$$

$$F'(x) = -\frac{1}{x^2} \int_0^x f(t)dt + \frac{f(x)}{x} \tag{26}$$

$$F''(x) = \frac{2}{x^3} \int_0^x f(t)dt - \frac{1}{x^2} \frac{d}{dx} \int_0^x f(t)dt + \frac{f'(x)x - f(x)}{x^2} \tag{27}$$

$$F''(x) = \frac{2}{x^3} \int_0^x f(t)dt - \frac{f(x)}{x^2} + \frac{f'(x)x - f(x)}{x^2} \tag{28}$$

$$F''(x) = \frac{2}{x^3} \int_0^x f(t)dt - \frac{2f(x)}{x^2} + \frac{f'(x)x}{x^2} \tag{29}$$

$$F''(x) = \frac{2}{x^3} (\int_0^x f(t)dt - f(x)x + \frac{f'(x)x^2}{2}) \tag{30}$$

$$= \frac{2}{x^3} (\int_0^x f(t) - f(x) - f'(x)(t - x)dt) \tag{31}$$

The term $f(t) - f(x) - f'(x)(t - x)$ is always positive because of the definition below for any convex function f

$$f(t) \geq f(x) + f'(x)(t - x) \tag{32}$$

Therefore, since the domain of F is confined to all positive numbers, the hessian of F is always greater than zero.

2.5 Exercise 3.6

- The epigraph of a function is a halfspace if the function is affine. Since the function is linear and infinite, its epigraph is a halfspace.
- If a function is linear then its epigraph is a convex cone.
- If a function is piecewise affine, then its epigraph is a polyhedron.

2.6 Exercise 3.15

2.6.1 part a

Showing that for $x > 0, u_0(x) = \lim_{\alpha\to 0} u_\alpha(x)$

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha} \tag{33}$$

L'hopital :)

$$\lim \frac{x^\alpha - 1}{\alpha} = \lim x^\alpha \log x \tag{34}$$

This goes to $\log x$ as $\alpha \to 0$

2.6.2 Part b

Firstly, u_α is monotonically increasing because the first derivative is

$$u'_\alpha = x^{\alpha-1} \tag{35}$$

Which is always positive. The function is concave since the hessian is

$$u''_\alpha = (\alpha - 1)x^{\alpha-2} \tag{36}$$

Since $0 < \alpha \leq 1$, the function is affine for $\alpha = 1$ and concave for all other values of α. The equation $u_\alpha(1) = 0$ is also satisfied since $1^\alpha = 1\forall\alpha$

2.7 Exercise 3.16

Finding whether the functions are convex, concave, quasiconcave, or quasiconvex.

2.7.1 Part b

$$f(x_1, x_2) = x_1 x_2 \tag{37}$$

$$\nabla^2 f = \begin{bmatrix} 01 \\ 10 \end{bmatrix} \tag{38}$$

In reducing to row echelon form, rows 1 and 2 would need to exchange and therefore the matrix is not positive semidefinite and not convex. It is not concave since the negative of the matrix is not positive semidefinite. It is quasiconcave but I didn't really know that.

2.7.2 Part c

$$f(x_1, x_2) = \frac{1}{x_1 x_2} \tag{39}$$

$$\nabla^2 f = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix} \tag{40}$$

This function is convex if $\frac{2}{x_1^3 x_2} \geq 0$ and $\frac{3}{x_1^4 x_2^4} \geq 0$. Since $x_1, x_2 \in \mathbb{R}^2_{++}$, the function is convex.

2.7.3 Part d

$$f(x_1, x_2) = \frac{x_1}{x_2} \tag{41}$$

$$\nabla^2 f = \begin{bmatrix} 0 & \frac{-1}{x^2} \\ \frac{-1}{x^2} & \frac{2x_1}{x_2^3} \end{bmatrix} \tag{42}$$

This function is not convex or concave. I still don't really get quasiconvex or quasiconcave tbh.

2.7.4 Part e

$$f(x_1, x_2) = \frac{x_1^2}{x_2} \tag{43}$$

I know this is convex because I watched the lecture but here the hessian

$$\nabla^2 f = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} \tag{44}$$

This function is convex when its hessian is positive semidefinite, which is satisfied when $\frac{2}{x_2} \geq 0$, $\frac{2x_1^2}{x_2^3} \geq 0$ and $\frac{4x_1^2}{x_2^4} - \frac{4x_1^2}{x_2^4} \geq 0$. These are always true and therefore f is convex.

2.8 Exercise 3.18

2.9 Exercise 3.24

2.10 Exercise 3.36