# My Solutions for Exercises of Deep Learning Fundamentals by Bishop

Andres Espinosa

May 10, 2024

## Contents

# 1 The Deep Learning Revolution

## 1.1 No Exercises

# 2 Probabilities

### Exercise 2.1

Bayes rule

$$P[C = 1|T = 1] = \frac{P[T = 1|C = 1] * P[C = 1]}{P[T = 1|C = 1] * P[C = 1] + P[T = 1|C = 0] * P[C = 0]} \tag{1}$$

$$P[C = 1|T = 1] = \frac{0.90 * 0.001}{0.90 * 0.001 + 0.03 * 0.999} = 0.0292 \tag{2}$$

Given that the test result was positive, there is a 2.92% chance that you have cancer.

## 2.1 Exercise 2.2

Not attempted

### Exercise 2.3

$$p(\mathbf{y}) = \int p_{\mathbf{u},\mathbf{v}}\,(\mathbf{u}, \mathbf{y} - \mathbf{u})d\mathbf{u} \tag{3}$$

$$= \int p_{\mathbf{u}}(\mathbf{u})p_{\mathbf{v}}(\mathbf{y} - \mathbf{u})d\mathbf{u} \tag{4}$$

## 2.2 Exercise 2.4

Not attempted

### Exercise 2.5

Exponential:

$$p(x|\lambda) = \lambda e^{-\lambda x} \tag{5}$$

Laplace:

$$p(x|\mu, \gamma) = \frac{1}{2\gamma} e^{-\frac{|x-\mu|}{\gamma}} \tag{6}$$

Verifying that the exponential distribution is normalized:

$$p(x|\lambda) = \lambda e^{-\lambda x} \tag{7}$$

$$\int_0^\infty \lambda e^{-\lambda x} = -e^{-\lambda x}\Big|_0^\infty = \frac{1}{e^\infty} + \frac{1}{e^0} \tag{8}$$

$$= 1 \tag{9}$$

Verifying the laplace distribution:

$$p(x|\mu, \gamma) = \frac{1}{2\gamma} e^{-\frac{|x-\mu|}{\gamma}} \tag{10}$$

$$\begin{cases} \frac{1}{2\gamma} e^{-\frac{x-\mu}{\gamma}} & \text{if } x \geq \mu \\ \frac{1}{2\gamma} e^{-\frac{-x+\mu}{\gamma}} & \text{if } x < \mu \end{cases} \tag{11}$$

$$\int_{\mu}^{\infty} \frac{1}{2\gamma} e^{-\frac{x-\mu}{\gamma}} = \frac{1}{2} e^{-\frac{x-\mu}{\gamma}} \Big|_{\mu}^{\infty} = -\frac{1}{2}(e^{-\infty} - e^{0}) \tag{12}$$

$$= \frac{1}{2} \tag{13}$$

$$\int_{-\infty}^{\mu} \frac{1}{2\gamma} e^{-\frac{-x+\mu}{\gamma}} = \frac{1}{2} e^{-\frac{-x+\mu}{\gamma}} \Big|_{-\infty}^{\mu} = \frac{1}{2}(e^{0} - e^{-\infty}) \tag{14}$$

$$= \frac{1}{2} \tag{15}$$

$$\frac{1}{2} + \frac{1}{2} = 1 \tag{16}$$

## 2.3 Exercise 2.6

Not attempted

## 2.4 Exercise 2.7

$$P(x|D) = \frac{1}{N} \sum_{n=1}^{N} \delta(x - x_n) \tag{17}$$

$$E[f] = \int p(x) f(x) dx \tag{18}$$

$$\text{Substituting:} \tag{19}$$

$$E[f] = \int \frac{1}{N} \sum_{n=1}^{N} \delta(x - x_n) f(x) dx \tag{20}$$

$$E[f] = \frac{1}{N} \sum_{n=1}^{N} \int_{x_n-\varepsilon}^{x_n+\varepsilon} \delta(x - x_n) f(x) dx \tag{21}$$

$$E[f] = \frac{1}{N} \sum_{n=1}^{N} f(x_n) \int_{x_n-\varepsilon}^{x_n+\varepsilon} \delta(x - x_n) dx \tag{22}$$

$$E[f] = \frac{1}{N} \sum_{n=1}^{N} f(x_n) \tag{23}$$

## 2.5 Exercise 2.8

Not attempted

## 2.6 Exercise 2.9

$$cov[X, y] = E_{x,y}[xy] - E[x]E[y] \tag{24}$$

If x and y are independent, the joint distribution is equal to the product of the marginals. $p(x, y) = p(x)p(y)$. If $E_{x,y}[xy] = E[x]E[y]$, then the covariance will be zero.

## 2.7 Exercise 2.10

Not attempted

### 2.8 Exercise 2.11

Proving $E[x] = E_y[E_x[x|y]]$:

$$E_x[x|y] = \int p(x|y)xdx \tag{25}$$

$$\text{Substituting:} \tag{26}$$

$$E[x] = E_y[\int p(x|y)xdx] \tag{27}$$

$$E[x] = \int E_y[p(x|y)]xdx \tag{28}$$

$$E[x] = \int\int p(x|y)xp(y)dxdy \tag{29}$$

$$E[x] = \int\int \frac{p(x,y)}{p(y)}xp(y)dxdy \tag{30}$$

$$E[x] = \int\int p(x,y)xdxdy \tag{31}$$

$$E[x] = E[x] \tag{32}$$

### 2.9 Exercise 2.12

Not attempted

### 2.10 Exercise 2.13

$$E[x] = \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{1}{2\sigma^2}(x-\mu)^2}dx \tag{33}$$

$$\text{Change of variables } z = \frac{x-\mu}{\sigma}, \sigma dz = dx \tag{34}$$

$$E[x] = \int_{-\infty}^{\infty} \sigma\frac{\sigma z+\mu}{\sqrt{2\pi\sigma^2}}e^{\frac{1}{2}z^2}dz \tag{35}$$

$$E[x] = \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ze^{\frac{1}{2}z^2} + \frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{1}{2}z^2} \tag{36}$$

$$E[x] = \frac{\sigma}{\sqrt{2\pi}} * 0 + \frac{\mu}{\sqrt{2\pi}} * \sqrt{2\pi} \tag{37}$$

$$E[x] = \mu \tag{38}$$

### 2.11 Exercise 2.14

Not attempted

## 2.12 Exercise 2.15

Solving for $\mu_{ml}$:

$$\log p(x|\mu,\sigma^2) = \frac{-1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{N}{2}\log\sigma^2 - \frac{N}{2}\log 2\pi \tag{39}$$

$$\frac{d}{d\mu} logp(x|\mu,\sigma^2) = \frac{1}{2\sigma^2}\sum_{n=1}^{N}2(x_n-\mu) \tag{40}$$

$$0 = \frac{1}{\sigma^2}\sum_{n=1}^{N}(x_n-\mu) \tag{41}$$

$$0 = \sum_{n=1}^{N}x_n - \sum_{n=1}^{N}\mu \tag{42}$$

$$N\mu = \sum_{n=1}^{N}x_n \tag{43}$$

$$\mu_{ml} = \frac{1}{n}\sum_{n=1}^{N}x_n \tag{44}$$

Solving for $\sigma_{ml}$:

$$\log p(x|\mu,\sigma^2) = \frac{-1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{N}{2}\log\sigma^2 - \frac{N}{2}\log 2\pi \tag{45}$$

$$\frac{d}{d\sigma^2} logp(x|\mu,\sigma^2) = \frac{1}{2\sigma^4}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{N}{2\sigma^2} \tag{46}$$

$$\frac{N}{2\sigma^2} = \frac{1}{2\sigma^4}\sum_{n=1}^{N}(x_n-\mu)^2 \tag{47}$$

$$\sigma_{ml}^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n-\mu_{ml})^2 \tag{48}$$

## 2.13 Exercise 2.16

not attempted

## 2.14 Exercise 2.17

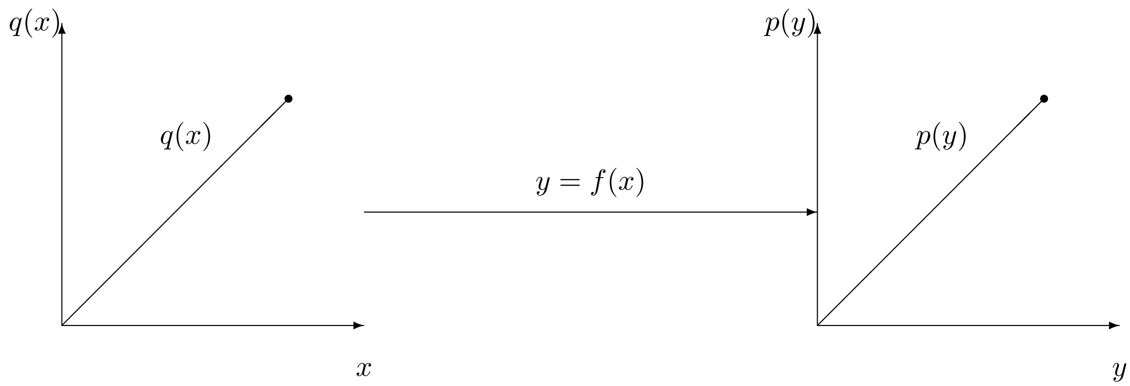
Finding expectation of $\hat{\sigma}^2$

$$
\begin{aligned}
E[\hat{\sigma}^2] &= E[\frac{1}{N}\sum_{n=1}^{N}(x_n-\mu)^2] && (49)\\
&= \frac{1}{N}\sum_{n=1}^{N}E[x_n^2-2x_n\mu+\mu^2] && (50)\\
&= \frac{1}{N}\sum_{n=1}^{N}E[x_n^2]-E[2x_n\mu]+E[\mu^2] && (51)\\
&= \frac{1}{N}\sum_{n=1}^{N}E[x_n^2]-2E[x_n]E[x_n]+E[\mu^2] && (52)\\
&= \frac{1}{N}\sum_{n=1}^{N}E[x_n^2]-E[x_n]^2 && (53)\\
&= \frac{1}{N}\sum_{n=1}^{N}\mu^2+\sigma^2-\mu^2 && (54)\\
&= \sigma^2 && (55)
\end{aligned}
$$

### 2.15 Exercise 2.18

Not attempted

### 2.16 Exercise 2.19




### 2.17 Exercise 2.20

Not attempted

## 2.18 Exercise 2.21

Showing $h(p^2) = 2h(p)$:

$$
\begin{aligned}
h(p) &= h(p(x_1)) + h(p(x_2)) + \cdots + h(p(x_n)) && (56)\\
h(p^2) &= h(x_1^2) + h(x_2^2) + \cdots + h(x_n^2) && (57)\\
&\because h(x) = -\log_2 p(x), && (58)\\
h(p^2) &= 2h(x_1) + 2h(x_2) + \cdots + 2h(x_n) && (59)\\
h(p^2) &= 2h(p) && (60)
\end{aligned}
$$

This can be applied to any exponent which inclues any choice of $n$ integer or $\frac{n}{m}$ positive rational number.

$$
\begin{aligned}
h(p^x) &= xh(p) \; \forall \; Q^+ && (61)\\
&\therefore && (62)\\
h(p) &\propto lnp && (63)
\end{aligned}
$$

## 2.19 Exercise 2.22

Not attempted

## 2.20 Exercise 2.23

Not attempted

## 2.21 Exercise 2.24

Not attempted

## 2.22 Exercise 2.25

Not attempted

## 2.23 Exercise 2.26

Not attempted

## 2.24 Exercise 2.27

Not attempted

## 2.25 Exercise 2.28

Not attempted

## 2.26 Exercise 2.29

Not attempted

### 2.27 Exercise 2.30

Not attempted

### 2.28 Exercise 2.31

Not attempted

### 2.29 Exercise 2.32

Not attempted

### 2.30 Exercise 2.33

Not attempted

### 2.31 Exercise 2.34

Not attempted

### 2.32 Exercise 2.35

Not attempted

### 2.33 Exercise 2.36

Not attempted

### 2.34 Exercise 2.37

Not attempted

### 2.35 Exercise 2.38

Not attempted

### 2.36 Exercise 2.39

Not attempted

### 2.37 Exercise 2.40

Not attempted

### 2.38 Exercise 2.41

Not attempted

# 3 Standard Distributions

## 3.1 Exercise 3.1

Proving $\sum_{x=0}^{1} p(x|\mu) = 1$:

$$Bern(x|\mu) = \mu^x(1-\mu)^{1-x} \tag{64}$$
$$\sum_{x=0}^{1} p(x|\mu) = \mu^0(1-\mu)^1 + \mu^1(1-\mu)^0 \tag{65}$$
$$= 1 - \mu + \mu \tag{66}$$
$$= 1 \tag{67}$$

Proving $E[x] = \mu$:

$$E[x] = \sum_{x=0}^{1} p(x|\mu) * x \tag{68}$$
$$E[x] = 0 * \mu^0(1-\mu)^1 + 1 * \mu^1(1-\mu)^0 \tag{69}$$
$$= 0 * (1-\mu) + 1 * \mu \tag{70}$$
$$= \mu \tag{71}$$

Proving $var[x] = \mu(1-\mu)$:

$$Var[x] = \sum_i (x_i - \mu)^2 p(x_i) \tag{72}$$
$$= (0-\mu)^2(1-\mu) + (1-\mu)^2\mu \tag{73}$$
$$= \mu^2(1-\mu) + (1-\mu)^2\mu \tag{74}$$
$$= \mu(1-\mu)(\mu + (1-\mu)) \tag{75}$$
$$= \mu(1-\mu)(1) \tag{76}$$
$$= \mu(1-\mu) \tag{77}$$

Solving for entropy of the bernoulli distribution proves simple as there are two probabilities, $p_1 = \mu, p_2 = 1 - \mu$. Therefore the entropy must be:

$$H[x] = -\mu ln\mu - (1-\mu)ln(1-\mu) \tag{78}$$

## 3.2 Exercise 3.2

Not attempted

### 3.3 Exercise 3.3

Normalizing the binomial distribution $\binom{N}{m}\mu^m(1-\mu)^{N-m}$:

$$\text{First step is proving } \binom{N}{m} + \binom{N}{m-1} = \binom{N+1}{m}: \tag{79}$$

$$\binom{N}{m} + \binom{N}{m-1} = \frac{N!}{(N-m)!m!} + \frac{N!}{(N-m+1)!(m-1)!} \tag{80}$$

$$= \frac{N!}{(N-m)!m!} * \frac{N-m+1}{N-m+1} + \frac{N!}{(N-m+1)!(m-1)!} * \frac{m}{m} \tag{81}$$

$$= \frac{N!(N+1-m)}{(N+1-m)!m!} + \frac{N!m}{(N-m+1)!m!} \tag{82}$$

$$= \frac{(N+1)! - N!m + N!m}{(N+1-m)!m!} \tag{83}$$

$$= \frac{(N+1)!}{(N+1-m)!m!} \tag{84}$$

$$= \binom{N+1}{m} \tag{85}$$

With this information we would like to prove that $(1+x)^N = \sum_{m=0}^{N}\binom{N}{m}x^m$
N=2:

$$\binom{2}{0}x^0 + \binom{2}{1}x^1 + \binom{2}{2}x^2 = 1 + 2x + x^2 = (1+x)^2 \tag{86}$$

N=3:

$$\binom{3}{0}x^0 + \binom{3}{1}x^1 + \binom{3}{2}x^2\binom{3}{3}x^3 = 1 + 3x + 3x^2 + x^3 = (1+x)^3 \tag{87}$$

N=4

$$\binom{4}{0}x^0 + \binom{4}{1}x^1 + \binom{4}{2}x^2\binom{4}{3}x^3 + \binom{4}{4}x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 = (1+x)^4 \tag{88}$$

By induction: $\forall x \in \mathbb{R}$ and $N \in \mathbb{N}$

$$(1+x)^N = \sum_{m=0}^{N}\binom{N}{m}x^m \tag{89}$$

Normalizing the binomial distribution:

$$\sum_{m=0}^{N} \binom{N}{m} \mu^m (1-\mu)^{N-m} = 1 \tag{90}$$

$$= (1-\mu)^N \sum_{m=0}^{N} \binom{N}{m} \mu^m (1-\mu)^{-m} \tag{91}$$

$$= (1-\mu)^N \sum_{m=0}^{N} \binom{N}{m} (\frac{\mu}{1-\mu})^m \tag{92}$$

$$= (1-\mu)^N (1 + \frac{\mu}{1-\mu})^N \tag{93}$$

$$= ((1-\mu)(1 + \frac{\mu}{1-\mu}))^N \tag{94}$$

$$= (1 + \frac{\mu}{1-\mu} - \mu - \frac{\mu^2}{1-\mu}) \tag{95}$$

$$= ((1-\mu) + \frac{\mu}{1-\mu}(1-\mu))^N \tag{96}$$

$$= (1-\mu+\mu)^N \tag{97}$$

$$= 1 \tag{98}$$

### 3.4 Exercise 3.4

Not attempted

### 3.5 Exercise 3.5

Finding the mode of the multivariate gaussian. The mode of a distribution is the same as the maximum of the probability density function. For the multivariate gaussian, this is:

$$\mathcal{N}(\mathbf{x}|\mu, \mathbf{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\mathbf{\Sigma}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)} \tag{99}$$

$$\frac{d}{d\mathbf{x}} \mathcal{N}(\mathbf{x}|\mu, \mathbf{\Sigma}) \text{ ignoring normalization constant for brevity} \tag{100}$$

$$0 = -\frac{1}{2} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)} * \frac{d}{d\mathbf{x}} (\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu) \tag{101}$$

$$0 = -\frac{1}{2} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)} * 2\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu) \tag{102}$$

$$\mathbf{x} = \mu \tag{103}$$

### 3.6 Exercise 3.6

Not attempted

### 3.7 Exercise 3.7

Finding Kullback-Leibler divergence between $q(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\mu_{\mathbf{q}}, \mathbf{\Sigma_q})$ and $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\mu_{\mathbf{p}}, \mathbf{\Sigma_p})$

$$KL(p||q) = -\int p(x) ln \frac{q(x)}{p(x)} dx \tag{104}$$

$$KL(q(x)||p(x)) = -\int \mathcal{N}(\mathbf{x}|\mu_{\mathbf{q}}, \mathbf{\Sigma_q}) ln \frac{\mathcal{N}(\mathbf{x}|\mu_{\mathbf{p}}, \mathbf{\Sigma_p})}{\mathcal{N}(\mathbf{x}|\mu_{\mathbf{q}}, \mathbf{\Sigma_q})} \tag{105}$$

$$= -\int \mathcal{N}(\mathbf{x}|\mu_{\mathbf{q}}, \mathbf{\Sigma_q})(ln\mathcal{N}(\mathbf{x}|\mu_{\mathbf{p}}, \mathbf{\Sigma_p}) - ln\mathcal{N}(\mathbf{x}|\mu_{\mathbf{q}}, \mathbf{\Sigma_q})) \tag{106}$$

$$\text{definition: } ln\mathcal{N}(\mathbf{x}|\mu_{\mathbf{p}}, \mathbf{\Sigma_p}) = -\frac{D}{2}ln(2\pi) - \frac{1}{2}ln|\mathbf{\Sigma_p}| - \frac{1}{2}(\mathbf{x} - \mu_{\mathbf{p}})^T \mathbf{\Sigma_p}^{-1}(\mathbf{x} - \mu_{\mathbf{p}}) \tag{107}$$

$$= -\frac{1}{2}\int \mathcal{N}(\mathbf{x}|\mu_{\mathbf{q}}, \mathbf{\Sigma_q})(-ln|\mathbf{\Sigma_p}| + ln|\mathbf{\Sigma_q}| - (\mathbf{x} - \mu_{\mathbf{p}})^T \mathbf{\Sigma_p}^{-1}(\mathbf{x} - \mu_{\mathbf{p}}) + (\mathbf{x} - \mu_{\mathbf{q}})^T \mathbf{\Sigma_q}^{-1}(\mathbf{x} - \mu_{\mathbf{q}})) \tag{108}$$

$$\text{Splitting integral into three} \tag{109}$$

$$\text{First: } -\frac{1}{2}ln\frac{|\mathbf{\Sigma_q}|}{|\mathbf{\Sigma_p}|}\int \frac{1}{(2\pi)^{D/2}|\mathbf{\Sigma_q}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{q}})^T \mathbf{\Sigma_q}^{-1}(\mathbf{x}-\mu_{\mathbf{q}})} \tag{110}$$

$$\text{Second: } \frac{1}{2}\int \frac{1}{(2\pi)^{D/2}|\mathbf{\Sigma_q}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{q}})^T \mathbf{\Sigma_q}^{-1}(\mathbf{x}-\mu_{\mathbf{q}})}(\mathbf{x} - \mu_{\mathbf{p}})^T \mathbf{\Sigma_p}^{-1}(\mathbf{x} - \mu_{\mathbf{p}}) \tag{111}$$

$$\text{Third: } -\frac{1}{2}\int \frac{1}{(2\pi)^{D/2}|\mathbf{\Sigma_q}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{q}})^T \mathbf{\Sigma_q}^{-1}(\mathbf{x}-\mu_{\mathbf{q}})}(\mathbf{x} - \mu_{\mathbf{p}})^T \mathbf{\Sigma_p}^{-1}(\mathbf{x} - \mu_{\mathbf{p}}) \tag{112}$$

$$\text{Evaluating first integral: } -\frac{1}{2}ln\frac{|\mathbf{\Sigma_q}|}{|\mathbf{\Sigma_p}|} * 1 \tag{113}$$

$$\text{Evaluating second integral: } \frac{1}{2}(\mu_{\mathbf{p}} - \mu_{\mathbf{q}})^T \mathbf{\Sigma_p}^{-1}(\mu_{\mathbf{p}} - \mu_{\mathbf{q}}) \tag{114}$$

$$\text{Evaluating third integral: } -\frac{1}{2}(\mu_{\mathbf{p}} - \mu_{\mathbf{q}})^T \mathbf{\Sigma_p}^{-1}(\mu_{\mathbf{p}} - \mu_{\mathbf{q}}) + \frac{1}{2}Tr(\mathbf{\Sigma_p^{-1}}\mathbf{\Sigma_q}) \tag{115}$$

I am not sure about this one it is quite hard

### 3.8 Exercise 3.8

Not attempted

### 3.9 Exercise 3.9

Not attempted

### 3.10 Exercise 3.10

Not attempted

### 3.11 Exercise 3.11

Not attempted

### 3.12 Exercise 3.12

Not attempted

### 3.13 Exercise 3.13

Not attempted

### 3.14 Exercise 3.14

Not attempted

### 3.15 Exercise 3.15

Not attempted

### 3.16 Exercise 3.16

Not attempted

### 3.17 Exercise 3.17

Not attempted

### 3.18 Exercise 3.18

Not attempted

### 3.19 Exercise 3.19

Not attempted

### 3.20 Exercise 3.20

Not attempted

### 3.21 Exercise 3.21

Not attempted

### 3.22 Exercise 3.22

Not attempted

### 3.23 Exercise 3.23

Not attempted

### 3.24 Exercise 3.24

Not attempted

### 3.25 Exercise 3.25

Not attempted

### 3.26 Exercise 3.26

Not attempted

### 3.27 Exercise 3.27

Not attempted

### 3.28 Exercise 3.28

Not attempted

### 3.29 Exercise 3.29

Not attempted

### 3.30 Exercise 3.30

Not attempted

### 3.31 Exercise 3.31

Not attempted

### 3.32 Exercise 3.32

Not attempted

### 3.33 Exercise 3.33

Not attempted

### 3.34 Exercise 3.34

Not attempted

### 3.35 Exercise 3.35

Not attempted

### 3.36 Exercise 3.36

Not attempted

### 3.37 Exercise 3.37

Deriving a maximimum likelihood estimator for a histogram-like density model. Space $\mathbf{x}$ is divided into fixed regions with density $p(\mathbf{x}) = h_i \ \forall \ i$ regions. Volume $i$ is denoted $\Delta_i$. $N$ total observations such that $n_i$ observations are in region $i$.

$$p(x_i) = \frac{n_i}{N\Delta_i} \tag{116}$$

### 3.38 Exercise 3.38

Not attempted